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WELL-POSEDNESS AND REGULARITY FOR A PARABOLIC-HYPERBOLIC PENROSE-FIFE PHASE FIELD SYSTEM

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Abstract. This work is concerned with the study of an initial boundary value problem for a non-conserved phase field system arising from the Penrose-Fife approach to the kinetics of phase transitions. The system couples a nonlinear parabolic equation for the absolute temperature with a nonlinear hyperbolic equation for the phase variable χ, which is characterized by the presence of an inertial term multiplied by a small positive coefficient μ. This feature is the main consequence of supposing that the response of χ to the generalized force (which is the functional derivative of a free energy potential and arises as a consequence of the tendency of the free energy to decay towards a minimum) is subject to delay. We first obtain well-posedness for the resulting initial-boundary value problem in which the heat flux law contains a special function of the absolute temperature ϑ, i.e. $\alpha(\vartheta) \sim \vartheta - 1/\vartheta$. Then we prove convergence of any family of weak solutions of the parabolic-hyperbolic model to a weak solution of the standard Penrose-Fife model as $\mu \searrow 0$. However, the main novelty of this paper consists in proving some regularity results on solutions of the parabolic-hyperbolic system (including also estimates of Moser type) that could be useful for the study of the longterm dynamics.

Keywords: Penrose-Fife model, hyperbolic equation, continuous dependence, regularity

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1. Introduction

In this paper we consider a modification of the thermodynamically consistent model for the description of the kinetics of phase transitions proposed by O. Penrose and P. Fife in [9], [21], and [22]. Hence, let us introduce the state variables describing the phase transitions: the absolute temperature $\vartheta\colon Q := \Omega \times (0,T) \to \mathbb{R}$ and the order parameter $\chi\colon Q \to \mathbb{R}$, where T is the reference time and $\Omega \subset \mathbb{R}^N$ $(N \leqslant 3)$ is a bounded connected domain with smooth boundary Γ. Then we consider an energy balance equation of the form

$$(\vartheta + \lambda(\chi))_t - \Delta\alpha(\vartheta) = m \quad \text{in } Q, \tag{1.1}$$

where the subscript t stands for the time derivative, m is a heat source term, $\vartheta+\lambda(\chi)$ accounts for the internal energy of the system with $\lambda(\cdot)$ representing the latent heat density of the phase transition; observe that here it may have a quadratic growth so that second order phase transitions can be taken into account (cf. [3, Section 4] for more details on this subject). Moreover, α in (1.1) has the form

$$\alpha(r) = k_1 r - k_2 r^{-1} \quad \forall\, r > 0 \tag{1.2}$$

with k_1 and k_2 positive constants, and (1.1) is coupled with the hyperbolic equation governing the evolution of the phase variable χ, which may be written as

$$\mu\chi_{tt} + \chi_t - \Delta\chi + g(\chi) + \lambda'(\chi)\vartheta^{-1} = 0 \quad \text{in } Q, \tag{1.3}$$

where the subscript tt stands for the second time derivative, g represents a third-degree polynomial function with a positive leading coefficient. An example of g can be the derivative of the double-well potential, i.e. $g(r) = r^3 - r - \lambda'(r)\vartheta_c^{-1}$, with $r \in \mathbb{R}$ and ϑ_c the critical temperature of the system.

Note that the well-posedness for the same kind of problem with $\alpha(r) \sim -1/r$ in (1.1) has been examined in [6]. In our approach, due to the presence of the term $k_1\vartheta$ in the function α (cf. (1.2)), we are able to get more information on the ϑ-variable than the ones obtained by Colli, Grasselli and Ito in [6]. However, we emphasize that this paper brings a further contribution consisting in the deduction of some regularity results entailing, in particular, an $L^\infty(Q)$ bound on the temperature field ϑ.

Moreover, laws like (1.2), which ensure coercivity in ϑ, could be helpful in order to show dissipativity properties for the solution to an initial-boundary value problem for the system (1.1)–(1.3). Hence, these considerations lead us to address the question of the existence of an absorbing set (and then of a global attractor) for the semigroup associated with the problem considered in a suitable phase space. This will be the aim of a forthcoming paper (cf. [24]).

As regards the corresponding linearized problem (obtained by linearizing ϑ^{-1} around the critical value ϑ_c and referred to as the Caginalp model (cf. [4]), it has been considered (also from the long time behaviour's point of view) in [14], [15], [16], [17]. However, we observe that laws like $\alpha(r) \sim -1/r$, which turn out to be satisfactory for low and intermediate temperatures, do not look acceptable for the higher ones, when $\alpha(r) \sim r$ (this choice of α corresponds to the standard Fourier law) better decribes the evolution of the system. Hence, these considerations lead us to introduce a law like (1.2) in the energy balance (1.1).

Regarding the phase equation (1.3), let us summarize here the main novelty of this approach, that is, the presence of the inertial term $\mu\chi_{tt}$ in (1.3). In fact, the original law describing the evolution of χ in the framework of the Penrose-Fife models was

$$\chi_t - \Delta\chi + g(\chi) + \lambda'(\chi)\vartheta^{-1} = 0 \quad \text{in } Q. \tag{1.4}$$

We may note that (1.4), which can be considered as a limiting case of (1.3), says that the response of χ to the generalized force $\delta\mathcal{F}/\delta\chi$ is istantaneous, i.e.

$$\chi_t = -\frac{1}{\vartheta}\frac{\delta\mathcal{F}}{\delta\chi},$$

where $\delta\mathcal{F}/\delta\chi$ denotes the functional derivative of $\mathcal{F}$ (the free energy functional) with respect to χ and $\mathcal{F}$ has the form

$$\mathcal{F}(\vartheta,\chi) = \int_\Omega \Big\{\vartheta - \vartheta\ln\vartheta + \frac{\vartheta}{2}|\nabla\chi|^2 + \vartheta\hat{g}(\chi) + \lambda(\chi)\Big\}\,\mathrm{d}x,$$

where $\hat{g}$ is a primitive of g. The derivation of the energy balance (1.1) can be found in [3, Chap. 4, pp. 168–169]. The typical conditions coupled with (1.1) and (1.4) are the Cauchy ones for ϑ and χ,

$$\vartheta(0) = \vartheta_0 \quad \text{in } \Omega, \tag{1.5}$$

$$\chi(0) = \chi_0 \quad \text{in } \Omega \tag{1.6}$$

along with the boundary conditions

$$(\alpha(\vartheta)_{\mathbf{n}} + \gamma\alpha(\vartheta) = h \quad \text{on } \Sigma := \Gamma\times(0,T), \tag{1.7}$$

$$\chi_{\mathbf{n}} = 0 \quad \text{on } \Sigma. \tag{1.8}$$

Here the subscript $\mathbf{n}$ stands for the outward normal derivative to Γ, γ is the positive heat-transmission coefficient, $h\colon \Sigma\to\mathbb{R}$ has the form $\gamma\alpha(\vartheta_\Gamma)$, ϑ_Γ being the outside temperature on the boundary. In this way we find a boundary value problem coupling (1.1) and (1.4)–(1.8) which has been widely studied in literature (cf., e.g., [7], [8], [18], and [19]).

Equation (1.3) comes from the recent supposition that in some situations (as, e.g., in the melt of He^4 crystals) the response of χ to the generalized force $\delta\mathcal{F}/\delta\chi$ is subject to a delay expressed by a suitable time dependent relaxation kernel k (cf. [11], [17], [25] and references therein), i.e.

$$\chi_t = -\int_{-\infty}^{t} k(t-s)\frac{\delta\mathcal{F}}{\delta\chi}\,\mathrm{d}s.$$

If we choose k as a decreasing exponential of the form $k(t) = \frac{1}{\mu}\mathrm{e}^{-t/\mu}$ ($t \geqslant 0$) for some positive coefficient μ sufficiently small, we find exactly (1.3). Note that, as $\mu \searrow 0$, $k(t) \to \delta_0(t)$ where δ_0 is the Dirac mass at 0, so that (1.3) reduces to the standard phase equation (1.4).

After these considerations, in this paper we denote by (P_μ) the problem of finding a pair (ϑ, χ) solving equations (1.1) and (1.3) coupled with boundary conditions (1.7)–(1.8) and Cauchy conditions (1.5)–(1.6) along with the additional initial condition

$$\chi_t(0) = \chi_1 \quad \text{in } \Omega, \tag{1.9}$$

which is needed due to the hyperbolic character of (1.9).

An outline of our work follows. The first result is related to the existence of a weak solution to (P_μ) in case of quadratic latent heat function λ (which models second order phase transitions) and strictly positive μ. The proof is contained in Section 3. Then, in Section 4, we prove convergence as $\mu \searrow 0$ of any family of solutions $(\vartheta_\mu, \chi_\mu)$ of (P_μ) to a weak solution of the corresponding initial boundary value problem which couples (1.1) and (1.4)–(1.8). Section 5 is devoted to showing regularity results (also with estimates of Moser type) on solutions of (P_μ) ($\mu > 0$) under further assumptions on the data. Our further results are two continuous dependence theorems. The former, which entails uniqueness for $N = 1$, is proved in Section 6. The latter, whose proof is contained in Section 7, guarantees that, in case of λ linear, (P_μ) has a unique solution in every spatial dimension ($N = 2, 3$).

2. Main results

Consider the initial-boundary value problem (1.1), (1.3), (1.5)–(1.9). We make the following general assumptions on the known functions appearing in (1.1), (1.3) and (1.7):

$$\lambda \in C^2(\mathbb{R}), \tag{2.1}$$

$$\lambda'' \in L^\infty(\mathbb{R}), \tag{2.2}$$

$$\alpha(r) = kr - k/r \quad \forall\, r \in (0, +\infty), \tag{2.3}$$

$$g \in C^1(\mathbb{R}), \tag{2.4}$$

$$\exists\, \tau_1, \tau_2 > 0\colon\ |g(r)| \leqslant \tau_1 |r|^3 + \tau_2 \quad \forall\, r \in \mathbb{R}, \tag{2.5}$$

$$\lim_{r \to \pm\infty} g(r) = \pm\infty, \tag{2.6}$$

with k a given positive constant. Note that, thanks to (2.4)–(2.5), g could be the derivative of a multiple-well potential.

Remark 2.1. First, let us note that under these assumptions on g, it is easy to see that there exists a primitive $\hat{g}$ of g such that

$$0 \leqslant \hat{g}(r) \leqslant \tau_3 |r|^4 + \tau_4 \quad \forall\, r \in \mathbb{R}, \tag{2.7}$$

for some positive constants τ_3, τ_4. Then, we remark that the form of α (1.2) can be reduced to the simplified one (cf. (2.3)), that is to the case in which $k_1 = k_2 = k$. Indeed, introducing $\Theta := \beta\vartheta$ for $\beta = \sqrt{k_1/k_2}$, we may rewrite $\alpha(\vartheta) = \widetilde{\alpha}(\Theta) = k(\Theta - \Theta^{-1})$ and $\widetilde{\lambda}(s) = \beta\lambda(s)$ with $k = \sqrt{k_1 k_2}$. Hence, with an abuse of notation let us take this $\widetilde{\alpha}$ and $\widetilde{\lambda}$ as our α and λ, respectively, in the rest of the paper. Notice that with this simple change of variables we are able to deal also with a more general form of α (cf. (1.2)) and not only with the one introduced in (2.3), used here and in the sequel only for simplicity of notation.

Moreover, let us suppose the following regularity of the data of our system:

$$m \in L^2(Q), \tag{2.8}$$

$$h \in L^2(\Sigma), \tag{2.9}$$

$$\vartheta_0 \in L^2(\Omega), \quad \vartheta_0 > 0 \ \text{ a.e. in } \Omega, \tag{2.10}$$

$$\ln \vartheta_0 \in L^1(\Omega), \tag{2.11}$$

$$\chi_0 \in H^1(\Omega), \tag{2.12}$$

$$\chi_1 \in L^2(\Omega). \tag{2.13}$$

The next step is to give a variational formulation of the problem (1.1), (1.3), (1.5)–(1.9). To this end, we use the notation $(\cdot,\cdot)$ both for the scalar product in $H := L^2(\Omega)$ and in $(L^2(\Omega))^N$, also denoted by H, and $|\cdot|$ for the corresponding norm. For the sake of convenience, $V := H^1(\Omega)$ will be endowed with the inner product $(\!(\cdot,\cdot)\!)$, defined by

$$(\!(v_1, v_2)\!) := \int_\Omega \nabla v_1 \nabla v_2 + \gamma \int_\Gamma v_1 v_2 \quad \forall\, v_1, v_2 \in V, \tag{2.14}$$

where γ is the positive heat-transmission coefficient appearing in the boundary condition (1.7). Define $W := H^2(\Omega)$ and let us denote by $\langle\cdot,\cdot\rangle$ also the duality pairing between V' and V. We identify H with a subspace of V', as usual, so that

$\langle u, v\rangle = (u, v)$ for all $u \in H$ and for all $v \in V$. Moreover, we may denote the scalar product in $L^2(\Gamma)$ by $(\cdot,\cdot)_\Gamma$.

Next, we define the Riesz isomorphism $J\colon V \to V'$, and the scalar product in V', respectively, by

$$\langle Jv_1, v_2\rangle := (\!(v_1, v_2)\!) \quad \forall\, v_1, v_2 \in V, \tag{2.15}$$

$$(\!(w_1, w_2)\!)_* := \left\langle w_1, J^{-1} w_2\right\rangle \quad \forall\, w_1, w_2 \in V'. \tag{2.16}$$

Let us observe that the norm in V related to the inner product defined above (which will be denoted by $\|\cdot\|$) is equivalent to the usual norm in V. Similar considerations hold also for V' and we use the notation $\|\cdot\|_*$ for the norm in V' related to the inner product (2.16).

Finally, let $f \in L^2(0,T;V')$ be defined by

$$\langle f(t), v\rangle := \int_\Omega m(t)v + \gamma \int_\Gamma h(t)v \quad \forall\, v \in V \text{ and for a.e. } t \in (0,T). \tag{2.17}$$

Then we are ready to state the rigorous formulation of the problem (1.1), (1.3), (1.5)–(1.9) (for a strictly positive coefficient μ).

P r o b l e m (P_μ). Find a pair (ϑ, χ) such that

$$\vartheta \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \quad \vartheta > 0 \text{ a.e. in } Q, \tag{2.18}$$

$$\vartheta^{-1} \in L^2(0,T;V), \tag{2.19}$$

$$\chi \in C^1([0,T];H) \cap C^0([0,T];V) \cap H^2(0,T;V'), \tag{2.20}$$

$$(\vartheta + \lambda(\chi))_t + kJ(\vartheta - \vartheta^{-1}) = f \quad \text{in } V', \text{ a.e. in } (0,T), \tag{2.21}$$

$$\langle \mu\chi_{tt} + \chi_t, v\rangle + (\nabla\chi, \nabla v) + (g(\chi) + \lambda'(\chi)\vartheta^{-1}, v) = 0 \quad \forall\, v \in V, \text{ a.e. in } (0,T), \tag{2.22}$$

$$\vartheta(\cdot,0) = \vartheta_0, \quad \chi(\cdot,0) = \chi_0, \quad \chi_t(\cdot,0) = \chi_1 \quad \text{a.e. in } \Omega. \tag{2.23}$$

R e m a r k 2.2. Note that the first initial condition in (2.23) holds almost everywhere in Ω due to the weak continuity of $t \mapsto \vartheta(t)$ from $[0,T]$ to H. Moreover, by comparison with (2.22), it follows that $\chi_{tt} \in L^2(0,T;H) + C^0([0,T];V')$ (cf. also [6, Remark 2.2] for further details on this point).

Our first result is

Theorem 2.3. *Suppose that* (2.1)–(2.13) *hold. Then, for any* $0 < \mu < 1$, *problem* (P_μ) *has at least one solution* $(\vartheta^\mu, \chi^\mu)$.

The next theorem is a regularity result for solutions of (P_μ).

Theorem 2.4. *Suppose that the same hypotheses as in Theorem 2.3 hold. Under the further regularity assumptions on the data*

$$\chi_0 \in W, \quad (\chi_0)_{\mathbf{n}} = 0 \ \textit{on} \ \Gamma, \tag{2.24}$$

$$\chi_1 \in V, \tag{2.25}$$

$$\lambda' \in L^\infty(\mathbb{R}), \tag{2.26}$$

and if we suppose that there exists a positive constant C_1 *such that the inequality*

$$|g'(r)| \leqslant C_1(1 + |r|^2) \tag{2.27}$$

holds for all $r \in \mathbb{R}$, *then the* χ^μ *component of every solution* $(\vartheta^\mu, \chi^\mu)$ *of* (P_μ) *has additional regularity*

$$\chi^\mu \in W^{1,\infty}(0, T; V) \cap L^\infty(0, T; W) \hookrightarrow C^0(\overline{Q}) \cap C^0([0, T]; V). \tag{2.28}$$

Moreover, if we suppose (in addition to (2.1)–(2.13) *and* (2.24)–(2.27)*) that*

$$\vartheta_0,\ 1/\vartheta_0 \in L^\infty(\Omega), \quad m \in L^2(0, T; L^6(\Omega)), \quad h \in L^\infty(\Sigma), \tag{2.29}$$

then every solution $(\vartheta^\mu, \chi^\mu)$ *of* (P_μ) *has additional regularity*

$$\vartheta^\mu \in L^\infty(Q), \quad -\frac{1}{\vartheta^\mu} \in L^\infty(Q). \tag{2.30}$$

Finally, assume that the hypotheses (2.1)–(2.13), (2.24)–(2.27), (2.29) *and the assumptions*

$$f \in W^{1,1}(0, T; V'), \tag{2.31}$$

$$\alpha(\vartheta_0) \in V \tag{2.32}$$

are satisfied. Then every solution $(\vartheta^\mu, \chi^\mu)$ *of* (P_μ) *has further regularity*

$$\chi^\mu_{tt} \in L^\infty(0, T; H), \quad \vartheta^\mu \in H^1(0, T; H), \quad k\vartheta^\mu - \frac{k}{\vartheta^\mu} \in L^\infty(0, T; V). \tag{2.33}$$

Remark 2.5. Let us observe that condition (2.31) holds true if $m \in W^{1,1}(0, T; H)$ and $h \in L^2(0, T; H^{1/2}(\Gamma)) \cap W^{1,1}(0, T; H^{-1/2}(\Gamma))$. Finally, note that in the case $N = 1$ it is possible (with a slight modification of the fourth regularity estimate obtained in Section 5) to prove that regularity (2.33) also holds if (2.29) is not satisfied.

Consider now the formal limit problem (corresponding to the case $\mu = 0$ in (P_μ)).

Problem (P_0). Find a pair (ϑ, χ) satisfying

$$
\begin{aligned}
&(2.34)\qquad \vartheta \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \quad \vartheta > 0 \text{ a.e. in } Q,\\
&(2.35)\qquad \vartheta^{-1} \in L^2(0,T;V),\\
&(2.36)\qquad \chi \in H^1(0,T;H) \cap L^2(0,T;W) \hookrightarrow C^0([0,T];V),\\
&(2.37)\qquad (\vartheta + \lambda(\chi))_t + kJ(\vartheta - \vartheta^{-1}) = f \quad \text{in } V', \text{ a.e. in } (0,T),\\
&(2.38)\qquad \chi_t - \Delta\chi + g(\chi) + \lambda'(\chi)\vartheta^{-1} = 0 \quad \text{a.e. in } Q,\\
&(2.39)\qquad \chi_{\mathbf{n}} = 0 \quad \text{a.e. on } \Sigma,\\
&(2.40)\qquad \vartheta(\cdot,0) = \vartheta_0, \quad \chi(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega.
\end{aligned}
$$

Then we can prove the following

Theorem 2.6. *Suppose that* (2.1)–(2.13) *are satisfied. Moreover, let* $\mu \in (0, \mu_0]$, $\mu_0 > 0$ *being fixed. Then there exists a positive constant* R *(independent of* μ*) such that for any solution* $(\vartheta^\mu, \chi^\mu)$ *to* (P_μ) *we have*

$$
\begin{aligned}
(2.41)\qquad &\|\vartheta^\mu\|_{H^1(0,T;V')\cap L^\infty(0,T;H)\cap L^2(0,T;V)} + \|(\vartheta^\mu)^{-1}\|_{L^2(0,T;V)}\\
&+\sqrt{\mu}\|\chi_t^\mu\|_{L^\infty(0,T;H)} + \|\chi_t^\mu\|_{L^2(0,T;H)} + \|\chi^\mu\|_{L^\infty(0,T;V)} \leqslant R.
\end{aligned}
$$

Moreover, it follows that the unique solution (ϑ, χ) *to* (P_0) *is the (weak) limit of the sequence* $\{(\vartheta^\mu, \chi^\mu)\}$*, where* $(\vartheta^\mu, \chi^\mu)$ *is an arbitrary solution to* (P_μ)*, i.e., we have that the convergences*

$$
\begin{aligned}
&(2.42)\quad \vartheta^\mu \to \vartheta && \textit{weakly star in } L^\infty(0,T;H),\\
&(2.43)\quad \vartheta^\mu \to \vartheta && \textit{weakly in } H^1(0,T;V') \cap L^2(0,T;V),\\
&(2.44)\quad \vartheta^\mu \to \vartheta && \textit{strongly in } C^0([0,T];V') \cap L^2(0,T;H),\\
&(2.45)\quad 1/\vartheta^\mu \to 1/\vartheta && \textit{weakly in } L^2(0,T;V),\\
&(2.46)\quad \mu\chi_t^\mu \to 0 && \textit{strongly in } C^0([0,T];H),\\
&(2.47)\quad \chi^{\mu_k} \to \chi && \textit{weakly in } H^1(0,T;H) \textit{ and weakly star in } L^\infty(0,T;V),\\
&(2.48)\quad \chi^\mu \to \chi && \textit{strongly in } C^0(0,T;L^4(\Omega))
\end{aligned}
$$

hold as $\mu \searrow 0$.

Remark 2.7. Theorem 2.6 yields the existence of a solution to (P_0). Uniqueness follows from [23, Theorem 3.3] (cf. Remark 2.10 below for a comparison with other existing results).

The next theorem gives a conditional continuous dependence result for solutions of (P_μ) with $\mu > 0$ in the case of $N = 2, 3$, and a non conditional result in the case of $N = 1$ or if $N = 2, 3$ but only with more regular data (cf. (2.24)–(2.27), (2.29), and Theorem 2.4).

Theorem 2.8. *Assume that hypotheses* (2.1)–(2.6), (2.26) *(if $N = 2, 3$), and* (2.27) *hold. Moreover, take two sets of data $\{\vartheta_{0i}, \chi_{0i}, \chi_{1i}, m_i, h_i\}$, $i = 1, 2$, under the assumptions* (2.8)–(2.13) *and denote by (ϑ_i, χ_i) the corresponding solution to problem* (P_μ). *Assume an additional condition*

$$u_i := -\vartheta_i^{-1} \in L^2(0, T; L^\infty(\Omega)), \quad i = 1, 2, \tag{2.49}$$

and let the inequality

$$\begin{aligned} \max\{&\|\chi_1\|_{L^\infty(0,T;V)}, \|\chi_2\|_{L^\infty(0,T;V)}, \|\vartheta_1\|_{L^\infty(0,T;H)}, \|\vartheta_2\|_{L^\infty(0,T;H)}, \\ &\|u_1\|_{L^2(0,T;L^\infty(\Omega))}, \|u_2\|_{L^2(0,T;L^\infty(\Omega))}, \|(\chi_1)_t\|_{L^2(0,T;H)}, \|(\chi_2)_t\|_{L^2(0,T;H)}\} \leqslant M_1 \end{aligned} \tag{2.50}$$

hold for a positive constant M_1. Then there exists a positive constant $D_1 = D_1(M_1)$ also depending on T, Ω, γ, μ, λ, and C_1 (cf. hypothesis (2.27)*) such that*

$$\begin{aligned} &\|\vartheta_1 - \vartheta_2\|^2_{L^2(0,T;H)} + \|\vartheta_1 - \vartheta_2\|^2_{L^\infty(0,T;V')} + \|(\chi_1 - \chi_2)_t\|^2_{L^\infty(0,T;H)} \\ &\quad + \|1 * [k(u_1 - u_2) + k(\vartheta_1 - \vartheta_2)]\|^2_{L^\infty(0,T;V)} + \|\chi_1 - \chi_2\|^2_{L^\infty(0,T;V)} \\ &\leqslant D_1(\|\vartheta_{01} - \vartheta_{02}\|^2_* + \|\chi_{01} - \chi_{02}\|^2 + |\chi_{11} - \chi_{12}|^2 + \|f_1 - f_2\|^2_{L^2(0,T;V')}), \end{aligned} \tag{2.51}$$

where f_i is the datum corresponding to m_i, h_i, $i = 1, 2$, according to formula (2.17). *In particular, if $N = 1$, then problem* (P_μ) *has a unique solution.*

Making stronger hypotheses on λ (basically we ask λ to be an affine function), we have the following result, which entails uniqueness of solution for problem (P_μ) also in the case of $N = 2, 3$.

Theorem 2.9. *Let the hypotheses* (2.1)–(2.6), (2.27), *and*

$$\lambda(r) = r \quad \forall\, r \in \mathbb{R} \tag{2.52}$$

hold. Moreover, take two sets of data $\{\vartheta_{0i}, \chi_{0i}, \chi_{1i}, m_i, h_i\}$, $i = 1, 2$, under the assumptions (2.8)–(2.13), *denote by (ϑ_i, χ_i) the corresponding solution to problem* (P_μ), *and choose a positive constant M_2 such that*

$$\max\{\|\chi_1\|_{L^\infty(0,T;V)}, \|\chi_2\|_{L^\infty(0,T;V)}\} \leqslant M_2. \tag{2.53}$$

Then, setting $u_i = -\vartheta_i^{-1}$, the following continuous dependence result holds:

$$
\begin{aligned}
(2.54)\quad & \|\vartheta_1-\vartheta_2\|^2_{L^2(0,T;H)\cap L^\infty(0,T;V')} + \|1*[k(u_1-u_2)+k(\vartheta_1-\vartheta_2)]\|^2_{L^\infty(0,T;V)}\\
&+\|(\chi_1-\chi_2)_t\|^2_{L^\infty(0,T;V')}+\|1*(\chi_1-\chi_2)\|^2_{L^\infty(0,T;V)}+\|\chi_1-\chi_2\|^2_{L^\infty(0,T;H)}\\
&\leqslant D_2(\|\vartheta_{01}-\vartheta_{02}\|^2_*+|\chi_{01}-\chi_{02}\|^2+\|\chi_{11}-\chi_{12}\|^2_*+\|f_1-f_2\|^2_{L^2(0,T;V')})
\end{aligned}
$$

for a positive constant $D_2 = D_2(M_2)$ also depending on T, Ω, γ, μ, λ, and C_1, where f_i is the datum corresponding to m_i and h_i, $(i = 1,2)$ according to formula (2.17).

Remark 2.10. We have to observe that the existence result for a weak formulation of (P_0) was given in [7] for a more general expression of the heat flux law and the uniqueness result was obtained in [8] under a quite similar assumption on the heat flux law but only in the case of strong regularity assumptions on the data. Moreover, in [23], this result has been improved, the uniqueness and continuous dependence result has been shown under the same hypotheses on the data as in the existence theorem.

Let us recall at this point some useful inequalities that we will often use in the sequel without recalling them. Let us start by recalling that, by the continuity of the trace operator (in this setting) from V to $L^2(\Gamma)$, there exists a positive constant C_Γ (depending only on Ω and γ) such that

$$(2.55)\qquad \|v\|^2_{L^2(\Gamma)} \leqslant C_\Gamma \|v\|^2 \quad \forall\, v\in V.$$

Let us also note that, as Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\leqslant 3)$, there exists a positive constant C_S depending only on Ω and γ such that

$$(2.56)\qquad \|v\|_{L^p(\Omega)} \leqslant C_S\|v\| \quad \forall\, v\in V,\ \ 1\leqslant p\leqslant 6.$$

We widely use also the elementary inequality

$$(2.57)\qquad r-\ln(r) \geqslant \frac{1}{3}(r+|\ln(r)|) \quad \forall\, r\in(0,+\infty),$$

and Young's inequality in the forms

$$(2.58)\qquad ab\leqslant \eta a^2+\frac{1}{4\eta}b^2,\quad ab\leqslant \eta a^q+\frac{q-1}{q}\Big(\frac{1}{\eta q}\Big)^{1/(q-1)}b^{q/(q-1)}$$

with $a,b,\eta\in(0,+\infty)$ and $q\in(1,+\infty)$. Moreover, let us recall a particular case of the Gagliardo-Nirenberg inequality (cf., e.g., [10]), holding true in our $\Omega\subset\mathbb{R}^N$ $(N\leqslant 3)$, i.e.,

$$(2.59)\qquad \|\nabla u\|_{L^6(\Omega)} \leqslant C_a|\Delta u|+C_b|u|,$$

which holds for some positive constants C_a and C_b.

3. Proof of Theorem 2.3

This proof is split into several steps. First, we regularize problem (P_μ) and construct and solve a suitable sequence of approximating problems (P_μ^n), $n \in \mathbb{N}$, then we establish some estimates for the solutions (ϑ^n, χ^n) of this sequence of problems (the subscript μ is omitted for simplicity of notation) that will allow us to pass to the limit as n goes to $+\infty$, getting finally a solution to our problem (P_μ). In this section we will use the same symbol C for positive constants that may be different from each other and may depend on the data of the problem, but which are independent of n and μ. First of all, recalling that $\alpha(\vartheta) = k(\vartheta - \vartheta^{-1})$, we introduce functions

$$\varrho(s) := \alpha^{-1}(s), \quad \hat{\alpha}(s) := \int_1^s \alpha(r)\,\mathrm{d}r = \frac{k}{2}s^2 - k\ln s - \frac{k}{2} \quad \forall\, s > 0, \tag{3.1}$$

and an auxiliary variable $w := \alpha(\vartheta)$. Observe that ϱ is well-defined, because α is an increasing function. Next, we truncate ϱ from above and from below and add the outcome to νI, where $\nu > 0$ is a small parameter and I stands for the identity operator. Thus, we obtain a bi-Lipschitz continuous function. As in [7, Section 3], we consider the sequence of functions and graphs

$$\varrho_n(z) := \begin{cases} \varrho(-n) & \text{if } z \in (-\infty, -n], \\ \varrho(z) & \text{if } z \in (-n, n), \\ \varrho(n) & \text{if } z \in [n, +\infty). \end{cases} \quad \text{and} \quad \alpha^n := \varrho_n^{-1}, \tag{3.2}$$

Observe that α_n acts on a bounded closed interval contained in $(0, +\infty)$. Now, we choose a sequence of real numbers $\{\nu_n\}$ such that

$$0 < \nu_{n+1} \leqslant \nu_n \leqslant 1 \ \ \forall\, n \in \mathbb{N}, \quad \text{and} \quad \nu_n \searrow 0 \ \text{as} \ n \nearrow +\infty. \tag{3.3}$$

Our approximation consists in taking $\upsilon_n := \nu_n w + \varrho_n(w)$ instead of ϑ in (2.21). We must suitably arrange the first initial condition in (2.23). To this aim we introduce the same regularization as that used in [7, Lemma 3.1]; in fact the proof of that lemma is the same in our case. Let us recall this result for the reader's convenience. Let us introduce auxiliary functions

$$\zeta_{1,n}(r) := 1 - (\varrho_n(r))^{-1}, \quad \zeta_{2,n}(r) := \varrho_n(r) - 1 \quad \forall\, r \in \mathbb{R}. \tag{3.4}$$

Note that $\zeta_{1,n}$ and $\zeta_{2,n}$ are bounded, Lipschitz continuous, and increasing functions. Moreover, we may note that $\zeta_{i,n}(0) = 0$ and

$$\widehat{\zeta_{i,n}} := \int_0^r \zeta_{i,n}(s)\,\mathrm{d}s \geqslant 0 \quad \forall\, r \in \mathbb{R}, \ \ i = 1, 2. \tag{3.5}$$

Lemma 3.1. *Let* (2.10)–(2.11) *and* (3.1) *hold, and let* $w_0 := \alpha(\vartheta_0)$, $n \in \mathbb{N}$,

$$\nu_n := (1 + n + \varrho(n) + \varrho(-n)^{-1})^{-2}, \quad \vartheta_{0n} := \varrho_n(w_0), \quad w_{0n} := \alpha(\vartheta_{0n}).$$

Then there exists a positive constant C, *depending only on* $|\vartheta_0|$, $\|\ln \vartheta_0\|_{L^1(\Omega)}$, k *and on* $|\Omega|$, *such that*

$$|\vartheta_{0n}| + \|\ln(\vartheta_{0n})\|_{L^1(\Omega)} + \nu_n |w_{0n}|^2 + \nu_n \sum_{i=1}^{2} \|\widehat{\zeta_{1,n}}(w_{0n})\|_{L^\infty(\Omega)} \leqslant C \quad \forall\, n \in \mathbb{N}. \tag{3.6}$$

Moreover, (3.3) *is satisfied,* $\vartheta_{0n} = \varrho_n(w_{0n})$ *a.e. in* Ω, *and*

$$\nu_n w_{0n} + \vartheta_{0n} \to \vartheta_0 \quad \textit{strongly in } H \textit{ as } n \nearrow \infty.$$

Now, let us approximate λ and g (cf., e.g., [6, Section 3]). We set

$$\lambda_n(r) := \begin{cases} \lambda(-n) + \lambda'(-n)(r+n) & \text{if } r < -n, \\ \lambda(r) & \text{if } -n \leqslant r \leqslant n, \quad n \in \mathbb{N}, \\ \lambda(n) + \lambda'(n)(r-n) & \text{if } r > n. \end{cases} \tag{3.7}$$

Note that

$$\lambda_n \in C^{1,1}(\mathbb{R}), \quad \lambda_n', \lambda_n'' \in L^\infty(\mathbb{R}), \quad \text{and } \lambda_n \to \lambda \text{ a.e. in } \mathbb{R}. \tag{3.8}$$

Moreover, we have (cf. (2.1)–(2.2))

$$|\lambda_n'(r)| \leqslant C_\lambda(1 + |r|) \quad \forall\, r \in \mathbb{R}, \tag{3.9}$$

$$|\lambda_n(r)| \leqslant C_\lambda(1 + |r|^2) \quad \forall\, r \in \mathbb{R} \text{ and } \forall\, n \in \mathbb{N}, \tag{3.10}$$

where C_λ (possibly different from line to line) are positive constants depending only on λ. Finally, let us approximate g and its primitive $\hat{g}$ by g_n and $\hat{g}_n$ (cf. [6, Section 3]):

$$g_n \in C^{0,1}(\mathbb{R}), \tag{3.11}$$

$$g_n \to g \quad \text{uniformly on the compact subsets of } \mathbb{R}, \tag{3.12}$$

$$g_n = \widehat{(g_n)}', \tag{3.13}$$

$$0 \leqslant \hat{g}_n(r) \leqslant \hat{g}(r), \quad |g_n(r)| \leqslant |g(r)| \ \forall\, r \in \mathbb{R}. \tag{3.14}$$

We are ready now to formulate the approximate problem for any $n \in \mathbb{N}$.

Problem (P_μ^n). Find $\vartheta_n \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $\chi_n \in W^{2,\infty}(0,T;V') \cap C^1([0,T];H) \cap C^0([0,T];V)$, and the auxiliary unknowns $w_n \in L^2(0,T;V)$, $\upsilon_n \in H^1(0,T;V')$ fulfilling

$$(3.15)\quad \vartheta_n = \varrho_n(w_n), \quad \vartheta_n > 0 \text{ a.e. in } Q, \quad \upsilon_n = \nu_n w_n + \vartheta_n \quad \text{a.e. in } Q,$$

$$(3.16)\quad \vartheta_n^{-1} \in L^2(0,T;V),$$

$$(3.17)\quad (\upsilon_n + \lambda_n(\chi_n))_t + J(w_n) = f \quad \text{in } V' \text{ a.e. in } (0,T),$$

$$(3.18)\quad \langle \mu(\chi_n)_{tt}, v\rangle + ((\chi_n)_t, v) + (\nabla\chi_n, \nabla v) + (g_n(\chi_n) + \lambda_n'(\chi_n)\vartheta_n^{-1}, v) = 0 \quad \forall\, v \in V, \text{ a.e. in } (0,T),$$

$$(3.19)\quad \upsilon_n(\cdot,0) = \vartheta_{0n} + \nu_n w_{0n}, \quad \chi_n(\cdot,0) = \chi_0, \quad (\chi_n)_t(\cdot,0) = \chi_1 \quad \text{a.e. in } \Omega.$$

Existence and uniqueness for (P_μ^n)**.** We can proceed like in [6, Section 3], i.e. we can apply a fixed-point theorem to the contractive mapping S (that we will define in a moment) into the Banach space

$$(3.20)\qquad H_T = L^2(0,T;H) \times C^0([0,T];H).$$

Fix $(\overline{w}_n, \overline{\chi}_n) \in H_T$ and then consider the Cauchy problem

$$(3.21)\qquad \langle \mu(\chi_n)_{tt} + (\chi_n)_t, v\rangle + (\nabla\chi_n, \nabla v) + (\chi_n, v) = (\mathcal{G}(\overline{w}_n, \overline{\chi}_n), v) \quad \forall\, v \in V, \text{ a.e. in } (0,T),$$

$$(3.22)\qquad \chi_n(0) = \chi_0, \quad (\chi_n)_t(0) = \chi_1 \quad \text{a.e. in } \Omega,$$

$$(3.23)\qquad \mathcal{G}(\overline{w}_n, \overline{\chi}_n) = \overline{\chi}_n - g_n(\overline{\chi}_n) - \lambda_n'(\overline{\chi}_n)(\varrho_n(\overline{w}_n))^{-1} \in L^\infty(0,T;H).$$

As is well known, there is a unique solution $\chi_n \in W^{2,\infty}(0,T;V') \cap C^1([0,T];H) \cap C^0([0,T];V)$ to (3.21)–(3.22) (cf., for example, [27, Lemma 4.1, p. 76]). Therefore, we may apply for instance [7, Lemma 3.4], which gives us a unique solution $w_n \in C^0([0,T];H) \cap L^2(0,T;V)$ to

$$(3.24)\quad \langle (\upsilon_n)_t, v\rangle + (\!(w_n, v)\!) = -\langle (\lambda_n(\chi_n))_t - f, v\rangle \quad \forall\, v \in V, \quad \text{a.e. in } (0,T),$$

$$(3.25)\quad \upsilon_n = \nu_n w_n + \varrho_n(w_n),$$

$$(3.26)\quad \upsilon_n(\cdot,0) = \upsilon_{0n} \quad \text{a.e. in } \Omega.$$

Hence, let us introduce a mapping S from H_T into itself such that

$$(3.27)\qquad S(\overline{w}_n, \overline{\chi}_n) := (w_n, \chi_n).$$

We find that S is a contraction of H_T into itself; in fact, if we consider $(\overline{w}_n^j, \overline{\chi}_n^j) \in H_T$, $j = 1,2$ and the corresponding (w_n^j, χ_n^j), just using the same techniques as that employed in [6, pp. 11–12], we can find a positive constant Λ_n blowing up as n goes to $+\infty$ such that the estimate

$$\|w_n^1 - w_n^2\|^2_{L^2(0,t;H)} + \|\chi_n^1 - \chi_n^2\|^2_{C^0([0,t];H)} \leqslant \Lambda_n \int_0^t (\|\overline{w}_n^1 - \overline{w}_n^2\|^2_{L^2(0,s;H)} + \|\overline{\chi}_n^1 - \overline{\chi}_n^2\|^2_{C^0(0,s;H)}) \tag{3.28}$$

holds for any $t \in [0,T]$. The reader may refer directly to [6, pp. 11–12] for explicit calculations leading to (3.28). Thus, from (3.28), we deduce that for any fixed $n \in \mathbb{N}$ we can find an integer $m = m(n)$ such that S^m is a contraction in H_T. Therefore, thanks to the contraction principle, S has a unique fixed point in H_T, i.e. (P_μ^n) has a unique solution.

We derive now some a priori estimates (independent of $n \in \mathbb{N}$ and $\mu > 0$) for the solution of problem (P_μ^n). Hence, let C be a positive constant which may vary from line to line and may depend on all the data of the problem except of n and μ. Let us point out that the following estimates for equation (3.17) are formal. Indeed, we know that $\upsilon_n \in H^1(0,T;V')$, but we would need to know that w_n and ϑ_n separately belong to $H^1(0,T;V')$ (so that $w_n(\cdot,0) = w_{0n}$ and $\vartheta_n(\cdot,0) = \vartheta_{0n}$ would hold a.e. in Ω), which would require further approximation of f and w_0. However, let us proceed here in a formal way and refer to [7, p. 321] and references therein for more details on this subject.

First a priori estimate. Testing equation (3.17) by $\zeta_{1,n}(w_n)$ and integrating it over $(0,t)$ with $0 \leqslant t \leqslant T$, we obtain the equality

$$\int_0^t [\langle(\upsilon_n)_t, \zeta_{1,n}(w_n)\rangle + \langle J(w_n), \zeta_{1,n}(w_n)\rangle] = \int_0^t \langle f, \zeta_{1,n}(w_n)\rangle - \int_0^t\int_\Omega \lambda_n'(\chi_n)(\chi_n)_t\zeta_{1,n}(w_n). \tag{3.29}$$

Moreover, using $w_n = \alpha(\vartheta_n)$ where $|w_n| \leqslant n$, we get

$$\langle(\upsilon_n)_t, \eta_{1,n}(w_n)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\nu_n\widehat{\zeta_{1,n}}(w_n) + \vartheta_n - \ln(\vartheta_n)), \tag{3.30}$$

$$\langle J(w_n), \zeta_{1,n}(w_n)\rangle = (\!(w_n - \alpha(\vartheta_n), \zeta_{1,n}(w_n))\!) + (\!(\alpha(\vartheta_n), \zeta_{1,n}(w_n))\!), \tag{3.31}$$

$$\int_\Omega \nabla(w_n - \alpha(\vartheta_n))\nabla\zeta_{1,n}(w_n) = 0, \quad \int_\Gamma (w_n - \alpha(\vartheta_n))\zeta_{1,n}(w_n) \geqslant 0, \tag{3.32}$$

which hold a.e. in $(0,T)$, and

$$-\int_0^t\int_\Omega \nabla\alpha(\vartheta_n)\nabla\Big(\frac{1}{\vartheta_n}\Big) \geqslant C\int_0^t\int_\Omega \Big|\nabla\Big(\frac{1}{\vartheta_n}\Big)\Big|^2, \tag{3.33}$$

$$\int_0^t\int_\Gamma \gamma\alpha(\vartheta_n)(1 - 1/\vartheta_n)) \geqslant k\gamma\int_0^t\int_\Gamma \Big(\frac{1}{\vartheta_n}\Big)^2 - C - k\gamma\int_0^t\int_\Gamma \frac{1}{\vartheta_n}. \tag{3.34}$$

Since $f \in L^2(0,T;V')$, $\zeta_{1,n}(w_n) = 1 - 1/\vartheta_n$ (see (3.4) and (3.15)) and $\widehat{\zeta_{1,n}}$ is non-negative by (3.5), we can obtain from (3.29), by applying (3.30)–(3.34), (2.57), (3.6), (2.14) and Young's inequality, that

$$\begin{aligned}(3.35)\quad \|\widehat{\zeta_{1,n}}(w_n(t))\|_{L^1(\Omega)} + \frac{1}{3}\|\vartheta_n(t)\|_{L^1(\Omega)} + \frac{1}{3}\|\ln\vartheta_n(t)\|_{L^1(\Omega)} + C\|\vartheta_n^{-1}\|^2_{L^2(0,t;V)} \\ \leqslant C + \int_0^t\int_\Omega \lambda_n'(\chi_n)(\chi_n)_t\Big(\frac{1}{\vartheta_n} - 1\Big).\end{aligned}$$

Then we set $v = (\chi_n)_t$ in (3.18), integrate over $(0,t)$ and add to both sides (where the subscript s stands for the derivative w.r.t. the time variable $s \in (0,t)$)

$$(3.36)\qquad |\chi_n(t)|^2 = |\chi_0|^2 + 2\int_0^t ((\chi_n)_t(s), \chi_n(s)),$$

in order to find the full V-norm of $\chi_n(t)$ on the left-hand side in (3.18). Hence, applying Young's inequality, we get

$$\begin{aligned}(3.37)\qquad \frac{\mu}{2}|(\chi_n)_t(t)|^2 + \frac{1}{2}\|(\chi_n)_t\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\chi_n(t)\|^2 + \int_\Omega \hat{g}_n(\chi_n(t)) \\ \leqslant C\Big(\mu|\chi_1|^2 + \|\chi_0\| + \int_\Omega \hat{g}_n(\chi_0) + \|\chi_n\|^2_{L^2(0,t;H)}\Big) \\ - \int_0^t\int_\Omega \lambda_n'(\chi_n(s))(\chi_n(s))_t\vartheta_n^{-1}(s).\end{aligned}$$

Finally, summing up (3.35) and (3.37), using (3.14), (2.12), (2.13), (3.9) and applying Young's inequality, we arrive at

$$\begin{aligned}(3.38)\qquad \mu|(\chi_n)_t(t)|^2 + \|(\chi_n)_t\|^2_{L^2(0,t;H)} + \|\chi_n(t)\|^2 + \|\vartheta_n(t)\|_{L^1(\Omega)} \\ + \|\ln\vartheta_n(t)\|_{L^1(\Omega)} + \|\vartheta_n^{-1}\|^2_{L^2(0,t;V)} \\ \leqslant C\Big(1 + \int_0^t |\chi_n(s)|^2\Big).\end{aligned}$$

Employing a standard version of Gronwall's lemma and recalling (2.4), we obtain our first a priori estimate, yielding for all $t \in [0,T]$ that

$$\begin{aligned}(3.39)\qquad \mu|(\chi_n)_t(t)|^2 + \|(\chi_n)_t\|^2_{L^2(0,t;H)} + \|\chi_n(t)\|^2 + \|\vartheta_n(t)\|_{L^1(\Omega)} \\ + |g(\chi_n(t))|^2 + \|\ln\vartheta_n(t)\|_{L^1(\Omega)} + \|\vartheta_n^{-1}\|^2_{L^2(0,t;V)} \leqslant C.\end{aligned}$$

Second a priori estimate. Test now (3.17) by w_n, integrate it over $(0,t)$ and recall (2.15) along with the definition of $\|\cdot\|$ to show that

$$(3.40)\qquad \int_0^t [\langle(v_n)_t, w_n\rangle + \|w_n\|^2] = \int_0^t \langle f, w_n\rangle - \int_0^t\int_\Omega \lambda_n'(\chi_n)(\chi_n)_t w_n.$$

Since we will also consider equation (3.40) with $\lambda_n'(\chi_n)(\chi_n)_t$ replaced by some other functions, we will keep $\lambda_n'(\chi_n)(\chi_n)_t$ in what follows as long as possible. Now use the following identity to treat the left-hand side in (3.40):

$$\langle (v_n)_t, w_n\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\frac{1}{2}\nu_n w_n^2 + \hat{\alpha}(\vartheta_n)\Big), \tag{3.41}$$

which holds with α and not α_n thanks to the fact that, by definition (3.1)–(3.2), $\int_1^r \alpha_n(\tau)\,\mathrm{d}\tau = \hat{\alpha}(r)$ for all $r\in[\varrho(-n),\varrho(n)]$. Invoking Hölder's and Young's inequalities, we observe that

$$\begin{aligned} -\int_0^t\int_\Omega \lambda_n'(\chi_n)(\chi_n)_t w_n &\leqslant C\int_0^t \|\lambda_n'(\chi_n)(\chi_n)_t\|_{L^{6/5}(\Omega)}\|w_n\| \\ &\leqslant \eta\|w_n\|^2_{L^2(0,t;V)} + C/\eta\|\lambda_n'(\chi_n)(\chi_n)_t\|_{L^2(0,t;L^{6/5}(\Omega))} \end{aligned} \tag{3.42}$$

holds for every positive η. Since $f\in L^2(0,T;V')$, from (3.40), using (3.41), (3.42), (3.1), (3.6), (3.39) and choosing properly η, we obtain

$$\begin{aligned} &\frac{1}{2}\nu_n|w_n(t)|^2 + \frac{k}{2}|\vartheta_n(t)|^2 + \frac{1}{2}\|w_n\|^2_{L^2(0,t;V)} \\ &\quad\leqslant C(1+\|\lambda_n'(\chi_n)(\chi_n)_t\|_{L^2(0,t;L^{6/5}(\Omega))}). \end{aligned} \tag{3.43}$$

By virtue of (3.39) and (3.9) we deduce by using the generalized Hölder's inequality that

$$\begin{aligned} \|\lambda_n'(\chi_n)(\chi_n)_t\|^2_{L^2(0,t;L^{6/5}(\Omega))} &\leqslant C\|\lambda_n'(\chi_n)(\chi_n)_t\|^2_{L^2(0,t;L^{3/2}(\Omega))} \\ &\leqslant C\int_0^t \|(|\chi_n|+1)(\chi_n)_t\|^2_{L^{3/2}(\Omega)} \\ &\leqslant C\int_0^t \|\,|\chi_n|+1\|^2_{L^6(\Omega)}\|(\chi_n)_t\|^2 \\ &\leqslant (\|\chi_n\|^2_{L^\infty(0,t;V)}+1)\|(\chi_n)_t\|^2_{L^2(0,t;H)} \leqslant C. \end{aligned} \tag{3.44}$$

Hence, by (3.43), we finally obtain the second a priori bound

$$\nu_n\|w_n\|_{L^\infty(0,T;H)} + \|\vartheta_n\|_{L^\infty(0,T;L^2(\Omega))} + \|w_n\|_{L^2(0,T;V)} \leqslant C. \tag{3.45}$$

Moreover, again using the same techniques together with (3.42) and (3.44), it is possible to show that

$$\|\lambda_n'(\chi_n)/\vartheta_n\|_{L^2(0,T;L^3(\Omega))} \leqslant C. \tag{3.46}$$

Third a priori estimate. Just comparing the terms in (3.17)–(3.18), remembering (3.19), (3.44), (2.15), (3.39), (3.45), (3.19), (3.46), the fact that $f \in L^2(0,T;V')$ and the assumption on the approximating initial data (cf. Lemma 3.1), we obtain

$$\|\upsilon_n\|^2_{H^1(0,T;V')} + \mu^2\|(\chi_n)_{tt}\|^2_{L^2(0,T;V')} \leqslant C. \tag{3.47}$$

Fourth a priori estimate. In order to establish this estimate, first we multiply (3.17) by $\zeta_{2,n}(w_n)$, integrate it over $(0,t)$, and use the analogues of (3.31)–(3.32) written for $\zeta_{2,n}(w_n) = \vartheta_n - 1$ in order to get the inequality

$$\begin{aligned} \langle(\upsilon_n)_t, \zeta_{2,n}(w_n)\rangle + \int_0^t\int_\Omega \nabla\alpha(\vartheta_n)\nabla\vartheta_n + \int_0^t\int_\Gamma \gamma\alpha(\vartheta_n)\zeta_{2,n}(w_n) \\ \leqslant -\int_0^t\int_\Omega \lambda_n'(\chi_n)(\chi_n)_t(\vartheta_n - 1) + \int_0^t \langle f, \vartheta_n - 1\rangle\,. \end{aligned} \tag{3.48}$$

Then, we rewrite the left-hand side of (3.48) using the definitions of α (3.1) and of w_n in (3.15) together with the obvious inequality $r^2 - 2r \geqslant 2^{-1}(r^2 - 4)$, which holds for any $r \in \mathbb{R}$, as follows:

$$\langle(\upsilon_n)_t, \zeta_{2,n}(w_n)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\Big(\nu_n\widehat{\zeta_{2,n}}(w_n) + \frac{1}{2}\vartheta_n^2 - \vartheta_n\Big), \tag{3.49}$$

$$\int_0^t\int_\Omega \nabla\alpha(\vartheta_n)\nabla\vartheta_n \geqslant C\int_0^t\int_\Omega |\nabla\vartheta_n|^2, \tag{3.50}$$

$$\int_0^t\int_\Gamma \gamma\alpha(\vartheta_n)\zeta_{2,n}(w_n) \geqslant C\int_0^t\int_\Gamma \vartheta_n^2 - C. \tag{3.51}$$

After combining (3.48) with (3.49)–(3.51), (3.39), (3.6) and (2.14), we arrive at the same estimate as in (3.42) just with w_n replaced by ϑ_n for sufficiently small η. Using also the fact that $f \in L^2(0,T;V')$ and Young's inequality, we end up with

$$\begin{aligned} \nu_n\|\widehat{\zeta_{2,n}}(w_n)(t)\|_{L^1(\Omega)} + |\vartheta_n(t)|^2 + \|\vartheta_n\|^2_{L^2(0,t;V)} \\ \leqslant C(1 + \|\lambda_n'(\chi_n)(\chi_n)_t\|_{L^2(0,t;L^{6/5}(\Omega))}). \end{aligned} \tag{3.52}$$

Finally, because of (3.44), we get the bound

$$\|\vartheta_n\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leqslant C. \tag{3.53}$$

Passage to the limit as $n \nearrow \infty$. Now, we will deduce from the previous a priori estimates some convergences for the solution of problem (P_μ^n). These convergences will be valid for suitable subsequences. From (3.39), (3.45), (3.47) and (3.53) we deduce that there exist a pair (ϑ, χ) and a function w such that

$$
\begin{array}{lll}
(3.54) & \vartheta_n \to \vartheta & \text{weakly star in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;V), \\
(3.55) & w_n \to w & \text{weakly in } L^2(0,T;V), \\
(3.56) & \chi_n \to \chi & \text{weakly star in } W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V), \\
(3.57) & (\chi_n)_{tt} \to \chi_{tt} & \text{weakly in } L^2(0,T;V')
\end{array}
$$

as n goes to $+\infty$. Moreover, from (3.3), (3.15), (3.45) and (3.54)–(3.55) we deduce that

$$
\begin{array}{lll}
(3.58) & \upsilon_n \to \vartheta & \text{weakly in } H^1(0,T;V') \cap L^2(0,T;V), \\
 & \vartheta_n - \upsilon_n \to 0 & \text{strongly in } L^2(0,T;V).
\end{array}
$$

Then, applying the Aubin compactness lemma (cf. [20, Theorem 5.1, p. 58]) and the generalized Ascoli theorem (cf. [26, Corollary 4, p. 85]), we obtain also the strong convergences

$$
\begin{array}{lll}
(3.59) & \upsilon_n \to \vartheta & \text{strongly in } L^2(0,T;H) \cap C^0([0,T];V') \text{ and a.e. in } Q, \\
(3.60) & \chi_n \to \chi & \text{strongly in } C^0([0,T];L^4(\Omega)) \text{ and a.e. in } Q, \\
(3.61) & \vartheta_n \to \vartheta & \text{strongly in } L^2(0,T;H) \text{ and a.e. in } Q
\end{array}
$$

as n goes to $+\infty$. Now, we may pass to the limit in problem (P_μ^n) as n goes to $+\infty$. In fact, observe that

$$
\varrho_n(v) \to \varrho(v) \quad \text{strongly in } L^2(0,T;H)
$$

as n goes to $+\infty$ and for any $v \in L^2(0,T;H)$ such that $\varrho(v) \in L^2(0,T;H)$. Hence, recalling (3.2), (3.15), (3.55) and (3.61), taking into account the monotonicity of ϱ_n and the maximal monotonicity of the graph ϱ and using [2, Prop. 2.5, p. 27], we find that

$$
(3.62) \qquad \vartheta > 0 \quad \text{and} \quad w = \alpha(\vartheta) = k(\vartheta - \vartheta^{-1}) \quad \text{a.e. in } Q.
$$

Taking into account (3.39), (3.61) and (3.62), we may also deduce that

$$
(3.63) \qquad \frac{1}{\vartheta_n} \to \frac{1}{\vartheta} \quad \text{weakly in } L^2(0,T;V) \text{ and a.e. in } Q.
$$

On the other hand, on account of (2.1)–(2.2), (3.7)–(3.8) and (3.60), we have

$$\lambda_n'(\chi_n) \to \lambda'(\chi) \quad \text{strongly in } C^0([0,T];L^4(\Omega)) \tag{3.64}$$

as n goes to $+\infty$; therefore, combining (3.64) with (3.56), we get

$$\lambda_n'(\chi_n)(\chi_n)_t \to \lambda'(\chi)\chi_t \quad \text{weakly in } L^2(0,T;L^{4/3}(\Omega)) \\ \text{and weakly star in } L^\infty(0,T;L^{4/3}(\Omega)) \tag{3.65}$$

as n goes to $+\infty$. Since g_n converges uniformly to g on compact subsets of $\mathbb{R}$ (cf. (3.12)) and (3.60) holds, we have that

$$g_n(\chi_n) \to g(\chi) \quad \text{a.e. in } Q \tag{3.66}$$

as n goes to $+\infty$. Using (2.5) and (3.14), we get the estimate

$$\int_0^t |g_n(\chi_n)|^2 \leqslant \int_0^t |g(\chi_n)|^2 \leqslant \int_0^t (\tau_1|\chi_n|^3 + \tau_2)^2 \leqslant C(\|\chi_n\|^6_{L^\infty(0,T;V)} + 1).$$

Thus $\{g_n(\chi_n)\}$ is bounded in $L^2(0,T;H)$ and so, thanks to (3.66), we get (cf. [20, Lemma 1.3, p. 12])

$$g_n(\chi_n) \to g(\chi) \quad \text{weakly in } L^2(0,T;H) \text{ as } n \nearrow \infty. \tag{3.67}$$

Finally, (3.63) and (3.64) yield

$$\frac{\lambda_n'(\chi_n)}{\vartheta_n} \to \frac{\lambda'(\chi)}{\vartheta} \quad \text{weakly in } L^2(0,T;H) \text{ as } n \nearrow \infty. \tag{3.68}$$

Using the convergences of the sequences of the initial data (cf. Lemma 3.1), (3.55)–(3.58), (3.65), (3.67) and (3.68), we may pass to the limit in (3.17)–(3.19) as n goes to $+\infty$. Invoking also (3.62), we obtain a solution (ϑ,χ) to (2.21)–(2.23). Combining (3.54), (3.56)–(3.58), (3.63), (3.66) and (3.67), we conclude that (2.18), (2.19) and $\chi_{tt} \in L^2(0,T;V')$ hold and that (2.22) can be rewritten as a hyperbolic equation with right-hand side in $L^2(0,T;H)$. The regularity $\chi \in C^1([0,T];H) \cap C^0([0,T];V)$ follows from a standard argument for hyperbolic equations (cf., e.g., [27]). This concludes the proof of Theorem 2.3.

4. Proof of Theorem 2.6

Concerning the notation, during this section we will use the same as we did in the existence estimates of Section 3. We know that the solution $(\vartheta^\mu, \chi^\mu)$ of problem (P_μ) we have obtained from our approximation scheme certainly satisfies the a priori bound (2.41) due to (3.39), (3.45), (3.47) and (3.53) since all constants denoted by C in Section 3 are independent of μ. We now prove that any solution of Problem (P_μ) necessarily satisfies it.

First estimate. Like in [6, Section 4], we may observe that

$$g(\chi^\mu) \in C^0([0,T]; H), \tag{4.1}$$

$$\lambda'(\chi^\mu)(\vartheta^\mu)^{-1} \in L^2(0,T;H), \tag{4.2}$$

$$M_\mu := \lambda'(\chi^\mu)\chi^\mu_t \in L^2(0,T;L^{4/3}(\Omega)). \tag{4.3}$$

By truncation, we can take a sequence $\{M_n\} \subset L^2(0,T;H)$ such that

$$\|M_n\|_{L^2(0,T;L^{4/3}(\Omega))} \leqslant \|M_\mu\|_{L^2(0,T;L^{4/3}(\Omega))} \tag{4.4}$$

and

$$M_n \to M_\mu \quad \text{weakly in } \ L^2(0,T;L^{4/3}(\Omega)) \quad \text{and a.e. in } \ Q,$$

and consider the Cauchy problem (cf. (3.17) and (3.19))

$$\langle(\varrho_n(w_n) + \nu_n w_n)_t, v\rangle + (\!(w_n, v)\!) = (m - M_n, v) + (h, v)_\Gamma \qquad \forall\, v \in V, \ \text{a.e. in } \ (0,T), \tag{4.5}$$

$$w_n(0) = w_{0,n} \quad \text{a.e. in } \ \Omega \tag{4.6}$$

with ν_n defined as in Lemma 3.1. Clearly, there exists a unique $w_n \in C^0([0,T];H) \cap L^2(0,T;V)$ solving (4.5)–(4.6). Defining now ϑ_n and υ_n according to (3.15), we have $\vartheta_n \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and $\upsilon_n \in H^1(0,T;V')$. Moreover, because of (2.17), the formulas (3.29), (3.35), (3.40), (3.43), (3.48) and (3.52) still hold with M_n instead of $\lambda_n'(\chi_n)(\chi_n)_t$. Therefore, summing up the modified versions of (3.35), (3.43) and (3.52), we end up with

$$\begin{aligned}
&\|\widehat{\zeta_{1,n}}(w_n)(t)\|_{L^1(\Omega)} + \frac{1}{3}\|\vartheta_n(t)\|_{L^1(\Omega)} + \frac{1}{3}\|\ln\vartheta_n\|_{L^1(\Omega)} + C\|\vartheta_n^{-1}\|^2_{L^2(0,t;V)} \\
&\qquad + \frac{\nu_n}{2}|w_n(t)|^2 + \frac{k+1}{2}|\vartheta_n(t)|^2 + \frac{1}{2}\|w_n\|^2_{L^2(0,t;V)} \\
&\qquad + \nu_n\|\widehat{\zeta_{2,n}}(w_n)(t)\|_{L^1(\Omega)} + C\|\vartheta_n\|^2_{L^2(0,t;V)} \\
&\leqslant C(1 + \|M_n\|_{L^2(0,t;L^{6/5}(\Omega))}) + \int_0^t\!\!\int_\Omega M_n(\vartheta_n^{-1} - 1).
\end{aligned} \tag{4.7}$$

Considering now (3.42) with $\lambda_n'(\chi_n)(\chi_n)_t$ replaced by M_n and w_n replaced by $1/\vartheta_n$ for sufficiently small η and recalling also (4.4), we see that

$$
\begin{aligned}
(4.8)\qquad & \sqrt{\nu_n}\|w_n\|_{L^\infty(0,T;H)} + \|w_n\|_{L^2(0,T;V)} + \|(\vartheta_n)^{-1}\|_{L^2(0,T;V)} \\
& + \|\vartheta_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\ln\vartheta_n\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C_\mu,
\end{aligned}
$$

where C_μ denotes a positive constant which depends on $\|M_\mu\|_{L^\infty(0,T;L^{4/3}(\Omega))}$ but is independent of n. Moreover, by comparison, we also have the bound

$$
(4.9)\qquad \|\nu_n w_n + \vartheta_n\|_{H^1(0,T;V')} \leqslant C_\mu.
$$

Passage to the limit as $n \nearrow \infty$. Arguing as in the "passage to the limit" subsection of Section 3, we obtain the convergences (as n goes to $+\infty$)

$$
\begin{aligned}
&(4.10)\quad \vartheta_n \to \vartheta^\mu && \text{weakly star in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;V),\\
&(4.11)\quad \vartheta_n \to \vartheta^\mu && \text{strongly in } C^0([0,T];V') \cap L^2(0,T;H),\\
&(4.12)\quad \vartheta_n^{-1} \to (\vartheta^\mu)^{-1} && \text{weakly in } L^2(0,T;V),\\
&(4.13)\quad w_n \to \alpha(\vartheta^\mu) && \text{weakly in } L^2(0,T;V),\\
&(4.14)\quad \nu_n w_n \to 0 && \text{strongly in } C^0([0,T];H),
\end{aligned}
$$

where ϑ^μ is the unique positive solution to the problem

$$
\begin{aligned}
&(4.15)\quad \langle \vartheta^\mu, v\rangle + (\!(\alpha(\vartheta^\mu), v)\!) = (m - M_\mu, v) + (h,v)_\Gamma \quad \forall\, v \in V, \text{ a.e. in } (0,T),\\
&(4.16)\quad \vartheta^\mu(\cdot,0) = \vartheta_0 \quad \text{a.e. in } \Omega.
\end{aligned}
$$

We observe now that by (4.14) and by the boundedness of $\{\vartheta_n\}$ in $L^\infty(0,T;H)$, we may deduce the following convergence that will be useful in the sequel:

$$
(4.17)\qquad \vartheta_n(t) \to \vartheta^\mu(t) \quad \text{weakly in } H \text{ and strongly in } V' \text{ as } n \nearrow \infty.
$$

Second estimate. Recalling (3.35) with M_n instead of $\lambda_n'(\chi_n)(\chi_n)_t$, using Young's inequality, (3.30)–(3.32) and (3.33)–(3.34), we obtain the inequality

$$
(4.18)\quad \frac{1}{3}\|\vartheta_n(t)\|_{L^1(\Omega)} + \frac{1}{3}\|\ln\vartheta_n(t)\|_{L^1(\Omega)} + C\int_0^t \|\vartheta_n^{-1}\|^2 \leqslant C + \int_0^t\!\!\int_\Omega M_n(\vartheta_n^{-1} - 1).
$$

Now, using (4.4), (4.12), (4.17) and the weak lower semicontinuity of the norm, we get from (4.18) the inequality

$$
(4.19)\qquad \frac{1}{3}\|\vartheta^\mu(t)\|_{L^1(\Omega)} + \frac{1}{3}\|\ln\vartheta^\mu(t)\|_{L^1(\Omega)} \leqslant C + \int_0^t\!\!\int_\Omega M_\mu\Big(\frac{1}{\vartheta^\mu} - 1\Big).
$$

Regarding the second variable χ^μ we know that χ^μ satisfies equation (2.22), hence we can formally (the argument may be made rigorous (cf., e.g., [5, Appendix]) take $v = \chi_t^\mu$ in (2.22) and integrate over $(0,t)$ getting the energy inequality

$$\begin{aligned} &\frac{\mu}{2}|\chi_t^\mu(t)|^2 + \frac{1}{2}|\nabla\chi^\mu(t)|^2 + \int_\Omega \hat{g}(\chi^\mu(t)) + \int_0^t |(\chi^\mu)_t(s)|^2 \\ &\leqslant \frac{\mu}{2}|\chi_1|^2 + \frac{1}{2}|\nabla\chi_0|^2 + \int_\Omega \hat{g}(\chi_0) - \int_0^t (M_\mu(s), (\vartheta^\mu)^{-1}(s)). \end{aligned} \tag{4.20}$$

Now, arguing as in Section 3, i.e. adding together (4.18), (4.20) and (3.35) with χ_n replaced by χ^μ, using (2.12), (2.13) and (3.9) with λ'_n replaced by λ', and applying afterwards Young's inequality, we show that

$$\begin{aligned} &\mu|(\chi^\mu)_t(t)|^2 + \|(\chi^\mu)_t\|^2_{L^2(0,t;H)} + \|\chi^\mu(t)\|^2 + \|\vartheta^\mu(t)\|_{L^1(\Omega)} \\ &+\|\ln\vartheta^\mu(t)\|_{L^1(\Omega)} + \left\|\frac{1}{\vartheta^\mu}\right\|^2_{L^2(0,t;V)} \leqslant C\left(1 + \int_0^t |\chi^\mu|^2\right). \end{aligned} \tag{4.21}$$

Using Gronwall's lemma, we get a uniform bound for the left-hand side of (4.21). Recalling now the definition of M_μ and the first estimates in (3.44) with λ'_n replaced by λ and χ_n replaced by χ^μ, it turns out that we have derived also a uniform bound for $\|M_\mu\|_{L^2(0,T;L^{4/3}(\Omega))}$ independently of μ. Hence, we can replace C_μ with C in (4.8)–(4.9). Finally, by comparison with (2.21), we have that

$$\|\vartheta_t^\mu\|_{L^2(0,T;V')} \leqslant R, \tag{4.22}$$

and so, from (4.21)–(4.22), we deduce the desired estimate (2.41).

Passage to the limit as $\mu \searrow 0$. Recalling (4.3), combining (4.21) and (4.22) with (4.8) and (4.9) (with C instead of C_μ), we find (2.41); thus, we may find a subsequence $\{\mu_k\}$ that converges to 0 and a pair (ϑ,χ) such that the following convergences hold (for $k \nearrow \infty$):

$$\vartheta^{\mu_k} \to \vartheta \qquad \text{weakly star in } L^\infty(0,T;H), \tag{4.23}$$

$$\vartheta^{\mu_k} \to \vartheta \qquad \text{weakly in } H^1(0,T;V') \cap L^2(0,T;V), \tag{2.24}$$

$$\vartheta^{\mu_k} \to \vartheta \qquad \text{strongly in } C^0([0,T];V') \cap L^2(0,T;H), \tag{4.25}$$

$$\frac{1}{\vartheta^{\mu_k}} \to \frac{1}{\vartheta} \qquad \text{weakly in } L^2(0,T;V), \tag{4.26}$$

$$
\begin{array}{lll}
(4.27) & \alpha(\vartheta^{\mu_k}) \to \alpha(\vartheta) & \text{weakly in } L^2(0,T;V), \\
(4.28) & \mu_k \chi_t^{\mu_k} \to 0 & \text{strongly in } C^0([0,T];H), \\
(4.29) & \chi^{\mu_k} \to \chi & \text{weakly star in } L^\infty(0,T;V), \\
(4.30) & \chi^{\mu_k} \to \chi & \text{weakly in } H^1(0,T;H), \\
(4.31) & \chi^{\mu_k} \to \chi & \text{strongly in } C^0(0,T;L^{6-\varepsilon}(\Omega)) \quad \forall\, \varepsilon > 0.
\end{array}
$$

We may first integrate (2.22) written for $(\vartheta^{\mu_k}, \chi^{\mu_k})$ in time over $(0,t)$, then we may write also (2.21) for the pair $(\vartheta^{\mu_k}, \chi^{\mu_k})$ and, finally, thanks to (4.23)–(4.31), we may pass to the limit as k goes to $+\infty$, arguing as in Section 3 for the nonlinearities and deducing that the limit pair (ϑ, χ) satisfies a.e. in $(0,T)$

$$
\begin{array}{ll}
(4.32) & \langle(\vartheta + \lambda(\chi))_t, v\rangle + (\!(\alpha(\vartheta), v)\!) = (m, v) + (h, v)_\Gamma \quad \forall\, v \in V, \\
(4.33) & (\chi, v) + (1 * \nabla\chi, \nabla v) + (1 * (g(\chi) + \lambda'(\chi)\vartheta^{-1}, v) = (\chi_0, v) \quad \forall\, v \in V, \\
(4.34) & \vartheta(\cdot, 0) = \vartheta_0 \quad \text{a.e. in } \Omega.
\end{array}
$$

On the other hand, owing to (2.1)–(2.5), (2.12), (4.26), (4.29)–(4.31), we deduce from (4.33) that

$$
\begin{array}{ll}
(4.35) & (\chi_t, v) + (\nabla\chi, \nabla v) + (g(\chi) + \lambda'(\chi)\vartheta^{-1}, v) = 0 \quad \forall\, v \in V, \ \text{a.e. in } (0,T), \\
(4.36) & \chi(\cdot, 0) = \chi_0 \quad \text{a.e. in } \Omega.
\end{array}
$$

Finally, we may recover the regularity of χ (2.36) by comparison with (4.35). This proves that the pair (ϑ, χ) solves problem (P_0) and the uniqueness of solutions to (P_0) proves that the whole family $\{(\vartheta^\mu, \chi^\mu)\}$ converges to (ϑ, χ) as μ goes to 0, in the sense of (4.23)–(4.31). This concludes the proof of Theorem 2.6.

5. Proof of Theorem 2.4

Throughout this section, we will formally perform the regularity estimates for problem (P_μ). We will omit the indices μ for simplicity of notation and use here the already known estimates for the solution of problem (P_μ). Moreover, only within this section, the positive constants appearing in the estimates (denoted by C) are allowed to depend on all the data of the problem (including μ) since we are interested here (cf. Theorem 2.4) in proving a regularity result concerning solutions to problem (P_μ) for $\mu > 0$.

First regularity estimate. We introduce now further regularity assumptions on the data (2.24)–(2.27) in order to prove (2.28). Under these hypotheses, we want to take $v = -\Delta\chi_t$ as a test function in equation (2.22). In order to do that we have to know that it is an admissible test function. Hence, we may proceed writing the equation (2.22) as

$$\mu\chi_{tt} + \chi_t - \Delta\chi = s \quad \text{in } V' \text{ and a.e. in } (0,T) \text{ with } \chi_{\mathbf{n}} = 0 \text{ on } \Sigma, \tag{5.1}$$

where, thanks to (2.19), (2.20) and (2.26), the function s belongs to $L^2(0,T;V)$. Moreover, we can proceed by regularizing s with some $s_n \in H^2(0,T;V)$, the data χ_0 and χ_1 with some $\chi_{i,n} \in H^3(\Omega) \cap W_{\mathbf{n}}$ $(i = 0,1)$ with $W_{\mathbf{n}} := \{w \in W\colon w_{\mathbf{n}} = 0$ on $\Gamma\}$. Then, using [12, Teorema 4.4, p. 661], we obtain that the solution of (5.1) is in $C^2(0,T;V) \cap C^3(0,T;H)$, and so $-\Delta\chi_t \in C^0([0,T];H)$ as desired. Whence, we can proceed now formally testing (2.22) with $-\Delta\chi_t$, integrating some terms by parts in space (using the boundary condition (1.8) and the initial conditions in (2.23)), and integrating it again, but in time, over $(0,t)$ for $t \in (0,T)$, getting

$$\begin{aligned}&\frac{\mu}{2}\|\nabla\chi_t(t)|^2 + \|\nabla\chi_t\|^2_{L^2(0,t;H)} + \frac{1}{2}|\Delta\chi(t)|^2 \\ &\leqslant \frac{\mu}{2}|\nabla\chi_1|^2 + \frac{1}{2}|\Delta\chi_0|^2 + \int_0^t\int_\Omega g(\chi)\Delta\chi_t + \int_0^t\int_\Omega \lambda'(\chi)\vartheta^{-1}\Delta\chi_t.\end{aligned} \tag{5.2}$$

In order to estimate the last two integrals, we have to use (2.18)–(2.20). To deal with the term containing g, we apply Young's inequality and (2.59):

$$\begin{aligned}\left|\int_0^t\int_\Omega g'(\chi)\nabla\chi\nabla\chi_t\right| &\leqslant C\int_0^t\int_\Omega (1+\chi^2)|\nabla\chi|\,|\nabla\chi_t| \\ &\leqslant C + \frac{1}{4}\|\nabla\chi_t\|^2_{L^2(0,t;H)} + C\int_0^t \|\chi\|^2_{L^6(\Omega)}\|\nabla\chi\|_{L^6(\Omega)}|\nabla\chi_t| \\ &\leqslant C + \frac{1}{4}\|\nabla\chi_t\|^2_{L^2(0,t;H)} + C\int_0^t (|\Delta\chi| + |\chi|)|\nabla\chi_t| \\ &\leqslant C + \frac{1}{2}\|\nabla\chi_t\|^2_{L^2(0,t;H)} + C\|\Delta\chi\|^2_{L^2(0,t;H)}.\end{aligned} \tag{5.3}$$

Regarding the term containing λ, we have to use (2.26) as follows:

$$\begin{aligned}\int_0^t\int_\Omega \lambda'(\chi)\vartheta^{-1}\Delta\chi_t &= -\int_0^t\int_\Omega \vartheta^{-1}\lambda''(\chi)\nabla\chi\nabla\chi_t - \int_0^t\int_\Omega \lambda'(\chi)\nabla\vartheta^{-1}\nabla\chi_t \\ &\leqslant C\int_0^t \|\vartheta^{-1}\|_{L^6(\Omega)}\|\nabla\chi\|_{L^6(\Omega)}|\nabla\chi_t| + C\int_0^t |\nabla\vartheta^{-1}|\,|\nabla\chi_t| \\ &\leqslant C + \frac{1}{4}\|\nabla\chi_t\|^2_{L^2(0,t;H)} + \|\Delta\chi\|^2_{L^2(0,t;H)}.\end{aligned} \tag{5.4}$$

Combining (5.2) with (5.3) and (5.4) allows us to apply a standard version of Gronwall's lemma to show that

$$\mu\|\nabla\chi_t\|^2_{L^\infty(0,T;H)} + \|\nabla\chi_t\|^2_{L^2(0,T;H)} + \|\Delta\chi\|^2_{L^\infty(0,T;H)} \leqslant C \tag{5.5}$$

and, consequently,

$$\|\chi\|_{L^\infty(Q)} \leqslant C. \tag{5.6}$$

Second regularity estimate. We take now into account further regularity assumptions given in (2.29). Let K be a positive constant such that the bound

$$1 + \frac{2}{k} + \|m\|_{L^2(0,T;L^6(\Omega))} + \|h\|_{L^\infty(\Sigma)} + |\Gamma| \leqslant K + K|\Gamma| \tag{5.7}$$

holds. Now, in order to prove an estimate for ϑ in $L^\infty(Q)$, we use Moser's technique (cf., e.g., [1] and [19]). First of all, let us set $\ell := m - \lambda'(\chi)\chi_t$. Observe that, thanks to (5.6), (2.20) and (5.7), we have that ℓ is bounded in $L^2(0,T;L^6(\Omega))$. Moreover, let $p \in (1,+\infty)$, multiply (2.21) by ϑ^p and integrate over Ω, obtaining

$$\begin{aligned}\frac{1}{p+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta^{p+1} + k\int_\Omega \nabla\vartheta\nabla\vartheta^p + \frac{4pk}{(p-1)^2}\int_\Omega |\nabla\vartheta^{\frac{p-1}{2}}|^2 + \gamma k\int_\Gamma \vartheta^{p+1}\\ \leqslant \gamma k\int_\Gamma \vartheta^{p-1} + \gamma\int_\Gamma h\vartheta^p + \int_\Omega \ell\vartheta^p.\end{aligned} \tag{5.8}$$

Simply using (5.7) and the generalized Young inequality (2.58) first with the exponents $p/(p-1)$ and p and then with the exponents $(p+1)/p$ and $p+1$, we deduce that

$$\gamma\Big(k\int_\Gamma \vartheta^{p-1} + \int_\Gamma h\vartheta^p\Big) \leqslant \gamma K\Big(\int_\Gamma \vartheta^p + 1\Big) \leqslant \frac{\gamma kp}{2(p+1)}\int_\Gamma \vartheta^{p+1} + \gamma K^{2p+3}. \tag{5.9}$$

Hence, we deduce from (5.8) multiplied by $p+1$ (neglecting the nonnegative term $k\int_\Omega \nabla\vartheta\nabla\vartheta^p$) the inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta^{p+1} + \frac{4pk(p+1)}{(p-1)^2}\int_\Omega |\nabla\vartheta^{\frac{p-1}{2}}|^2 + \gamma\frac{k}{2}p\int_\Gamma \vartheta^{p+1}\\ \leqslant \gamma(p+1)K^{2p+3} + (p+1)\int_\Omega \ell\vartheta^p.\end{aligned} \tag{5.10}$$

Moreover, using the generalized Young inequality, we can observe that $\int_\Gamma \vartheta^{p+1} \geqslant \int_\Gamma \vartheta^{p-1} - |\Gamma|$ and so the last two integrals on the left-hand side of (5.10) can be bounded from below as follows:

$$\frac{4pk(p+1)}{(p-1)^2}\int_\Omega |\nabla\vartheta^{\frac{p-1}{2}}|^2 + \gamma\frac{k}{2}p\int_\Gamma \vartheta^{p+1} \geqslant C\|\vartheta^{\frac{p-1}{2}}\|^2 - \gamma\frac{k}{2}|\Gamma|.$$

Hence, owing to Hölder's and Young's inequalities, we obtain from (5.10) (using (2.29)) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \vartheta^{p+1} + \overline{C}\Big(\int_\Omega (\vartheta^{\frac{p-1}{2}})^6\Big)^{2/6} \leqslant CK^{2p+3}(p+1) + (p+1)\int_\Omega |\ell|\vartheta^p. \tag{5.11}$$

Next, we infer from Hölder's and Young's inequalities that

$$(p+1)\int_\Omega |\ell|\vartheta^p \leqslant (p+1)\|\ell\|_{L^6(\Omega)}\|\vartheta^{\frac{p-1}{2}}\|_{L^6(\Omega)}\|\vartheta^{\frac{p+1}{2}}\|_{L^{3/2}(\Omega)} \tag{5.12}$$

$$\leqslant \frac{\overline{C}}{2}\Big(\int_\Omega \vartheta^{3(p-1)}\Big)^{1/3} + C(p+1)^2\|\ell\|^2_{L^6(\Omega)}\Big(\int_\Omega \vartheta^{3(p+1)/4}\Big)^{4/3}. \tag{5.13}$$

We now consider sequences (p_n) and (σ_n) of real numbers defined by

$$p_0 = 2, \quad p_{n+1} = \frac{4}{3}p_n, \quad \sigma_n = 2p_n, \quad n \in \mathbb{N}.$$

Then we have that $\sigma_{n+1} = 4/3\sigma_n$. Now, letting $n \in \mathbb{N}$ and taking $p = p_{n+1} - 1$ in (5.11), it follows from (5.13)

$$\sup_{t\in(0,T)}\int_\Omega \vartheta^{p_{n+1}}(t) \leqslant C\sigma^2_{n+1}\max\Big\{K^{\sigma_{n+1}}, \sup_{t\in(0,T)}\Big(\int_\Omega \vartheta^{p_n}(t)\Big)^{4/3}\Big\}.$$

Hence, using Lemma 8.1 in the Appendix with $(\gamma_n) = \sup_{t\in(0,T)}\|\vartheta(t)\|^{p_n}_{L^{p_n}(\Omega)}$, $(\delta_n) = (\sigma_n)$, $a = 4/3$, $c = 0$, $b = 2$ and $C_1 = K$, we get that there exists a positive constant $\tilde{C}$ independent of n such that

$$\sup_{t\in(0,T)}\|\vartheta(t)\|^{1/2}_{L^{p_n}(\Omega)} \leqslant \tilde{C}, \quad \forall\, n \in \mathbb{N}. \tag{5.14}$$

Taking $n \nearrow \infty$, we immediately obtain that ϑ is bounded in $L^\infty(Q)$.

Third regularity estimate. We may employ the same Moser's technique in order to establish the $L^\infty(Q)$ bound for $1/\vartheta$ multiplying (2.21) by $-\vartheta^{-p}$ and integrating over Ω.

Fourth regularity estimate. During this estimate we will use strongly the previous ones. In fact, to obtain this estimate, we need the regularity assumptions (2.24)–(2.27), (2.29) and (2.31)–(2.32) on the data. Then, let us multiply (2.21) by $(\vartheta_t/\vartheta^2 + \vartheta_t)$ and add the resulting equation to the time derivative of (2.22) tested with z_t, where we have set $z := \chi_t$. Doing that, we obtain the following equality, which holds for all $t \in [0,T]$:

$$
\begin{aligned}
(5.15)\ \int_\Omega \Big(\vartheta_t^2 + \frac{\vartheta_t^2}{\vartheta^2}\Big)(t) + \frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\vartheta - \vartheta^{-1})(t)\|^2 + \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z_t(t)|^2 + |z_t(t)|^2 \\
+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla z(t)|^2 = \Big\langle f(t), \Big(\vartheta_t + \frac{\vartheta_t}{\vartheta^2}\Big)(t)\Big\rangle - \int_\Omega \lambda'(\chi(t))z(t)\Big(\frac{\vartheta_t}{\vartheta^2} + \vartheta_t\Big)(t) \\
- \int_\Omega g'(\chi(t))z(t)z_t(t) - \int_\Omega \frac{\lambda''(\chi(t))}{\vartheta(t)} z(t)z_t(t) + \int_\Omega \lambda'(\chi(t))\frac{\vartheta_t}{\vartheta^2}(t)z_t(t).
\end{aligned}
$$

Now, we integrate it over $(0,t)$ with $t \in [0,T]$ and then estimate the terms on the right-hand side. First, we may integrate by parts (in time) the term containing f and use hypotheses (2.31)–(2.32) as follows:

$$
\begin{aligned}
(5.16)\quad & \int_0^t \Big\langle f, \vartheta_t + \frac{\vartheta_t}{\vartheta^2}\Big\rangle \\
&= -\int_0^t \langle f_t, \vartheta - \vartheta^{-1}\rangle + \langle f(t), (\vartheta - \vartheta^{-1})(t)\rangle - \langle f(0), (\vartheta - \vartheta^{-1})(0)\rangle \\
&\leqslant \int_0^t \|f_t\|_* \|\vartheta - \vartheta^{-1}\| + \frac{k}{4}\|(\vartheta - \vartheta^{-1})(t)\|^2 \\
&\quad + C\|f\|^2_{C^0([0,T];V')} + \|(\vartheta - \vartheta^{-1})(0)\|^2.
\end{aligned}
$$

In the sequel, we set

$$
\begin{aligned}
J_1(t) &:= \int_0^t \int_\Omega \lambda'(\chi(t))z(t)\Big(\frac{\vartheta_t}{\vartheta^2} + \vartheta_t\Big)(t), \\
J_2(t) &:= -\int_0^t \int_\Omega \frac{\lambda''(\chi(t))}{\vartheta(t)} z(t)z_t(t), \\
J_3(t) &:= \int_\Omega \lambda'(\chi(t))\frac{\vartheta_t}{\vartheta^2}(t)z_t(t).
\end{aligned}
$$

Let us use Hölder's inequality and the first regularity estimate for z (cf. (5.5)) with the continuous embedding of V in $L^4(\Omega)$, and (2.26), in order to get the following estimate for $J_1(t)$:

$$
\begin{aligned}
(5.17)\qquad |J_1(t)| &\leqslant C\Big(\int_0^t \|z\|_{L^4(\Omega)}\Big|\frac{\vartheta_t}{\vartheta}\Big|\,\|\vartheta^{-1}\|_{L^4(\Omega)} + \int_0^t |z|\,|\vartheta_t|\Big) \\
&\leqslant \frac{1}{2}\Big(\int_0^t \Big|\frac{\vartheta_t}{\vartheta}\Big|^2 + \int_0^t |\vartheta_t|^2\Big) + C\int_0^t \|\vartheta^{-1}\|^2.
\end{aligned}
$$

Then we use (2.2) and the third regularity estimate in order to obtain

$$
(5.18)\qquad |J_1(t)| \leqslant \frac{1}{2}\int_0^t |z_t|^2 + C\int_0^t |z|^2.
$$

Now, in order to estimate the term containing g in (5.15), we have to use Hölder's inequality, (2.19), (2.36) and the estimate (2.41), in fact (since $\chi \in L^\infty(Q)$ due to (5.6)) arriving at

$$\int_0^t\int_\Omega g'(\chi)zz_t \leqslant C_1\int_0^t\int_\Omega (1+\chi^2)|zz_t| \leqslant \frac{1}{2}\int_0^t |z_t|^2 + C\int_0^t |z|^2. \tag{5.19}$$

Finally, from (2.26), Hölder's inequality the third regularity estimate on ϑ^{-1} it follows that

$$|J_3(t)| \leqslant C\int_0^t \left|\frac{\vartheta_t}{\vartheta}\right| \|\vartheta^{-1}\|_{L^\infty(\Omega)}|z_t| \leqslant \delta\int_0^t \left|\frac{\vartheta_t}{\vartheta}\right|^2 + C_\delta\int_0^t \|\vartheta^{-1}\|^2_{L^\infty(\Omega)}|z_t|^2 \tag{5.20}$$

for all $\delta > 0$ and for a positive constant C_δ. Now (5.15), thanks to (5.16)–(5.20) and provided δ is sufficiently small, becomes

$$\begin{aligned}\int_0^t\int_\Omega\Big(\vartheta_t^2 + \frac{\vartheta_t^2}{\vartheta^2}\Big) + \|(\vartheta-\vartheta^{-1})(t)\|^2 + \mu|z_t(t)|^2 + \|z_t\|^2_{L^2(0,t;H)} + \|z(t)\|^2 \\ \leqslant C\Big(1 + \int_0^t \|z\|^2 + \int_0^t \|z_t\|^2 + \int_0^t \|\vartheta - \vartheta^{-1}\|^2\Big),\end{aligned} \tag{5.21}$$

where we have added up to both sides of (5.15) (integrated in time) the term

$$\frac{1}{2}|z(t)|^2 = \frac{1}{2}|z(0)|^2 + \int_0^t (z_s(s), z(s))$$

in order to obtain the full V-norm of z on the left-hand side. Moreover, applying a standard version of Gronwall's lemma to (5.21), we have the regularities (2.33) (for solutions of (P_μ) with a strictly positive coefficient μ).

This concludes both the proof of Theorem 2.4 and this section.

6. Proof of Theorem 2.8

Suppose, throughout this section, that (ϑ_i, χ_i) for $i = 1, 2$ solves problem (P_μ) with the data ϑ_{0i}, χ_{0i}, χ_{1i}, f_i instead of ϑ_0, χ_0, χ_1, f, respectively, and be $u_i := -\vartheta_i^{-1}$. Set, for this section,

$$\begin{aligned}\chi = \chi_1 - \chi_2, \quad \vartheta = \vartheta_1 - \vartheta_2, \quad u = u_1 - u_2, \quad f = f_1 - f_2, \\ \chi_0 = \chi_{01} - \chi_{02}, \quad \chi^1 = \chi_{11} - \chi_{12}, \quad \vartheta_0 = \vartheta_{01} - \vartheta_{02}.\end{aligned} \tag{6.1}$$

Then we have

$$\langle(\vartheta+\lambda(\chi_1)-\lambda(\chi_2))_t,v\rangle+k(\!(\vartheta+u,v)\!)=(f,v)\quad\forall\, v\in V,\ \text{ a.e. in } (0,T), \tag{6.2}$$

$$\begin{aligned}&\langle\mu\chi_{tt},v\rangle+(\chi_t,v)+(\nabla\chi,\nabla v)+(g(\chi_1)-g(\chi_2)-\lambda'(\chi_1)u_1\\&\qquad+\lambda'(\chi_2)u_2,v)=0\quad\forall\, v\in V,\ \text{ a.e. in } (0,T),\end{aligned} \tag{6.3}$$

$$\vartheta(\cdot,0)=\vartheta_0,\quad \chi(\cdot,0)=\chi_0,\quad \chi_t(\cdot,0)=\chi^1\quad\text{a.e. in }\Omega. \tag{6.4}$$

Let us integrate (6.2) in time over $(0,t)$, take $v=k(\vartheta+u)$, and integrate in time once more. Then we have the equality

$$\begin{aligned}&k\|\vartheta\|^2_{L^2(0,t;H)}+k\int_0^t(\vartheta,u)+\frac12\|1*[k\vartheta+ku](t)\|^2\\&=k\int_0^t\langle\vartheta_0+\lambda(\chi_{01})-\lambda(\chi_{02})+1*f,\vartheta+u\rangle+k\int_0^t\!\!\int_\Omega(\lambda(\chi_1)-\lambda(\chi_2))(\vartheta+u).\end{aligned} \tag{6.5}$$

First, we observe that (cf., e.g., [8]) there exists a positive constant $\bar d$ such that

$$(\vartheta,u)\geqslant\bar d\int_\Omega\frac{|u|^2}{1+|u_1|^2+|u_2|^2}, \tag{6.6}$$

and, thanks to (2.26), using Young's inequality we obtain (cf. also [6, (5.9)–(5.11)])

$$\begin{aligned}-\int_0^t\!\!\int_\Omega k(\lambda(\chi_1)-\lambda(\chi_2))u&\leqslant C\int_0^t\!\!\int_\Omega|\chi||u|\\&\leqslant\frac{\bar d}{2}\int_\Omega\frac{|u|^2}{1+|u_1|^2+|u_2|^2}+C\int_\Omega(1+|u_1|^2+|u_2|^2)|\chi|^2\\&\leqslant\frac{\bar d}{2}\int_\Omega\frac{|u|^2}{1+|u_1|^2+|u_2|^2}+\|\chi\|^2_{L^2(0,t;H)}+\sum_{i=1}^2\int_0^t\|u_i\|^2_{L^\infty(\Omega)}|\chi|^2.\end{aligned} \tag{6.7}$$

Note that, if $N=1$, there is no need of (2.26) since $C^0([0,T];V)\hookrightarrow C^0(\overline{Q})$. In this case, the constant C depends on M_1 as well. Moreover, using Young's inequality, we get

$$-\int_0^t\!\!\int_\Omega(\lambda(\chi_1)-\lambda(\chi_2))k\vartheta\leqslant\delta\|\vartheta\|^2_{L^2(0,t;H)}+C_\delta\|\chi\|^2_{L^2(0,t;H)} \tag{6.8}$$

for all $\delta>0$ and for a positive constant C_δ. Finally, we have to integrate by parts in time the term containing f in (6.5) and to use Young's inequality once more, to arrive at

$$\text{(6.9)}\qquad \int_0^t \langle 1*f, k\vartheta + ku\rangle = \langle (1*f)(t), 1*[k\vartheta+ku](t)\rangle - \int_0^t \langle f, 1*(k\vartheta+ku)\rangle$$
$$\leqslant \frac{1}{4}\|[1*(k\vartheta+ku)](t)\|^2 + C\|(1*f)(t)\|_*^2$$
$$+ \|f\|^2_{L^2(0,t;V')} + \int_0^t \|1*(k\vartheta+ku)\|^2.$$

Hence, using the estimates (6.6)–(6.9) and choosing δ sufficiently small, (6.5) becomes

$$\text{(6.10)}\qquad \|\vartheta\|^2_{L^2(0,t;H)} + \int_0^t \frac{|u|^2}{1+|u_1|^2+|u_2|^2} + \|[1*(k\vartheta+ku)](t)\|^2$$
$$\leqslant C\Bigg(\|\vartheta_0\|_*^2 + |\chi_0|^2 + \|f\|^2_{L^2(0,t;V')} + \int_0^t \|1*(k\vartheta+ku)\|^2$$
$$+ \|\chi\|^2_{L^2(0,t;H)} + \sum_{i=1}^2 \int_0^t \|u_i\|^2_{L^\infty(\Omega)}|\chi|^2\Bigg),$$

where the constant C depends also on M_1.

Now, let us take $v = \chi_t$ in (6.3) and then integrate it over $(0,t)$ to obtain

$$\text{(6.11)}\qquad \frac{\mu}{2}|\chi_t(t)|^2 + \|\chi_t\|^2_{L^2(0,t;H)} + \frac{1}{2}|\nabla\chi(t)|^2$$
$$\leqslant -\int_0^t (g(\chi_1)-g(\chi_2),\chi_t) + \int_0^t (\lambda'(\chi_1)u_1 - \lambda'(\chi_2)u_2, \chi_t)$$
$$+ \frac{\mu}{2}|\chi^1|^2 + \frac{1}{2}|\nabla\chi_0|^2.$$

We notice here that this argument is only formal since $\chi_t(t)$ does not belong to V, but it may be made rigorous for example by using the same techniques as that employed in [5].

Now, in order to estimate the terms on the right-hand side in (6.11), we may proceed exactly like in [6, (5.14)–(5.16)]. Let us recall here the procedure performed in order to get a bound for the second integral on the right-hand side in (6.11) because it is just here that the assumption (2.49) is necessary. Indeed, first we may write

$$\text{(6.12)}\qquad \int_0^t (\lambda'(\chi_1)u_1 - \lambda'(\chi_2)u_2, \chi_t)$$
$$= \int_0^t\int_\Omega u_1(\lambda'(\chi_1)-\lambda'(\chi_2))\chi_t + \int_0^t\int_\Omega \lambda'(\chi_2)u\chi_t.$$

Then, let us estimate the two integrals in (6.12) separately, using (2.26), (2.49) and Young's inequality, as follows:

$$\int_0^t \int_\Omega u_1(\lambda'(\chi_1) - \lambda'(\chi_2))\chi_t \leqslant \delta \int_0^t |\chi_t|^2 + C_\delta \int_0^t \|u_1\|^2_{L^\infty(\Omega)} |\chi|^2 \tag{6.13}$$

for all positive δ and for a positive constant C_δ. Moreover, in order to estimate the second integral in (6.12), we may observe that, thanks also to (6.6) and (2.50), we have

$$\begin{aligned}\int_0^t \int_\Omega \lambda'(\chi_2) u \chi_t &\leqslant C \int_0^t \int_\Omega |u|\,|\chi_t| \\ &\leqslant C \int_0^t \int_\Omega \frac{|u|}{\sqrt{1+|u_1|^2+|u_2|^2}} \sqrt{1+|u_1|^2+|u_2|^2}|\chi_t| \\ &\leqslant \frac{\bar d}{4} \int_0^t \int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|^2} + C \int_0^t \sum_{i=1}^2 (1+\|u_i\|^2_{L^\infty(\Omega)})|\chi_t|^2.\end{aligned} \tag{6.14}$$

Further, adding to both sides of (6.11) the term

$$\frac{1}{2}|\chi(t)|^2 = \frac{1}{2}|\chi_0|^2 + \int_0^t (\chi_t, \chi)$$

in order to recover the full V-norm of $\chi(t)$ on the left-hand side, using (6.12)–(6.14) and choosing δ sufficiently small, we get

$$\begin{aligned}&\frac{\mu}{2}|\chi_t(t)|^2 + \|\chi_t\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\chi(t)\|^2 \\ &\quad\leqslant C\Big(\frac{\mu}{2}|\chi^1|^2 + \|\chi_0\|^2 + \|\chi_t\|^2_{L^2(0,t;H)} \\ &\qquad + \|\chi\|^2_{L^2(0,t;V)}\Big) + \frac{\bar d}{4} \int_0^t \int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|^2} \\ &\qquad + C \int_0^t \sum_{i=1}^2 \|u_i\|^2_{L^\infty(\Omega)} |\chi_t|^2,\end{aligned} \tag{6.15}$$

where the constant C depends also on M_1.

Finally, let us set $v = J^{-1}\vartheta$ in (6.2) and integrate over $(0,t)$ to obtain, thanks to (2.15)–(2.16) and to the monotonicity of the function $\vartheta \mapsto -1/\vartheta$ from $(0,+\infty)$ into $(-\infty,0)$,

$$\begin{aligned}&\frac{1}{2}\|\vartheta(t)\|^2_* + k\|\vartheta\|^2_{L^2(0,t;H)} \\ &\quad\leqslant \frac{1}{2}\|\vartheta_0\|^2_* + \int_0^t \langle f, J^{-1}\vartheta\rangle \\ &\qquad - \int_0^t \int_\Omega (\lambda'(\chi_1)(\chi_1)_t - \lambda'(\chi_2)(\chi_2)_t) J^{-1}\vartheta.\end{aligned} \tag{6.16}$$

Now, write down the last integral in (6.6) as

$$(6.17)\qquad -\int_0^t\int_\Omega(\lambda'(\chi_1)(\chi_1)_t-\lambda'(\chi_2)(\chi_2)_t)J^{-1}\vartheta = I_1+I_2,$$

where

$$(6.18)\qquad I_1 := -\int_0^t\int_\Omega J^{-1}(\vartheta)(\lambda'(\chi_1)-\lambda'(\chi_2))(\chi_1)_t,$$

$$(6.19)\qquad I_2 := -\int_0^t\int_\Omega \lambda'(\chi_2)J^{-1}(\vartheta)((\chi_1)_t-(\chi_2)_t),$$

and estimate the above two integrals separately, using (2.26) and (2.50), Young's and Hölder's inequalities as follows:

$$(6.20)\qquad \begin{aligned} I_1 &\leqslant C\int_0^t \|J^{-1}\vartheta\|_{L^4(\Omega)}\|\chi\|_{L^4(\Omega)}|(\chi_1)_t| \\ &\leqslant C\|\chi\|^2_{L^2(0,t;V)}+\int_0^t |(\chi_1)_t|^2\|\vartheta\|^2_*, \end{aligned}$$

$$(6.21)\qquad I_2 \leqslant C\int_0^t |J^{-1}\vartheta|\,|\chi_t| \leqslant C(\|\vartheta\|^2_{L^2(0,t;V')}+\|\chi_t\|^2_{L^2(0,t;H)}),$$

where the constant C depends also on M_1. Now, inserting (6.17)–(6.21) into (6.16), we get

$$(6.22)\qquad \begin{aligned} \|\vartheta(t)\|^2_*+\|\vartheta\|^2_{L^2(0,t;H)} &\leqslant C(\|\vartheta_0\|^2_*+\|f\|^2_{L^2(0,t;V')}+\|\vartheta\|^2_{L^2(0,t;V')} \\ &\quad +\|\chi\|^2_{L^2(0,t;V)}+\|\chi_t\|^2_{L^2(0,t;H)}+\int_0^t\|\vartheta\|^2_*|(\chi_1)_t|^2). \end{aligned}$$

Finally, we may collect estimates (6.10), (6.15), (6.22) arriving at

$$(6.23)\qquad \begin{aligned} &\|\vartheta(t)\|^2_*+\|\vartheta\|^2_{L^2(0,t;H)}+\int_0^t\int_\Omega\frac{|u|^2}{1+|u_1|^2+|u_2|^2}+\|1*(k\vartheta+ku)(t)\|^2 \\ &\quad +\mu|\chi_t(t)|^2+\|\chi_t\|^2_{L^2(0,t;H)}+\|\chi(t)\|^2 \\ &\leqslant C\Big(\|\vartheta_0\|^2_*+\mu|\chi^1|^2+\|\chi_0\|^2+\|f\|^2_{L^2(0,t;V')}+\int_0^t\|1*(k\vartheta+ku)\|^2 \\ &\quad +\|\vartheta\|^2_{L^2(0,t;V')}+\|\chi\|^2_{L^2(0,t;V)}+\|\chi_t\|^2_{L^2(0,t;H)}\Big)+\int_0^t\|\vartheta\|^2_*|(\chi_1)_t|^2 \\ &\quad +C\int_0^t\sum_{i=1}^2\|u_i\|^2_{L^\infty(\Omega)}|\chi|^2+C\int_0^t\sum_{i=1}^2\|u_i\|^2_{L^\infty(\Omega)}|\chi_t|^2 \end{aligned}$$

and, applying a standard version of Gronwall's lemma to (6.23) and recalling (2.50), we find the continuous dependence estimate (2.51), which yields the uniqueness of solutions to problem (P_μ) in the case of $N = 1$. Hence the proof of Theorem 2.8 is completed.

7. Proof of Theorem 2.9

Throughout this section, we will use the same notation as in the previous Section 6. Observe that, thanks to (2.52), (6.5) becomes

$$\tag{7.1} \begin{aligned} k\|\vartheta\|^2_{L^2(0,t;H)} + k\int_0^t (\vartheta, u) + \int_0^t\int_\Omega \chi(k\vartheta + ku) + \frac{1}{2}\|[1 * (k\vartheta + ku)](t)\|^2 \\ = \int_0^t \langle \vartheta_0 + \chi_0 + 1 * f, k\vartheta + ku\rangle \end{aligned}$$

and, integrating by parts (in time) the third term in (7.1), we obtain

$$\tag{7.2} \begin{aligned} -\int_0^t\int_\Omega \chi(k\vartheta + ku) &= \int_0^t\int_\Omega \chi_t[1 * (k\vartheta + ku)] - \int_\Omega \chi(t)[1 * (k\vartheta + ku)(t)] \\ &\leqslant \int_0^t \|\chi_t\|_*\|[1 * (k\vartheta + ku)]\| + C\|\chi(t)\|_*\|[1 * (k\vartheta + ku)](t)\|. \end{aligned}$$

Therefore, taking (6.6) and (7.2) into account, integrating by parts the terms containing f as in the previous section and using Young's and Hölder's inequalities, (7.1) becomes

$$\tag{7.3} \begin{aligned} k\|\vartheta\|^2_{L^2(0,t;H)} + \frac{1}{2}\|[1 * (k\vartheta + k)u](t)\|^2 + \int_0^t \frac{|u|^2}{1 + |u_1|^2 + |u_2|^2} \\ \leqslant C\Big(\|\vartheta_0\|_*^2 + \|\chi_0\|_*^2 + \|f\|^2_{L^2(0,t;V')} + \int_0^t \|\chi_t\|_*^2 + \int_0^t \|1 * (k\vartheta + ku)\|^2\Big). \end{aligned}$$

Now, we may consider equation (6.3) and proceed like in [6, Section 6]: we integrate (6.3) with respect to time over $(0, t)$, finding

$$\tag{7.4} \begin{aligned} \langle \mu\chi_t, v\rangle + (\chi, v) + (\nabla(1 * \chi), \nabla v) + (1 * (g(\chi_1) - g(\chi_2)) - 1 * u, v) \\ = \langle \mu\chi^1, v\rangle + (\chi_0, v) \quad \forall\, v \in V, \ \text{ a.e. in } (0, T). \end{aligned}$$

Set now $v = \chi(t)$ in (7.4) and integrate in time again in order to obtain

$$\tag{7.5} \begin{aligned} \frac{\mu}{2}|\chi(t)|^2 + \int_0^t |\chi|^2 + \frac{1}{2}|\nabla(1 * \chi)(t)|^2 \\ = -\int_0^t (1 * (g(\chi_1) - g(\chi_2)) - 1 * u, \chi) + \frac{\mu}{2}|\chi_0|^2 + \langle \mu\chi^1 + \chi_0, 1 * \chi(t)\rangle. \end{aligned}$$

First we can estimate the term $\int_0^t (1 * u, \chi)$ using the fact that

$$\int_0^t |1 * u|^2 \leqslant C \int_0^t |1 * (k\vartheta + ku)|^2 + C \int_0^t \Big(\int_0^s |\vartheta(\tau)|^2 \,\mathrm{d}\tau \Big). \tag{7.6}$$

Then we may recall the estimate of the term $\int_0^t (1 * (g(\chi_1) - g(\chi_2)), \chi)$ in (7.5) already performed in [6, (6.6)–(6.8)] because it is just the same in our case, obtaininig in this way (with (7.6)) the estimate

$$\begin{aligned} \frac{\mu}{2} |\chi(t)|^2 + \int_0^t |\chi|^2 + \frac{1}{2} |\nabla(1 * \chi)(t)|^2 \leqslant C \Big(|\chi_0|^2 + \|\chi^1\|_*^2 + \int_0^t \|\vartheta\|^2_{L^2(0,s;H)} \\ + \int_0^t |1 * (k\vartheta + ku)|^2 + \int_0^t |\nabla(1 * \chi)|^2 \Big) + C(M_2) \int_0^t |\chi|^2, \end{aligned} \tag{7.7}$$

thanks also to Young's inequality. Moreover, by comparison with (7.4) and thanks to (7.7), we obtain

$$\begin{aligned} \|\chi_t(t)\|_*^2 \leqslant C(M_2) \Big(& |\chi_0|^2 + \|\chi^1\|_*^2 + \int_0^t \|\vartheta\|^2_{L^2(0,s;H)} \\ & + \int_0^t |1 * (k\vartheta + ku)|^2 + \int_0^t |\chi|^2 + \int_0^t |\nabla(1 * \chi)|^2 \Big) \\ & + 2|1 * u(t)|^2. \end{aligned} \tag{7.8}$$

Next, we may take $v = J^{-1}\vartheta$ in (6.2), integrate over $(0,t)$ and treating the result as in (6.16)–(6.22) in the previous section, find the estimate

$$\|\vartheta(t)\|_*^2 + \|\vartheta\|^2_{L^2(0,t;H)} \leqslant C(\|\vartheta_0\|_*^2 + \|f\|^2_{L^2(0,t;V')} + \|\chi_t\|^2_{L^2(0,t;V')} + \|\vartheta\|^2_{L^2(0,t;V')}). \tag{7.9}$$

Finally, multiplying (7.8) by a sufficiently small constant and summing it up with (7.3), (7.7) and (7.9), using again (7.6) without the integral over $(0,t)$ in order to estimate $2|1 * u(t)|^2$ in (7.8), and applying a standard version of Gronwall's lemma, we recover immediately the estimate (2.54). Hence, Theorem 2.9 is completely proved.

8. Appendix

Let us recall here a lemma whose proof can be found in [1, p. 841].

Lemma 8.1. *Let $a > 1$, $b \geqslant 0$, $c \in \mathbb{R}$, $C_0 \geqslant 1$, $C_1 \geqslant 1$ and δ_0 be given numbers such that $\delta_0 + c(a-1)^{-1} > 0$. Moreover, consider the sequence $\{\delta_k\}_{k\geqslant 0}$ of real numbers defined by $\delta_{k+1} = a\delta_k + c$, $k \in \mathbb{N}$. If $\{\gamma_k\}_{k\geqslant 0}$ is a sequence of positive real numbers satisfying*

$$\begin{aligned}\gamma_0 &\leqslant C_1^{\delta_0},\\ \gamma_{k+1} &\leqslant C_0\delta_{k+1}^b \max\{C_1^{\delta_{k+1}}, \gamma_k^a\},\end{aligned}$$

then the sequence $\{\gamma_k^{1/\delta_k}\}_{k\geqslant 0}$ is bounded.

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