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ON AN ELASTO-DYNAMIC EVOLUTION EQUATION WITH NON DEAD LOAD AND FRICTION

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Abstract. In this paper, we are interested in the dynamic evolution of an elastic body, acted by resistance forces depending also on the displacements. We put the mechanical problem into an abstract functional framework, involving a second order nonlinear evolution equation with initial conditions. After specifying convenient hypotheses on the data, we prove an existence and uniqueness result. The proof is based on Faedo-Galerkin method.

Keywords: evolution equation, existence and uniqueness, Faedo-Galerkin method, friction, elasticity, dynamic process

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1. Introduction

Numerous problems in physics such as the perturbation of pressure of the gas in acoustics, the variation of potential in electromagnetism, the vibration of rods and beams in elasticity and viscoelasticity, or the motion of particles in quantum relativity are described by the propagation of waves and lead to second order evolution equations in time of the form

$$\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) + B\dot{\boldsymbol{u}}(t) = \boldsymbol{f}(t).$$

This explains the importance to study such problems and the literature in this field is rather extensive, see e.g. [7], [18], [19], [20]. In these papers, the operator B is nonlinear in general, and the operator A attached to the solution is assumed to be linear. Important pioneering works and abstract results to extend the linearity to nonlinear results have been obtained, and can be found in [1], [3], [4], [11], [12], [17]. In particular in [1], semigroup and monotone operators theory have been used.

In the present work, we are interested in the dynamic evolution of an elastic body. The new feature here is that the body motion is driven by forces which depend also on the displacements. This leads to a new mathematical model involving second-order nonlinear evolution equation of the form

$$\ddot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) + C\boldsymbol{u}(t) + B\dot{\boldsymbol{u}}(t) = \boldsymbol{f}(t) + \boldsymbol{g} \quad \text{a.e. on } (0,T),$$

with the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0.$$

The main novelty concerns then different assumptions on the second member $\boldsymbol{g}$, and on the operator C which represents a nonlinear perturbation of A, in order to obtain an existence and uniqueness result.

The paper is organized as follows. In the next Section 2, we set the problem in an abstract framework of evolution triple, where we specify assumptions on the operators and spaces. Then we state an existence and uniqueness result, Theorem 1. Similar results in various framework are known and can be found in the literature mentioned below. For the convenience of the reader, and following [11], the proof is presented in Section 3; it is based on Galerkin approximations, hemi-continuity and coercivity. Finally in Section 4, we show how to apply the abstract result to the existence and uniqueness of weak solution to the original contact problem, Theorem 2.

2. Statement of the problem

Let V and H be two separable Hilbert spaces. We denote in the sequel by V' the dual space of V. Identifying H with its own dual, we assume an evolution triple

$$V \subset H \subset V',$$

where the inclusions are dense and continuous. We use the notation $\langle\cdot,\cdot\rangle_{V'\times V}$ to represent the duality pairing between V' and V. Then we have

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{V'\times V} = (\boldsymbol{u}, \boldsymbol{v})_H \quad \forall\, \boldsymbol{u} \in H, \ \ \boldsymbol{v} \in V. \tag{2.1}$$

We make the following assumptions.

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in H; \tag{2.2}$$

$$\boldsymbol{f} \in L^2(0,T;H); \tag{2.3}$$

$$\boldsymbol{g} \in V' \ \text{ is independent of time.} \tag{2.4}$$

Let

$$\text{(2.5)} \qquad \begin{aligned} &A\colon\ V \to V' \ \text{ be linear, continuous, symmetric} \\ &\text{and satisfying the following coercivity condition:} \\ &\langle A\boldsymbol{u}, \boldsymbol{u}\rangle_{V'\times V} \geqslant c_A \|\boldsymbol{u}\|_V^2, \quad \forall\, \boldsymbol{u} \in V, \end{aligned}$$

for some constant $c_A > 0$.

Consider the perturbation operator

$$\text{(2.6)} \qquad \begin{gathered} C\colon\ V \to H, \\ \|C\boldsymbol{u} - C\boldsymbol{v}\|_H \leqslant \theta(\|\boldsymbol{u}\|_V, \|\boldsymbol{v}\|_V)\, \|\boldsymbol{u} - \boldsymbol{v}\|_V, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \end{gathered}$$

where $\theta\colon\ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ maps bounded subsets of $\mathbb{R}^+ \times \mathbb{R}^+$ into bounded subsets of $\mathbb{R}^+$. We remark that the last property implies that

$$\boldsymbol{u} \in L^\infty(0,T;V) \Longrightarrow C\boldsymbol{u} \in L^\infty(0,T;H),$$

where

$$(C\boldsymbol{u})(t) = C(\boldsymbol{u}(t)), \quad \text{a.e. } t \in (0,T).$$

Then we suppose that for all sequences $\boldsymbol{v}_n \in V$ and $\boldsymbol{v} \in V$,

$$\text{(2.7)} \qquad \begin{gathered} \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \quad \text{weakly in } V \\ \Longrightarrow C\boldsymbol{v}_n \rightharpoonup C\boldsymbol{v} \quad \text{weakly in } H \ \text{ for some subsequence.} \end{gathered}$$

Moreover, we suppose that

(2.8) There exists a differentiable function $G\colon\ V \to \mathbb{R}$ satisfying

(i) For all sequences $\boldsymbol{v}_n \in V$ and $\boldsymbol{v} \in V$,

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \ \text{ weakly in } \ V \Longrightarrow \liminf_{n\to\infty} G(\boldsymbol{v}_n) \geqslant G(\boldsymbol{v});$$

(ii) $\exists\, c_G, d_G \in \mathbb{R}, \quad \forall\, \boldsymbol{v} \in V, \quad G(\boldsymbol{v}) \geqslant c_G \|\boldsymbol{v}\|_V + d_G;$

(iii) $(C\boldsymbol{u}(t), \boldsymbol{u}'(t))_H = \dfrac{\mathrm{d}}{\mathrm{dt}} G(\boldsymbol{u}(t)), \quad \forall \boldsymbol{u} \in W^{1,2}(0,T;V), \ \text{a.e. } t \in (0,T).$

Let $B\colon\ H \to H$ satisfy

$$\text{(2.9)} \qquad B \ \text{ is monotone;}$$

$$\text{(2.10)} \qquad B \ \text{ is hemicontinuous,}$$

i.e. for all $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ in H,

$$(B(\boldsymbol{u}+\lambda\boldsymbol{v}),\boldsymbol{w})_H \longrightarrow (B\boldsymbol{u},\boldsymbol{w})_H, \quad \lambda\to 0;$$

$$\boldsymbol{v}\in L^2(0,T;H) \Longrightarrow B\boldsymbol{v}\in L^2(0,T;H), \tag{2.11}$$

where

$$(B\boldsymbol{v})(t) = B(\boldsymbol{v}(t)), \quad \text{a.e. } t\in(0,T);$$

$$B\colon\ L^2(0,T;H)\to L^2(0,T;H) \ \text{ is bounded,} \tag{2.12}$$

i.e. B maps bounded subsets in $L^2(0,T;H)$ into bounded subsets in $L^2(0,T;H)$.

For example, if B is continuous and $\|B\boldsymbol{v}\|_H \leqslant c(1+\|\boldsymbol{v}\|_H)$, $\forall\boldsymbol{v}\in H$, then the last three properties are satisfied.

Let us now consider the following abstract second order evolution problem.

Problem P_1. Find $\boldsymbol{u}\colon\ [0,T]\to V$ such that

$$\boldsymbol{u}''(t)+A\boldsymbol{u}(t)+C\boldsymbol{u}(t)+B\boldsymbol{u}'(t) = \boldsymbol{f}(t)+\boldsymbol{g} \quad \text{a.e. on } (0,T), \tag{2.13}$$

with the initial conditions

$$\boldsymbol{u}(0)=\boldsymbol{u}_0, \quad \boldsymbol{u}'(0)=\boldsymbol{v}_0, \tag{2.14}$$

and satisfying the regularity

$$\begin{gathered}\boldsymbol{u}\in L^\infty(0,T;V), \quad \boldsymbol{u}'\in L^\infty(0,T;H),\\ \frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}\in L^\infty(0,T;V').\end{gathered} \tag{2.15}$$

Our main result concerning the problem P_1 is stated as follows and is proved in the next section.

Theorem 1. *Assume that* (2.2)–(2.12) *hold, then there exists a unique solution* $\boldsymbol{u}$ *satisfying* (2.13)–(2.15). *Moreover,*

$$\boldsymbol{u}\colon\ [0,T]\to V \quad \textit{and} \quad \boldsymbol{u}'\colon\ [0,T]\to H \ \textit{ are continuous.} \tag{2.16}$$

3. Proof of Theorem 1

We are going to give the proof, first the uniqueness part then the existence part.

Uniqueness. We use the following well known lemma (the energy inequality).

Lemma 1. *Let A satisfying (2.5), $\boldsymbol{w} \in L^\infty(0,T;V)$, $\boldsymbol{w}' \in L^\infty(0,T;H)$, and $\boldsymbol{h} \in L^2(0,T;H)$ satisfy*

$$\boldsymbol{w}'' + A\boldsymbol{w} = \boldsymbol{h}, \quad \boldsymbol{w}(0) = 0, \ \ \boldsymbol{w}'(0) = 0. \tag{3.1}$$

Then we have

$$(A\boldsymbol{w}(t), \boldsymbol{w}(t))_{V'\times V} + \|\boldsymbol{w}'(t)\|_H^2 \leqslant 2\int_0^t (\boldsymbol{h}(s), \boldsymbol{w}'(s))_H \, \mathrm{d}s \quad \text{a.e. } t \in (0,T).$$

P r o o f. The main ideas of the proof can be found for example in [11, p. 22] for a slightly different framework. □

Now let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two solutions satisfying (2.13), (2.14) and (2.15). Put $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ and $\boldsymbol{h} = -(C\boldsymbol{u} - C\boldsymbol{v} + B\boldsymbol{u}' - B\boldsymbol{v}')$. Then $\boldsymbol{w}$ satisfies the hypotheses in Lemma 1, thus

$$(A\boldsymbol{w}(t), \boldsymbol{w}(t))_{V'\times V} + \|\boldsymbol{w}'(t)\|_H^2 \leqslant 2\int_0^t (\boldsymbol{h}(s), \boldsymbol{w}'(s))_H \, \mathrm{d}s \quad \text{a.e. } t \in (0,T).$$

By (2.5), (2.6) and (2.9), we deduce that for some constant c and a.e. $t \in (0,T)$,

$$\begin{aligned}\|\boldsymbol{w}(t)\|_V^2 + \|\boldsymbol{w}'(t)\|_H^2 &\leqslant c\int_0^t \|\boldsymbol{w}(s)\|_V \|\boldsymbol{w}'(s)\|_H \, \mathrm{d}s \\ &\leqslant \frac{c}{2}\int_0^t (\|\boldsymbol{w}(s)\|_V^2 + \|\boldsymbol{w}'(s)\|_H^2) \, \mathrm{d}s.\end{aligned}$$

We conclude by Gronwall's lemma that $\boldsymbol{w} = 0$, which completes the uniqueness part in Theorem 1.

Existence. We follow again here the main ideas in [11, p. 8 and p. 38].

Let $\{\boldsymbol{e}_n,\, n \geqslant 1\}$ be a Hilbert basis of V. Using the Theorem of Caratheodory (see e.g. [20, p. 1044), we can define the sequence

$$\boldsymbol{u}_n(t) = \sum_{j=1}^n c_{nj}(t)\boldsymbol{e}_j, \quad c_{nj} \in W^{2,2}(0,T), \quad \forall\, t \in [0,T], \ \forall n \geqslant 1$$

which satisfies

$$\begin{aligned}(\boldsymbol{u}_n''(t), \boldsymbol{e}_j)_{V'\times V} + (A\,\boldsymbol{u}_n(t), \boldsymbol{e}_j)_{V'\times V} + (C\boldsymbol{u}_n(t), \boldsymbol{e}_j)_{V'\times V} + (B\boldsymbol{u}_n'(t), \boldsymbol{e}_j)_{V'\times V} \\ = (\boldsymbol{f}(t), \boldsymbol{e}_j)_H + (\boldsymbol{g}, \boldsymbol{e}_j)_{V'\times V}, \ \ \forall\, j = 1, \ldots, n, \ \ \forall\, n \geqslant 1,\end{aligned} \tag{3.2}$$

and

$$\text{(3.3)} \qquad \boldsymbol{u}_n(0) := \boldsymbol{u}_{n0} \to \boldsymbol{u}_0 \text{ strongly in } V, \quad \boldsymbol{u}_n'(0) := \boldsymbol{v}_{n0} \to \boldsymbol{v}_0 \text{ strongly in } H.$$

Then we have

$$\text{(3.4)} \quad \begin{aligned} &(\boldsymbol{u}_n''(t), \boldsymbol{w})_{V' \times V} + (A\boldsymbol{u}_n(t), \boldsymbol{w})_{V' \times V} + (C\boldsymbol{u}_n(t), \boldsymbol{w})_{V' \times V} + (B\boldsymbol{u}_n'(t), \boldsymbol{w})_{V' \times V} \\ &\qquad = (\boldsymbol{f}(t), \boldsymbol{w})_H + (\boldsymbol{g}, \boldsymbol{w})_{V' \times V}, \quad \forall \boldsymbol{w} \in \operatorname{span}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n), \ \forall n \geqslant 1. \end{aligned}$$

Putting then $\boldsymbol{w} = \boldsymbol{u}_n'(t)$ in (3.4), we get

$$\begin{aligned} &(\boldsymbol{u}_n'', \boldsymbol{u}_n')_{V' \times V} + (A\boldsymbol{u}_n, \boldsymbol{u}_n')_{V' \times V} + (C\boldsymbol{u}_n, \boldsymbol{u}_n')_{V' \times V} + (B\boldsymbol{u}_n', \boldsymbol{u}_n')_{V' \times V} \\ &\qquad = (\boldsymbol{f}, \boldsymbol{u}_n')_H + (\boldsymbol{g}, \boldsymbol{u}_n')_{V' \times V}, \quad \forall n \geqslant 1. \end{aligned}$$

Now using (2.3), (2.5), (2.8) and (2.11), we integrate the last relation over $(0, t)$, and obtain for all $t \in [0, T]$,

$$\text{(3.5)} \quad \begin{aligned} &\frac{1}{2}\|\boldsymbol{u}_n'\|_H^2 - \frac{1}{2}\|\boldsymbol{v}_{n0}\|_H^2 + \frac{1}{2}(A\boldsymbol{u}_n(t), \boldsymbol{u}_n(t))_{V' \times V} - \frac{1}{2}(A\boldsymbol{u}_{n0}, \boldsymbol{u}_{n0})_{V' \times V} \\ &\qquad + G(\boldsymbol{u}_n(t)) - G(\boldsymbol{u}_{n0}) + \int_0^t (B\boldsymbol{u}_n'(s), \boldsymbol{u}_n'(s))_H \, \mathrm{d}s \\ &\quad = \int_0^t (\boldsymbol{f}(s), \boldsymbol{u}_n'(s))_H \, \mathrm{d}s + (\boldsymbol{g}, \boldsymbol{u}_n(t))_{V' \times V} - (\boldsymbol{g}, \boldsymbol{u}_n(0))_{V' \times V}. \end{aligned}$$

By (2.9), we have

$$\text{(3.6)} \quad \begin{aligned} &\frac{1}{2}\|\boldsymbol{u}_n'\|_H^2 - \frac{1}{2}\|\boldsymbol{v}_{n0}\|_H^2 + \frac{1}{2}(A\boldsymbol{u}_n(t), \boldsymbol{u}_n(t))_{V' \times V} - \frac{1}{2}(A\boldsymbol{u}_{n0}, \boldsymbol{u}_{n0})_{V' \times V} \\ &\qquad + G(\boldsymbol{u}_n(t)) - G(\boldsymbol{u}_{n0}) \\ &\quad \leqslant \int_0^t (\boldsymbol{f}(s), \boldsymbol{u}_n'(s))_H \, \mathrm{d}s - \int_0^t (B(0), \boldsymbol{u}_n'(s))_H \, \mathrm{d}s \\ &\qquad + (\boldsymbol{g}, \boldsymbol{u}_n(t))_{V' \times V} - (\boldsymbol{g}, \boldsymbol{u}_n(0))_{V' \times V}. \end{aligned}$$

Using then

$$(\boldsymbol{f}(s), \boldsymbol{u}_n'(s))_H \leqslant \frac{1}{2}(\|\boldsymbol{f}(s)\|_H^2 + \|\boldsymbol{u}_n'(s)\|_H^2),$$

and

$$-(B(0), \boldsymbol{u}_n'(s))_H \leqslant \frac{1}{2}(\|B(0)\|_H^2 + \|\boldsymbol{u}_n'(s)\|_H^2),$$

and by (2.5), (2.8) and (3.3), we deduce after some algebra that for some constant $c > 0$,

$$\|\boldsymbol{u}_n'(t)\|_H^2 + \|\boldsymbol{u}_n(t)\|_V^2 \leqslant c + c \int_0^t \|\boldsymbol{u}_n'(s)\|_H^2 \,\mathrm{d}s.$$

Thus by Gronwall's inequality,

$$\|\boldsymbol{u}_n'(t)\|_H \leqslant c, \quad \|\boldsymbol{u}_n(t)\|_V \leqslant c, \quad \forall\, t \in [0,T]. \tag{3.7}$$

Then for a subsequence,

$$\begin{aligned} \boldsymbol{u}_n &\rightharpoonup \boldsymbol{u} \quad \text{in } L^\infty(0,T;V) \text{ weakly star,} \\ \boldsymbol{u}_n' &\rightharpoonup \boldsymbol{u}' \quad \text{in } L^\infty(0,T;H) \text{ weakly star.} \end{aligned} \tag{3.8}$$

Now letting $\varphi \in \mathcal{D}(0,T)$, putting $\boldsymbol{w} = \varphi(t)\boldsymbol{e}_j$ in (3.4) and integrating the relation over $(0,T)$, we have for all $n \geqslant 1$,

$$\begin{aligned} &-\int_0^T (\boldsymbol{u}_n'(t), \varphi'(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t + \int_0^T (A\boldsymbol{u}_n(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t \\ &+ \int_0^T (C\boldsymbol{u}_n(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t + \int_0^T (B\boldsymbol{u}_n'(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t \\ &\qquad = \int_0^T (\boldsymbol{f}(t), \varphi(t)\boldsymbol{e}_j)_H \,\mathrm{d}t + \int_0^T (\boldsymbol{g}, \varphi(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t. \end{aligned} \tag{3.9}$$

By (2.5) and (3.8), we have

$$\begin{aligned} -\int_0^T (\boldsymbol{u}_n'(t), \varphi'(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t &= -\int_0^T (\boldsymbol{u}_n'(t), \varphi'(t)\boldsymbol{e}_j)_H \,\mathrm{d}t \\ &\to -\int_0^T (\boldsymbol{u}'(t), \varphi'(t)\boldsymbol{e}_j)_H \,\mathrm{d}t, \\ \int_0^T (A\boldsymbol{u}_n(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t &= \int_0^T (\varphi(t)A\boldsymbol{e}_j, \boldsymbol{u}_n(t))_{V'\times V} \,\mathrm{d}t \\ &\to \int_0^T (A\boldsymbol{u}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t. \end{aligned}$$

Now from (3.7) and (3.8), it can be shown that a.e. $t \in [0,T]$,

$$\boldsymbol{u}_n(t) \rightharpoonup \boldsymbol{u}(t) \text{ weakly in } V, \quad \boldsymbol{u}_n'(t) \rightharpoonup \boldsymbol{u}'(t) \text{ weakly in } H. \tag{3.10}$$

Thus using (2.6), (2.7) and Lebesgue's dominated convergence, we deduce that

$$\begin{aligned} \int_0^T (C\boldsymbol{u}_n(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t &= \int_0^T (C\boldsymbol{u}_n(t), \varphi(t)\boldsymbol{e}_j)_H \,\mathrm{d}t \\ &\to \int_0^T (C\boldsymbol{u}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V} \,\mathrm{d}t. \end{aligned}$$

On the other hand, by (3.7) and (2.12), $(B\boldsymbol{u}_n')$ is bounded in $L^2(0,T;H)$, thus

$$B\boldsymbol{u}_n' \rightharpoonup \boldsymbol{\chi} \quad \text{weakly in } L^2(0,T;H). \tag{3.11}$$

This implies

$$\int_0^T (B\,\boldsymbol{u}_n'(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \to \int_0^T (\boldsymbol{\chi}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t.$$

Now passing to the limit in (3.9) when $n \to +\infty$, we obtain $\forall\, j \geqslant 1$, $\forall\, \varphi \in \mathcal{D}(0,T)$,

$$\begin{aligned}
&-\int_0^T (\boldsymbol{u}'(t), \varphi'(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t + \int_0^T (A\boldsymbol{u}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
&+\int_0^T (C\boldsymbol{u}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t + \int_0^T (\boldsymbol{\chi}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
=&\int_0^T (\boldsymbol{f}(t), \varphi(t)\boldsymbol{e}_j)_H\,\mathrm{d}t + \int_0^T (\boldsymbol{g}, \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t,
\end{aligned}$$

or

$$\begin{aligned}
&\left(-\int_0^T \boldsymbol{u}'(t)\varphi'(t)\,\mathrm{d}t, \boldsymbol{e}_j\right)_{V'\times V} + \left(\int_0^T A\boldsymbol{u}(t)\varphi(t)\,\mathrm{d}t, \boldsymbol{e}_j\right)_{V'\times V} \\
&+\left(\int_0^T C\boldsymbol{u}(t)\varphi(t)\,\mathrm{d}t, \boldsymbol{e}_j\right)_{V'\times V} + \left(\int_0^T \boldsymbol{\chi}(t)\varphi(t)\,\mathrm{d}t, \boldsymbol{e}_j\right)_{V'\times V} \\
=&\left(\int_0^T \boldsymbol{f}(t)\varphi(t)\,\mathrm{d}t, \boldsymbol{e}_j\right)_H + \Big(\int_0^T \varphi(t)\boldsymbol{g}\,\mathrm{d}t, \boldsymbol{e}_j)_{V'\times V}.
\end{aligned}$$

This means that

$$\boldsymbol{u}'' + A\boldsymbol{u} + C\boldsymbol{u} + \boldsymbol{\chi} = \boldsymbol{f} + \boldsymbol{g} \quad \text{in } \mathcal{D}'(0,T;V').$$

Thus

$$\boldsymbol{u}'' \in L^2(0,T;V'),$$

and then

$$\boldsymbol{u}'' + A\boldsymbol{u} + C\boldsymbol{u} + \boldsymbol{\chi} = \boldsymbol{f} + \boldsymbol{g} \quad \text{a.e. in } (0,T). \tag{3.12}$$

Let us verify now the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{and} \quad \boldsymbol{u}'(0) = \boldsymbol{v}_0.$$

From (3.8), we see that

$$\boldsymbol{u}_n(t) \rightharpoonup \boldsymbol{u}(t) \quad \text{weakly in } H, \ \forall\, t \in [0,T].$$

Thus by (3.3) and the continuous imbedding $V \subset H$, we obtain

$$\boldsymbol{u}_n(0) \rightharpoonup \boldsymbol{u}(0) \ \text{ weakly in } H \quad \text{and} \quad \boldsymbol{u}_n(0) \to \boldsymbol{u}_0 \ \text{ strongly in } H,$$

which shows that $\boldsymbol{u}(0) = \boldsymbol{u}_0$.

To continue, let $\varphi \in C^1([0,T])$ be such that $\varphi(T) = 0$, put $\boldsymbol{w} = \varphi(t)\boldsymbol{e}_j$ in (3.4) and integrate the relation over $(0,T)$; this gives for all $n \geqslant 1$,

$$\begin{aligned}
&-(\boldsymbol{u}_n'(0), \varphi(0)\boldsymbol{e}_j)_{V'\times V} - \int_0^T (\boldsymbol{u}_n'(t), \varphi'(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
&+ \int_0^T (A\boldsymbol{u}_n(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t + \int_0^T (C\boldsymbol{u}_n(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
&+ \int_0^T (B\boldsymbol{u}_n'(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
=& \int_0^T (\boldsymbol{f}(t), \varphi(t)\boldsymbol{e}_j)_H\,\mathrm{d}t + \int_0^T (\boldsymbol{g}, \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t.
\end{aligned}$$

Using (2.5), (2.7) and (3.8), and letting $n \to +\infty$, we get

$$\begin{aligned}
&-(\boldsymbol{v}_0, \varphi(0)\boldsymbol{e}_j)_H - \int_0^T (\boldsymbol{u}'(t), \varphi'(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t + \int_0^T (A\boldsymbol{u}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
&+ \int_0^T (C\boldsymbol{u}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t + \int_0^T (B\boldsymbol{u}'(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
=& \int_0^T (\boldsymbol{f}(t), \varphi(t)\boldsymbol{e}_j)_H\,\mathrm{d}t + \int_0^T (\mathbf{g}, \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t.
\end{aligned}$$

Using now (3.12), multiplying by $\varphi(t)\boldsymbol{e}_j$ and integrating over $(0,T)$, we get

$$\begin{aligned}
&-(\boldsymbol{u}'(0), \varphi(0)\boldsymbol{e}_j)_H - \int_0^T (\boldsymbol{u}'(t), \varphi'(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t + \int_0^T (A\boldsymbol{u}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
&+ \int_0^T (C\boldsymbol{u}(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t + \int_0^T (B\boldsymbol{u}'(t), \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t \\
&\qquad = \int_0^T (\boldsymbol{f}(t), \varphi(t)\boldsymbol{e}_j)_H\,\mathrm{d}t + \int_0^T (\mathbf{g}, \varphi(t)\boldsymbol{e}_j)_{V'\times V}\,\mathrm{d}t.
\end{aligned}$$

Thus for all $\varphi \in C^1([0,T])$ such that $\varphi(T) = 0$, we have

$$(\boldsymbol{u}'(0), \varphi(0)\boldsymbol{e}_j)_H = (\boldsymbol{v}_0, \varphi(0)\boldsymbol{e}_j)_H,$$

which implies clearly that $\boldsymbol{u}'(0) = \boldsymbol{v}_0$. $\square$

The last step which now remains is to show that $\boldsymbol{\chi} = B\boldsymbol{u}'$, then the conclusion in Theorem 1 follows immediately from (3.12). Then by (3.10) and (3.5) we deduce that

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{u}'(t)\|_H^2 &- \frac{1}{2}\|\boldsymbol{v}_0\|_H^2 + \frac{1}{2}(A\boldsymbol{u}(t), \boldsymbol{u}(t))_{V'\times V} - \frac{1}{2}(A\boldsymbol{u}_0, \boldsymbol{u}_0)_{V'\times V} \\ &+ G(\boldsymbol{u}(t)) - G(\boldsymbol{u}_0) + \liminf_{n\to+\infty}\int_0^t (B\boldsymbol{u}_n'(s), \boldsymbol{u}_n'(s))_H \,\mathrm{d}s \\ \leqslant \int_0^t (\boldsymbol{f}(s), \boldsymbol{u}'(s))_H \,\mathrm{d}s &+ (\mathbf{g}, \boldsymbol{u}(t))_{V'\times V} - (\mathbf{g}, \boldsymbol{u}_0)_{V'\times V}.\end{aligned} \tag{3.13}$$

We now use a well known result (the energy inequality), which in fact improves Lemma 1 in the uniqueness part, and is given as follows (see e.g. [14]).

Lemma 2. *Let A satisfy (2.5), $\boldsymbol{w} \in L^\infty(0,T;V)$, $\boldsymbol{w}' \in L^\infty(0,T;H)$, $\boldsymbol{h} \in L^2(0,T;H)$, $\boldsymbol{L} \in V'$, $\boldsymbol{w}_0 \in V$, and $\boldsymbol{w}_1 \in H$ satisfy*

$$\boldsymbol{w}'' + A\boldsymbol{w} = \boldsymbol{h} + \boldsymbol{L}, \quad \boldsymbol{w}(0) = \boldsymbol{w}_0, \;\; \boldsymbol{w}'(0) = \boldsymbol{w}_1. \tag{3.14}$$

Then we have

$$\begin{aligned}\boldsymbol{w}\colon [0,T] \to V \quad &\textit{is continuous,} \\ \boldsymbol{w}'\colon [0,T] \to H \quad &\textit{is continuous,}\end{aligned}$$

and for all $t \in [0,T]$,

$$\begin{aligned}\frac{1}{2}(A\boldsymbol{w}(t), \boldsymbol{w}(t))_{V'\times V} &+ \frac{1}{2}\|\boldsymbol{w}'(t)\|_H^2 \\ = \frac{1}{2}(A\boldsymbol{w}_0, \boldsymbol{w}_0)_{V'\times V} &+ \frac{1}{2}\|\boldsymbol{w}_1\|_H^2 + \int_0^t (\boldsymbol{h}(s), \boldsymbol{w}'(s))_H \,\mathrm{d}s \\ &+ (\boldsymbol{L}, \boldsymbol{w}(t))_{V'\times V} - (\boldsymbol{L}, \boldsymbol{w}_0)_{V'\times V}.\end{aligned}$$

Applying Lemma 2 to the function $\boldsymbol{u}$, we obtain (2.16) and

$$\begin{aligned}\frac{1}{2}(A\boldsymbol{u}(t), \boldsymbol{u}(t))_{V'\times V} &+ \frac{1}{2}\|\boldsymbol{u}'(t)\|_H^2 \\ = \frac{1}{2}(A\boldsymbol{u}_0, \boldsymbol{u}_0)_{V'\times V} &+ \frac{1}{2}\|\boldsymbol{v}_0\|_H^2 \\ &+ \int_0^t (\boldsymbol{f}(s) - \boldsymbol{\chi} - C\boldsymbol{u}(s), \boldsymbol{u}'(s))_H \,\mathrm{d}s \\ &+ (\mathbf{g}, \boldsymbol{u}(t))_{V'\times V} - (\mathbf{g}, \boldsymbol{u}_0)_{V'\times V}, \quad \forall\, t \in [0,T].\end{aligned}$$

From (3.13) and (3.15), we deduce that

$$\liminf_{n\to+\infty}\int_0^T (B\boldsymbol{u}_n'(s),\boldsymbol{u}_n'(s))_{V'\times V}\,\mathrm{d}s \leqslant \int_0^T (\boldsymbol{\chi}(s),\boldsymbol{u}'(s))_H\,\mathrm{d}s \quad \text{a.e. } t\in(0,T).$$

Then by (3.8) and (3.11), we have for all $\boldsymbol{v}\in L^2(0,T;H)$,

$$\begin{aligned} 0 &\leqslant \liminf_{n\to+\infty}\int_0^T (B\boldsymbol{u}_n'(s)-B\boldsymbol{v}(s),\boldsymbol{u}_n'(s)-\boldsymbol{v}(s))_{V'\times V}\,\mathrm{d}s \\ &\leqslant \int_0^T (\boldsymbol{\chi}(s)-B\boldsymbol{v}(s),\boldsymbol{u}'(s)-\boldsymbol{v}(s))_H\,\mathrm{d}s. \end{aligned}$$

Taking now $\boldsymbol{v}=\boldsymbol{u}'-\lambda\boldsymbol{w}$, $\boldsymbol{w}\in L^2(0,T;H)$, $\lambda>0$, we have

$$\int_0^T (\boldsymbol{\chi}(s)-B(\boldsymbol{u}'(s)-\lambda\boldsymbol{w}(s)),\boldsymbol{w}(s))_H\,\mathrm{d}s\geqslant 0.$$

Letting $\lambda\to 0$, we get by (2.10)

$$\int_0^T (\boldsymbol{\chi}(s)-B\boldsymbol{u}'(s),\boldsymbol{w}(s))_H\,\mathrm{d}s\geqslant 0,$$

this for all $\boldsymbol{w}\in L^2(0,T;H)$.

Clearly the last inequality implies that $\boldsymbol{\chi}=B\boldsymbol{u}'$. □

4. Application to contact problem

We now give an application in elastic contact mechanics. Situations of contact between deformable bodies are very common in industry and everyday life. In particular, the study of dynamic contact problems concerns the existence and uniqueness of solutions of second-order evolution equations, and necessitates abstract tools such as maximal monotone operators, fixed points and variational methods. Because of the importance of the contact problems, there exists an extensive literature on various aspects of the subject, including mathematical and numerical analysis, see e.g. [5], [6], [8], [9], [10], [13], [15], [16]. In particular, a dynamic problem with adhesive contact was studied in [5], and unilateral dynamic contact problems for viscoelastic and thermo-viscoelastic bodies were analyzed in [9], [10].

Here, we study the dynamic evolution of an elastic body, which is acted by forces depending also on the displacements. The weak formulation of this mechanical problem leads to an abstract second-order evolution equation that we met in the previous section.

Let us describe now the mechanical contact problem. An elastic body occupies a domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary Γ and unit outer normal vector $\boldsymbol{\nu}$. The boundary is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $\operatorname{meas}(\Gamma_1) > 0$. Let $T > 0$ and $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$. There are volume forces of density $\boldsymbol{f}_0 - \alpha\, j(\boldsymbol{u}) - \beta\, b(\dot{\boldsymbol{u}})$ acting in $\Omega \times (0, T)$, with $\boldsymbol{f}_0$ which is independent of the displacement $\boldsymbol{u}$, i.e. which corresponds to a dead load, whereas $-\alpha\, j(\boldsymbol{u})$ and $-\beta\, b(\dot{\boldsymbol{u}})$ express a resistance to the motion. A surface traction of density $\boldsymbol{f}_2$ independent of time acts on Γ_2. The body may come into contact with a foundation, over the potential contact surface Γ_3. Consider then the following mechanical problem.

P r o b l e m P_2. Find a displacement field $\boldsymbol{u}\colon\ \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$\varrho\, \ddot{\boldsymbol{u}} = \operatorname{Div} \mathcal{A}\varepsilon(\boldsymbol{u}) + \boldsymbol{f}_0 - \alpha\, j(\boldsymbol{u}) - \beta\, b(\dot{\boldsymbol{u}}) \quad \text{in } \Omega \times (0, T), \tag{4.1}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{4.2}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{4.3}$$

$$\sigma_\nu = -k u_\nu, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{T} \quad \text{on } \Gamma_3 \times (0, T), \tag{4.4}$$

$$\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(\cdot, 0) = \boldsymbol{v}_0 \quad \text{in } \Omega. \tag{4.5}$$

Here the dots above a quantity represent derivatives of the quantity with respect to the time variable, $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$, and the summation convention over repeated indices is adopted. We recall that (4.1) represents the dynamic equation of the motion, where $\varrho = \varrho(\boldsymbol{x}) \geqslant 0$ is the mass density, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{1 \leqslant i,\, j \leqslant d}$, with $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ denotes the linearized strain tensor, $\mathcal{A}$ the elastic stress tensor, Div the *divergence* operators, defined by $\operatorname{Div} \boldsymbol{\sigma} = (\partial \sigma_{ij}/\partial x_j)_{1 \leqslant i \leqslant d}$, for any second order symmetric tensors $\boldsymbol{\sigma} = (\sigma_{ij})_{1 \leqslant i,\, j \leqslant d}$ on $\mathbb{R}^d$, $\alpha\, j(\boldsymbol{u})$ represents a spring tension with coefficient $\alpha = \alpha(\boldsymbol{x}) \geqslant 0$, the latter may be for instance proportional to a power of the displacement, and $\beta\, \dot{\boldsymbol{u}}$ is a resistance to the motion proportional to the velocity with coefficient $\beta = \beta(\boldsymbol{x}) \geqslant 0$. The equations (4.2) and (4.3) recall the fixed and the surface boundary conditions. The equation (4.4) represents the contact condition on Γ_3, where the normal stress σ_ν is proportional to the normal displacement u_ν with coefficient $k = k(\boldsymbol{x}) \geqslant 0$, and the tangential traction $\boldsymbol{\sigma}_\tau$ is defined by $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{x})$, which is supposed to be independent of time. Finally in (4.5), $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$ denotes the initial displacement and $\boldsymbol{v}_0 = \boldsymbol{v}_0(\boldsymbol{x})$ the initial velocity field.

In order to give the variational formulation of the problem P_2 and to study the existence and uniqueness of weak solutions, we now introduce different spaces, operators and make assumptions on the data. Consider

$$H = L^2(\Omega)^d, \quad Q = \{\boldsymbol{\sigma} = (\sigma_{ij}),\ 1 \leqslant i, j \leqslant d \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

with the canonical inner product $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$,

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^d \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1\}.$$

In the evolution triple

$$V \subset H \subset V',$$

the spaces H and V are endowed with the following inner product (since $\operatorname{meas}\Gamma_1 > 0$, Korn's inequality holds, see e.g. [16, p. 79]):

$$(\!(\boldsymbol{u},\boldsymbol{v})\!)_H = (\varrho\boldsymbol{u},\boldsymbol{v})_H, \quad (\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q,$$

where we assume

$$\varrho \in L^\infty(\Omega), \quad \varrho(\boldsymbol{x}) \geqslant \varrho_* > 0 \ \text{ a.e. } \ \boldsymbol{x} \in \Omega. \tag{4.6}$$

Consider now $A\colon V \to V'$ defined by

$$(A\,\boldsymbol{u},\boldsymbol{w})_{V'\times V} = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + \int_{\Gamma_3} k u_\nu w_\nu \,\mathrm{d}\Gamma,$$

where the elastic tensor $\mathcal{A} = (a_{ijkh})\colon \Omega \times S_d \to S_d$, with S_d the space of second order symmetric tensors on $\mathbb{R}^d$, satisfies

$$\begin{cases} \text{(a) } a_{ijkh} \in L^\infty(\Omega); \\ \text{(b) } \mathcal{A}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{A}\boldsymbol{\tau} \quad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau} \in S_d, \ \text{ a.e. in } \ \Omega; \\ \text{(c) there exists } \ m_{\mathcal{A}} > 0 \ \text{ such that} \\ \qquad \mathcal{A}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geqslant m_{\mathcal{A}}\|\boldsymbol{\tau}\|^2 \quad \forall\,\boldsymbol{\tau} \in S_d, \ \text{ a.e. in } \ \Omega, \end{cases} \tag{4.7}$$

and

$$k \in L^\infty(\Gamma_3), \quad k \geqslant 0. \tag{4.8}$$

Let us define $B\colon H \to H$ by: for all $\boldsymbol{v} \in H$,

$$B(\boldsymbol{v})(\boldsymbol{x}) = \frac{\beta(\boldsymbol{x})b(\boldsymbol{v}(\boldsymbol{x}))}{\varrho(\boldsymbol{x})}, \quad \text{a.e. } \boldsymbol{x} \in \Omega,$$

with

$$\beta \in L^\infty(\Omega), \quad \beta \geqslant 0, \tag{4.9}$$

and $b\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies for some constant $c_b > 0$,

$$b \text{ is continuous monotone and } |b(\boldsymbol{v})| \leqslant c_b(1 + |\boldsymbol{v}|), \quad \forall\, \boldsymbol{v} \in \mathbb{R}^d. \tag{4.10}$$

Define $C\colon V \to H$ by

$$C(\boldsymbol{v})(\boldsymbol{x}) = \frac{\alpha(\boldsymbol{x})\, j(\boldsymbol{v}(\boldsymbol{x}))}{\varrho(\boldsymbol{x})}, \quad \text{a.e. } \boldsymbol{x} \in \Omega, \ \forall\, \boldsymbol{v} \in V,$$

where

$$\alpha \in L^\infty(\Omega), \quad \alpha \geqslant 0, \tag{4.11}$$

and $j\colon \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be (Fréchet) differentiable and satisfies

$$\|dj(\boldsymbol{v})\|_{L(\mathbb{R}^d)} \leqslant c_2 + c_2|\boldsymbol{v}|^m, \quad \forall\, \boldsymbol{v} \in \mathbb{R}^d, \tag{4.12}$$

for some constant $c_2 > 0$ (recall that the linear mapping $dj(\boldsymbol{v}) \in L(\mathbb{R}^d)$ is the Fréchet derivative of j at $\boldsymbol{v}$) and m satisfies

$$\begin{cases} 0 \leqslant m & \text{if } d = 1 \text{ or } d = 2; \\ 0 \leqslant m \leqslant \dfrac{2}{d-2} & \text{if } d > 2. \end{cases} \tag{4.13}$$

Moreover, assume that there exists $J\colon \mathbb{R}^d \to \mathbb{R}$ (Fréchet) differentiable satisfying

$$\begin{cases} \exists\, c_J, d_J \in \mathbb{R}, & \forall\, \boldsymbol{v} \in \mathbb{R}^d, \ J(\boldsymbol{v}) \geqslant c_J|\boldsymbol{v}| + d_J; \\ \nabla J(\boldsymbol{v}) = j(\boldsymbol{v}), & \forall\, \boldsymbol{v} \in \mathbb{R}^d, \end{cases} \tag{4.14}$$

where $\nabla J(\boldsymbol{v})$ stands for the gradient of J at $\boldsymbol{v}$.

Let us note that from (4.12) there exist some constants $c_3 > 0$ and $c_4 > 0$ satisfying

$$|j(\boldsymbol{v})| \leqslant c_3(1 + |\boldsymbol{v}| + |\boldsymbol{v}|^{m+1}), \quad \forall\, \boldsymbol{v} \in \mathbb{R}^d, \tag{4.15}$$

and

$$|j(\boldsymbol{v}_1) - j(\boldsymbol{v}_2)| \leqslant c_4|\boldsymbol{v}_1 - \boldsymbol{v}_2|(1 + |\boldsymbol{v}_1|^m + |\boldsymbol{v}_2|^m), \quad \forall\, \boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^d. \tag{4.16}$$

Then using Sobolev embeddings (see e.g. [2, p. 168]) and (4.15), we check that for all $\boldsymbol{v} \in V$, $C(\boldsymbol{v}) \in H$.

Let us also mention that from (4.12) and (4.15) there exists some constant $c_5 > 0$ satisfying

$$|J(\boldsymbol{v})| \leqslant c_5 + c_5|\boldsymbol{v}|(1 + |\boldsymbol{v}| + |\boldsymbol{v}|^{m+1}), \quad \forall\, \boldsymbol{v} \in \mathbb{R}^d. \tag{4.17}$$

For the special case of $d = 1$, we remark that the assumption (4.14) becomes

$$\exists\, c_J \in \mathbb{R}, \;\; \forall t \in \mathbb{R}, \qquad \int_0^t j(s)\,\mathrm{d}s \geqslant c_J \quad \text{and} \quad j'(t) \geqslant 0,$$

where we take

$$J(t) = \int_0^t j(s)\,\mathrm{d}s, \quad \forall\, t \in \mathbb{R}.$$

Finally an example of a function j for which the assumptions (4.12) and (4.14) are verified is given by

$$j(\boldsymbol{v}) = |\boldsymbol{v}|^m \boldsymbol{v}, \quad \forall\, \boldsymbol{v} \in \mathbb{R}^d,$$

where we define

$$J(\boldsymbol{v}) = \frac{|\boldsymbol{v}|^{m+2}}{m+2}, \quad \forall\, \boldsymbol{v} \in \mathbb{R}^d.$$

To continue, define

$$\boldsymbol{f}(t) = \frac{\boldsymbol{f}_0(t)}{\varrho},$$

with

$$\boldsymbol{f}_0 \in L^2(0, T; H). \tag{4.18}$$

Finally, define

$$(\mathbf{g}, \boldsymbol{w})_{V' \times V} = \int_{\Gamma_2} \boldsymbol{f}_2 \boldsymbol{w}\,\mathrm{d}\Gamma + \int_{\Gamma_3} \boldsymbol{T} \boldsymbol{w}_\tau\,\mathrm{d}\Gamma,$$

where

$$\boldsymbol{f}_2 \in L^2(\Gamma_2)^d, \tag{4.19}$$

$$\boldsymbol{T} \in L^2(\Gamma_3)^d. \tag{4.20}$$

Keeping the above notation, multiplying in the problem P_2 by test functions $\boldsymbol{v} \in V$, we deduce the following weak formulation of the problem P_2:

Problem PV_2. Find $\boldsymbol{u}\colon [0,T] \to V$ such that

$$(\!(\ddot{\boldsymbol{u}}(t), \boldsymbol{v})\!)_H + (A\boldsymbol{u}(t), \boldsymbol{v})_{V'\times V} + (\!(C\boldsymbol{u}(t), \boldsymbol{v})\!)_H + (\!(B\dot{\boldsymbol{u}}(t), \boldsymbol{v})\!)_H$$
$$= (\!(\boldsymbol{f}(t), \boldsymbol{v})\!)_H + (\boldsymbol{g}, \boldsymbol{v})_{V'\times V}, \quad \text{on } (0,T), \ \forall \boldsymbol{v} \in V,$$

with the initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0.$$

We recognize here a second order evolution problem studied in Section 2. Rewrite then the hypotheses on initial data

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in H. \tag{4.21}$$

Applying again Theorem 1, we have then the following statement.

Theorem 2. *Assume that the assumptions* (4.6)–(4.14) *and* (4.18)–(4.21) *hold, then there exists a unique solution* $\boldsymbol{u}$ *to the problem* PV_2 *with the regularity*

$$\boldsymbol{u} \in L^\infty(0,T;V), \quad \boldsymbol{u}' \in L^\infty(0,T;H),$$
$$\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2} \in L^\infty(0,T;V').$$

Proof. We have to check now the assumptions (2.5)–(2.12).

Assumption (2.5) follows from (4.7) and Korn's inequality (see e.g. [16], p. 79).

Assumption (2.6). From the definition of the operator C and (4.16), we get for some constant c, for all $\boldsymbol{u}, \boldsymbol{v} \in V$, a.e. $x \in \Omega$,

$$|C\boldsymbol{u}(x) - C\boldsymbol{v}(x)|^2 \leqslant c|\boldsymbol{u}(x) - \boldsymbol{v}(x)|^2(1 + |\boldsymbol{u}(x)|^m + |\boldsymbol{v}(x)|^m)^2.$$

By Hölder's inequality, we have

$$\|C\boldsymbol{u} - C\boldsymbol{v}\|_H \leqslant c\|\boldsymbol{u} - \boldsymbol{v}\|_{L^{2p}(\Omega)}(1 + \||\boldsymbol{u}|^m\|_{L^{2q}(\Omega)} + \||\boldsymbol{v}|^m\|_{L^{2q}(\Omega)}),$$

where $p^{-1} + q^{-1} = 1$.

Now consider the case of $d > 2$, by using the continuous embedding $H^1(\Omega) \subset L^{2^*}(\Omega)$ and $0 \leqslant m \leqslant \frac{2}{d-2}$, and taking $p = \frac{2^*}{2} = \frac{d}{d-2}$ (then $q = \frac{d}{2}$), we obtain

$$\|C\boldsymbol{u} - C\boldsymbol{v}\|_H \leqslant c\|\boldsymbol{u} - \boldsymbol{v}\|_V(1 + \|\boldsymbol{u}\|_V^m + \|\boldsymbol{v}\|_V^m).$$

For the case of $d = 2$, we can suppose that $m > 0$, using the continuous embedding $H^1(\Omega) \subset L^q(\Omega)$, $q \geqslant 2$, and the fact that there exists $p > 1$ such that $m \geqslant \frac{1}{q} = 1 - \frac{1}{p}$, we deduce that

$$\|C\boldsymbol{u} - C\boldsymbol{v}\|_H \leqslant c\|\boldsymbol{u} - \boldsymbol{v}\|_V (1 + \|\boldsymbol{u}\|_V^m + \|\boldsymbol{v}\|_V^m).$$

Finally for the last case of $d = 1$, the same previous inequality holds by using the continuous embedding $H^1(\Omega) \subset L^\infty(\Omega)$.

This completes the verification of (2.6).

Assumption (2.7). Let $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ weakly in V. By the Sobolev compact embedding $V \subset H$ (see e.g. [2, p. 169]), we have $\boldsymbol{v}_n \to \boldsymbol{v}$ strongly in H for some subsequence, then $C\boldsymbol{v}_n(\boldsymbol{x}) \to C\boldsymbol{v}(\boldsymbol{x})$ a.e. in Ω, as $\|C\boldsymbol{v}_n\|_H$ is bounded, we conclude by classical arguments that $C\boldsymbol{v}_n \rightharpoonup C\boldsymbol{v}$ weakly in H.

Assumption (2.8). Define now

$$G(\boldsymbol{v}) = \int_\Omega \alpha(\boldsymbol{x}) J(\boldsymbol{v}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}, \quad \forall \boldsymbol{v} \in V.$$

As for the operator C, using (4.17), Hölder's inequality, and the Sobolev continuous embedding, we verify that for all $\boldsymbol{v} \in V$,

$$\int_\Omega |\alpha(\boldsymbol{x}) J(\boldsymbol{v}(\boldsymbol{x}))|\,\mathrm{d}\boldsymbol{x} \leqslant c + c\|\boldsymbol{v}\|_V (1 + \|\boldsymbol{v}\|_V + \|\boldsymbol{v}\|_V^{m+1}).$$

Thus the function $G\colon V \to \mathbb{R}$ is well defined and bounded. Clearly, from (4.14) and Fatou's lemma, the assumptions (2.8 (i))–(2.8 (iii)) are satisfied.

For example for (2.8 (i)), let $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ weakly in V; we have $\boldsymbol{v}_n \to \boldsymbol{v}$ strongly in H for some subsequence, thus $\boldsymbol{v}_n(\boldsymbol{x}) \to \boldsymbol{v}(\boldsymbol{x})$ a.e. in Ω, and by the continuity of J, $J(\boldsymbol{v}_n(\boldsymbol{x})) \to J(\boldsymbol{v}(\boldsymbol{x}))$ a.e. in Ω. Since G is bounded and $\boldsymbol{v}_n$ bounded in V, we can apply Fatou's lemma and obtain the desired result.

Finally, assumptions (2.9)–(2.11) follow immediately from the definition of B and (4.9). $\square$

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