Václav Finěk
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QUADRATURE FORMULAS BASED ON THE SCALING FUNCTION*

Václav Finěk, Liberec

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Abstract. The scaling function corresponding to the Daubechies wavelet with two vanishing moments is used to derive new quadrature formulas. This scaling function has the smallest support among all orthonormal scaling functions with the properties $M_2 = M_1^2$ and $M_0 = 1$. So, in this sense, its choice is optimal. Numerical examples are given.

Keywords: Daubechies wavelet, quadrature formula

MSC 2000: 65T60, 65D32

1. Introduction

Wavelets play a large role in analysis as well as in numerical methods. Under certain assumptions (for more detail see [11], [6]), they provide one-point quadrature formulas to evaluate scalar products with the scaling function in the form $\int f(x)\varphi(x)\,\mathrm{d}x$. Then it seems to be an attractive idea to employ these wavelets for the construction of quadrature formulas based on the scaling functions which make this effective evaluation of the scalar products possible. The attraction of this idea consists in the fact that when some further properties (which wavelets or scaling function should satisfy) are added the number of the scaling functions having their support at least partly in the interval $[0,1]$ increases only to $2^J + R - 1$ (on the level of resolution J), where R is the length of the support. And at the same time the order of error increases too.

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How fast the accuracy of these formulas grows is one of many interesting questions arising in this context.

In the present paper the Daubechies wavelet with two vanishing moments is employed to introduce the construction of such quadrature formulas. The choice of this wavelet is due to the fact that its scaling function has the smallest support among all orthonormal scaling functions with the properties $M_2 = M_1^2$ and $M_0 = 1$ (see Lemma 8). And these properties are essential for the construction of the above mentioned one-point quadrature formulas. As to the construction, first, the partition of unity is applied. Subsequently, the relation $M_1^2 = M_2$ is brought into play to evaluate scalar products effectively. Here, the problem with the evaluation of the scalar products at the endpoints of the interval arises. In Lemma 9, it is proposed how to solve this problem without destroying the efficiency of this method and in Theorem 10 it is proved that the derived formula is exact for polynomials up to degree 3 and consequently the order of error is 2^{-4J} where J denotes the resolution level. In this case, the accuracy of the whole quadrature formula is one order better then the accuracy of the corresponding one-point formula for evaluation of the scalar products. At the end of this paper, a slightly modified quadrature formula for periodic functions is derived. This formula has the same accuracy as the original one. Numerical experiments show that especially the modified quadrature formula provides very interesting results and exceeds expectations. For details see Tab. 2.

A similar idea, using the scaling function to construct quadrature formulas, has appeared in [7], where the partition of unity was also applied and the fact that the discrete approximations are exact for polynomials up to degree n (which holds in the case that the scaling function reproduces polynomials up to degree n) was used to evaluate the scalar products and to construct quadrature formulas. However, for instance for the Daubechies wavelet with two vanishing moments, this approach leads to the accuracy two order worse than the accuracy attained by the formulas introduced in this paper.

This paper is organized as follows. First, some basic properties of wavelets, especially the Daubechies wavelets, are briefly summarized (see [1], [2], [4], [8], [12]). In the next two sections, some known facts about evaluation of scaling moments (see [10], [5]) and also about evaluation of the scalar product (see [11], [6]) are repeated. Furthermore, the fourth section also contains the proof that the scaling function corresponding to the Daubechies wavelet with two vanishing moments has the smallest support among all orthonormal scaling functions with the properties $M_2 = M_1^2$ and $M_0 = 1$. In the fifth section, the quadrature formula is derived and its exactness for polynomials up to degree 3 is proved. In the last part of this text, the quadrature formula for periodic functions is inferred and numerical examples are given to verify the theory.

2. Preliminaries

Now, some fundamentals of the wavelet theory—definitions and properties—are briefly reviewed.

Definition 1. Any function $\psi \in L_2(\mathbb{R})$ which generates an orthonormal basis of the space $L_2(\mathbb{R})$ by the system of translations and dilations

$$\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$$

is called a wavelet.

Let wavelet subspaces be denoted by $W_j := \operatorname{span}\{\psi_{j,k};\ k \in \mathbb{Z}\}$, i.e.

$$L_2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Definition 2. Let the following four conditions be satisfied:

i) $\bigcup_j V_j$ is dense in $L^2(\mathbb{R})$,

ii) $\bigcap_j V_j = \{0\}$,

iii) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots,$

iv) $f(x) \in V_j \Leftrightarrow f(2^{-j}x) \in V_0$,

v) exists a function $\varphi \in L_2(\mathbb{R})$ such that $\{\varphi(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_0.

Then the sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is called a multiresolution analysis and φ is its corresponding scaling function.

Let $\varphi_{j,k}$ be defined by

$$\varphi_{j,k}(x) := 2^{j/2}\varphi(2^j x - k) \quad j,k \in \mathbb{Z}. \tag{1}$$

Then iii), iv) imply that $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_j.

Furthermore, let it be assumed that the spaces W_j are orthogonal complements of V_j in V_{j+1}, so that $V_j \bigoplus W_j = V_{j+1}$ for all $j \in \mathbb{Z}$. This property implies that the scaling function φ satisfies the scaling equation (the scaling identity)

$$\varphi(x) = \sum_{k\in\mathbb{Z}} h_k \varphi(2x - k), \tag{2}$$

where h_k are called the corresponding *scaling parameters*, and the wavelet ψ satisfies the wavelet equation

$$\psi(x) = \sum_{k\in\mathbb{Z}} g_k \varphi(2x - k); \tag{3}$$

g_k are called *wavelet parameters*.

Now, the properties of the scaling parameters of the Daubechies wavelets are going to be summarized. Let p be a positive integer and let the following four conditions be satisfied:

i) $h_k = 0 \ \forall\, k \notin \{0, 1, \dots, 2p-1\}$,

ii) $\delta_{m,0} = 2^{-1} \sum_{j=0}^{2p-1} h_j h_{2m+j}$ for $1-p \leqslant m \leqslant p-1$,

iii) $\sum_{k=0}^{2p-1} h_k = 2$,

iv) $\sum_{k=0}^{2p-1} (-1)^k h_k k^n = 0$ for $0 \leqslant n \leqslant p-1$.

Examples for systems $\{h_k\}$ satisfying the above conditions for $1 \leqslant p \leqslant 10$ can be found in [3]. In [8], it is proved that for fixed p, there exists only one linearly independent scaling function φ which satisfies the scaling equation with scaling parameters $\{h_k\}$. Furthermore, the support of these function is contained in $[0, 2p-1]$ and according to the condition iv) this wavelets are also called the (Daubechies) wavelets with p vanishing moments (for more detail see for instance [3], [8]). As usual, the scaling function can be constructed so that

$$\int_0^{2p-1} \varphi(x)\,\mathrm{d}x = 1.$$

In addition to the previous properties, let the corresponding wavelet parameters g_k be chosen such that

$$g_k := (-1)^k h_{2p-k-1}.$$

Then the support of the wavelets is contained in $[0, 2p-1]$, too. Applying the wavelet equation (3) and the condition iv) yields

$$\int_0^{2p-1} \psi(x)\,\mathrm{d}x = 2^{-1} \sum_{k=0}^{2p-1} g_k \int_0^{2p-1} \varphi(x)\,\mathrm{d}x = 2^{-1} \sum_{k=0}^{2p-1} g_k = 0.$$

Consider an indicator function $\chi_{[a,b]}$. This function is an element of $L^2(\mathbb{R})$. Since the integral of ψ vanishes, the integrals of $\psi_{j,k}$ are also zero except if their support contains a or b. Now, if a tends to $-\infty$ and b to $+\infty$, then the constant function 1 can be expressed in terms of $\varphi(x-k)$. Furthermore,

$$\int_{-\infty}^{\infty} \varphi(x-k)\,\mathrm{d}x = 1$$

is independent of k, hence the partition of unity is obtained:

$$1 = \sum_{k \in \mathbb{Z}} \varphi(x-k). \tag{4}$$

So, the constant function can be expressed as the sum of translations of the scaling function.

3. Evaluations of the scaling moments

Let the continuous scaling moments M_n and the discrete scaling moments m_n be denoted by

$$M_n := \int_0^{2p-1} x^n \varphi(x)\,\mathrm{d}x \quad \text{and} \quad m_n := \sum_{k=0}^{2p-1} h_k k^n.$$

The continuous moments can be expressed as linear combinations of the discrete moments. Here, the discrete moments are only used to evaluate the continuous ones. In literature, there are various versions of the formula in dependence on the definition of the scaling equation. Here, the following one is employed:

Lemma 3. *Let us assume that only finitely many h_k do not vanish and $M_0 = 1$. Then for any $n \in \mathbb{N}$, $n > 0$, the following recursion holds:*

$$M_n = \frac{1}{2^{n+1}-2} \sum_{j=1}^{n} \binom{n}{j} m_j M_{n-j}. \tag{5}$$

P r o o f. Using substitution, the binomial formula and the interchange of summation and integration yields

$$\begin{aligned}
\int_0^{2p-1} x^n \varphi(x)\,\mathrm{d}x &= \sum_0^{2p-1} h_k \int_0^{2p-1} x^n \varphi(2x-k)\,\mathrm{d}x \\
&= \frac{1}{2^{n+1}} \sum_0^{2p-1} h_k \int_0^{2p-1} (y+k)^n \varphi(y)\,\mathrm{d}y \\
&= \frac{1}{2^{n+1}} \sum_{j=0}^{n} \binom{n}{j} \int_0^{2p-1} y^{n-j} \varphi(y)\,\mathrm{d}y \sum_0^{2p-1} h_k k^j \\
&= \frac{1}{2^{n+1}} \sum_{j=0}^{n} \binom{n}{j} m_j M_{n-j}.
\end{aligned}$$

This implies immediately

$$M_n = \frac{1}{2^{n+1}-2} \sum_{j=1}^{n} \binom{n}{j} m_j M_{n-j},$$

which proves Lemma 3. □

However, formula (5) for the evaluation of moment integrals can not be applied for evaluation of these scaling moments when the support of the scaling function exceeds the interval. For example, in the case of the Daubechies wavelet with two vanishing moments ($p = 2$), two integrals on both sides have support outside the interval $[0, 3]$. These integrals must be examined separately. For other wavelets, the procedure is similar.

Let us denote

$$A_n := \int_0^2 x^n \varphi(x+1)\,\mathrm{d}x \quad \text{and} \quad B_n := \int_0^1 x^n \varphi(x+2)\,\mathrm{d}x.$$

To evaluate these scaling moments, the scaling equation (2) is applied and consequently the system of linear equations must be solved. The moments at the other endpoint can be expressed as a linear combination of already known moments. Let us denote

$$C_n := \int_1^3 x^n \varphi(x-1)\,\mathrm{d}x \quad \text{and} \quad D_n := \int_2^3 x^n \varphi(x-2)\,\mathrm{d}x.$$

After applying the scaling equation and some elementary arrangements, the following two lemmas are obtained:

Lemma 4. *For any $n \in \mathbb{N}$,*

$$(2^{n+1} - h_1)A_n - h_0 B_n = h_3 \sum_{i=0}^{n} \binom{n}{i} M_{n-i} + h_2 M_n$$
$$-h_3 A_n + (2^{n+1} - h_2) B_n = 0.$$

Lemma 5. *For any $n \in \mathbb{N}$,*

$$C_n = \sum_{i=0}^{n} \binom{n}{i} (M_{n-i} - 3^i B_{n-i}) \quad \textit{and} \quad D_n = \sum_{i=0}^{n} \binom{n}{i} (2^i M_{n-i} - 3^i A_{n-i}).$$

(For more detail see [5].) The solution of such systems of linear equations was studied e.g. in [10].

4. Evaluation of the scalar product

The evaluation of the scalar products was studied e.g. in [11], where it was shown that the relation $M_2 = M_1^2$ is valid for the scaling function with compact support which reproduces polynomials up to degree 2. Consequently, this relation is an essential tool used to derive one point quadrature formulas. In the paper [6], the assumption of compact support is not required and new interesting relations among scaling moments are derived. Unfortunately, the next relations cannot improve the accuracy of the quadrature formulas without additional assumptions, which lead to an extension of the support. The next lemma and the proof are cited from [11], [6] to make the paper self-contained and to support better understanding of the assumption $M_2 = M_1^2$.

Lemma 6. *Let $M_2 = M_1^2$ and let f be a polynomial up to degree 2, then*

$$\int f(x)\varphi_{j,k}(x)\,\mathrm{d}x = \sqrt{2^{-j}}\,f\Big(\frac{M_1+k}{2^j}\Big). \tag{6}$$

P r o o f. Substitution (1) leads to

$$\int f(x)\varphi_{j,k}(x)\,\mathrm{d}x = \sqrt{2^j}\int f(x)\varphi(2^jx-k)\,\mathrm{d}x = \sqrt{2^{-j}}\int f\Big(\frac{x+k}{2^j}\Big)\varphi(x)\,\mathrm{d}x.$$

Since f is a polynomial, it can be replaced by its Taylor expansion

$$f\Big(\frac{x+k}{2^j}\Big) = f\Big(\frac{\alpha+k}{2^j}\Big) + 2^{-j}(x-\alpha)f'\Big(\frac{\alpha+k}{2^j}\Big) + \frac{2^{-2j}}{2}(x-\alpha)^2 f''\Big(\frac{\alpha+k}{2^j}\Big)$$

(f'' is a constant function.) Thus

$$\begin{aligned}\int f(x)\varphi_{j,k}(x)\,\mathrm{d}x &= \sqrt{2^{-j}}\Big(f\Big(\frac{\alpha+k}{2^j}\Big) + 2^j(M_1-\alpha)f'\Big(\frac{\alpha+k}{2^j}\Big)\Big)\\ &\quad + \sqrt{2^{-j}}\,\frac{2^{-2j}}{2}(M_2-2M_1\alpha+\alpha^2)f''\Big(\frac{\alpha+k}{2^j}\Big).\end{aligned}$$

Finally, choosing $\alpha = M_1$ and using the relation $M_2 = M_1^2$ we complete the proof.

$\square$

This quadrature formula holds e.g. for the scaling function corresponding to the Daubechies wavelets with three and more vanishing moments ($p \geqslant 3$). This was proved in [11]. However simple computations show that this formula is also true for the corresponding scaling function to the Daubechies wavelet with only two vanishing moments too (see Remark 7). This fact is about to be put in use in the next part.

R e m a r k 7. The corresponding scaling parameters to the Daubechies wavelet with two vanishing moments are

$$h_0 = \frac{1+\sqrt{3}}{4}, \quad h_1 = \frac{3+\sqrt{3}}{4}, \quad h_2 = \frac{3-\sqrt{3}}{4}, \quad h_3 = \frac{1-\sqrt{3}}{4}, \tag{7}$$

and the corresponding moments

$$M_1 = \frac{3-\sqrt{3}}{2}, \quad A_0 = \frac{7-3\sqrt{3}}{12}, \quad B_0 = \frac{5-3\sqrt{3}}{12}. \tag{8}$$

Using Lemma 3 and elementary computations we conclude

$$M_1^2 = \frac{1}{4}m_1^2 = 3 - \frac{3}{2}\sqrt{3} \quad \text{and} \quad M_2 = \frac{1}{6}(m_1^2 + m_2) = 3 - \frac{3}{2}\sqrt{3}.$$

So, the assumption from Lemma 6 is fulfilled.

In the next lemma, optimality of this scaling function is going to be proved in the sense of the smallest support while preserving orthogonality and the property $M_2 = M_1^2$.

Lemma 8. *The scaling function corresponding to the Daubechies wavelet with two vanishing moments has the smallest support among all orthonormal scaling functions with the properties $M_2 = M_1^2$ and $M_0 = 1$.*

P r o o f. First, the number of the scaling parameters is even. By contradiction— let the number of the scaling parameters be odd and at the same time $h_0 \neq 0$ and $h_{2p} \neq 0$. Applying the scaling equation (2) and using the orthogonality of translations of the scaling function gives $\sum_{k=0}^{2p} h_k\, h_{k-2l} = 0$ for all $l \in \mathbb{Z}$. Since $2p$ is even, $h_{2p}\, h_0 = 0$ is obtained and this is a contradiction.

In Remark 7, it was shown that for the scaling function corresponding to the Daubechies wavelet with two vanishing moments ($p = 2$) $M_2 = M_1^2$ holds. So, it remains to examine the case $p = 1$. In this case, the scaling equation (2) yields

$$h_0\, \varphi(0) = \varphi(0) \quad \text{and} \quad h_1\, \varphi(1) = \varphi(1).$$

Since $\varphi(0) = \varphi(1) = 0$ leads to the null function, either h_0 or h_1 is equal to one. Thus, the condition $M_0 = 1$ implies $\sum_{k=0}^{1} h_k = 2$ and the second parameter is also equal one. From these parameters the Haar function is obtained and this function has not the property $M_2 = M_1^2$. $\square$

Before coming to the main part of this text, the following auxiliary lemma is going to be proved.

Lemma 9. *For the scaling function corresponding to the Daubechies wavelet with two vanishing moments the equality*

$$2^{-j/2}\sum_{k=-2}^{3\cdot 2^j-1}\int_0^3 x^n\varphi_{j,k}(x)\,\mathrm{d}x=\sum_{k=-2}^{3\cdot 2^j-1}\int x^n\varphi_{j,k}(x)\,\mathrm{d}x\int_0^3\varphi_{j,k}(x)\,\mathrm{d}x \tag{9}$$

holds for $n\in\{0,1,2\}$.

P r o o f. It is sufficient to prove this equality only for $j=0$. Moreover, the support of $x^n\varphi_{0,0}(x)$ is in $[0,3]$ and hence

$$\int_0^3 x^n\varphi_{0,0}(x)\,\mathrm{d}x=\int x^n\varphi_{0,0}(x)\,\mathrm{d}x\int_0^3\varphi_{0,0}(x)\,\mathrm{d}x.$$

Thus, it remains to prove

$$\sum_{k=-2,-1,1,2}\int_0^3 x^n\varphi_{0,k}(x)\,\mathrm{d}x=\sum_{k=-2,-1,1,2}\int x^n\varphi_{0,k}(x)\,\mathrm{d}x\int_0^3\varphi_{0,k}(x)\,\mathrm{d}x, \tag{10}$$

or equivalently,

$$\begin{aligned}A_n+B_n+C_n+D_n&=B_0\sum_{i=0}^{n}\binom{n}{i}(-2)^iM_{n-i}\\&\quad+A_0\sum_{i=0}^{n}\binom{n}{i}(-1)^iM_{n-i}+(1-B_0)\sum_{i=0}^{n}\binom{n}{i}M_{n-i}\\&\quad+(1-A_0)\sum_{i=0}^{n}\binom{n}{i}(2)^iM_{n-i}.\end{aligned} \tag{11}$$

Now, Lemma 4 and 5 are used. They imply

$$A_n+B_n=\frac{1}{2^{n+1}-1}\left(h_3\sum_{i=0}^{n}\binom{n}{i}M_{n-i}+h_2\,M_n\right) \tag{12}$$

and

$$C_n+D_n=\sum_{i=0}^{n}\binom{n}{i}(M_{n-i}+2^i\,M_{n-i}-3^i\,(A_{n-i}+B_{n-i})). \tag{13}$$

Using (11), (12) and (13) we obtain for $n = 0$

$$2 = A_0 + B_0 + C_0 + D_0 = B_0 + A_0 + (1 - B_0) + (1 - A_0) = 2.$$

In a similar way, we can proceed for $n = 1$ and $n = 2$. $\square$

Now, all is ready to prove the theorem about the quadrature formula.

Theorem 10. *Let φ be the scaling function corresponding to the Daubechies wavelet with two vanishing moments. Then*

$$\begin{aligned}(14)\qquad 2^j \int_0^3 f(x)\,\mathrm{d}x \approx{}& \frac{5-3\sqrt{3}}{12} f\Big(\frac{M_1-2}{2^j}\Big) + \frac{7-3\sqrt{3}}{12} f\Big(\frac{M_1-1}{2^j}\Big) \\ &+ \sum_{k=0}^{3\cdot 2^j-3} f\Big(\frac{M_1+k}{2^j}\Big) + \frac{7+3\sqrt{3}}{12} f\Big(\frac{M_1+3\cdot 2^j-2}{2^j}\Big) \\ &+ \frac{5+3\sqrt{3}}{12} f\Big(\frac{M_1+3\cdot 2^j-1}{2^j}\Big)\end{aligned}$$

is exact for polynomials up to degree three.

P r o o f. First, the partition of unity on interval [0,3] is required in the form

$$\sum_{k=-2}^{3\cdot 2^j-1} \varphi_{j,k} = 2^{j/2}.$$

This is derived in a similar way as (4). Then, this partition of unity is added to the integrand

$$\begin{aligned}(15)\qquad \int_0^3 f(x)\,\mathrm{d}x &= 2^{-j/2} \int_0^3 f(x) \sum_{k=-2}^{3\cdot 2^j-1} \varphi_{j,k}(x)\,\mathrm{d}x \\ &= 2^{-j/2} \sum_{k=-2}^{3\cdot 2^j-1} \int_0^3 f(x)\varphi_{j,k}(x)\,\mathrm{d}x \\ &\approx \sum_{k=-2}^{3\cdot^j-1} \int f(x)\varphi_{j,k}(x)\,\mathrm{d}x \int_0^3 \varphi_{j,k}(x)\,\mathrm{d}x.\end{aligned}$$

For polynomials of degree at most 2, the last approximation can be replaced by equality (see Lemma 9). Then the first integral in (15) is approximated by

$$(16)\qquad \int f(x)\varphi_{j,k}(x)\,\mathrm{d}x \approx 2^{-j/2} f\Big(\frac{M_1+k}{2^j}\Big),$$

the second is evaluated exactly (see Lemmas 4, 5 and 8) and finally the quadrature formula (14) is obtained. Lemmas 6 and 9 imply that this quadrature formula is exact for polynomials of degree at most 2. Then it remains to prove that this quadrature formula is exact also for polynomials of degree 3. It is again sufficient to prove this assertion only for $j = 0$. (Because for $j > 0$, the composite quadrature formula is obtained. It means that the quadrature formula for $j > 1$ can be obtained by applying the quadrature formula for $j = 0$ to subintervals.) Thus

$$\begin{aligned}\int_0^3 x^3\,\mathrm{d}x &\approx B_0(M_1-2)^3 + A_0(M_1-1)^3 + M_1^3 + (1-B_0)(M_1+1)^3\\ &\quad + (1-A_0)(M_1+2)^3\\ &= 3M_1^3 + 9M_1^2(1-A_0-B_0) + M_1(15-9A_0+9B_0) + 9(1-A_0-B_0)\end{aligned}$$

and the evaluation of this term gives $\frac{81}{4}$. □

6. Numerical examples

To verify the above developed theory, the quadrature formula (14) is tested on some examples. Let the exact integrals be denoted by

$$I := \int_0^3 f(x)\,\mathrm{d}x$$

and their approximations by

$$\begin{aligned}2^J I_J = {}&\frac{5-3\sqrt{3}}{12} f\Big(\frac{M_1-2}{2^J}\Big) + \frac{7-3\sqrt{3}}{12} f\Big(\frac{M_1-1}{2^J}\Big)\\ &+ \sum_{k=0}^{3\cdot 2^J-3} f\Big(\frac{M_1+k}{2^J}\Big) + \frac{7+3\sqrt{3}}{12} f\Big(\frac{M_1+3\cdot 2^J-2}{2^J}\Big)\\ &+ \frac{5+3\sqrt{3}}{12} f\Big(\frac{M_1+3\cdot 2^J-1}{2^J}\Big).\end{aligned} \tag{17}$$

Let $f \in C^4(-2^{1-J},3)$, then the Taylor expansion can be applied. Lemma 10 implies that the first three derivatives in the Taylor expansion vanish and from (15), (16) the error estimate

$$I - I_J = \frac{2^{-5J}}{24}\sum_{k=-2}^{3\cdot 2^J-1}\int_0^3 f''''\Big(\frac{\xi_{J,k}(x)+k}{2^J}\Big)(x-M_1)^4\varphi(x)\,\mathrm{d}x$$

is obtained where $\xi_{J,k}(x)$ is between x and M_1.

The obtained errors $2^{4J}|I - I_J|$ are summarized in Tab. 1. These results are consistent with the above presented theory and the error order $O(2^{-4J})$ agrees with the expectations.

J	$2^{4J}\|I - I_J\|$			$2^{2J}\|I - I_J^*\|$	$2^J\|I - I_J^*\|$
	x^4	e^{-2x}	e^{-10x}	$\sin\frac{1}{3}\pi x + 1$	e^{-10x}
0	0.35	0.112845	13989.657185	0.070489	0.098235
1	0.35	0.064682	114.765024	0.068964	0.157706
2	0.35	0.049869	19.110266	0.068593	0.176710
3	0.35	0.043940	9.268845	0.068501	0.165481
4	0.35	0.041277	6.662768	0.068478	0.152199
5	0.35	0.040013	5.682111	0.068472	0.143660
6	0.35	0.039398	5.253701	0.068471	0.138953
7	0.35	0.039094	5.053157	0.068470	0.136497

Table 1. Numerical results.

For the third integral, the first approximated values are strongly influenced by the fact that the quadrature formula takes two values outside the interval $[0, 3]$ and in this case these values are extremely large. Then, these two points approach step by step the interval $[0, 3]$ and at the same time their influence is decreasing.

For periodic functions with period 3, the first function value and the last but one in (17) are the same and the second is the same as the last one. So, their coefficients add up and only the last two values are used. Thus, the following slightly modified quadrature formula is obtained:

$$2^J I_J^* = \sum_{k=0}^{3\cdot 2^J - 1} f\Big(\frac{M_1 + k}{2^J}\Big).$$

The assumed error order is $O(2^{-4J})$ for periodic functions from $C^4(0, 3)$ with period 3 and $O(2^{-J})$ for functions from $C^1(0, 3)$. The numerical results exceed expectations. The integrals of $\sin\frac{1}{3}2k\pi x + 1$ and $\cos\frac{1}{3}2k\pi x + 1$ were for $k = 1, 2, 4, 5$ integrated exactly (with an error $2^{4J}|I - I_J^*| < 10^{-10}$ for $J = 0, \ldots, 7$). Other ($k = 3, 6$) errors obtained are summarized in Tab. 1 and Tab. 2.

J	$2^{4J}\lvert I - I_J^*\rvert$				
	$\sin 2\pi x + 1$	$\sin 4\pi x + 1$	$\cos 2\pi x + 1$	$\cos 4\pi x + 1$	$\cos 8\pi x + 1$
0	2.237504	2.980942	1.998393	0.337618	2.924010
1	$< 10^{-10}$	35.800072	$< 10^{-10}$	31.974284	5.401881
2	$< 10^{-10}$	$< 10^{-10}$	$< 10^{-10}$	$< 10^{-10}$	511.588549
3–7	$< 10^{-10}$	$< 10^{-10}$	$< 10^{-10}$	$< 10^{-10}$	$< 10^{-10}$

Table 2. Numerical results for 3-periodic functions.

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Author's address: *V. Finěk*, Technical University of Liberec, Faculty of Education, Hálkova 6, CZ-461 17 Liberec 1, Czech Republic, e-mail: `finek@karel.troja.mff.cuni.cz`.