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SPARSE FINITE ELEMENT METHODS FOR OPERATOR EQUATIONS WITH STOCHASTIC DATA*

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Abstract. Let $A\colon V \to V'$ be a strongly elliptic operator on a d-dimensional manifold D (polyhedra or boundaries of polyhedra are also allowed). An operator equation $Au = f$ with stochastic data f is considered. The goal of the computation is the mean field and higher moments $\mathcal{M}^1 u \in V$, $\mathcal{M}^2 u \in V \otimes V$, …, $\mathcal{M}^k u \in V \otimes \ldots \otimes V$ of the solution.

We discretize the mean field problem using a FEM with hierarchical basis and N degrees of freedom. We present a Monte-Carlo algorithm and a deterministic algorithm for the approximation of the moment $\mathcal{M}^k u$ for $k \geqslant 1$.

The key tool in both algorithms is a "sparse tensor product" space for the approximation of $\mathcal{M}^k u$ with $O(N(\log N)^{k-1})$ degrees of freedom, instead of N^k degrees of freedom for the full tensor product FEM space.

A sparse Monte-Carlo FEM with M samples (i.e., deterministic solver) is proved to yield approximations to $\mathcal{M}^k u$ with a work of $O(MN(\log N)^{k-1})$ operations. The solutions are shown to converge with the optimal rates with respect to the Finite Element degrees of freedom N and the number M of samples.

The deterministic FEM is based on deterministic equations for $\mathcal{M}^k u$ in $D^k \subset \mathbb{R}^{kd}$. Their Galerkin approximation using sparse tensor products of the FE spaces in D allows approximation of $\mathcal{M}^k u$ with $O(N(\log N)^{k-1})$ degrees of freedom converging at an optimal rate (up to logs).

For nonlocal operators wavelet compression of the operators is used. The linear systems are solved iteratively with multilevel preconditioning. This yields an approximation for $\mathcal{M}^k u$ with at most $O(N(\log N)^{k+1})$ operations.

Keywords: wavelet compression of operators, random data, Monte-Carlo method, wavelet finite element method

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1. Introduction

We analyze the Finite Element solution of operator equations $Au = f$ where the data f are random fields, i.e. measurable maps from a probability space into a set of admissible data for the operator A.

We mention only diffusion problems with stochastic source terms or vibrations with random forcing terms. Such equations also arise when equations with random operators A are solved by perturbation expansions.

The simplest approach for the numerical solution of $Au = f$ is Monte Carlo (MC) simulation, i.e. generating a large number M of data samples f_j with prescribed statistics, and solving, possibly in parallel, for the corresponding solution ensemble $\{u_j = A^{-1}f_j\colon\ j = 1, \ldots, M\}$. Statistical moments and probabilities of the random solution u are approximated from $\{u_j\}$. Convergence of the MC method as the number M of samples increases is ensured (for suitable sampling) by the central limit theorem. The MC method allows in general only the convergence rate $O(M^{-1/2})$.

If statistical moments, i.e., the mean field and higher order moments of the random solution u, are of interest, one can exploit the linearity of the equation $Au = f$ to derive a deterministic equation for the kth moment of the random solution. For the Laplace equation with stochastic data this approach is due to I. Babuška [1]. This deterministic equation and its Finite Element (FE) solution were investigated in [24], [25] in the case when A is an elliptic partial differential operator. It was shown that the kth moment of the solution could be computed in a complexity comparable to that of a FE solution for the mean field problem by the use of sparse tensor products of standard FE spaces for which a hierarchical basis is available. Let us mention that the use of sparse tensor product approximations is a well known device in high dimensional numerical integration [26], multivariate approximation [27], and in complexity theory [28].

In the present paper we are also interested in the case where A is a nonlocal operator, such as a strongly elliptic pseudodifferential operator. For example, we can first consider a boundary value problem for an elliptic differential operator and stochastic boundary data, then the boundary integral formulation leads to a problem $Au = f$ where A is an integral operator. As in the case of local operators, sparse tensor products of standard FE spaces allow deterministic approximation of the kth moment of the random solution u with relatively few degrees of freedom; however, to achieve optimal computational complexity, the Galerkin matrix for the operator A must also be compressed, or sparsified. The design and the numerical analysis of deterministic and stochastic solution algorithms that obtain the kth moment of the random solution of nonlocal operator equations with random data in log-linear complexity in the number N of degrees of freedom for the mean field problem is one purpose of the present paper.

We apply the methods to the numerical solution of Dirichlet and Neumann problems for the Laplace or Helmholtz equation with stochastic data. Using a wavelet Galerkin discretization we form sparse tensor products of the trial spaces to obtain well conditioned, sparse representations of stiffness matrices for the operator A as well as for wavelet discretizations of its k-fold tensor product which is the operator arising in the kth moment problem.

We analyze the impact of the operator compression on the accuracy of functionals of the Galerkin solution such as far field evaluations of the random potential at a point. For example, means and variances of the potential at a point can be computed with accuracy $O(N^{-p})$ for any fixed order p for random boundary data with known second moments in $O(N)$ complexity where N denotes the number of degrees of freedom on the boundary.

The outline of the paper is as follows:

In Section 2, we describe the operator equations considered and derive the deterministic problems for the higher moments, generalizing [25]. We establish the Fredholm property for the tensor product operator and regularity estimates for the statistical moments in anisotropic Sobolev spaces with mixed highest derivative. Section 3 introduces definition and basic properties of wavelet Finite Element Methods. Section 4 is devoted to the analysis of Sparse Tensor Product Monte Carlo Finite Element Methods for the computation of k-point correlation functions of the random solution. Section 5 addresses the deterministic numerical solution of the moment equations, in particular the impact of various matrix compressions on the accuracy of the approximated moments, the preconditioning of the product operator and the solution algorithm. Section 6 contains examples from Finite and Boundary Element Methods.

2. Operator equations with stochastic data

We consider the operator equation

$$Au = f \tag{2.1}$$

where A is a bounded linear operator from a separable Hilbert space V into its dual V'.

The operator A is a differential or pseudodifferential operator of order ϱ on a bounded d-dimensional manifold D which may be closed or have a boundary. For a closed manifold and $s \geqslant 0$ we define $\tilde{H}^s(D) := H^s(D)$ as the usual Sobolev space.

For a manifold D with boundary we assume that it can be extended to a closed manifold $\tilde{D}$ and define

$$\tilde{H}^s(D) := \{u|_D\colon\ u \in H^s(\tilde{D}),\ u|_{\tilde{D}\setminus D} = 0\}$$

with the induced norm. If D is a bounded domain in $\mathbb{R}^d$ we use $\tilde{D} := \mathbb{R}^d$. We now assume that $V = \tilde{H}^{\varrho/2}(D)$. In the case of a second order differential operator this means that we have Dirichlet boundary conditions (other boundary conditions can be treated in an analogous way, but we want to simplify the presentation).

The manifold D may be smooth, but we also consider the case that D is a polyhedron in $\mathbb{R}^d$, or the boundary of a polyhedron in $\mathbb{R}^{d+1}$, or a part of the boundary of a polyhedron.

For the deterministic operator A in (2.1) we assume strong ellipticity in the sense that there exists $\alpha > 0$ and a compact operator $T\colon V \to V'$ such that the Gårding inequality

$$\forall\, v \in V\colon\ \langle (A+T)v, v\rangle \geqslant \alpha \|v\|_V^2 \tag{2.2}$$

holds. For the deterministic algorithm in Section 5 we need the slightly stronger assumption that T' is smoothing with respect to a scale of smoothness spaces (see (5.3) below). Here and in what follows, $\langle\cdot,\cdot\rangle$ denotes the $V' \times V$ duality pairing. We assume also that

$$\ker A = \{0\}, \tag{2.3}$$

which implies that for every $f \in V'$, (2.1) admits a unique solution $u \in V$ and, moreover, that $A^{-1}\colon V' \to V$ is continuous, i.e. there is $C_A > 0$ such that for all $f \in V'$ we have

$$\|u\|_V = \|A^{-1}f\|_V \leqslant C_A \|f\|_{V'}.$$

We consider (2.1) for data f and a solution u which are random fields. By this we mean mappings from (Ω, Σ, P), a σ-finite probability space, into separable Hilbert spaces V' and V, respectively.

We define a random field f with values in a separable Hilbert space X as a mapping $f\colon \Omega \to X$ which maps events $E \in \Sigma$ to Borel sets in X (the Borel σ-algebra of X is generated by the open sets of X).

Note that the mapping $f\colon \Omega \to X$ induces a measure $\tilde{P}$ on X.

We say that a random field $u\colon \Omega \to X$ is in the Bochner space $L^1(\Omega, X)$ if $\omega \mapsto \|u(\omega)\|_X$ is measurable and integrable so that $\|u\|_{L^1(\Omega,X)} := \int_\Omega \|u(\omega)\|_X \,\mathrm{d}P(\omega)$ is finite. In this case the Bochner integral

$$\mathbb{E}u := \int_\Omega u(\omega)\,\mathrm{d}P(\omega) \in X$$

exists and we have

$$\|\mathbb{E}u\|_X \leqslant \|u\|_{L^1(\Omega,X)}. \tag{2.4}$$

Let $k \geqslant 1$. We say that a random field $u\colon \Omega \to X$ is in the Bochner space $L^k(\Omega, X)$ if $\omega \mapsto \|u(\omega)\|_X^k$ is measurable and integrable so that $\|u\|_{L^k(\Omega,X)}^k = \int_\Omega \|u(\omega)\|_X^k \, \mathrm{d}P(\omega)$ is finite. Note that $L^k(\Omega, X) \supset L^l(\Omega, X)$ for $k < l$ as can be seen from the Cauchy-Schwarz inequality.

Let B denote a continuous linear mapping from X to another separable Hilbert space Y. For a random field $u \in L^k(\Omega, X)$ this mapping defines a random variable $v(\omega) = Bu(\omega)$, and we have that $v \in L^k(\Omega, Y)$ and

$$\|Bu\|_{L^k(\Omega,Y)} \leqslant C\|u\|_{L^k(\Omega,X)}. \tag{2.5}$$

Furthermore, we have

$$B \int_\Omega u \, \mathrm{d}P(\omega) = \int_\Omega Bu \, \mathrm{d}P(\omega). \tag{2.6}$$

We are interested in statistics of the random solution u of (2.1) and, in particular, in statistical moments. To define them, for any $k \in \mathbb{N}$ we need the k-fold tensor product space

$$X^{(k)} = \underbrace{X \otimes \ldots \otimes X}_{k\text{-times}},$$

and equip it with the natural norm $\| \circ \|_{X^{(k)}}$. It has the property that

$$\|u_1 \otimes \ldots \otimes u_k\|_{X^{(k)}} = \|u_1\|_X \ldots \|u_k\|_X \tag{2.7}$$

holds for every $u_1, \ldots, u_k \in X$ (see [6] for more on norms on tensor product spaces). For a random field $u \in L^k(\Omega, X)$ we now consider the random field $u^{(k)}$ defined by $u(\omega) \otimes \ldots \otimes u(\omega)$. Then $u^{(k)} = u \otimes \ldots \otimes u \in L^1(\Omega, X^{(k)})$:

$$\begin{aligned} \|u^{(k)}\|_{L^1(\Omega,X^{(k)})} &= \int_\Omega \|u(\omega) \otimes \ldots \otimes u(\omega)\|_{X^{(k)}} \, \mathrm{d}P(\omega) \\ &= \int_\Omega \|u(\omega)\|_X \ldots \|u(\omega)\|_X \, \mathrm{d}P(\omega) = \|u\|_{L^k(\Omega,X)}^k. \end{aligned} \tag{2.8}$$

Hence we can now define the moment $\mathcal{M}^k u$ as the expectation of $u \otimes \ldots \otimes u$:

Definition 2.1. For $u \in L^k(\Omega, X)$ with some integer $k \geqslant 1$, the kth moment of $u(\omega)$ is defined by

$$\mathcal{M}^k u = \mathbb{E}[\underbrace{u \otimes \ldots \otimes u}_{k\text{-times}}] = \int_{\omega\in\Omega} \underbrace{u(\omega) \otimes \ldots \otimes u(\omega)}_{k\text{-times}} \, \mathrm{d}P(\omega) \in X^{(k)}. \tag{2.9}$$

Note that (2.4) gives

$$\|\mathcal{M}^k u\|_X \leqslant \|u\|^k_{L^k(\Omega,V)}. \tag{2.10}$$

We now consider the operator equation $Au = f$ where $f \in L^k(\Omega, V')$ is given with $k \geqslant 1$. Since $A^{-1}\colon V' \to V$ is continuous we obtain from (2.5) that $u \in L^k(\Omega, V)$ and

$$\|u\|_{L^k(\Omega,V)} \leqslant C\|f\|_{L^k(\Omega,V')}.$$

Note that this implies that the moment $\mathcal{M}^k u$ exists and satisfies

$$\|\mathcal{M}^k u\|_{V^{(k)}} \leqslant \|u\|^k_{L^k(\Omega,V)}.$$

Remark 2.2. Note that our definitions of the moments $\mathcal{M}^k u$ as Bochner integrals coincide for $k = 1, 2$ with the definition of the expectation and covariance (for a centered random variable): In, e.g., [4, Def. 2.2.7] the expectation is defined as a mapping $X' \to \mathbb{R}$, and the covariance is a mapping $X' \times X' \to \mathbb{R}$. In the case of a reflexive space these objects can be identified with elements of X and $X \otimes X$, respectively, and coincide with $\mathcal{M}^1 u$ and $\mathcal{M}^2 u$.

Remark 2.3. Since $A^{-1} : V' \to V$ in (2.1) is, by (2.2) and (2.3), bijective a measure P on the space V' of data induces, via $P \circ A$, a measure $\tilde{P}$ on the space V of solutions to (2.1).

An example for a measure P on X' is the Gaussian measure Γ (see, e.g., [16] for probability measures over X and, in particular, [4], [14] for Gaussian measures on function spaces). If $P = \Gamma$ is Gaussian over V' and A in (2.1) is linear, $\tilde{\Gamma}$ is also Gaussian over V (e.g. [4, Lemma 2.2.2]).

Since a Gaussian measure is completely determined by mean and covariance, hence only $\mathcal{M}^k u$ for $k = 1, 2$ are of interest in this case.

We now consider the tensor product operator $A^{(k)} = A \otimes \ldots \otimes A$ (k times) which maps $V^{(k)}$ to $(V')^{(k)}$. For $v \in V$ and $g := Av$ we obtain that $A^{(k)} v \otimes \ldots \otimes v = g \otimes \ldots \otimes g$. Consider a random field $u \in L^k(\Omega, V)$ and let $f := Au \in L^k(\Omega, V')$. Then the tensor product $u^{(k)} = u \otimes \ldots u$ (k times) is in the space $L^1(\Omega, V^{(k)})$, and we obtain from (2.5) using $B = A^{(k)}$ that

$$A^{(k)} u^{(k)} = f^{(k)}$$

and $f^{(k)} \in L^1(\Omega, (V')^{(k)})$. Now (2.6) implies for the expectations

$$A^{(k)} \mathcal{M}^k u = \mathcal{M}^k f. \tag{2.11}$$

In the case $k = 1$ this is just the equation $A\mathbb{E}u = \mathbb{E}f$ for the mean field. Note that this equation provides a way to compute the moments $\mathcal{M}^k u$ in a deterministic way. We will investigate the numerical approximation in Section 5. This is an alternative to the Monte-Carlo approximation of the moments which will be considered in Section 4.

In the deterministic approach, explicit knowledge of the joint probability densities of f and of the probability measure P is not required to determine the order k statistics of the random solution u from order k statistics of f.

For nonlinear operator equations, associated systems of moment equations require a closure hypothesis which must be additionally imposed and verified. For the linear operator equation (2.1), however, a closure hypothesis is not necessary as (2.11) holds.

To establish solvability of (2.11), we consider for operators $A_i \in \mathcal{L}(V_i, V_i')$, $i = 1, \ldots, k$, the tensor product operator $A_1 \otimes A_2 \otimes \ldots \otimes A_k$:

Proposition 2.4. *For an integer $k > 1$, let V_i, $i = 1, \ldots, k$, be Hilbert spaces with duals V_i' and let $A_i \in \mathcal{L}(V_i, V_i')$ be injective and satisfy a Gårding inequality, i.e., there are compact $T_i \in \mathcal{L}(V_i, V_i')$ and $\alpha_i > 0$ such that*

$$\forall\, v \in V_i \colon\ \langle (A_i + T_i)v, v\rangle \geqslant \alpha_i \|v\|^2_{V_i}, \tag{2.12}$$

where $\langle \cdot, \cdot \rangle$ denotes the $V_i' \times V_i$ duality pairing.

Then the product operator $\mathcal{A} = A_1 \otimes A_2 \otimes \ldots \otimes A_k \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ where $\mathcal{V} = V_1 \otimes V_2 \otimes \ldots \otimes V_k$ and $\mathcal{V}' = (V_1 \otimes V_2 \otimes \ldots \otimes V_k)' \cong V_1' \otimes V_2' \otimes \ldots \otimes V_k'$ is injective and, for every $f \in \mathcal{V}'$, the problem $\mathcal{A}u = f$ admits a unique solution u with

$$\|u\|_{\mathcal{V}} \leqslant C \|f\|_{\mathcal{V}'}.$$

Proof. The injectivity and the Gårding inequality (2.12) imply the bounded invertibility of A_i for each i. This implies the bounded invertibility of $\mathcal{A}$ on $\mathcal{V}' \to \mathcal{V}$ since we can write

$$\mathcal{A} = (A_1 \otimes I^{(k-1)}) \circ (I \otimes A_2 \otimes I^{(k-2)}) \circ \ldots \circ (I^{(k-1)} \otimes A_k)$$

where $I^{(j)}$ denotes the j-fold tensor product of the identity operator on the appropriate V_i. Note that each factor in the composition is invertible. □

To apply this result to (2.11), we require the special case

$$A^{(k)} := \underbrace{A \otimes A \otimes \ldots \otimes A}_{k\text{-times}} \in \mathcal{L}(V^{(k)}, (V')^{(k)}) = \mathcal{L}(V^{(k)}, (V^{(k)})'). \tag{2.13}$$

Theorem 2.5. *If A in* (2.1) *satisfies* (2.2), (2.3), *then for every $k > 1$ the operator $A^{(k)} \in (V^{(k)}, (V')^{(k)})$ is injective on $V^{(k)}$ and the equation*

$$A^{(k)} Z = \mathcal{M}^k f \tag{2.14}$$

has for every $f \in L^k(\Omega, V')$ a unique solution $Z \in V^{(k)}$.

This solution coincides with the kth moment $\mathcal{M}^k u$ of the random field in (2.9)*: $Z = \mathcal{M}^k u$.*

Proof. By (2.10), the assumption $f \in L^k(\Omega, V')$ ensures that $\mathcal{M}^k f \in (V')^{(k)}$. The unique solvability of (2.14) follows immediately from Theorem 2.4 and the assumptions (2.2) and (2.3). The identity $Z = \mathcal{M}^k u$ follows from (2.11) and from the uniqueness of the solution of (2.14). □

The numerical analysis of approximation schemes for (2.14) will require a regularity theory for (2.14). To this end we introduce a smoothness scale $(Y_s)_{s \geqslant 0}$ for the data f with $Y_0 = V'$ and $Y_s \subset Y_t$ for $s > t$. We assume that we have a corresponding scale $(X_s)_{s \geqslant 0}$ of "smoothness spaces" for the solutions with $X_0 = V$ and $X_s \subset X_t$ for $s > t$, so that $A^{-1}\colon Y_s \to X_s$ is continuous.

In the case of a smooth closed manifold we can use $Y_s = H^{-\varrho/2+s}(D)$ and $X_s = H^{\varrho/2+s}(D)$. For differential operators with smooth coefficients and a manifold with a smooth boundary we can use $Y_s = H^{-\varrho/2+s}(D)$ and $X_s = \tilde{H}^{\varrho/2} \cap H^{\varrho/2+s}(D)$. Note that in other cases (a pseudodifferential operator on a manifold with boundary, or a differential operator on a domain with nonsmooth boundary) the spaces X_s will contain functions which are singular at the boundary.

Theorem 2.6. *Assume* (2.2), (2.3) *and that there is $s_0 > 0$ such that $A^{-1}\colon Y_s \to X_s$ is continuous for $0 \leqslant s \leqslant s_0$. Then we have for all $k \geqslant 1$ and for $0 \leqslant s \leqslant s_0$*

$$\|\mathcal{M}^k u\|_{X_s^{(k)}} \leqslant C \|\mathcal{M}^k f\|_{Y_s^{(k)}} \leqslant C \|f\|^k_{L^k(\Omega, Y_s)}. \tag{2.15}$$

Proof. If (2.2), (2.3) hold the operator $A^{(k)}$ is invertible and $\mathcal{M}^k u = (A^{(k)})^{-1}(\mathcal{M}^k f)$ holds with $\|\mathcal{M}^k u\|_{X_0^{(k)}} \leqslant C_k \|\mathcal{M}^k f\|_{Y_0^{(k)}}$. To prove (2.15), from (2.12) and from $A^{(k)} = (A \otimes I^{(k-1)})(I \otimes A^{(k-1)})$ we get that

$$(I \otimes A^{(k-1)})(\mathcal{M}^k u) = (A \otimes I^{(k-1)})^{-1}(\mathcal{M}^k f) = (A^{-1} \otimes I^{(k-1)})(\mathcal{M}^k f). \tag{2.16}$$

Applying here the a-priori estimate for A,

$$\|A^{-1}f\|_{X_s} \leqslant C_s\|f\|_{Y_s}, \quad 0 \leqslant s \leqslant s_0, \tag{2.17}$$

we obtain

$$\begin{aligned}\|(I \otimes A^{(k-1)})(\mathcal{M}^k u)\|_{X_s \otimes X_0^{(k-1)}} &= \|(A^{-1} \otimes I^{(k-1)})\mathcal{M}^k f\|_{X_s \otimes Y_0^{(k-1)}} \\ &\leqslant C_s\|\mathcal{M}^\ell f\|_{Y_s \otimes Y_0^{(\ell-1)}}.\end{aligned}$$

Writing in (2.16) $A^{(k-1)} = (A \otimes I^{(k-2)})(I \otimes A^{(k-2)})$ and reasoning in the same fashion, we get from (2.17)

$$\|(I^{(2)} \otimes A^{(k-2)})(\mathcal{M}^k u)\|_{X_s^{(2)} \otimes X_0^{(k-2)}} \leqslant C_s^2\|\mathcal{M}^k f\|_{Y_s^{(2)} \otimes Y_0^{(k-2)}}.$$

Iterating this argument proves (2.15). □

3. Discretization

In order to obtain a finite dimensional problem we need to discretize in both Ω and D.

For D we will use finite element spaces $V_\ell \subset V$.

3.1. Nested finite element spaces

The Galerkin approximation of (2.1) is based on a sequence $\{V_\ell\}_{\ell=0}^\infty$ of subspaces of V of dimension $N_\ell = \dim V_\ell < \infty$ which are dense in V, i.e. $V = \overline{\bigcup_{\ell \geqslant 0} V_\ell}$, and nested, i.e.

$$V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_\ell \subset V_{\ell+1} \subset \ldots \subset V. \tag{3.1}$$

We assume that for functions u in the smoothness spaces X_s with $s \geqslant 0$ we have an *approximation rate* of the form

$$\inf_{v \in V_\ell} \|u - v\|_V \leqslant C N_\ell^{-s/d}\|u\|_{X_s}. \tag{3.2}$$

3.2. Finite elements with uniform mesh refinement

We will now describe examples of the subspaces V_ℓ for subspaces which satisfy the assumptions of Section 3.1. We introduce finite elements which are only continuous across element boundaries. These elements are suitable for operators of order $\varrho < 3$. Let $P_p(K)$ denote the polynomials of degree $\leqslant p$ on a set K.

Let us first consider the case of a bounded polyhedron $D \subset \mathbb{R}^d$. Let $\mathcal{T}_0$ be a partition of D into simplices K which is regular. Let $\{\mathcal{T}_\ell\}_{\ell=0}^\infty$ be the sequence of partitions obtained by uniform mesh refinement: We can bisect the edges of $\mathcal{T}^k$ and obtain a new partition into simplices which belong to finitely many congruency classes. We set $V_\ell = S^p(D, \mathcal{T}_\ell) = \{u \in C^0(\overline{D})\colon u|_K \in P_p(K)\ \forall K \in \mathcal{T}_\ell\}$ and $h_\ell = \max\{\mathrm{diam}(K)\colon K \in \mathcal{T}_\ell\}$.

Then $N_\ell = \dim V_\ell = O(h_\ell^{-d})$ as $\ell \to \infty$. With $V = \tilde{H}^{\varrho/2}(D)$ and $X_s = H^{\varrho/2+s}(D)$, standard finite element approximation results give that (3.2) holds for $s \in [0, p+1-\frac{1}{2}\varrho]$ and

$$\inf_{v \in V_\ell} \|u - v\|_V \leqslant C N^{-s/d} \|u\|_{X_s}.$$

For the case that D is the boundary $D = \partial\mathcal{D}$ of a polyhedron $\mathcal{D} \subset \mathbb{R}^{d+1}$ we can define finite element spaces in the same way as above and obtain the same convergence rates.

For a d-dimensional domain $D \subset \mathbb{R}^d$ with a smooth boundary we can first divide D into pieces D_J which can be mapped to a simplex S by smooth mappings $\Phi_J\colon D_J \to S$ (which must be C^0 compatible where two pieces D_J, $D_{J'}$ touch). Then we can define on D finite elements functions which on D_J are of the form $g \circ \Phi_J$ where g is a polynomial.

For a d-dimensional smooth surface $D \subset \mathbb{R}^{d+1}$ we can similarly divide D into pieces which can be mapped to simplices in $\mathbb{R}^d$, and again define finite elements using these mappings.

3.3. Wavelet basis for V_l

We will need a hierarchical basis for the nested spaces $V_0 \subset \ldots \subset V_L$: We start with a basis $\{\psi_j^0\}_{j=1,\ldots,N_0}$ for the space V_0. We write the finer spaces V_l with $l > 0$ as a direct sum $V_l = V_{l-1} \oplus W_l$ with a suitable space W_l with basis functions $\{\psi_j^l\}_{j=1,\ldots,M_l}$. Therefore we have that $V_L = V_0 \oplus W_1 \oplus \ldots \oplus W_L$, and $\{\psi_j^l\colon l = 0, \ldots, L,\ j = 1, \ldots, M_l\}$ is a hierarchical basis for V_L where $M_0 := N_0$:

(P1) $V_\ell = \mathrm{span}\{\psi_j^\ell\colon 1 \leqslant j \leqslant M_k,\ 0 \leqslant k \leqslant \ell\}$.

Let us define $N_\ell := \dim V_\ell$ and $N_{-1} := 0$, then we have $M_\ell := N_\ell - N_{\ell-1}$ for $\ell = 0, 1, 2, \ldots, L$.

Property (P1) is in principle sufficient for the formulation and implementation of the sparse MC-Galerkin method and the deterministic sparse Galerkin method. In order to obtain an algorithm with log-linear complexity we will need that the hierarchical basis satisfies additional properties (P2)–(P5) of a *wavelet basis*. This will allow us to perform matrix compression for nonlocal operators, and to obtain optimal preconditioning for the iterative linear system solver.

(P2) *Small support:* $\mathrm{diam}\,\mathrm{supp}(\psi_j^\ell) = O(2^{-\ell})$.

(P3) *Energy norm stability:* there is a constant $C_B > 0$ independent of L, such that for all $v^L = \sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} v_j^\ell \psi_j^\ell(x) \in V_L$ we have

$$C_B^{-1} \sum_{\ell=0}^{\ell} \sum_{j=1}^{M_\ell} |v_j^\ell|^2 \leqslant \|v\|_V^2 \leqslant C_B \sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} |v_j^\ell|^2. \tag{3.3}$$

(P4) Wavelets ψ_j^ℓ with $\ell \geqslant \ell_0$ have *vanishing moments* up to order $p^* \geqslant p - \varrho$:

$$\int \psi_j^\ell(x) x^\alpha \, \mathrm{d}x = 0, \quad 0 \leqslant |\alpha| \leqslant p^*, \tag{3.4}$$

except possibly for wavelets where the closure of the support intersects the boundary ∂D or the boundaries of the coarsest mesh. In the case of mapped finite elements we require vanishing moments for the polynomial function $\psi_j^\ell \circ \Phi_J^{-1}$.

(P5) *Decay of coefficients for "smooth" functions in X_s:* There is $C > 0$ independent of L such that for $u \in X_s$ we have

$$\sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} |u_j^\ell|^2 \, 2^{2\ell s} \leqslant C L^\nu \|u\|_{X_s}^2, \quad \nu = \begin{cases} 0 & \text{for } 0 \leqslant s < p + 1 - \frac{1}{2}\varrho, \\ 1 & \text{for } s = p + 1 - \frac{1}{2}\varrho. \end{cases} \tag{3.5}$$

A function $u \in V$ has a wavelet expansion $\sum_{\ell=0}^{L} \sum_{j=1}^{\infty} u_j^\ell \, \psi_j^\ell$. We define the projection $P_L \colon V \to V_L$ by truncating this wavelet expansion of u, i.e.,

$$P_L u := \sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} u_j^\ell \psi_j^\ell. \tag{3.6}$$

Because of the stability (P3) and the approximation property (3.2) we obtain that the wavelet projection P_L is quasioptimal: For $0 \leqslant s \leqslant s_0$ and $u \in X_s$ we have

$$\|u - P_L u\|_V \leqslant C N_L^{-s/d} \|u\|_{X_s}. \tag{3.7}$$

In the case of finite elements which are only continuous across the element boundaries one method for constructing wavelets satisfying these conditions is given in [9]. This approach gives wavelets for any degree $p \geqslant 1$ and vanishing moments up to any order p^*, in any dimension; the stability condition (P3) is satisfied for $\varrho < 3$ on smooth manifolds, and for $\varrho \leqslant 2$ on Lipschitz manifolds. The construction is explicitly carried out for piecewise linear ones ($p = 1$) with p' up to 5 for $d = 1, 2$, which allows $\varrho \geqslant -4$; in the case $d = 3$ the case $p' = 1$ is shown, which allows $\varrho \geqslant 0$.

3.4. Tensor product spaces: Full grid and sparse grid subspaces

We want to compute an approximation for $\mathcal{M}^k u \in V \otimes \ldots \otimes V = V^{(k)}$. Therefore we need a suitable finite dimensional subspace of $V^{(k)}$. The simplest choice is the tensor product space $V_L \otimes \ldots \otimes V_L = V_L^{(k)}$, but this space has dimension N_L^k which is not practical for $k > 1$.

A reduction in cost is possible by using the so-called sparse tensor products.

We now define the k-fold *sparse tensor product space* $\hat{V}_L^{(k)}$ by

$$\hat{V}_L^{(k)} = \sum_{\substack{\underline{\ell} \in \mathbb{N}_0^k \\ |\underline{\ell}| \leqslant L}} V_{\ell_1} \otimes \ldots \otimes V_{\ell_k} \tag{3.8}$$

where we denote by $\underline{\ell}$ the vector $(\ell_1, \ldots, \ell_k) \in \mathbb{N}_0^k$ and by $|\underline{\ell}| = \ell_1 + \ldots + \ell_k$ its length. We can write $\hat{V}$ as a direct sum by using the complement spaces W_l:

$$\hat{V}_L^{(k)} = \sum_{\substack{\underline{\ell} \in \mathbb{N}_0^k \\ |\underline{\ell}| \leqslant L}} W_{\ell_1} \otimes \ldots \otimes W_{\ell_k}. \tag{3.9}$$

We define the sparse projection operator $\hat{P}_L^{(k)} \colon V^{(k)} \to \hat{V}_L^{(k)}$ by truncating the wavelet expansion:

$$(\hat{P}_L^{(k)} v)(x) := \sum_{\substack{0 \leqslant \ell_1 + \ldots + \ell_k \leqslant L \\ 1 \leqslant j_\nu \leqslant M_{\ell_\nu},\, \nu = 1, \ldots, k}} v_{j_1 \ldots j_k}^{\ell_1 \ldots \ell_j} \psi_{j_1}^{\ell_1}(x_1) \ldots \psi_k^{\ell_k}(x_k). \tag{3.10}$$

In terms of the projections $Q_\ell := P_\ell - P_{\ell-1}$, $\ell = 0, 1, \ldots$ and $P_{-1} := 0$ we can express $\hat{P}_L^{(k)}$ as

$$\hat{P}_L^{(k)} = \sum_{0 \leqslant \ell_1 + \ldots + \ell_k \leqslant L} Q_{\ell_1} \otimes \ldots \otimes Q_{\ell_k}. \tag{3.11}$$

The approximation property of sparse grid spaces $\hat{V}_L^{(k)}$ was established for example in [25, Proposition 4.2], [11], [22].

Proposition 3.1.

$$\begin{aligned} &\inf_{v \in \hat{V}_L^{(k)}} \|U - v\|_{V^{(k)}} \\ &\qquad \leqslant C(k) \begin{cases} N_L^{-s/d} \|U\|_{X_s^{(k)}} & \textit{if } 0 \leqslant s < p + 1 - \frac{1}{2}\varrho, \\ N_L^{-s/d} L^{(k-1)/2} \|U\|_{X_s^{(k)}} & \textit{if } s = p + 1 - \frac{1}{2}\varrho. \end{cases} \end{aligned} \tag{3.12}$$

The stability property (P3) implies the following result (see, e.g., [22]):

Lemma 3.2 (Properties of $\hat{P}_L^{(k)}$). *Assume* (P1)–(P5) *and that the component spaces* V_ℓ *of* $\hat{V}_L^{(k)}$ *have the approximation property* (3.2)*. Then for* $U \in V^{(k)}$ *we have stability:*

$$\|\hat{P}_L^{(k)} U\|_{V^{(k)}} \leqslant C\|U\|_{V^{(k)}}, \tag{3.13}$$

and hence for $U \in X_s^{(k)}$ *and* $0 \leqslant s \leqslant s_0$ *we have quasioptimal convergence of* $\hat{P}_L^{(k)} U$*:*

$$\|U - \hat{P}_L^{(k)} U\|_{V^{(k)}} \leqslant C(k) N_L^{-s/d} (\log N_L)^{(k-1)/2} \|U\|_{X_s^{(k)}}. \tag{3.14}$$

3.5. Galerkin method for space discretization

We first consider the discretization of the problem $Au(\omega) = f(\omega)$ for a fixed ω. In the Monte-Carlo method this problem will be approximatively solved for many values of $\omega \in \Omega$.

The Galerkin discretization of (2.1) reads: find $u_L(\omega) \in V_L$ such that

$$\langle A u_L(\omega), v_L\rangle = \langle f(\omega), v_L\rangle \quad \forall\, v_L \in V_L,\ \ P\text{-a.e. } \omega \in \Omega. \tag{3.15}$$

It is well known that the injectivity (2.3) of A, the Gårding inequality (2.2) and the density of the sequence $\{V_\ell\}_{\ell=0}^{\infty}$ imply that there exists $L_0 > 0$ such that for $L \geqslant L_0$ problem (3.15) has a unique solution $u_L(\omega)$. Furthermore, the *inf-sup condition* holds (see, e.g., [12]): There exists $c_S > 0$ such that for all $L \geqslant L_0$

$$\inf_{0 \neq u \in V_L} \sup_{0 \neq v \in V_L} \frac{\langle Au, v\rangle}{\|u\|_V \|v\|_V} \geqslant \frac{1}{c_S} > 0. \tag{3.16}$$

The inf-sup condition then implies quasioptimality of the approximations $u_L(\omega)$ for $L \geqslant L_0$ (see [2]): There is $C > 0$ such that

$$\forall\, L \geqslant L_0\colon\ \|u(\omega) - u_L(\omega)\|_V \leqslant C \inf_{v \in V_L} \|u(\omega) - v\|_V \quad P\text{-a.e. } \omega \in \Omega. \tag{3.17}$$

From (3.17) and (3.2), we obtain an asymptotic error estimate: Let $\sigma := \min\{s_0, p + 1 - \frac{1}{2}\varrho\}$. There is $C > 0$ such that for $0 < s \leqslant \sigma$

$$\forall\, L \geqslant L_0\colon\ \|u(\omega) - u_L(\omega)\|_V \leqslant C N_L{}^{-s/d} \|u\|_{X_s} \quad P\text{-a.e. } \omega \in \Omega. \tag{3.18}$$

4.1. Monte-Carlo error for continuous problem

For a random variable Y let us denote the mean of M independent identically distributed random variables as $\overline{Y}^M$:

$$\overline{Y}^M(\omega_1,\dots,\omega_M) := \frac{1}{M}(Y(\omega_1)+\dots+Y(\omega_M)).$$

The simplest approach to the numerical solution of (2.1) for $f \in L^0(\Omega, V')$ is Monte Carlo (MC) simulation. Let us first consider the situation without discretization of V. We generate M data samples $f(\omega_j)$, $j = 1,2,\dots,M$, of $f(\omega)$ and find the solutions $u(\omega_j) \in V$ of the problems

$$Au(\omega_j) = f(\omega_j), \quad j = 1,\dots,M. \tag{4.1}$$

We then approximate the kth moment $\mathcal{M}^k u$ by the mean $\overline{E}^M_{\mathcal{M}^k u}$ of $u(\omega_j)\otimes\dots\otimes u(\omega_j)$:

$$\overline{E}^M_{\mathcal{M}^k u} := \overline{u\otimes\dots\otimes u}^M = \frac{1}{M}\sum_{j=1}^M u(\omega_j)\otimes\dots\otimes u(\omega_j). \tag{4.2}$$

It is well known that the Monte-Carlo error decreases as $M^{-1/2}$ in a probabilistic sense if the variance of $\mathcal{M}^k u$ exists. Otherwise we obtain a lower rate:

Theorem 4.1. *Let $k \geqslant 1$. Assume that $f \in L^{\alpha k}(\Omega, V')$ for some $\alpha \in (1,2]$. For $M \geqslant 1$ samples we define the sample mean $\overline{E}^M_{\mathcal{M}^k u}$ as in (4.2). Then there exists C such that for every $M \geqslant 1$ and every $0 < \varepsilon < 1$*

$$P\biggl(\|\overline{E}^M_{\mathcal{M}^k u} - \mathcal{M}^k u\|_{V^{(k)}} \leqslant C\frac{\|f\|^k_{L^{\alpha k}(\Omega,V')}}{\varepsilon^{1/\alpha}M^{1-1/\alpha}}\biggr) \geqslant 1-\varepsilon. \tag{4.3}$$

Proof. In the case $\alpha = 2$ this is a consequence of the Chebyshev inequality. In the general case $1 \leqslant \alpha \leqslant 2$ we proceed as follows: We define $U\colon \Omega \to V^{(k)}$ by $U := u\otimes\dots\otimes u$ and $Y\colon \Omega^M \to V^{(k)}$ by $Y := \overline{U}^M - E(U) = \overline{E}^M_{\mathcal{M}^k u} - \mathcal{M}^k u$. We then have for any $\lambda > 0$

$$\begin{aligned}
\|Y\|^\alpha_{L^\alpha(\Omega^M,V^{(k)})} &= \int_{\Omega^M} \|Y(\omega)\|^\alpha_{V^{(k)}} \,\mathrm{d}P^M(\omega)\\
&\geqslant \int_{\{\omega\colon \|Y(\omega)\|_{V^{(k)}} \geqslant \lambda\}} \|Y(\omega)\|^\alpha_{V^{(k)}} \,\mathrm{d}P^M(\omega)\\
&\geqslant \lambda^\alpha \int_{\{\omega\colon \|Y(\omega)\|_{V^{(k)}} \geqslant \lambda\}} 1 \,\mathrm{d}P^M(\omega) = \lambda^\alpha P^M(\|Y(\omega)\|_{V^{(k)}} \geqslant \lambda).
\end{aligned}$$

We now use Lemma 4.2 (given below) and obtain

$$P^M(\|Y(\omega)\|_{V^{(k)}} < \lambda) \geqslant 1 - \left(\frac{\|Y\|_{L^\alpha(\Omega^M, V^{(k)})}}{\lambda}\right)^\alpha \geqslant 1 - \left(\frac{c\|U\|_{L^\alpha(\Omega, V^{(k)})}}{\lambda M^{1-1/\alpha}}\right)^\alpha.$$

Now (4.3) follows by using

$$\varepsilon := \left(\frac{c\|U\|_{L^\alpha(\Omega, V^{(k)})}}{\lambda M^{1-1/\alpha}}\right)^\alpha \Longrightarrow \lambda = \frac{c\|U\|_{L^\alpha(\Omega, V^{(k)})}}{\varepsilon^{1/\alpha} M^{1-1/\alpha}}.$$

Note that we have $\|U\|_{L^\alpha(\Omega, V^{(k)})} = \|u\|^k_{L^{\alpha k}(\Omega, V)} \leqslant C\|f\|^k_{L^{\alpha k}(\Omega, V')}$. □

It remains to show

Lemma 4.2. *Assume $U\colon \Omega \to V$ is a random variable with values in a Hilbert space V. Then we have for $1 \leqslant \alpha \leqslant 2$*

$$\|\overline{U}^M - E(U)\|_{L^\alpha(\Omega^M, V)} \leqslant C M^{-(1-\alpha^{-1})} \|U\|_{L^\alpha(\Omega, V)}. \tag{4.4}$$

Proof. For $\alpha = 2$ let $W = U - E(U)$, then we obtain (4.4) using

$$\begin{aligned}
\|\overline{W}^M\|^2_{L^2(\Omega, V)} &= \int_{\Omega^M} M^{-2}(W(\omega_1) + \ldots + W(\omega_M), W(\omega_1) + \ldots + W(\omega_M))\,\mathrm{d}P(\omega) \\
&= M^{-2}\left(\int_\Omega \|W(\omega_1)\|^2_V\,\mathrm{d}P(\omega_1) + \ldots + \int_\Omega \|W(\omega_M)\|^2_V\,\mathrm{d}P(\omega_M)\right) \\
&= M^{-1}\|W\|^2_{L^2(\Omega, V)}
\end{aligned}$$

since $E(W) = 0$ and $\|W\|_{L^2(\Omega, V)} \leqslant \|U\|_{L^2(\Omega, V)}$. For $\alpha = 1$ we have

$$\begin{aligned}
\|\overline{U}^M\|_{L^1(\Omega, V)} &\leqslant \int_{\Omega^M} M^{-1}(\|U(\omega_1)\|_V + \ldots + \|U(\omega_M)\|_V)\,\mathrm{d}P^M(\omega) \\
&= M^{-1}\left(\int_\Omega \|U(\omega_1)\|_V\,\mathrm{d}P(\omega_1) + \ldots + \int_\Omega \|U(\omega_M)\|_V\,\mathrm{d}P(\omega_M)\right) \\
&= \|U\|_{L^1(\Omega, V)}.
\end{aligned}$$

Then we obtain the statement for $1 \leqslant \alpha \leqslant 2$ by interpolation. □

This shows that we can obtain a rate of $M^{-1/2}$ in a probabilistic sense for the Monte-Carlo method. A finer estimate can be obtained using the law of the iterated logarithm, see, e.g., [16, Chapt. 8] for the vector valued case.

Lemma 4.3. *Assume that V is a separable Hilbert space and that $X \in L^2(\Omega, V)$. Then*

$$\limsup_{M\to\infty} \frac{\|\overline{X}^M - E(X)\|_V}{(2M^{-1}\log\log M)^{1/2}} \leqslant \|X - E(X)\|_{L^2(\Omega,V)} \quad \text{with probability } 1. \tag{4.5}$$

Proof. The classical law of iterated logarithm (see, e.g., [5]) states for a real valued random variable Y that

$$\limsup_{M\to\infty} \frac{|\overline{Y}^M - E(Y)|^2}{2M^{-1}\log\log M} = \operatorname{Var} Y \quad \text{with probability } 1. \tag{4.6}$$

Let $Z := X - E(X)$. As V is separable we can assume without loss of generality that $V = \ell^2$. Let e^j for $j = 1, 2, \dots$ denote the standard basis of ℓ^2. Then $Y := (e^j, Z) = Z_j$ is a real valued random variable and we have (4.6) with $\operatorname{Var} Y = (e^j \otimes e^j, \mathcal{M}^2 Z) = (\mathcal{M}^2 Z)_{j,j}$. Now we add these estimates for $j = 1, 2, \dots$ and obtain

$$\limsup_{M\to\infty} \frac{\sum\limits_{j=1}^{\infty} |Z_j|^2}{(2M^{-1}\log\log M)^{1/2}} \leqslant \sum_{j=1}^{\infty} (\mathcal{M}^2 Z)_{j,j} \quad \text{with probability } 1. \tag{4.7}$$

Then

$$\sum_{j=1}^{\infty} (\mathcal{M}^2 Z)_{j,j} = E\Bigl(\sum_{j=1}^{\infty} (Z \otimes Z)_{j,j}\Bigr) = \int_\Omega \sum_{j=1}^{\infty} |Z_j|^2 \,\mathrm{d}P(\omega) = \|Z\|^2_{L^2(\Omega,V)}.$$

□

Applying Lemma 4.3 to $X = u \otimes \dots \otimes u$ gives, due to $\|u \otimes \dots \otimes u\|_{L^2(\Omega, V^{(k)})} = \|u\|^k_{L^{2k}(\Omega,V)} \leqslant C\|f\|^k_{L^{2k}(\Omega,V')}$, the following result:

Theorem 4.4. *Let $f \in L^{2k}(\Omega, V')$. Then*

$$\limsup_{M\to\infty} \frac{\|\overline{E}^M_{\mathcal{M}^k u} - \mathcal{M}^k u\|_{V^{(k)}}}{(2M^{-1}\log\log M)^{1/2}} \leqslant C\|f\|^k_{L^{2k}(\Omega,V')} \quad \text{with probability } 1. \tag{4.8}$$

4.2. Monte-Carlo Galerkin method and error of sparse moment approximation

We now use the Galerkin method with the subspaces $V_L \subset V$ to solve (4.1) approximately and obtain Galerkin approximations $u_L(\omega_j)$. Then the mean

$$\overline{E}^{M,L}_{\mathcal{M}^k u} := \frac{1}{M} \sum_{j=1}^{M} u_L(\omega_j) \otimes \dots \otimes u_L(\omega_j) \tag{4.9}$$

yields an approximation for $\mathcal{M}^k u$. However, one needs $O(N_L^k)$ of memory and $O(MN_L^k)$ of operations to compute this mean, which is usually not practical for $k > 1$.

Therefore we propose to use the sparse approximation

$$\hat{E}^{M,L}_{\mathcal{M}^k u} := \hat{P}_L^{(k)} \overline{E}^{M,L}_{\mathcal{M}^k u} \tag{4.10}$$

which needs only a memory of $O(N_L^k (\log N_L)^{k-1})$. By (3.6), $u_L(\omega_j)$ can then be represented as

$$u_L(\omega_j) = \sum_{\ell=0}^{L} \sum_{k=1}^{M_\ell} U_k^\ell(\omega_j) \psi_k^\ell \tag{4.11}$$

and we can compute the *sparse tensor product MC estimate* of $\mathcal{M}^k u$ with $\hat{P}_L^{(k)}$ given in (3.10) as

$$\hat{E}^{M,L}_{\mathcal{M}^k u} = \frac{1}{M} \sum_{j=1}^{M} \hat{P}_L^{(k)} [u_L(\omega_j) \otimes \ldots \otimes u_L(\omega_j)] \in V_L^{(k)}. \tag{4.12}$$

It can be computed in $O(MN_L(\log N_L)^{k-1})$ operations since $\hat{P}_L^{(k)}[u_L(\omega_j) \otimes \ldots \otimes u_L(\omega_j)]$ can be computed in $O(N_L(\log N_L)^{k-1})$ operations: For each j we first compute $u_L(\omega_j)$ in the wavelet basis and then form $\hat{P}_L^{(k)}[u_L(\omega_j) \otimes \ldots \otimes u_L(\omega_j)]$ using

$$\hat{P}_L^{(k)}(v \otimes \ldots \otimes v) = \sum_{\substack{0 \leqslant \ell_1 + \ldots + \ell_k \leqslant L \\ 1 \leqslant j_\nu \leqslant M_{\ell_\nu},\, \nu=1,\ldots,k}} v_{j_1}^{l_1} \ldots v_{j_k}^{l_k} \psi_{j_1}^{l_1} \ldots \psi_{j_k}^{l_k}.$$

The following result addresses the convergence of the sparse MC-Galerkin approximation of $\mathcal{M}^k u$. Recall that $\sigma := \min\{s_0, p + 1 - \frac{1}{2}\varrho\}$ with s_0 as in Theorem 2.6.

Theorem 4.5. *Assume that $f \in L^k(\Omega, Y_s) \cap L^{\alpha k}(\Omega, V')$ for some $\alpha \in (1, 2]$ and some $s \in (0, \sigma]$. Then there is $C(k) > 0$ such that for all $M \geqslant 1$, $L \geqslant L_0$ and all $0 < \varepsilon < 1$ we have*

$$P(\|\hat{E}^{M,L}_{\mathcal{M}^k u} - \mathcal{M}^k u\|_{V^{(k)}} < \lambda) \geqslant 1 - \varepsilon$$

with

$$\lambda = C(k)[C N_L{}^{-s/d} (\log N_L)^{(k-1)/2} \|f\|^k_{L^k(\Omega, Y_s)} + \varepsilon^{-1/\alpha} M^{-(1-\alpha^{-1})} \|f\|^k_{L^{\alpha k}(\Omega, V')}].$$

P r o o f. We estimate

$$\begin{aligned}
&\|\hat{E}^{M,L}_{\mathcal{M}^k u} - \mathcal{M}^k u\|_{V^{(k)}} \\
&\quad = \Big\| \frac{1}{M} \sum_{j=1}^{M} \hat{P}_L^{(k)} [u_L(\omega_j) \otimes \ldots \otimes u_L(\omega_j)] - E(u \otimes \ldots \otimes u) \Big\|_{V^{(k)}} \\
&\quad \leqslant \frac{1}{M} \sum_{j=1}^{M} \Big\| \hat{P}_L^{(k)} [u_L(\omega_j) \otimes \ldots \otimes u_L(\omega_j) - u(\omega_j) \otimes \ldots \otimes u(\omega_j)] \Big\|_{V^{(k)}} \\
&\qquad + \Big\| \frac{1}{M} \sum_{j=1}^{M} \hat{P}_L^{(k)} [u(\omega_j) \otimes \ldots \otimes u(\omega_j)] - E(\hat{P}_L^{(k)} [u \otimes \ldots \otimes u]) \Big\|_{V^{(k)}} \\
&\qquad + \|(I - \hat{P}_L^{(k)}) \mathcal{M}^k u\|_{V^{(k)}}.
\end{aligned}$$

The last term is estimated using (3.14), Theorem 2.6 for $0 \leqslant s \leqslant s_0$ by

$$\|(I - \hat{P}_L^{(k)}) \mathcal{M}^k u\|_{V^{(k)}} \leqslant C(k) C N_L{}^{-s/d} (\log N_L)^{(k-1)/2} \|\mathcal{M}^k f\|_{Y_s^{(k)}}.$$

For the first term, we use (3.13) and (3.18) with a tensor product argument. For the second term, the statistical error, by (3.13) it suffices to establish a bound for

$$\Big\| E([u \otimes \ldots \otimes u]) - \frac{1}{M} \sum_{j=1}^{M} [u(\omega_j) \otimes \ldots \otimes u(\omega_j)] \Big\|_{V^{(k)}} = \|\mathcal{M}^k u - \overline{E}^M_{\mathcal{M}^k u}\|_{V^{(k)}},$$

which was estimated in Theorem 4.1. □

We can get a sharper result by using the law of iterated logarithms for estimating the statistical error (second term) in the previous proof:

Theorem 4.6. *Let $k \geqslant 1$ and $f \in L^{2k}(\Omega, V') \cap L^k(\Omega, Y_s)$ with $s \in (0, \sigma]$. Then*

$$\begin{aligned}
\|\hat{E}^{M,L}_{\mathcal{M}^k u} - \mathcal{M}^k u\|_{V^{(k)}} &\leqslant C N_L^{-s/d} (\log N_L)^{(k-1)/2} \|f\|^k_{L^k(\Omega, Y^s)} \\
&\quad + C a_M M^{-1/2} (\log \log M)^{1/2} \|f\|^k_{L^{2k}(\Omega, V')}
\end{aligned} \tag{4.13}$$

where $\limsup\limits_{M \to \infty} a_M \leqslant 1$ with probability 1.

This means that with probability 1 there are at most finitely many values of M where the error estimate (4.13) is not satisfied with $a_M = 1 + \varepsilon$.

R e m a r k 4.7. Note that all results in this section also hold in the case of a stochastic operator $A(\omega)$.

Let X now denote the space of bounded linear mappings $V \to V'$. Assume that $A\colon \Omega \to X$ is measurable (with respect to Borel sets of X) and that there exists C, $\alpha > 0$ and a compact $T\colon V \to V'$ such that

$$\|A(\omega)\|_V \leqslant C \quad \text{almost everywhere,} \tag{4.14}$$

$$\langle (A(\omega) + T)u, u\rangle \geqslant \alpha \|u\|_V^2 \quad \text{almost everywhere.} \tag{4.15}$$

Let $k \geqslant 1$. Then $f \in L^k(\Omega, V')$ implies $u = A^{-1}f \in L^k(\Omega, V)$ and $\mathcal{M}^k u \in V^{(k)}$. Also $f \in L^k(\Omega, Y_\delta)$ implies $u = A^{-1}f \in L^k(\Omega, X_\delta)$ and $\mathcal{M}^k u \in X_\delta^{(k)}$. Therefore all proofs in this section still apply.

5. Deterministic Galerkin approximation of moments

5.1. Sparse Galerkin approximation of $\mathcal{M}^k u$

We now approximate $\mathcal{M}^k u$ by using the Galerkin method for (2.14). If we use the full tensor product space $V_L^{(k)}$ the inf-sup condition of the discrete operator for $L \geqslant L_0$ follows directly from the inf-sup condition (3.16) by a tensor product argument. The regularity estimate for the kth moment $\mathcal{M}^k u$,

$$\|\mathcal{M}^k u\|_{X_s^{(k)}} \leqslant C_{\ell,s} \|\mathcal{M}^k f\|_{Y_s^{(k)}}, \quad 0 \leqslant s \leqslant s_0, \ \ k \geqslant 1, \tag{5.1}$$

then allows us to obtain convergence rates. However, this method is very expensive since we have to set up and solve a linear system with N_L^k unknowns.

We can reduce this complexity by using the sparse tensor product space $\hat{V}_L^{(k)}$. We define the sparse Galerkin approximation $\hat{Z}_L$ of $\mathcal{M}^k u$ as follows:

$$\text{find } \hat{Z}_L \in \hat{V}_L^{(k)} \text{ such that } \langle A^{(k)} \hat{Z}_L, v\rangle = \langle \mathcal{M}^k f, v\rangle \quad \forall\, v \in \hat{V}_L^{(k)}. \tag{5.2}$$

We first consider the case when the operator A is coercive, i.e., (2.2) holds with $T = 0$. Then also $A^{(k)}\colon V^{(k)} \to (V')^{(k)}$ is coercive, and the stability of the Galerkin method with $\hat{V}_L^{(k)}$ follows directly from $\hat{V}_L^{(k)} \subset V^{(k)}$.

In the case of $T \neq 0$ the stability on the sparse space $\hat{V}_L^{(k)}$ is not obvious: We know that $(A+T)\otimes \ldots \otimes (A+T)$ is coercive, but $(A+T)\otimes\ldots\otimes(A+T) - A\otimes\ldots\otimes A$ is not compact. Therefore we require some additional assumptions.

We assume that (2.2) holds with the additional requirement that $T'\colon V \to V'$ is smoothing with respect to the scale of spaces X_s, Y_s, and we also assume that the adjoint operator $A'\colon V \to V'$ possesses a regularity property: We assume that there exists $\delta > 0$ such that

$$T'\colon V \to Y_\delta \quad \text{is continuous,} \tag{5.3}$$

$$(A')^{-1}\colon Y_\delta \to X_\delta \ \text{ is continuous.} \tag{5.4}$$

Due to the indefiniteness of A we have to modify the sparse grid space: Let $L_0 \geqslant 0$ and $L \geqslant L_0$. We define a space $\hat{V}^{(k)}_{L,L_0}$ with $\hat{V}^{(k)}_{L} \subset \hat{V}^{(k)}_{L,L_0} \subset \hat{V}^{(k)}_{L+(k-1)L_0}$ as follows:

Definition 5.1. Let $S^1_{L,L_0} := \{0,\dots,L\}$. Let S^k_{L,L_0} be the set of indices $l \in \mathbb{N}_0^k$ satisfying the following conditions:

$$l_1 + \ldots + l_k \leqslant L + (k-1)L_0, \tag{5.5}$$

$$(l_{i_1},\dots,l_{i_{k-1}}) \in S^{k-1}_{L,L_0} \text{ if } i_1,\dots,i_{k-1} \text{ are different indices in } \{1,\dots,k\}. \tag{5.6}$$

Then we define

$$\hat{V}^{(k)}_{L,L_0} := \sum_{l \in S^k_{L,L_0}} W^{l_1} \otimes \ldots \otimes W^{l_k}. \tag{5.7}$$

Let $J_{L_0} := \{0,1,\dots,L_0\}$. Then S^k_{L,L_0} has the the following subsets:

$$J^k_{L_0}, \quad J^{k-1}_{L_0} \times S^1_{L,L_0}, \quad J^{k-2}_{L_0} \times S^2_{L,L_0}, \quad \dots, \quad J_{L_0} \times S^{k-1}_{L,L_0}.$$

Therefore $\hat{V}^{(k)}_{L,L_0}$ contains the following subspaces:

$$V^{(k)}_{L_0}, \quad V^{(k-1)}_{L_0} \otimes \hat{V}^{(1)}_{L,L_0}, \quad V^{(k-2)}_{L_0} \otimes \hat{V}^{(2)}_{L,L_0}, \quad \dots, \quad V_{L_0} \otimes \hat{V}^{(k-1)}_{L,L_0}. \tag{5.8}$$

We will choose a certain fixed $L_0 > 0$ and consider the sequence of spaces $\hat{V}^{(k)}_{L,L_0}$ with L going to infinity. Since $\hat{V}^{(k)}_{L} \subset \hat{V}^{(k)}_{L,L_0} \subset \hat{V}^{(k)}_{L+(k-1)L_0}$ we see that $\dim \hat{V}^{(k)}_{L,L_0}$ grows with the same rate as $\dim \hat{V}^{(k)}_{L}$ if $L \to \infty$.

We then have the following discrete stability property:

Theorem 5.2. *Assume that A and T satisfy* (2.2) *and* (5.3), (5.4). *Then there exist $L_0 \in \mathbb{N}$ and $c_S > 0$ such that for all $L \geqslant L_0$,*

$$\inf_{0 \neq u \in \hat{V}^{(k)}_{L,L_0}} \sup_{0 \neq v \in \hat{V}^{(k)}_{L,L_0}} \frac{\langle A^{(k)}u, v\rangle}{\|u\|_{V^{(k)}} \|v\|_{V^{(k)}}} \geqslant \frac{1}{c_S} > 0. \tag{5.9}$$

In the case $T = 0$ this holds with $L_0 = 0$.

Proof. Note that with $A_0 := A + T$ we have, with $\langle \cdot,\cdot\rangle$ denoting the $V \times V'$ resp. the $V' \times V$ duality,

$$\langle A_0 u, u\rangle = \langle u, (A' + T')u\rangle = \langle u, A'v\rangle = \langle Au, v\rangle$$

if we define $v = (A')^{-1}(A' + T')u = (I + \tilde{K})u$ with $\tilde{K} := (A')^{-1}T'$. By the assumptions (5.3), (5.4) we have that $\tilde{K}\colon V \to V_\delta$ is continuous.

Let $u \in \hat{V}_{L,L_0}$ be arbitrary. We have to find $v \in \hat{V}_{L,L_0}$ such that $\langle A^{(k)}u, v\rangle \geqslant c\|u\|^2_{V^{(k)}}$ and $\|v\|_{V^{(k)}} \leqslant C\|u\|_{V^{(k)}}$. Let $w := (I+\hat{K})\otimes\ldots\otimes(I+\hat{K})u$, then

$$\langle A^{(k)}u, w\rangle = \langle A_0^{(k)}u, u\rangle \geqslant \alpha^k\|u\|^2_{V^{(k)}}.$$

We now use $v := \hat{P}^{(k)}_{L,L_0}w$ with the projector $\hat{P}^{(k)}_{L,L_0}\colon\ V^{(k)} \to \hat{V}^{(k)}_{L,L_0}$ defined analogously to (3.10). With $\hat{K} := (I+\hat{K})\otimes\ldots\otimes(I+\hat{K}) - I$ and $E := w - v = (I - \hat{P}^{(k)}_{L,L_0})v$ we have

$$\begin{aligned}(5.10)\quad \langle A^{(k)}u, v\rangle = \langle A^{(k)}u, w\rangle - \langle A^{(k)}u, E\rangle &= \langle A_0^{(k)}u, u\rangle - \langle A^{(k)}u, E\rangle \\ &\geqslant \alpha^k\|u\|^2_{V^{(k)}} - c\|u\|_{V^{(k)}}\|E\|_{V^{(k)}}.\end{aligned}$$

With $u \in \hat{V}^{(k)}_{L,L_0}$ we get $(I - \hat{P}^{(k)}_{L,L_0})u = 0$ and

$$E = (I - \hat{P}^{(k)}_{L,L_0})(I+\hat{K})u = (I - \hat{P}^{(k)}_{L,L_0})\hat{K}u.$$

Note that $\hat{K}$ is a sum of 2^k-1 terms of the form $K_1\otimes\ldots\otimes K_k$ where each K_j is either I or $\tilde{K}$, and $K_j = \tilde{K}$ for at least one j. We consider one of these terms which contains a tensor product of k factors, where $i \geqslant 1$ factors are equal to $\tilde{K}$, and the remaining $k-i$ factors are equal to I. Without loss of generality we can assume that we have to estimate for $g := \tilde{K}^{(i)}\otimes I^{(k-i)}u$ the approximation error $\|(I - \hat{P}^{(k)}_{L,L_0})g\|_{V^{(k)}}$. Since $u \in \hat{V}^{(k)}_{L,L_0} \subset V^{(i)}\otimes\hat{V}^{(k-i)}_{L,L_0}$ (where we use the convention that the factor $\hat{V}^{(0)}_{L,L_0}$ is omitted) we obtain that $g \in V^{(i)}\otimes\hat{V}^{(k-i)}_{L,L_0}$ (where the factor $V^{(0)}_L$ is omitted). Since $V^{(i)}_{L_0}\otimes\hat{V}^{(k-i)}_{L,L_0} \subset \hat{V}^{(k)}_{L,L_0}$ we have that

$$(5.11)\quad \|(I-\hat{P}^{(k)}_{L,L_0})g\|_{V^{(k)}} \leqslant C\|(I-P^{(i)}_{L_0}\otimes\hat{P}^{(k-i)}_{L,L_0})g\|_{V^{(k)}} = C\|(I-P^{(i)}_{L_0}\otimes I^{(k-i)})g\|_{V^{(k)}}$$

where the last equality follows from $g \in V^{(i)}\otimes\hat{V}^{(k-i)}_{L,L_0}$. Now we use that $\hat{V}^{(i)}_{L_0} \subset V^{(i)}_{L_0}$, then the approximation result (3.12) for $\hat{V}^{(i)}_{L_0}$ gives

$$(5.12)\qquad \|(I - P^{(i)}_{L_0})f\|_{V^{(i)}} \leqslant CN_{L_0}^{-\delta/d}\|f\|_{X^{(i)}_\delta}.$$

As

$$(I - P^{(i)}_{L_0}\otimes I^{(k-i)})g = (I - P^{(i)}_{L_0})\otimes I^{(k-i)}g$$

we get from (5.11), (5.12) that

$$\|(I - \hat{P}^{(k)}_{L,L_0})g\|_{V^{(k)}} \leqslant CN^{-\delta/d}\|g\|_{X^{(i)}_\delta\otimes V^{(k-i)}} \leqslant CN_{L_0}^{-\delta/d}\|u\|_{V^{(k)}}$$

where we used $g = \tilde{K}^{(i)}\otimes I^{(k-i)}$ and the continuity of $\tilde{K}\colon\ V \to X_\delta$.

Since all $2^k - 1$ terms in E can be estimated in this way, we obtain

$$\|E\| \leqslant (2^k - 1) C N_{L_0}^{-\delta/d} \|u\|_{V^{(k)}}$$

and (5.10) gives

$$\langle A^{(k)} u, v \rangle \geqslant (\alpha^k - c(2^k - 1) C N_{L_0}^{-\delta/d}) \|u\|_{V^{(k)}}^2.$$

Therefore we can choose $L_0 > 0$ so that $\langle A^{(k)} u, v\rangle \geqslant \frac{1}{2}\alpha^k \|u\|_{V^{(k)}}^2$. As $\hat{K}\colon V \to V$ and $\hat{P}_{L,L_0}^{(k)}\colon V^{(k)} \to V^{(k)}$ are continuous we have that $\|v\|_{V^{(k)}} \leqslant c\|u\|_{V^{(k)}}$. □

The inf-sup condition now implies quasioptimal convergence (see [2]), and therefore the convergence rate is given by the approximation rate:

Theorem 5.3. *Assume* (2.2), (2.3).

(i) *Let $f \in L^k(\Omega, V')$. Then there is $L_0 > 0$ such that for all $L \geqslant L_0$ the sparse Galerkin approximation $\hat{Z}_L \in \hat{V}_{L,L_0}^{(k)}$ of $\mathcal{M}^k u$ is uniquely defined and converges quasioptimally, i.e. there is $C > 0$ such that for all $L \geqslant L_0$ we have*

$$\|\mathcal{M}^k u - \hat{Z}_L\|_{V^{(k)}} \leqslant C \inf_{v \in \hat{V}_{L,L_0}^{(k)}} \|\mathcal{M}^k u - v\|_{V^{(k)}} \to 0 \quad \text{as } L \to \infty.$$

(ii) *Assume that $f \in L^k(\Omega, Y_s)$ and the approximation property* (3.2). *Then for $0 \leqslant s \leqslant \sigma := \min\{s_0, p + 1 - \frac{1}{2}\varrho\}$ we have*

$$\|\mathcal{M}^k u - \hat{Z}_L\|_{V^{(k)}} \leqslant C(k) N_L^{-s/d} (\log N_L)^{(k-1)/2} \|f\|_{Y_s}. \tag{5.13}$$

Proof. The quasioptimality is a consequence of the inf-sup condition (5.9). Note that $\{V_\ell\}_{\ell=0}^\infty$ is dense in V, hence $\{V_\ell^{(k)}\}_{\ell=0}^\infty$ is dense in $V^{(k)}$. Since $V_{\tilde{L}}^{(k)} \subset \hat{V}_L^{(k)} \subset \hat{V}_{L,L_0}^{(k)}$ if $\tilde{L} < L/k$ we obtain that the sequence $\{\hat{V}_{\ell,L_0}^{(k)}\}_{\ell=L_0}^\infty$ is dense in $V^{(k)}$, which implies the convergence in (i). The convergence rate in (ii) follows from Proposition 3.1. □

5.2. Matrix compression

In the case when A is a differential operator the number of nonzero entries in the stiffness matrix for the standard FEM basis is $O(N)$. Therefore we can compute a matrix-vector product with $O(N)$ operations.

In the case of an integral equation the operator A is nonlocal and all elements of the stiffness matrix are nonzero. Then the cost of a matrix-vector product is $O(N^2)$, which increases the complexity of the algorithm. It is well known that one can improve the complexity to $O(N(\log N)^c)$ by using the so-called matrix compression with wavelets, see, e.g., [23], [8]. We will discuss this in this and the next section.

In the *compression step*, we replace most of the entries $A_{JJ'}$ of the stiffness matrix $\mathbf{A}^L$ with zeros, yielding an approximate stiffness matrix $\tilde{\mathbf{A}}^L$. The stiffness matrix $\mathbf{A}^L$ as well as its compressed variant $\tilde{\mathbf{A}}^L$ induce mappings from V^L to $(V^L)'$ which we denote by $\mathcal{A}_L$ and $\tilde{\mathcal{A}}_L$, respectively. We will require $\mathcal{A}_L$ and $\tilde{\mathcal{A}}_L$ to be close in the following sense: for certain values $s, s' \in [0,\sigma]$ with $\sigma = p + 1 - \frac{1}{2}\varrho$ and for $u \in X_s$, $v \in X_{s'}$ we have

$$(5.14)\qquad |\langle(\mathcal{A}_L - \tilde{\mathcal{A}}_L)P^L u, P^L v\rangle| \leqslant c(s,s')N_L^{-(s+s')/d}(\log N_L)^{q(s,s')}\|u\|_{X_s}\|v\|_{X_{s'}}$$

with $c(s,s') > 0$ and $q(s,s') \geqslant 0$ independent of L. The following result collects some properties of the corresponding approximate solutions.

Proposition 5.4. *Assume* (2.2) *and* (2.3).

1) *If* (5.14) *holds for* $(s,s') = (0,0)$ *with* $q(0,0) = 0$ *and* $c(0,0)$ *sufficiently small then there is* $L_0 > 0$ *such that for every* $L \geqslant L_0$, $(\tilde{\mathcal{A}}^L)^{-1}$ *exists and is uniformly bounded, i.e.*

$$(5.15)\qquad \forall\, L \geqslant L_0\colon\ \|(\tilde{\mathcal{A}}_L)^{-1}\|_{(V^L)' \to V^L} \leqslant C$$

for some C *independent of* L.

2) *If, in addition to the assumptions in* 1), (5.14) *holds with* $(s,s') = (\sigma,0)$, *then*

$$(5.16)\qquad \|(A^{-1} - (\tilde{\mathcal{A}}_L)^{-1})f\|_V \leqslant CN_L^{-\sigma/d}(\log N_L)^{q(\sigma,0)}\|f\|_{Y_\sigma}.$$

3) *Let* $g \in V'$ *be such that the solution* $\varphi \in V$ *of* $A'\varphi = g$ *belongs to* X_σ. *If, in addition to the assumptions in* 1) *and* 2), (5.14) *holds also for* $(s,s') = (0,\sigma)$ *and for* $(s,s') = (\sigma,\sigma)$, *then*

$$(5.17)\qquad |\langle g, (A^{-1} - (\tilde{\mathcal{A}}_L)^{-1})f\rangle| \leqslant CN_L^{-2\sigma/d}\cdot(\log N_L)^{\max\{q(0,\sigma)+q(\sigma,0),q(\sigma,\sigma)\}}$$

where $C = C(f,g)$.

Proof. 1) The Gårding inequality (2.2), the injectivity (2.3) and the density in V of the subspace sequence $\{V^L\}_L$ imply the discrete inf-sup condition (3.16).

Using (5.14) with $v^L \in V^L$ and $(s,s') = (0,0)$ we obtain due to (3.16)

$$\|\tilde{\mathcal{A}}_L v^L\|_{(V^L)'} \geqslant \|Av^L\|_{(V^L)'} - \|(A - \tilde{\mathcal{A}}_L)v^L\|_{(V^L)'} \geqslant c_s^{-1}\|v^L\|_V - Cc(0,0)\|v^L\|_V.$$

This implies that for $c(0,0) < 1/(2Cc_s)$ there is $L_0 > 0$ such that for all $L \geqslant L_0$

$$\|v^L\|_V \leqslant \frac{c_s}{2}\|\tilde{\mathcal{A}}_L v^L\|_{(V^L)'} \quad \forall\, v^L \in V^L, \tag{5.18}$$

whence (5.15).

2) Let $f \in Y_\sigma$ and $u = A^{-1}f$, $\tilde{u}^L = (\tilde{\mathcal{A}}_L)^{-1}f$ for $L \geqslant L_0$. We have

$$\|u - \tilde{u}^L\|_V \leqslant \|u - P^L u\|_V + \|P^L u - \tilde{u}^L\|_V.$$

Using (3.16) and $\langle \tilde{\mathcal{A}}_L u^L, v^L\rangle = \langle Au, v^L\rangle$ for all $v^L \in V^L$, we get

$$\|P^L u - \tilde{u}^L\|_V \leqslant C\|\tilde{\mathcal{A}}_L(P^L u - \tilde{u}^L)\|_{(V^L)'} = C\|\tilde{\mathcal{A}}_L P^L u - Au\|_{(V^L)'},$$

which yields

$$\|u - \tilde{u}^L\|_V \leqslant \|u - P^L u\|_V + C\|A(u - P^L u)\|_{(V^L)'} + C\|(A - \tilde{\mathcal{A}}_L)P^L u\|_{(V^L)'}. \tag{5.19}$$

Here, the first two terms are estimated using the stability (P3) and (3.6) which imply

$$\|u - P^L u\|_V \leqslant C \inf_{v \in V^L} \|u - v\|_V \leqslant C(N_L)^{-\sigma/d}\|u\|_{X_\sigma}, \tag{5.20}$$

and the continuity $A\colon V \to V'$. The third term in (5.19) is estimated by (5.14) for $(s, s') = (\sigma, 0)$ and $P^L v^L = v^L$ for all $v^L \in V^L$:

$$|\langle (A - \tilde{\mathcal{A}}_L)P^L u, v^L\rangle| \leqslant Cc(\sigma,0)N_L^{-\sigma/d}(\log N_L)^{q(\sigma,0)}\|u\|_{X_\sigma}\|v\|_V. \tag{5.21}$$

3) To show (5.17), we let $\varphi^L := P^L\varphi$ for $\varphi = (A')^{-1}g \in X_\sigma$ and $u = A^{-1}f$, $\tilde{u}^L = (\tilde{\mathcal{A}}_L)^{-1}f$ for $L \geqslant L_0$. Then

$$|\langle g, u - \tilde{u}^L\rangle| = |\langle \varphi, A(u - \tilde{u}^L)\rangle| \leqslant |\langle A(u - \tilde{u}^L), \varphi - \varphi^L\rangle| + |\langle A(u - \tilde{u}^L), \varphi^L\rangle|.$$

We estimate the first term by $C\|u - \tilde{u}^L\|_V\|\varphi - P^L\varphi\|_V$, which implies the bound (5.17) by virtue of (5.16) and (5.20). The second term is bounded as follows:

$$\begin{aligned}\langle A(u - \tilde{u}^L), \varphi^L\rangle &= \langle (\tilde{\mathcal{A}}_L - A)\tilde{u}^L, \varphi^L\rangle \\ &= \langle (\tilde{\mathcal{A}}_L - A)(\tilde{u}^L - P^L u), P^L_\varphi\rangle + \langle (\tilde{\mathcal{A}}_L - A)P^L u, P^L\varphi\rangle.\end{aligned}$$

Here we estimate the second term by (5.14) with $(s, s') = (\sigma, \sigma)$. For the first term, we use (5.14) with $(s, s') = (0, \sigma)$ to obtain

$$\begin{aligned}&|\langle (\tilde{\mathcal{A}}_L - A)P^L(\tilde{u}^L - P^L u), P^L\varphi\rangle| \\ &\qquad \leqslant CN_L^{-\sigma/d}(\log N_L)^{q(0,\sigma)}\|\tilde{u}^L - P^L u\|_V\|\varphi\|_{X_\sigma} \\ &\qquad \leqslant CN_L^{-\sigma/d}(\log N_L)^{q(0,\sigma)}(\|\tilde{u}^L - u\|_V + \|u - P^L u\|_V)\|\varphi\|_{X_\sigma}.\end{aligned}$$

Using here (5.16) and (5.20) completes the proof. □

5.3. Wavelet compression

We now explain how to obtain an approximate stiffness matrix $\tilde{\mathbf{A}}^L$ which on the one hand has only $O(N_L(\log N_L)^a)$ nonzero entries (out of N_L^2), and on the other hand satisfies the consistency condition (5.14).

Here we assume that the operator A is given in terms of a Schwartz kernel $k(x,y)$ as ($\Gamma = \partial D$ where D is a bounded Lipschitz polyhedron)

$$(A\varphi)(x) = \int_{y\in\Gamma} k(x,y)\varphi(y)\,\mathrm{d}\Gamma(y) \tag{5.22}$$

for $\varphi \in C_0^\infty(\Gamma)$ where $k(x,z)$ satisfies the Calderón-Zygmund estimates

$$|D_x^\alpha D_y^\beta\, k(x,y)| \leqslant C_{\alpha\beta}|x-y|^{-(d+\varrho+|\alpha|+|\beta|)}. \tag{5.23}$$

In the following, we combine the indices (ℓ, j) into a multiindex $J = (\ell, j)$ to simplify notation, and write ψ_J, $\psi_{J'}$, etc.

Due to the vanishing moment property (3.4) of the basis $\{\psi_J\}$, the entries $A^L_{JJ'} = \langle A\psi_J, \psi_{J'}\rangle$ of the moment matrix $\mathbf{A}^L$ with respect to the basis $\{\psi_J\}$ show fast decay (cf. [23]). Denote $S_J = \operatorname{supp}(\psi_J)$, $S_{J'} = \operatorname{supp}(\psi_{J'})$. Then we have the following decay estimate for $A_{JJ'}$ (see [23, Lemma 8.2.1]).

Proposition 5.5. *If the wavelets ψ_J, $\psi_{J'}$ satisfy the moment condition* (3.4) *and A satisfies* (5.22), (5.23)*, then*

$$|\langle A\psi_J, \psi_{J'}\rangle| \leqslant C\,\operatorname{dist}(S_J, S_{J'})^{-\gamma}\, 2^{-\gamma(\ell+\ell')/2} \tag{5.24}$$

where $\gamma := \varrho + d + 2 + 2(p^*+1) > 0$.

This can be exploited to truncate $\mathbf{A}^L$ to obtain a sparse matrix $\tilde{\mathbf{A}}^L$ with at most $O(N_L(\log N_L)^2)$ nonzero entries and such that (5.14) is true with $c(0,0)$ as small as desired, independent of L, $q(0,0) = 0$, and $q(0,\sigma) = q(\sigma,0) \leqslant \frac{3}{2}$, $q(\sigma,\sigma) \leqslant 3$, see [21], [23], [8], for example. The number of nonzero entries in a block $\tilde{\mathbf{A}}^L_{\ell,\ell'}$ of $\tilde{\mathbf{A}}^L$ is bounded by

$$nnz(\tilde{\mathbf{A}}^L_{\ell,\ell'}) \leqslant C(\min(\ell,\ell')+1)^d\, 2^{d\max(\ell,\ell')}. \tag{5.25}$$

Remark 5.6. For integral operators A an alternative approach for the efficient computation of matrix-vector products with the stiffness matrix $\mathbf{A}^L$ is given by the so-called cluster approximation. In this case one additionally assumes for the operator (5.22) that the kernel $k(x,z)$ is analytic in x and z for $z \neq 0$, and the size of its domain of analyticity is proportional to z. Under these assumptions, one can replace

$k(x,y)$ in (5.22) for $|x-y|$ sufficiently large by a so-called cluster approximation with degenerate kernels which are obtained by either truncated multipole expansions or polynomial interpolants of order $\log N_L$, allowing to apply the block $\tilde{\mathbf{A}}^L_{\ell,\ell'}$ to a vector in at most

$$C(\log N_L)^d\, 2^{d\max(\ell,\ell')}, \quad 0 \leqslant \ell, \ell' \leqslant L \tag{5.26}$$

operations. See, e.g., [20] for details.

5.4. Error analysis for sparse Galerkin with matrix compression

Based on the compressed stiffness matrix $\tilde{\mathbf{A}}^L$ and the corresponding operator $\tilde{\mathcal{A}}_L\colon V_L \to (V_L)'$ induced by it, we define the sparse tensor product approximation of $\mathcal{M}^k u$ with matrix compression analogous to (5.2) by: find $\tilde{Z}^k_L \in \hat{V}^{(k)}_{L,L_0}$ such that for all $v \in \hat{V}^{(k)}_{L,L_0}$

$$\langle \tilde{\mathcal{A}}^{(k)}_L \tilde{Z}^k_L, v\rangle = \langle \mathcal{M}^k f, v\rangle. \tag{5.27}$$

We prove bounds for the error $\tilde{Z}^k_L - \mathcal{M}^k u$.

Lemma 5.7. *Assume* (2.2), (2.3) *and that the spaces* V_L *as in Example* 3.2 *admit a hierarchic basis* $\{\psi^\ell_j\}_{\ell\geqslant 0}$ *satisfying* (P1)–(P5). *Assume further that the operator* $\tilde{\mathcal{A}}_L$ *in* (5.27) *satisfies the consistency estimate* (5.14) *for* $s = s' = 0$, $q(0,0) = 0$, *and with sufficiently small* $c(0,0)$.

Then there is $L_0 > 0$ *such that for all* $L \geqslant L_0$, *the* k*th moment problem with matrix compression,* (5.27), *admits a unique solution and the following error estimate holds:*

$$\begin{aligned} &\|\mathcal{M}^k u - \tilde{Z}^k_L\|_{V^{(k)}} \\ &\quad\leqslant C \inf_{v\in \hat{V}^{(k)}_L} \left\{ \|\mathcal{M}^k u - v\|_{V^{(k)}} + \sup_{0\neq w\in \hat{V}^{(k)}_L} \frac{|\langle (A^{(k)}_L - \tilde{\mathcal{A}}^{(k)}_L) v, w\rangle|}{\|w\|_{V^{(k)}}} \right\}. \end{aligned} \tag{5.28}$$

P r o o f. We show unique solvability of (5.27) for sufficiently large L. By Theorem 5.2 we have that (5.9) holds.

To show unique solvability of (5.27), we write

$$\begin{aligned} A^{(k)} - \tilde{\mathcal{A}}^{(k)}_L &= (A - \tilde{\mathcal{A}}_L) \otimes A^{(k-1)} + \tilde{\mathcal{A}}_L \otimes (A^{(k-1)} - \tilde{\mathcal{A}}^{(k-1)}_L) \\ &= (A - \tilde{\mathcal{A}}_L) \otimes A^{(k-1)} + \tilde{\mathcal{A}}_L \otimes (A - \tilde{\mathcal{A}}_L) \otimes A^{(k-2)} + \tilde{\mathcal{A}}^{(2)}_L \otimes (A^{(k-2)} - \tilde{\mathcal{A}}^{(k-2)}_L) \end{aligned}$$

and obtain, after iteration,

$$(5.29)\quad A^{(k)} - \tilde{\mathcal{A}}_L^{(k)} = (A - \tilde{\mathcal{A}}_L) \otimes A^{(k-1)} + \sum_{\nu=1}^{k-2} \tilde{\mathcal{A}}_L^{(\nu)} \otimes (A - \tilde{\mathcal{A}}_L) \otimes A^{(k-\nu-1)} + \tilde{\mathcal{A}}_L^{(k-1)} \otimes (A - \tilde{\mathcal{A}}_L)$$

(where the sum is omitted if $k = 2$). We get from (5.9) that for any $u \in \hat{V}_L^{(k)}$ there exists $v \in \hat{V}_L^{(k)}$ such that

$$(5.30)\quad \begin{aligned} \langle \tilde{\mathcal{A}}_L^{(k)} u, v\rangle &= \langle A^{(k)} u, v\rangle + \langle (\tilde{\mathcal{A}}_L^{(k)} - A^{(k)})u, v\rangle \\ &\geqslant \left[c_S^{-1} - \sup_{w \in \hat{V}_L^{(k)}} \sup_{\tilde{w} \in \hat{V}_L^{(k)}} \frac{\langle (\tilde{\mathcal{A}}_L^{(k)} - A^{(k)})w, \tilde{w}\rangle}{\|w\|_{V^{(k)}} \|\tilde{w}\|_{V^{(k)}}} \right] \|u\|_{V^{(k)}} \|v\|_{V^{(k)}}. \end{aligned}$$

To obtain an upper bound for the supremum, we admit $w, \tilde{w} \in V_{L,L_0}^{(k)} \supseteq \hat{V}_L^{(k)}$, use (5.29) and (5.14) with $s = s' = 0$ and $q(0,0) = 0$ to get

$$\|\tilde{\mathcal{A}}_L\|_{V_L \to (V_L)'} \leqslant \underbrace{\|A\|_{V \to V'}}_{c_A} + c(0,0)$$

and therefore estimate for any $w, \tilde{w} \in V_{L,L_0}^{(k)}$

$$(5.31)\quad \begin{aligned} &|\langle (\tilde{\mathcal{A}}_L^{(k)} - A^{(k)})w, \tilde{w}\rangle| \\ &\leqslant c(0,0) \left[c_A^{k-1} + \left(\sum_{\nu=1}^{k-2} (c_A + c(0,0))^\nu c_A^{k-\nu-1} \right) + (c_A + c(0,0))^{k-1} \right] \\ &\quad \times \|w\|_{V^{(k)}} \|\tilde{w}\|_{V^{(k)}} \\ &= c(0,0) \cdot C(A,k) \|w\|_{V^{(k)}} \|\tilde{w}\|_{V^{(k)}}. \end{aligned}$$

If $c(0,0)$ is sufficiently small, this implies due to (5.28) the stability of $\tilde{\mathcal{A}}_L^{(k)}$ on $\hat{V}_{L,L_0}^{(k)}$: there is $L_0 > 0$ such that

$$(5.32)\quad \inf_{0 \neq u \in \hat{V}_{L,L_0}^{(k)}} \sup_{0 \neq v \in \hat{V}_{L,L_0}^{(k)}} \frac{\langle \tilde{\mathcal{A}}_L^{(k)} u, v\rangle}{\|u\|_{V^{(k)}} \|v\|_{V^{(k)}}} \geqslant \frac{1}{2c_S} > 0$$

for all $L \geqslant L_0$, and hence the unique solvability of (5.27) for these L follows.

To prove (5.28), we proceed as in the proof of the first Strang Lemma (e.g. [7]). □

We now use this lemma to obtain the following convergence result:

Theorem 5.8. *Assume* (2.2), (2.3), $V = H^{\varrho/2}(\Gamma)$ *and that the subspaces* $\{V_\ell\}_{\ell=0}^{\infty}$ *are as in Example* 3.2, *and that in the smoothness spaces* $X_s = H^{\varrho/2+s}(\Gamma)$, $s \geqslant 0$, *the operator* $A\colon X_s \to Y_s$ *is bijective for* $0 \leqslant s \leqslant s_0$ *with some* $s_0 > 0$. *Assume further that a compression strategy for the matrix* $\mathbf{A}^L$ *in the hierarchic basis* $\{\psi_j^\ell\}$ *satisfying* (P1)–(P5) *is available with* (5.14) *for* $s' = 0$, $0 \leqslant s \leqslant \sigma = p + 1 - \frac{1}{2}\varrho$, $q(0,0) = 0$ *and with arbitrary small* $c(0,0)$, *independent of* L *for* $L \geqslant L_0$. *Then with* $\delta = \min\{p + 1 - \frac{1}{2}\varrho, s\}/d$, $0 \leqslant s \leqslant s_0$ *we have the error estimate*

$$\|\mathcal{M}^k u - \tilde{Z}_L^k\|_{V^{(k)}} \leqslant C(\log N_L)^{\min\{(k-1)/2, q(s,0)\}} N_L^{-\delta} \|\mathcal{M}^k f\|_{Y_s^{(k)}}. \tag{5.33}$$

P r o o f. We use (5.28) with the choice $v = \hat{P}_{L,L_0}^{(k)}$ and apply to $\|\mathcal{M}^k u - v\|_{V^{(k)}}$ the approximation result (2.6). We express the difference $A_L^{(k)} - \tilde{A}_L^{(k)}$ using (5.29). Then we obtain a sum of terms, each of which can be bounded using (5.14) and the continuity of $A_L^{(k)}$ and $\tilde{\mathcal{A}}_L^{(k)}$. This yields the following error bound:

$$\|\mathcal{M}^k u - \tilde{Z}_L^k\|_{V^{(k)}} \leqslant C[(\log N_L)^{(k-1)/2} N_L^{-\delta} + c(s,0)(\log N_L)^{q(s,0)} N_L^{-s/d}] \|\mathcal{M}^k u\|_{X_s^{(k)}}.$$

□

Theorem 5.8 addressed only the convergence of $\tilde{Z}_L^k$ in the "energy" norm $V^{(k)}$. In the applications which we have in mind, however, also functionals of the solution $\mathcal{M}^k u$ are of interest which we assume are given in the form $\langle G, \mathcal{M}^k u\rangle$ for some $G \in (V^{(k)})'$. We approximate such functionals by $\langle G, \tilde{Z}_L^k\rangle$.

Theorem 5.9. *With all assumptions as in Theorem* 5.8, *and, in addition, assuming that the adjoint problem*

$$(A^{(k)})'\Psi = G \tag{5.34}$$

admits a solution $\Psi \in X_{s'}^{(k)}$ *for some* $0 < s' \leqslant \sigma$ *and that the compression* $\tilde{\mathbf{A}}^L$ *of the stiffness matrix* $\mathbf{A}^L$ *satisfies* (5.14) *with* $s = s' = \sigma$, *we have*

$$|\langle G, \mathcal{M}^k u\rangle - \langle G, \tilde{Z}_L^k\rangle| \leqslant C(\log N_L)^{\min\{k-1, q(s,s')\}} N_L^{-(\delta+\delta')} \|\mathcal{M}^k f\|_{Y_s^{(k)}}$$

where $\delta = \min\{p + 1 - \frac{1}{2}\varrho, s\}/d$, $\delta' = \min\{p + 1 - \frac{1}{2}\varrho, s'\}/d$.

P r o o f. The proof is analogous to that of Theorem 5.4. 3), using the sparse approximation property (3.12) in place of (3.2). □

5.5. Iterative solution of the linear system

We solve the linear system (5.27) using iterative solvers and denote the matrix of this system by $\hat{\tilde{\mathbf{A}}}_L^{(k)}$. We will consider three different methods:

(1) If A is self-adjoint and (2.2) holds with $T = 0$ the matrix $\hat{\tilde{\mathbf{A}}}_L^{(k)}$ is Hermitian positive definite, and we use the conjugate gradient algorithm which requires one matrix-vector multiplication by the matrix $\hat{\tilde{\mathbf{A}}}_L^{(k)}$ per iteration.

(2) If A is not necessarily self-adjoint, but satisfies (2.2) with $T = 0$ we can use the GMRES algorithm with restarts every μ iterations. In this case $\hat{\tilde{\mathbf{A}}}_L^{(k)} + (\hat{\tilde{\mathbf{A}}}_L^{(k)})^H$ is positive definite. This requires two matrix-vector multiplications per iteration, one with $\hat{\tilde{\mathbf{A}}}_L^{(k)}$ and one with $(\hat{\tilde{\mathbf{A}}}_L^{(k)})^H$.

(3) In the general case when (2.2) is satisfied with some operator T we multiply the linear system by the matrix $(\hat{\tilde{\mathbf{A}}}_L^{(k)})^H$ and can then apply the conjugate gradient algorithm. This requires one matrix-vector multiplication with $\hat{\tilde{\mathbf{A}}}_L^{(k)}$ and one matrix-vector multiplication with $(\hat{\tilde{\mathbf{A}}}_L^{(k)})^H$ per iteration.

In order to achieve log-linear complexity it is essential that we never explicitly form the matrix $\hat{\tilde{\mathbf{A}}}_L^{(k)}$. Instead, we only store the matrix $\tilde{\mathbf{A}}^L$ for the mean field problem. We can then compute a matrix-vector product with $\hat{\tilde{\mathbf{A}}}_L^{(k)}$ (or $(\hat{\tilde{\mathbf{A}}}_L^{(k)})^H$) by an algorithm which multiplies parts of the coefficient vector by submatrices of $\tilde{\mathbf{A}}^L$, see Algorithm 5.10 in [25]. This requires $O((\log N_L)^{kd+2k-2} N_L)$ operations ([25, Theorem 5.12]).

Let us explain the algorithm in the case $k = 2$ and $L_0 = 0$: In this case a coefficient vector $\underline{u}$ has components $u_{j\,j'}^{l\,l'}$ where l, l' are the levels used for $\hat{V}_L^{(2)}$ (i.e., $l, l' \in \{0, \dots, L\}$ such that $l + l' \leqslant L + L_0$) and $j \in \{1, \dots, M_l\}$, $j' \in \{1, \dots, M_{l'}\}$. Let $\tilde{\mathbf{A}}^{L_1}$ denote the submatrix of $\tilde{\mathbf{A}}^L$ corresponding to levels $l, l' \leqslant L_1$. We can then compute the coefficients of the vector $\hat{\tilde{\mathbf{A}}}_L^{(k)} \underline{u}$ as follows where we overwrite at each step the current components with the result of a matrix-vector product:

- For $l = 0, \dots, L$, $j = 1, \dots, M_l$:
  multiply the column vector with components $(u_{j\,j'}^{l\,l'})_{\substack{l'=0\dots L-l \\ j'=0\dots M_{l'}}}$ by the matrix $\tilde{\mathbf{A}}^{L-l}$.
- For $l' = 0, \dots, L$, $j' = 1, \dots, M_l$:
  multiply the column vector with components $(u_{j\,j'}^{l\,l'})_{\substack{l=0\dots L-l \\ j=0\dots M_l}}$ by the matrix $\tilde{\mathbf{A}}^{L-l'}$.

We now analyze the convergence of the iterative solvers. The stability assumptions for the wavelet basis, the continuous and discrete operators imply the following results about the approximate stiffness matrix $\hat{\tilde{\mathbf{A}}}_L^{(k)}$:

Proposition 5.10. *Assume the basis $\{\psi_j^\ell\}$ satisfies (3.3) with c_B independent of L.*

(i) *Assume that* $\tilde{\mathcal{A}}_L$ *satisfies* (5.14) *for* $q(0,0) = 0$ *with sufficiently small* $c(0,0)$. *Then there is* C_2 *such that for all* L *the matrix* $\hat{\tilde{\mathbf{A}}}_L^{(k)}$ *of the problem* (5.27) *satisfies*

$$\|\hat{\tilde{\mathbf{A}}}_L^{(k)}\|_2 \leqslant C_2 < \infty. \tag{5.35}$$

(ii) *Assume additionally to the assumptions of* (i) *that* (2.2) *holds with* $T = 0$. *Then there is* $C_1 > 0$ *such that*

$$\lambda_{\min}(\tfrac{1}{2}(\hat{\tilde{\mathbf{A}}}_L^{(k)} + (\hat{\tilde{\mathbf{A}}}_L^{(k)})^H)) \geqslant C_1 > 0. \tag{5.36}$$

(iii) *Assume the discrete inf-sup condition* (5.9) *holds. Then we have with* C *independent of* L

$$\|(\hat{\tilde{\mathbf{A}}}_L^{(k)})^{-1}\|_2 \leqslant Cc_S. \tag{5.37}$$

Proof. Because of (3.3) the norm $\|v_L\|_{V^{(k)}}$ of $v_L \in \hat{V}_{L,L_0}^{(k)}$ is equivalent to the 2-vector-norm $\|\underline{v}\|_2$ of the coefficient vector $\underline{v}$. For (i) we obtain an arbitrarily small upper bound for the bilinear form with the operator $\mathcal{A} - \tilde{\mathcal{A}}_L$ with respect to the norm $\|v_L\|_{V^{(k)}}$. Since $\mathcal{A}$ is continuous we get an upper bound for the norm of $\tilde{\mathcal{A}}$ and therefore for the corresponding 2-matrix-norm.

In (ii) the bilinear form $\langle Av, v\rangle$ corresponds to the symmetric part of the matrix, and the lower bound corresponds to the smallest eigenvalue of the matrix. Since the norm of $\mathcal{A} - \tilde{\mathcal{A}}$ is arbitrarily small we also get lower bound for the compressed matrix.

In (iii) the inf-sup condition (5.9) states that for $L \geqslant L_0$ the solution operator mapping $(\hat{V}_{L,L_0}^{(k)})'$ to $\hat{V}_{L,L_0}^{(k)}$ is bounded by c_S. Because of the norm equivalence (3.3) this implies $\|(\hat{\tilde{\mathbf{A}}}_L^{(k)})^{-1}\|_2 \leqslant Cc_S$. □

For the method (1) with a self-adjoint positive definite operator A we have that $\lambda_{\max}/\lambda_{\min} \leqslant C_2/C_1 =: \kappa$ is bounded and independent of L, and obtain for the conjugate gradient iterates error estimates

$$\|\underline{u}^{(m)} - \underline{u}\|_2 \leqslant c\Big(1 - \frac{2}{\kappa^{1/2}+1}\Big)^m.$$

For the method (2) we obtain for the GMRES from [10] for the restarted GMRES method (e.g., with restart $\mu = 1$)

$$\|\underline{u}^{(m)} - \underline{u}\|_2 \leqslant c\Big(1 - \frac{1}{\kappa}\Big)^m.$$

For the method (3) we use the conjugate gradient method with the matrix $B := (\hat{\hat{\mathbf{A}}}_L^{(k)})^H \hat{\hat{\mathbf{A}}}_L^{(k)}$ and need the largest and smallest eigenvalue of this matrix. Now (5.37) states that $\lambda_{\min}(B) \geqslant (Cc_S)^{-2} > 0$. Therefore we have with $\tilde{\kappa} := C_2^2(Cc_S)^2$ that

$$\|\underline{u}^{(m)} - \underline{u}\|_2 \leqslant c\Big(1 - \frac{2}{\tilde{\kappa}^{1/2}+1}\Big)^m.$$

Note that the 2-vector norm $\|\underline{u}\|_2$ of the coefficient vector is equivalent to the norm $\|u\|_{V^{(k)}}$ of the corresponding function on $D \times \ldots \times D$. If we start with initial guess zero we therefore need a number M of iterations proportional to L to have an iteration error which is less than the Galerkin error. However, if we start on the coarsest mesh with initial guess zero, perform a sufficiently large (but independent of L) number M of iterations, use the resulting solution vector as starting vector on the next finer mesh, perform once more the same number M of iterations on this mesh, and so on, we obtain an approximate solution of the linear system with accuracy of the order of the discretization error in total work proportional to N, i.e. without the additional logarithmic factor L.

Therefore we have the following complexity result:

Proposition 5.11. *We can compute an approximation Z_L^k for $\mathcal{M}^k u$ using a fixed number m_0 of iterations such that*

$$\|Z_L^k - \mathcal{M}^k u\|_{V^{(k)}} \leqslant C N_L^{-s/d} L^\beta$$

where $\beta = \frac{1}{2}(k-1)$ in the case of a differential operator, $\beta = \min\{\frac{1}{2}(k-1), q(s,0)\}$ with $q(s,0)$ from (5.33) in the case when A is an integral operator. The total number of operations is $O(N(\log N)^{k-1})$ in the case of a differential operator. In the case of an integral operator we need at most $O(N(\log N)^{k+1})$ operations.

6. Examples: FEM and BEM for the Helmholtz equation

We now consider the Helmholtz equation in a domain $G \subset \mathbb{R}^n$ with boundary $\Gamma := \partial G$. We will discuss two ways of solving this equation with stochastic data: First we use the finite element approximation of the differential equation and apply our results for $D = G$ which is of dimension $d = n$.

Secondly, we consider the boundary integral formulation which is an integral equation on the boundary Γ. We discretize this equation and then apply our results for $D = \Gamma$ which is of dimension $d = n - 1$. In this case we can also allow exterior domains G as the computation is done on the bounded manifold Γ.

To keep the presentation simple we will just consider smooth boundaries and one type of boundary condition (Dirichlet condition for finite elements, Neumann condition for boundary elements). Other boundary conditions and operators can be treated in a similar way.

6.1. Finite element methods

Let $G \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. We consider the boundary value problem $V = H_0^1(G)$

$$(-\Delta - \kappa^2)u(\omega) = f(\omega) \quad \text{in } G, \quad u|_\Gamma = 0.$$

Here we have $V = H_0^1(G)$, $V' = H^{-1}(G)$, and the operator $A\colon\ V \to V'$ is defined by

$$\langle Au, v\rangle = \int_G (\nabla u \cdot \nabla v - \kappa^2 uv)\, \mathrm{d}x$$

and obviously satisfies the Gårding inequality $\langle Au, u\rangle \geqslant \|u\|_V^2 - (\kappa^2 + 1)\|u\|_{L^2(G)}^2$. The operator $-\Delta\colon\ V \to V'$ has eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots$ which converge to ∞. We need to assume that κ^2 is not one of the eigenvalues λ_j so that condition (2.3) is satisfied.

The spaces for smooth data for $s > 0$ are $Y_s = H^{-1+s}(G)$, the corresponding solution spaces are $X_s = H^{1+s}(G)$. We assume that the stochastic right-hand side function $f(\omega)$ satisfies $f \in L^k(\Omega, Y_s) = L^k(\Omega, H^{-1+s}(G))$ for some $s > 0$.

The space V_L has $N_L = O(h_L^{-d}) = O(2^{Ld})$ degrees of freedom and the sparse tensor product space $\hat{V}_{L,L_0}^{(k)}$ has $O(N_L(\log N_L)^{(k-1)})$ degrees of freedom. For $k \geqslant 1$ we can then numerically obtain a sparse grid approximation $Z_L^k \in \hat{V}_{L,L_0}^{(k)}$ for $\mathcal{M}^k u$ using a total of $O(N_L(\log N_L)^{(k-1)})$ operations satisfying the error estimate

$$\|Z_L^k - \mathcal{M}^k u\| \leqslant c h_L^p |\log(h_L)|^{(k-1)/2} \|f\|_{L^k(\Omega, Y_p)}^k$$

provided we have $f \in L^k(\Omega, Y_p)$.

6.2. Boundary element methods

We illustrate the preceding abstract results with the boundary reduction of the stochastic Neumann problem to a boundary integral equation of the first kind.

In a bounded domain $G \subset \mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial G$, we consider

$$(\Delta + \kappa^2)U = 0 \quad \text{in } D \tag{6.1}$$

with a wave number $\kappa^2 \in \mathbb{C}$ not an interior Neumann eigenvalue and boundary conditions

$$\gamma_1 U = n \cdot (\nabla U)|_\Gamma = \sigma \quad \text{on } \Gamma \tag{6.2}$$

where $\sigma \in L^k(\Omega, H^{-1/2}(\Gamma))$ with an integer $k \geqslant 1$ are given random boundary data, n is the exterior unit normal to Γ, and $H^s(\Gamma)$, $|s| \leqslant 1$, denotes the usual Sobolev spaces on Γ, see, e.g., [18]. We assume in (6.2) that P-a.s.

$$\langle \sigma, 1 \rangle = 0 \tag{6.3}$$

and, if $d = 2$, in (6.1) that

$$\operatorname{diam}(D) < 1. \tag{6.4}$$

Then the problem (6.1), (6.2) admits a unique solution $U \in L^k(\Omega, H^1(D))$ [24], [25].

For the boundary reduction, we define for $v \in H^{1/2}(\Gamma)$ the boundary integral operator

$$(Wv)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} e(x, y) v(y) \, \mathrm{d}s_y \tag{6.5}$$

with $e(x, y)$ denoting the fundamental solution of $-\Delta - \kappa^2$. The integral operator W is continuous (e.g. [18]),

$$W\colon \; H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \tag{6.6}$$

To reduce the stochastic Neumann problem (6.1), (6.2) to a boundary integral equation with $\sigma \in L^k(\Omega, H^{-1/2}(\Gamma))$ satisfying (6.3) a.s., we use a representation as a double layer potential R_2:

$$U(x, \omega) = (R_2\theta)(x, \omega) := -\int_{y \in \Gamma} \frac{\partial}{\partial n_y} e(x, y) \theta(y, \omega) \, \mathrm{d}s_y \tag{6.7}$$

where $\mathbb{E}_\theta$ satisfies for $\kappa \neq 0$ the BIE

$$W_1 \mathbb{E}_\theta = \mathbb{E}_\sigma \,, \tag{6.8}$$

with the hypersingular boundary integral operator $W_1 u := Wu + \langle u, 1 \rangle$.

We see that the mean field $\mathcal{M}^1 U$ can be obtained by solving the deterministic boundary integral equation (6.8). Based on the compression error analysis in Section 5.2, we obtain an approximate solution $E_\theta^L \in V^L$ in $O(N_L (\log N_L)^2)$ operations and memory with an error bound

$$\|E_\theta - E_\theta^L\|_{H^{1/2}(\Gamma)} \leqslant c N_L^{-(p+1/2)} (\log N_L)^{3/2} \|\sigma\|_{L^1(\Omega, H^{p+1}(\Gamma))}.$$

To determine the variance of the random solution U, second moments of θ are required. To derive boundary integral equations for them, we use that by Fubini's theorem, the operator $\mathcal{M}^2$ and the layer potential R_2 commute. For (6.1), (6.2) with $\sigma \in L^2(\Omega, H^{-1/2}(\Gamma))$, we obtain that $C_\theta = \mathcal{M}^2\theta$ satisfies for $\kappa = 0$ the BIE

$$(W_1 \otimes W_1)C_\theta = C_\sigma \quad \text{in } H^{1/2,1/2}(\Gamma \times \Gamma). \tag{6.9}$$

Here, the 'energy' space V equals $H^{1/2}(\Gamma)$ and $A = W_1$.

The unique solvability of the BIE (6.9) is ensured by

Proposition 6.1. *If $\kappa = 0$, the integral operator $W_1 \otimes W_1$ is coercive, i.e. there is $c_S > 0$ such that*

$$\forall\, C_\theta \in H^{1/2,1/2}(\Gamma \times \Gamma)\colon\ \langle (W_1 \otimes W_1)C_\theta, C_\theta\rangle \geqslant c_S \|C_\theta\|^2_{H^{1/2,1/2}(\Gamma\times\Gamma)}. \tag{6.10}$$

Proof. We prove (6.10). The operator W_1 is self-adjoint and coercive in $H^{1/2}(\Gamma)$ (e.g. [19], [13], [18]). Let $\{u_i\}_{i=1}^\infty$ denote an $H^{1/2}(\Gamma)$ orthonormal base in $H^{1/2}(\Gamma)$ consisting of eigenfunctions of W_1. Then, $\{u_i \otimes u_j\}_{i,j=1}^\infty$ is an orthonormal base in $H^{1/2,1/2}(\Gamma \times \Gamma)$ and we may represent any $C_\theta \in H^{1/2,1/2}(\Gamma\times\Gamma)$ in the form $C_\theta = \sum\limits_{i,j=1}^{\infty} c_{ij} u_i \otimes u_j$. For any $M < \infty$, consider $C_\theta^M = \sum\limits_{i,j=1}^{M} c_{ij} u_i \otimes u_j$. Then we calculate

$$\begin{aligned}\langle (W_1 \otimes W_1)C_\theta^M, C_\theta^M\rangle &= \Big\langle (W_1 \otimes W_1) \sum_{i,j=1}^{M} c_{ij} u_i \otimes u_j, \sum_{i',j'=1}^{M} c_{i'j'} u_{i'} \otimes u_{j'} \Big\rangle \\ &= \sum_{i,j=1}^{M} \lambda_i \lambda_j c_{ij}^2 \geqslant \lambda_1^2 \sum_{i,j=1}^{M} c_{ij}^2 = \lambda_1^2 \|C_\theta^M\|^2_{H^{1/2,1/2}(\Gamma\times\Gamma)}.\end{aligned}$$

Passing to the limit $M \to \infty$, we obtain (6.10). □

In the case $\kappa \neq 0$ we use that the integral operator W satisfies a Gårding inequality in $H^{1/2}(\Gamma)$ and obtain unique solvability of the BIE (6.9) for C_θ from Theorem 2.4, provided that W is injective, i.e. that κ is not an eigenvalue of the interior Neumann problem.

To compute the second moments of the random solution $U(x,\omega)$ at an interior point $x \in D$, we use

$$(\mathcal{M}^2 U)(x,x) = \mathcal{M}^2 R_2 \theta = (R_2 \otimes R_2)(\mathcal{M}^2\theta)$$

and obtain from Theorem 5.9 and the sparse tensor product approximation $\tilde{Z}_L^2$ of $\mathcal{M}^2\theta$ in $O(N_L(\log N_L)^3)$ operations and memory an approximation of $(\mathcal{M}^2U)(x,x)$ which satisfies, for smooth boundary Γ and data

$$\sigma \in L^2(\Omega, Y_{p+1/2}) = L^2(\Omega, H^{p+1}(\Gamma)),$$

at any interior point $x \in D$ the error bound

$$|(\mathcal{M}^2U)(x,x) - \langle R_2 \otimes R_2, \tilde{Z}_L^2\rangle| \leqslant c(x)(\log N_L)^3 N_L^{-2(p+1/2)} \|\sigma\|^2_{L^2(\Omega,H^{p+1}(\Gamma))}.$$

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