MULTISCALE CONVERGENCE AND REITERATED HOMOGENIZATION OF PARABOLIC PROBLEMS

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Abstract. Reiterated homogenization is studied for divergence structure parabolic problems of the form $\partial u_\varepsilon/\partial t - \operatorname{div}\bigl(a\bigl(x, x/\varepsilon, x/\varepsilon^2, t, t/\varepsilon^k\bigr)\nabla u_\varepsilon\bigr) = f$. It is shown that under standard assumptions on the function $a(x, y_1, y_2, t, \tau)$ the sequence $\{u_\varepsilon\}$ of solutions converges weakly in $L^2(0, T; H_0^1(\Omega))$ to the solution u of the homogenized problem $\partial u/\partial t - \operatorname{div}(b(x,t)\nabla u) = f$.

Keywords: reiterated homogenization, multiscale convergence, parabolic equation

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1. Introduction

In this paper we consider the homogenization problem for the following initial-boundary value problem:

$$(1)\qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \operatorname{div}\Bigl(a\Bigl(x, \dfrac{x}{\varepsilon}, \dfrac{x}{\varepsilon^2}, t, \dfrac{t}{\varepsilon^k}\Bigr)\nabla u_\varepsilon\Bigr) = f \quad \text{in } \Omega \times (0,T), \\ u_\varepsilon(x,0) = u_0(x), \\ u_\varepsilon(x,t) = 0 \quad \text{in } \partial\Omega \times (0,T), \end{cases}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, T and k are positive real numbers. Let us define $\Omega_T = \Omega \times (0,T)$ and $Y_\tau = Y_1 \times Y_2 \times (0,1)$, where $Y_1 = Y_2 = (0,1)^n$. We assume that the function $a = a(x, y_1, y_2, t, \tau)$ belongs to $C(\Omega_T; L^\infty_{\rm per}(Y_\tau))$ and satisfies the coercivity assumption

$$\alpha|\xi|^2 \leqslant a\xi\cdot\xi, \quad \forall\, \xi \in \mathbb{R}^n, \text{ a.e. in } \Omega_T \times Y_\tau.$$

With these structure conditions it is well-known that given $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$ there exists a unique solution $u_\varepsilon \in L^2(0,T;H_0^1(\Omega))$ to (1) with time derivative $\partial u_\varepsilon/\partial t \in L^2(0,T;H^{-1}(\Omega))$ for every fixed $\varepsilon > 0$.

The homogenization problem for (1) consists in studying the asymptotic behavior of the solutions u_ε as ε tends to zero.

Homogenization problems with more than one oscillating scale is referred to as reiterated homogenization and was first introduced in [3] for linear elliptic problems. More recently the linear elliptic problem was studied in [1] and the nonlinear monotone case was treated in [7]. A very elegant physical motivation is found in the fundamental paper [2] by Avellaneda on bounds for composite media where he constructs optimal bounds for a reiterated laminate structure using an effective medium theory. In the present report we prove a reiterated homogenization theorem (Theorem 5) for the parabolic problem (1). In particular, the proof of Theorem 5 will show how easy and powerful the two-scale and multi-scale convergence theory can be.

Throughout the paper we consider a sequence $\{\varepsilon_i\}$ of small positive numbers tending to zero which is denoted by $\{\varepsilon\}$. Any subsequence $\{\varepsilon'\}$ of the sequence $\{\varepsilon\}$ will also be denoted by $\{\varepsilon\}$.

The result of Theorem 5 is that the sequence of solutions $\{u_\varepsilon\}$ to the problem (1) converges weakly in $L^2(0,T;H_0^1(\Omega))$ to the solution u in $L^2(0,T;H_0^1(\Omega))$ to a homogenized problem of the form

$$(2)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(b(x,t)\nabla u) = f \quad \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x), \\ u(x,t) = 0 \quad \text{in } \partial\Omega \times (0,T), \end{cases}$$

where b depends on x and t but is no longer oscillating with ε. Indeed, b will also depend on k, but this will be clearly spelled out in Theorem 5.

As a warm up, in order to get a feeling for the interaction between the scales, we expand the solution u_ε to (1) in a multiple scales power series. Let us for the moment assume that

$$(3)\quad u_\varepsilon(x,t) = u\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},t,\frac{t}{\varepsilon^k}\Big) + \varepsilon u_1\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},t,\frac{t}{\varepsilon^k}\Big) + \varepsilon^2 u_2\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},t,\frac{t}{\varepsilon^k}\Big) + \ldots,$$

where all the u_is are assumed to be ε-periodic in $y_1 = x/\varepsilon$, ε^2-periodic in $y_2 = x/\varepsilon^2$ and ε^k-periodic in $\tau = t/\varepsilon^k$. The chain rule transforms the differential operators as

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \frac{1}{\varepsilon^k}\frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_1} + \frac{1}{\varepsilon^2}\frac{\partial}{\partial y_2}.$$

The divergence and gradient operators transform accordingly and we denote differentiation with respect to x, y_1 and y_2 by subscripts x, y_1 and y_2, respectively. In a standard way one can now insert the series (3) into the equation (1) and identify a hierarchy of equations of significant orders of ε. This is performed in Appendix at the end of the paper.

In Section 2 we give some preliminaries and present some well-known as well as some new results needed in the proof of the main result of the paper (Theorem 5) which is stated in Section 3 and proved in Section 4. The proof is lengthy but straightforward thanks to the preparatory Theorems 3 and 4.

2. Preliminaries

We will now recall the concept of multiscale convergence, see Allaire and Briane [1]. We will restrict ourselves to three spatial scales and two time scales as in the initial-boundary value problem (1) studied in this report.

Definition 1. A sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ is said to multi-scale converge (with three spatial scales) to $u = u(x, y_1, y_2)$ in $L^2(\Omega \times Y_1 \times Y_2)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\, \mathrm{d}x = \int_\Omega \int_{Y_1} \int_{Y_2} u(x, y_1, y_2)\varphi(x, y_1, y_2)\, \mathrm{d}x\, \mathrm{d}y_1\, \mathrm{d}y_2$$

for all functions $\varphi \in L^2(\Omega; C_{\mathrm{per}}(Y_1 \times Y_2))$.

Allaire and Briane proved the following compactness results:

Theorem 1. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then there exists a subsequence, still denoted by $\{u_\varepsilon\}$, and a function $u = u(x, y_1, y_2)$ in $L^2(\Omega\times Y_1\times Y_2)$ such that u_ε multi-scale converges to u.*

Theorem 2. *Let $\{u_\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$. Then there exist subsequences*

$$u_\varepsilon \to u \quad \textit{strongly in } L^2(\Omega)$$

and

$$\nabla u_\varepsilon \to \nabla_x u(x) + \nabla_{y_1} u_1(x, y_1) + \nabla_{y_2} u_2(x, y_1, y_2)$$

in the multi-scale sense, where $u \in H^1(\Omega)$, $u_1 \in L^2(\Omega; H^1_{\mathrm{per}}(Y_1))$ and $u_2 \in L^2(\Omega\times Y_1; H^1_{\mathrm{per}}(Y_2))$.

We can also consider bounded functions in L^2 depending on the time variable t.

Definition 2. A sequence $\{u_\varepsilon\}$ in $L^2(\Omega\times(0,T))$ is said to multi-scale converge in space-time with three spatial and two temporal scales if, for a constant $k>0$,

$$\lim_{\varepsilon\to 0}\int_\Omega\int_0^T u_\varepsilon(x,t)\varphi\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},t,\frac{t}{\varepsilon^k}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_\Omega\int_{Y_1}\int_{Y_2}\int_0^T\int_0^1 u(x,y_1,y_2,t,\tau)\varphi(x,y_1,y_2,t,\tau)\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}t\,\mathrm{d}\tau$$

where $u\in L^2(\Omega\times(0,T)\times Y_1\times Y_2\times(0,1))$ for all $\varphi\in L^2(\Omega\times(0,T);C_{\rm per}(Y_1\times Y_2\times(0,1)))$.

We have the following analogue of the compactness result of Theorem 2.

Proposition 1. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(0,T;H^1(\Omega))$ such that its distributional temporal derivative $\{u_\varepsilon'\}$ is a bounded sequence in $L^2(0,T;(H^1(\Omega))')$. Then $\{u_\varepsilon\}$ is compact in $L^2((0,T)\times\Omega)$ and there exist subsequences*

$$u_\varepsilon\to u\quad\textit{strongly in}\ \ L^2((0,T)\times\Omega)$$

and

$$\nabla u_\varepsilon\to\nabla_x u(x,t)+\nabla_{y_1}u_1(x,t,y_1)+\nabla_{y_2}u_2(x,t,y_1,y_2)$$

in the multi-scale sense, where $u\in L^2(0,T;H^1(\Omega))$, $u_1\in L^2((0,T)\times\Omega;H^1_{\rm per}(Y_1))$ and $u_2\in L^2((0,T)\times\Omega\times Y_1;H^1_{\rm per}(Y_2))$.

Proof. Since u_ε is bounded in $L^2(0,T;H^1_0(\Omega))$ and u_ε' is bounded in $L^2(0,T;H^{-1}(\Omega))$ with the initial datum u_0 in $L^2(\Omega)$ it is well-known that u_ε is compact in $L^2(0,T;L^2(\Omega))$. Moreover, since ∇u_ε is bounded in $L^2(0,T;L^2(\Omega,\mathbb{R}^n))$ the rest of the proof is completely analogous to the proof of Theorem 2 with the obvious changes of the function spaces. □

We also have the following multi-scale compactness in space and time.

Corollary 1 (Space-time). *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(0,T;H^1(\Omega))$ such that its distributional derivative $\{u_\varepsilon'\}$ is a bounded sequence in $L^2(0,T;(H^1(\Omega))')$. Then there exist subsequences*

$$u_\varepsilon\to u\quad\textit{strongly in}\ \ L^2((0,T)\times\Omega)$$

and

$$\nabla u_\varepsilon\to\nabla_x u(x,t)+\nabla_{y_1}u_1(x,t,y_1,\tau)+\nabla_{y_2}u_2(x,t,y_1,y_2,\tau)$$

in the multi-scale sense in space-time, where $u \in L^2((0,T);H^1(\Omega))$, $u_1 \in L^2((0,T)\times \Omega\times(0,1);H^1_{\rm per}(Y_1))$ *and* $u_2 \in L^2((0,T)\times\Omega\times Y_1\times(0,1);H^1_{\rm per}(Y_2))$.

Remark 1. If $\{u_\varepsilon\}$ is bounded in $H^1(0,T;H^1(\Omega))$, then the time derivative splits. By using test functions oscillating in time with frequency ε, i.e. $\varphi(x,t,\frac{t}{\varepsilon})$ the split yields the existence of a local function u_1 such that

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u}{\partial t} + \frac{\partial u_1}{\partial \tau}$$

in the multi-scale sense (in time), where $u \in H^1((0,T)\times\Omega)$ and $u_1 \in L^2((0,T); H^1_{\rm per}(0,1)\times H^1(\Omega))$. If we use instead test functions oscillating in time with frequency ε^2, i.e. $\varphi(x,t,\frac{t}{\varepsilon^2})$, then the split yields another local function u_2, i.e.,

$$\frac{\partial u_\varepsilon}{\partial t} \to \frac{\partial u}{\partial t} + \frac{\partial u_2}{\partial \tau}$$

in the multi-scale sense, where $u \in H^1((0,T)\times\Omega)$ and $u_2 \in L^2((0,T);H^1_{\rm per}(0,1)\times H^1(\Omega))$.

In this paper we will not use this observation.

Remark 2. The split of the time derivative is discussed in [8] and is proved analogously to the gradient split. In Theorem 5 we do not have $H^1(0,T;H^1(\Omega))$ a priori bounds, therefore there occurs no split in the time derivative. But as seen in Appendix, a formal expansion yields time split derivatives in the $k=1$ and $k=2$ cases. However, that is only formal and is never used since the local derivatives vanish when the equations are averaged over fast time.

We continue by stating and proving two theorems that will be crucial in the proof of the main Theorem 5. A similar result has been proved earlier in Holmbom [6].

Theorem 3. *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $H^1(\Omega)$ *and let* u *and* u_1 *be defined as in Theorem* 2. *Then*

$$\lim_{\varepsilon\to 0}\int_\Omega\Big(\frac{u_\varepsilon(x)-u(x)}{\varepsilon}\Big)\varphi\Big(x,\frac{x}{\varepsilon}\Big)\,{\rm d}x = \int_\Omega\int_{Y_1} u_1(x,y_1)\varphi(x,y_1)\,{\rm d}y_1\,{\rm d}x$$

for all $\varphi(x,y_1)=\varphi_1(x)\varphi_2(y_1)$ *where* $\varphi_1 \in C_0^\infty(\Omega)$ *and* $\varphi_2 \in C^\infty_{\rm per}(Y_1)$ *with mean value zero over* Y_1.

Proof. From Theorem 2, by choosing test functions $\psi(x,y)=\psi_1(x)\psi_2(y)$ in $C_0^\infty(\Omega;C^\infty_{\rm per}(Y_1;\mathbb{R}^n))$, $\psi_1\in C_0^\infty(\Omega)$, $\psi_2 \in C^\infty_{\rm per}(Y_1;\mathbb{R}^n)$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega(\nabla u_\varepsilon(x)-\nabla u(x))\cdot\psi_1(x)\psi_2\Big(\frac{x}{\varepsilon}\Big)\,{\rm d}x = \int_\Omega\int_{Y_1}\nabla u_1(x,y_1)\cdot\psi_1(x)\psi_2(y_1)\,{\rm d}y_1\,{\rm d}x.$$

The divergence theorem applied to both sides gives

$$\lim_{\varepsilon\to 0}\int_\Omega (u_\varepsilon(x)-u(x))\Big(\psi_2\Big(\frac{x}{\varepsilon}\Big)\operatorname{div}_x\psi_1(x)+\psi_1(x)\varepsilon^{-1}\operatorname{div}_{y_1}\psi_2\Big(\frac{x}{\varepsilon}\Big)\Big)\,\mathrm{d}x$$
$$=\int_\Omega\int_{Y_1} u_1(x,y_1)\psi_1(x)\operatorname{div}_{y_1}\psi_2(y_1)\,\mathrm{d}y_1\,\mathrm{d}x.$$

Taking into account the mean value zero condition over Y_1 for φ_2 we can apply the well-known Fredholm alternative and conclude that there exists a unique Y_1-periodic solution $\eta\in C^\infty_{\rm per}(Y_1)$ to

$$\begin{cases}\operatorname{div}_{y_1}(\nabla_{y_1}\eta)=\varphi_2, & \text{in } Y_1\\ \eta\in C^\infty_{\rm per}(Y_1;\mathbb{R}^n).\end{cases}$$

Now we simply let $\varphi_1=\psi_1$ and $\psi_2=\nabla_{y_1}\eta$ to obtain $\varphi_2=\operatorname{div}_{y_1}\psi_2$. The strong convergence of $\{u_\varepsilon\}$ in $L^2(\Omega)$ to u in Theorem 2 gives the result. □

As a consequence of Theorem 2 we can extend the result of Theorem 3 to the case of 3 scales and state the following:

Theorem 4. *Assume that $u_1(x,y)$ is of Carathéodory type and let $\{u_\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$. Further, let u,u_1,u_2 be defined by the limit in Theorem 2. Then*

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{u_\varepsilon(x)-u(x)-\varepsilon u_1(x,x/\varepsilon)}{\varepsilon^2}\,\varphi\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\,\mathrm{d}x$$
$$=\int_\Omega\int_{Y_1}\int_{Y_2} u_2(x,y_1,y_2)\varphi(x,y_1,y_2)\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2$$

in $L^2(\Omega\times Y_1;H^1_{\rm per}(Y_2))$ for $\varphi(x,y_1,y_2)=\varphi_1(x)\varphi_2(y_1)\varphi_3(y_2)$ where $\varphi_1\in C_0^\infty(\Omega)$ and $\varphi_2,\varphi_3\in C^\infty_{\rm per}(Y)$ with mean value zero over Y.

Remark 3. An example of a function which satisfies regularity conditions which allow a scaling of the function $u_1=u_1(x,y_1)$ is given in Cioranescu and Donato [4, Chapter 9]. Suppose

$$u_1(x,y_1)=\sum_{j=1}^n \omega_j(y_1)\frac{\partial u_0}{\partial x_j}(x)$$

where $\nabla_{y_1}\omega_i\in L^r(Y_1;\mathbb{R}^n)$, $i=1,\ldots,n$ and $\nabla_x u\in L^s(\Omega;\mathbb{R}^n)$ with $1\leqslant r,s<\infty$ and $1/r+1/s=1/2$. Then, for test functions $\varphi\in C_0^\infty(\Omega;C^\infty_{\rm per}(Y_1;\mathbb{R}^n))$,

$$\int_\Omega \nabla_{y_1}u_1\Big(x,\frac{x}{\varepsilon}\Big)\cdot\varphi\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\to\int_\Omega\int_{Y_1}\nabla_{y_1}u_1(x,y_1)\cdot\varphi(x,y)\,\mathrm{d}y_1\,\mathrm{d}x.$$

Remark 4. The result remains valid also for the case $r=s=2$, but then the two-scale convergence takes place in L^1.

P r o o f. Let us choose test functions $\psi \in C_0^\infty(\Omega; C_{\rm per}^\infty(Y_1 \times Y_2\colon\ \mathbb{R}^n))$. The result of Theorem 2 says that

$$\lim_{\varepsilon\to 0}\int_\Omega \Big(\nabla_x u_\varepsilon(x) - \nabla_x u(x) - \nabla_{y_1} u_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)\cdot \psi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\,\mathrm{d}x = \int_\Omega\int_{Y_1}\int_{Y_2} \nabla_{y_2} u_2(x,y_1,y_2)\cdot\psi(x,y_1,y_2)\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Integration by parts on both sides gives

$$\lim_{\varepsilon\to 0}\int_\Omega \Big(u_\varepsilon(x) - u(x) - \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)\Big((\mathrm{div}_x + \varepsilon^{-1}\,\mathrm{div}_{y_1} + \varepsilon^{-2}\,\mathrm{div}_{y_2})\psi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\Big)\,\mathrm{d}x = \int_\Omega\int_{Y_1}\int_{Y_2} u_2(x,y_1,y_2)\,\mathrm{div}_{y_2}\,\psi(x,y_1,y_2)\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

By Theorem 3

$$\lim_{\varepsilon\to 0}\int_\Omega \Big(u_\varepsilon(x) - u(x) - \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)\Big((\mathrm{div}_x + \varepsilon^{-1}\,\mathrm{div}_{y_1})\psi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\Big)\,\mathrm{d}x = 0.$$

Therefore

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{u_\varepsilon(x) - u(x) - \varepsilon u_1(x, x/\varepsilon)}{\varepsilon^2}\,\mathrm{div}_{y_2}\,\psi\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\Big)\,\mathrm{d}x = \int_\Omega\int_{Y_1}\int_{Y_2} u_2(x,y_1,y_2)\,\mathrm{div}_{y_2}\,\psi(x,y_1,y_2)\,\mathrm{d}x\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

Referring to Lemma 2.4 in [9] we can argue as in Theorem 3 and obtain any φ as $\varphi = \mathrm{div}_{y_2}\,\psi$. □

R e m a r k 5. If $\nabla_y u_1 \in L^r(Y; \mathbb{R}^n)$ and $\nabla_x u \in L^s(\Omega; \mathbb{R}^n)$ where $1 \leqslant r, s < \infty$, $1/r + 1/s = 1/2$, then the convergence in Theorem 4 takes place in L^2. However, since the limit u_2 is an element in $L^2(\Omega \times Y_1; H^1_{\rm per}(Y_2))$, this is just a technical argument.

3. The main result

Let us rewrite (1) in the variational formulation:
Find $u_\varepsilon \in L^2(0,T; H^1_0(\Omega))$ such that

$$(4)\quad -\int_{\Omega_T} u_\varepsilon(x,t)\frac{\partial\varphi(x,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} a\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k}\Big)\nabla u_\varepsilon(x,t)\cdot\nabla\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \varphi \in H^1(0,T; H^1_0(\Omega)), \quad u_\varepsilon(x,0) = u_0(x).$$

We first observe that by the structure conditions on $a(x, y_1, y_2, t, \tau)$ one immediately obtains the following a priori estimates (see e.g. [4, Chapter 11]:

$$\|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C \Big\| \frac{\partial u_\varepsilon}{\partial t} \Big\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C.$$

Defining the space

$$\mathcal{W} = \Big\{ v \mid v \in L^2(0,T;H_0^1(\Omega)),\ \ \frac{\partial v}{\partial t} \in L^2(0,T;H^{-1}(\Omega)) \Big\}$$

we find that $\|u_\varepsilon\|_{\mathcal{W}} \leqslant C$ where the norm is the usual graph norm.

The dynamics in the homogenized equations will be captured by considering test functions which capture the oscillations in time. Due to the spatial and temporal oscillations in the coefficient we expect u_ε to be of the form (3). In Appendix we use this multiple scales expansion in equation (1) to get an idea of which equations govern u, u_1 and u_2, respectively, in the homogenized system.

However, the proof of the homogenization theorem below is not based on the multiscale expansion. It is based on the compactness Theorems 3 and 4, together with test functions which are in resonance with the oscillating coefficients $a_\varepsilon = a(x, x/\varepsilon, x/\varepsilon^2, t, t/\varepsilon^k)$. Before stating and proving the reiterated homogenization theorem we introduce some notation and abreviations: We simply write a to denote $a(x, y_1, y_2, t, \tau)$ and u, u_1 and u_2 to denote $u(x,t)$, $u_1(x, y_1, t, \tau)$ and $u_2(x, y_1, y_2, t, \tau)$, respectively. We also write $\mathrm{d}y_\tau\,\mathrm{d}x_T$ to denote $\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t$. Moreover, we denote by φ_ε smooth oscillating test functions of the types $\varphi(x, x/\varepsilon, x/\varepsilon^2, t, t/\varepsilon^k)$, $\varphi(x, x/\varepsilon, x/\varepsilon^2, t)$, $\varphi(x, x/\varepsilon, t, t/\varepsilon^k)$ or $\varphi(x, x/\varepsilon, t)$ where the regularity of φ is strong enough to make sense of weak derivatives. We will use the short notation ~ -2 to denote the equation standing by the power ε^{-2}.

Theorem 5 (Reiterated homogenization)**.** *Let $\{u_\varepsilon\}$ be a sequence of solutions in $L^2(0,T;H_0^1(\Omega))$ of the initial-boundary value problem* (1)*. Then*

$$u_\varepsilon \to u, \quad \text{in } \mathcal{W} \text{ weakly},$$
$$a\Big(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, t, \frac{t}{\varepsilon^k}\Big)\nabla u_\varepsilon \to b(x,t)\nabla u, \quad \text{in } L^2(0,T;L^2(\Omega)^n) \text{ weakly},$$

where b is the homogenized coefficient defined by

$$b(x,t)\nabla u(x) = \int_{Y_\tau} a(x, y_1, y_2, t, \tau)[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}\tau$$

and $u \in \mathcal{W}$ solves the homogenized problem (2)*. The functions u, u_1 and u_2 satisfy a characteristic system of local equations of different order of ε. Depending on the* *value of the oscillation power k in the fast time variable, there are 7 different cases of systems of local equations, namely:*

The case $0 < k < 2$.

$$\begin{cases} \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \\ \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \end{cases}$$

where $u_1 \in L^2((0,T) \times \Omega \times (0,1); H^1_{\mathrm{per}}(Y_1))$ *and* $u_2 \in L^2((0,T) \times \Omega \times Y_1 \times (0,1); H^1_{\mathrm{per}}(Y_2))$.

The case $k = 2$.

$$\begin{cases} \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \\ \displaystyle -\int_{\Omega_T}\int_{Y_\tau} u_1(x, y_1, t, \tau)\frac{\partial \varphi}{\partial \tau}(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T \\ \displaystyle\quad + \int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \end{cases}$$

where $u_1 \in L^2((0,T) \times \Omega \times (0,1); H^1_{\mathrm{per}}(Y_1))$ *such that* $\partial u_1/\partial\tau \in L^2((0,T) \times \Omega \times (0,1); (H^1_{\mathrm{per}}(Y_1))')$ *and* $u_2 \in L^2((0,T) \times \Omega \times Y_1 \times (0,1); H^1_{\mathrm{per}}(Y_2))$.

The case $2 < k < 3$.

$$\begin{cases} \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \\ \displaystyle -\int_{\Omega_T}\int_{Y_\tau} u_1\frac{\partial \varphi}{\partial \tau}(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \\ \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \end{cases}$$

where $u_1 \in L^2((0,T) \times \Omega; H^1_{\mathrm{per}}(Y_1))$ *and* $u_2 \in L^2((0,T) \times \Omega \times Y_1 \times (0,1); H^1_{\mathrm{per}}(Y_2))$.

The case $k = 3$.

$$\begin{cases} \displaystyle -\int_{\Omega_T}\int_{Y_\tau} u_1\frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T \\ \displaystyle\quad + \int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \\ \displaystyle -\int_{\Omega_T}\int_{Y_\tau} u_2\frac{\partial \varphi}{\partial \tau}(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T \\ \displaystyle\quad + \int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0, \end{cases}$$

where $u_1 \in L^2((0,T)\times\Omega\times(0,1); H^1_{\mathrm{per}}(Y_1))$ *such that* $\partial u_1/\partial\tau \in L^2((0,T)\times\Omega\times(0,1); (H^1_{\mathrm{per}}(Y_1))')$ *and* $u_2 \in L^2((0,T)\times\Omega\times Y_1\times(0,1); H^1_{\mathrm{per}}(Y_2))$ *such that* $\partial u_2/\partial\tau \in L^2((0,T)\times\Omega\times Y_1\times(0,1); (H^1_{\mathrm{per}}(Y_2))')$.

The case $3 < k < 4$.

$$\begin{cases} -\displaystyle\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0, \\ \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0, \\ -\displaystyle\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0, \\ \displaystyle\int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\,\mathrm{d}y_2\,\mathrm{d}\tau\right]\cdot\nabla_{y_1}\varphi(x,y_1,t)\,\mathrm{d}y_1\,\mathrm{d}x\,\mathrm{d}t = 0, \end{cases}$$

where $u_1 \in L^2((0,T)\times\Omega; H^1_{\mathrm{per}}(Y_1))$ *and* $u_2 \in L^2((0,T)\times\Omega\times Y_1; H^1_{\mathrm{per}}(Y_2))$.

The case $k = 4$.

$$\begin{cases} -\displaystyle\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0, \\ -\displaystyle\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T \\ \quad +\displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0, \\ \displaystyle\int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\,\mathrm{d}y_2\,\mathrm{d}\tau\right]\cdot\nabla_{y_1}\varphi(x,y_1,t)\,\mathrm{d}y_1\,\mathrm{d}x\,\mathrm{d}t = 0, \end{cases}$$

where $u_1 \in L^2((0,T)\times\Omega; H^1_{\mathrm{per}}(Y_1))$ *and* $u_2 \in L^2((0,T)\times\Omega\times Y_1\times(0,1); H^1_{\mathrm{per}}(Y_2))$ *such that* $\partial u_2/\partial\tau \in L^2((0,T)\times\Omega\times Y_1\times(0,1); (H^1_{\mathrm{per}}(Y_2))')$.

The case $k > 4$.

$$\begin{cases} -\displaystyle\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0, \\ -\displaystyle\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial\varphi}{\partial\tau}(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0, \\ \displaystyle\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0, \\ \displaystyle\int_{\Omega_T}\left[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1}u_1 + \nabla_{y_2}u_2]\,\mathrm{d}y_2\,\mathrm{d}\tau\right]\cdot\nabla_{y_1}\varphi(x,y_1,t)\,\mathrm{d}y_1\,\mathrm{d}x\,\mathrm{d}t = 0, \end{cases}$$

where $u_1 \in L^2((0,T)\times\Omega; H^1_{\mathrm{per}}(Y_1))$ *and* $u_2 \in L^2((0,T)\times\Omega\times Y_1; H^1_{\mathrm{per}}(Y_2))$.

R e m a r k 6. The homogenized map b is derived in the usual way by separation of variables. Let us consider the variational form of the ε^{-2}-equation for the case $0 < k < 2$:

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

By virtue of linearity we can decouple variables:

$$u_2(x, t, y_1, y_2, \tau) = -(\nabla_x u(x, t) + \nabla_{y_1} u_1(x, t, y_1, \tau)) \cdot w_2(y_2, \tau).$$

We can now write the decoupled local ε^{-2}-equation as the parameter dependent (parameter τ) problem:

Find $w_2^k(\cdot, \tau) \in H^1_{\mathrm{per}}(Y_2)$ such that for almost every $\tau \in (0, 1)$

$$\int_{Y_2} a_{ij}(x, t, y_1, y_2, \tau)\left(\delta_{jk} - \frac{\partial w_2^k(y_2, \tau)}{\partial_{y_{2_j}}}\right)\frac{\partial\varphi(y_2)}{\partial_{y_{2_i}}}\,\mathrm{d}y_2 = 0$$

for all $\varphi \in H^1_{\mathrm{per}}(Y_2)$, and define

$$b^1_{ik}(x, t, y_1, \tau) = \int_{Y_2} a_{ij}(x, t, y_1, y_2, \tau)\left(\delta_{jk} - \frac{\partial w_2^k(y_2, \tau)}{\partial_{y_{2_j}}}\right)\mathrm{d}y_2.$$

The local decoupled ε^{-1}-equation can then be written (using the same traditional arguments as above):

Find $v_1^k(\cdot, \tau) \in H^1_{\mathrm{per}}(Y_1)$, such that for almost every $\tau \in (0, 1)$

$$\int_{Y_1} b^1_{ij}(x, t, y_1, y_2, \tau)\left(\delta_{jk} - \frac{\partial v_1^k(y_1, \tau)}{\partial_{y_{1_j}}}\right)\frac{\partial\varphi(y_1)}{\partial_{y_{1_i}}}\,\mathrm{d}y_1 = 0$$

for all $\varphi \in H^1_{\mathrm{per}}(Y_1)$.

Finally we define

$$b_{ik}(x, t) = \int_{Y_1}\int_0^1 b^1_{ij}(x, t, y_1, \tau)\left(\delta_{jk} - \frac{\partial v_1^k(y_1, \tau)}{\partial_{y_{1_j}}}\right)\mathrm{d}y_1\,\mathrm{d}\tau.$$

This procedure is standard and analogous for different cases. The existence and uniqueness of local solutions is carried out in [5] in the linear periodic case.

4. Proof of Theorem 5

The limit of the variational formulation (4) gives the variational form of the global problem:

Find $u \in L^2(0,T;H_0^1(\Omega))$ such that

$$(5)\qquad -\int_{\Omega_T} u(x,t)\frac{\partial\varphi(x,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T}\Big[\int_{Y_\tau} a(x,y_1,y_2,t,\tau)[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}\tau\Big]\cdot\nabla\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\Omega_T} f(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\quad\text{for all }\ \varphi\in L^2(0,T;H_0^1(\Omega)).$$

Next, choose test functions $\varphi_\varepsilon(x,t)=\varphi(x,x/\varepsilon,x/\varepsilon^2,t,t/\varepsilon^k)$. By the chain rule the variational formulation of (1) reads:

Find $u_\varepsilon \in L^2(0,T;H_0^1(\Omega))$ such that

$$(6)\qquad -\int_{\Omega_T} u_\varepsilon\Big(\frac{\partial\varphi_\varepsilon}{\partial t}+\frac{\varepsilon^{-k}\partial\varphi_\varepsilon}{\partial\tau}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon\nabla u_\varepsilon\cdot(\nabla_x+\varepsilon^{-1}\nabla_{y_1}+\varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\quad\forall\,\varphi_\varepsilon\in L^2(0,T;H_0^1(\Omega)),\ \ u_\varepsilon(x,0)=u_0(x).$$

Let us now case by case show that the local equations for u, u_1 and u_2 will appear as multiscale limits of (6) with appropriate choices of test functions φ_ε. As the formal analysis in Appendix shows, there are seven significant different cases for k to be considered: $0<k<2$, $k=2$, $2<k<3$, $k=3$, $3<k<4$, $k=4$ and $k>4$.

The case $0<k<2$.

Step 1. Let us consider (6). We choose test functions $\varphi_\varepsilon(x,t)=\varphi(x,x/\varepsilon,x/\varepsilon^2,t,t/\varepsilon^k)$. Multiplication by ε^2 on both sides of the equation and a limit passage yields the ~ -2 equation

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u+\nabla_{y_1}u_1+\nabla_{y_2}u_2]\cdot\nabla_{y_2}\varphi(x,y_1,y_2,t,\tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T=0.$$

Step 2. Choose test functions $\varphi_\varepsilon(x,t)=\varphi(x,x/\varepsilon,t,t/\varepsilon^k)$ and consider the equation

$$-\int_{\Omega_T} u_\varepsilon\Big(\frac{\partial\varphi_\varepsilon}{\partial t}+\varepsilon^{-k}\frac{\partial\varphi_\varepsilon}{\partial\tau}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon\nabla u_\varepsilon\cdot(\nabla_x+\varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε on both sides of the equation and a limit passage yields the ~ -1 equation

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0.$$

The case $k = 2$.

Step 1. We choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, x/\varepsilon^2, t, t/\varepsilon^2)$ and consider the equation

$$-\int_{\Omega_T} u_\varepsilon\Big(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-2}\frac{\partial\varphi_\varepsilon}{\partial\tau}\Big)\, \mathrm{d}x\, \mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon\nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t.$$

Multiplication by ε^2 on both sides of the equation and a limit passage in (8), using Corollary 1, yields the ~ -2 equation

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0.$$

Step 2. We choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, t, t/\varepsilon^2)$ and consider the difference between (6) and the weak limit (5):

$$-\int_{\Omega_T} (u_\varepsilon - u)\Big(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-2}\frac{\partial\varphi_\varepsilon}{\partial\tau}\Big)\, \mathrm{d}x\, \mathrm{d}t$$
$$+\int_{\Omega_T}\Big(a_\varepsilon\nabla u_\varepsilon - \int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\, \mathrm{d}y\, \mathrm{d}\tau\Big) \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t = 0.$$

Multiplication by ε^1 on both sides of the equation and a limit passage, where Theorem 3 is used in the first term yields the ~ -1 equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_1(x, y_1, t, \tau)\frac{\partial\varphi}{\partial\tau}(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T$$
$$+\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0.$$

The case $2 < k < 3$.

Step 1. Choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, x/\varepsilon^2, t)$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial t}\, \mathrm{d}x\, \mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon\nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t.$$

Multiplication by ε^2 and a limit passage yields the ~ -2 equation

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

Step 2. Choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, t, t/\varepsilon^2)$ and consider

$$-\int_{\Omega_T}\Big[(u_\varepsilon - u)\Big(\frac{\partial\varphi_\varepsilon}{\partial t} + \frac{\varepsilon^{-k}\partial\varphi_\varepsilon}{\partial\tau}\Big) - u\frac{\partial\varphi_\varepsilon}{\partial t}\Big]\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon\nabla u_\varepsilon\cdot(\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^{k-1} and a limit passage, where Theorem 3 is used, yields the $\sim -k+1$ equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_1\frac{\partial\varphi}{\partial\tau}(x, y_1, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

Hence $u_1 = u_1(x, y_1, t)$.

Step 3. Choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, t)$ and study the equation

$$-\int_{\Omega_T} u_\varepsilon\frac{\partial\varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon\nabla u_\varepsilon\cdot(\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε and a limit passage yields the ~ -1 equation

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_1}\varphi(x, y_1, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

The case $k = 3$.

Step 1. We consider again the difference between (6) and the weak limit (5), i.e.

$$-\int_{\Omega_T}(u_\varepsilon - u)\Big(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-3}\frac{\partial\varphi_\varepsilon}{\partial\tau}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T}\Big(a_\varepsilon\nabla u_\varepsilon - \int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\,\mathrm{d}y\,\mathrm{d}\tau\Big)$$
$$\times(\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = 0$$

Multiplication by ε^2 on both sides of the equation and a limit passage, where Theorem 3 is used, yields the ~ -2 equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T$$
$$+\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\cdot\nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

Step 2. We choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, t, t/\varepsilon^3)$. Scale $y_1 = x/\varepsilon$ in u_1, multiply (8) by ε and subtract this from the difference between (6) and (5). This gives

$$-\int_{\Omega_T}(u_\varepsilon - u - \varepsilon u_1)\Big(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-3}\frac{\partial\varphi_\varepsilon}{\partial\tau}\Big)\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T}\varepsilon u_1\frac{\partial\varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T}\left(a_\varepsilon\nabla u_\varepsilon - \int_{Y_\tau} a[\cdot]\,\mathrm{d}y\,\mathrm{d}\tau - \varepsilon\int_{Y_2} a[\cdot]\,\mathrm{d}y_2\right)\cdot(\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

where $a[\cdot] = a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] = a(x, y_1, y_2, t, \tau)[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]$. Multiplication by ε^1 on both sides of the equation and a limit passage, where Theorem 4 is used, yields the ~ -1 equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial \varphi}{\partial \tau}(x, y_1, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T$$
$$+\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\cdot\nabla_{y_1}\varphi(x, y_1, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

The case $3 < k < 4$.

Step 1. We again choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, x/\varepsilon^2, t, t/\varepsilon^k)$. However, now we consider

$$-\int_{\Omega_T}(u_\varepsilon - u)\Big(\frac{\partial\varphi_\varepsilon}{\partial t} + \varepsilon^{-k}\frac{\partial\varphi_\varepsilon}{\partial\tau}\Big) - u\frac{\partial\varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon\nabla u_\varepsilon\cdot(\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^{k-1} on both sides of the equation and a limit passage, where Theorem 3 is used, yields the $\sim -k+1$ equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

From this we conclude that $u_1 = u_1(x, y_1, t)$, i.e. it is independent of τ.

Step 2. Choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, x/\varepsilon^2, t)$ and consider the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^2 and a limit passage yields the ~ -2 equation

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

Step 3. Choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, t, t/\varepsilon^k)$, scale $y_1 = x/\varepsilon$ in u_1 and consider

$$-\int_{\Omega_T} (u_\varepsilon - u - \varepsilon u_1)\Big(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-4}\frac{\partial \varphi_\varepsilon}{\partial \tau}\Big) - (u + \varepsilon u_1)\frac{\partial \varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^{k-2} on both sides of the equation and a limit passage, where Theorem 4 is used, yields the $\sim -k+2$ equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

Hence $u_2 = u_2(x, y_1, y_2, t)$, i.e. it is independent of τ.

Step 4. Next we choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, t)$ and consider the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^1 on both sides of the equation and a limit passage yields the ~ -1 equation

$$\int_{\Omega_T}\Big[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\,\mathrm{d}y_2\,\mathrm{d}\tau\Big]\cdot \nabla_{y_1}\varphi(x, y_1, t)\,\mathrm{d}y_1\,\mathrm{d}x\,\mathrm{d}t = 0.$$

The case $k = 4$.

Step 1. We choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, x/\varepsilon^2, t, t/\varepsilon^4)$. Then we consider

$$-\int_{\Omega_T} (u_\varepsilon - u)\Big(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-4}\frac{\partial \varphi_\varepsilon}{\partial \tau}\Big) - u\frac{\partial \varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^3 on both sides of the equation and a limit passage, where Theorem 3 is used, yields the ~ -3 equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

From this we conclude that $u_1 = u_1(x, y_1, t)$, i.e. it is independent of τ.

Step 2. Scale $y_1 = x/\varepsilon$ in u_1 and consider

$$-\int_{\Omega_T} (u_\varepsilon - u - \varepsilon u_1)\Big(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-4}\frac{\partial \varphi_\varepsilon}{\partial \tau}\Big) - (u + \varepsilon u_1)\frac{\partial \varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^2 on both sides of the equation and a limit passage, where Theorem 4 is used, yields the ~ -2 equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T$$
$$+\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\,\mathrm{d}y_\tau\,\mathrm{d}x_T = 0.$$

Step 3. Next we choose test functions $\varphi_\varepsilon(x, t) = \varphi(x, x/\varepsilon, t)$. This yields

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^1 on both sides of the equation and a limit passage yields the ~ -1 equation

$$\int_{\Omega_T}\Big[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2]\,\mathrm{d}y_2\,\mathrm{d}\tau\Big] \cdot \nabla_{y_1}\varphi(x, y_1, t)\,\mathrm{d}y_1\,\mathrm{d}x\,\mathrm{d}t = 0.$$

The case $k > 4$.

Step 1. We choose test functions $\varphi_\varepsilon(x, t) = \varphi(x, x/\varepsilon, x/\varepsilon^2, t, t/\varepsilon^k)$, and we consider

$$-\int_{\Omega_T} (u_\varepsilon - u)\Big(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-4}\frac{\partial \varphi_\varepsilon}{\partial \tau}\Big) - u\frac{\partial \varphi_\varepsilon}{\partial t}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

Multiplication by ε^{k-1} on both sides of the equation and a limit passage, where Theorem 3 is used, yields the $\sim -k+1$ equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_1 \frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0.$$

From this we conclude that $u_1 = u_1(x, y_1, t)$, i.e. it is independent of τ.

Step 2. Scale $y_1 = x/\varepsilon$ in u_1 and consider

$$-\int_{\Omega_T} (u_\varepsilon - u - \varepsilon u_1)\Big(\frac{\partial \varphi_\varepsilon}{\partial t} + \varepsilon^{-k}\frac{\partial \varphi_\varepsilon}{\partial \tau}\Big) - (u + \varepsilon u_1)\frac{\partial \varphi_\varepsilon}{\partial t}\, \mathrm{d}x\, \mathrm{d}t$$
$$+\int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t.$$

Multiplication by ε^{k-2} on both sides of the equation and a limit passage, where Theorem 4 is used, yields the $\sim -k+2$ equation

$$-\int_{\Omega_T}\int_{Y_\tau} u_2 \frac{\partial \varphi}{\partial \tau}(x, y_1, y_2, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0.$$

From this we conclude that $u_2 = u_2(x, y_1, y_2, t)$, i.e. it is independent of τ.

Step 3. Next we choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, x/\varepsilon^2, t)$ and consider the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\, \mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1} + \varepsilon^{-2}\nabla_{y_2})\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t.$$

Multiplication by ε^2 on both sides and a limit passage yields the ~ -2 equation

$$\int_{\Omega_T}\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] \cdot \nabla_{y_2}\varphi(x, y_1, y_2, t, \tau)\, \mathrm{d}y_\tau\, \mathrm{d}x_T = 0.$$

Step 4. Next we choose test functions $\varphi_\varepsilon(x,t) = \varphi(x, x/\varepsilon, t)$ and consider the equation

$$-\int_{\Omega_T} u_\varepsilon \frac{\partial \varphi_\varepsilon}{\partial t}\, \mathrm{d}x\, \mathrm{d}t + \int_{\Omega_T} a_\varepsilon \nabla u_\varepsilon \cdot (\nabla_x + \varepsilon^{-1}\nabla_{y_1})\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t = \int_{\Omega_T} f\varphi_\varepsilon\, \mathrm{d}x\, \mathrm{d}t.$$

Multiplication by ε^1 on both sides of the equation and a limit passage yields the ~ -1 equation

$$\int_{\Omega_T}\Big[\int_{Y_\tau} a[\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2] dy_2\, \mathrm{d}\tau\Big] \cdot \nabla_{y_1}\varphi(x, y_1, t)\, \mathrm{d}y_1\, \mathrm{d}x\, \mathrm{d}t = 0.$$

□

R e m a r k 7. Theorem 5 easily generalizes to the case of N spatial scales and more than one temporal scale. The difference is that the number of intervals to be studied increases. Also, one needs to prove a generalization of Theorem 4 to the case of N scales.

R e m a r k 8. In the present paper we have analyzed a prototype problem in order to understand analytically the mechanism when more fine scales are added to the problem. We see that the occurrence of phenomena like resonances increases and we can obtain a variety of local effects, which in the end has a large impact on the global behaviour of the solution. Especially we note that by adding spatial scales the problem becomes more and more sensitive to a perturbation with respect to the number k.

5. Appendix: Multiple scales expansions

Let us revisit the expansion (3). By the chain rule we have

$$\frac{\partial u_\varepsilon}{\partial t} = \Big(\frac{\partial}{\partial t} + \varepsilon^{-k}\frac{\partial}{\partial \tau}\Big)(u + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)$$

and

$$\begin{aligned}-\operatorname{div}(a\nabla u_\varepsilon) = &-(\operatorname{div}_x + \varepsilon^{-1}\operatorname{div}_{y_1} + \varepsilon^{-2}\operatorname{div}_{y_2})[a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \\ &+ \varepsilon(\nabla_x u_1 + \nabla_{y_1} u_2 + \nabla_{y_2} u_3) + \ldots)].\end{aligned}$$

The three relevant powers of ε to study are -2, -1 and 0. Below we will use the fact that we can not verify the existence of the terms $\nabla_x u_1, \nabla_{y_1} u_2$ and $\nabla_{y_2} u_3$ in L^2 by the multiscale compactness Theorem 5. We therefore omit their contribution also in the formal expansion. With higher regularity they might exist and this would lead to a more complex array of local problems. We just point out in the cases $k = 1$ and $k = 2$ that there occur, formally, two time derivatives in the zero order equation. However, the local time derivative vanishes after averaging in local time. Compare with Remark 1 where this is explained and with Remark 2 above. The structure of the hierarchy of equations will depend on $k > 0$. It turns out that there are 7 significantly different cases to consider, namely: $0 < k < 2$, $k = 2$, $2 < k < 3$, $k = 3$, $3 < k < 4$, $k = 4$ and $k > 4$. We choose $k = 1$ for the case $0 < k < 2$ in order to point out the above remark.

$k = 1$

~ -2: $\quad -\operatorname{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;$

~ -1: $\quad \partial u/\partial\tau - \operatorname{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;$

~ 0: $\quad \partial u/\partial t + \partial u_1/\partial\tau - \operatorname{div}_x(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.$

$k = 2$

$$\begin{aligned}
&\sim -2\colon && \partial u/\partial \tau - \operatorname{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim -1\colon && \partial u_1/\partial \tau - \operatorname{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim 0\colon && \partial u/\partial t + \partial u_2/\partial \tau - \operatorname{div}_{x}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.
\end{aligned}$$

$2 < k < 3$

$$\begin{aligned}
&\sim -k\colon && \partial u/\partial \tau = 0;\\
&\sim -2\colon && -\operatorname{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim -k+1\colon && \partial u_1/\partial \tau = 0;\\
&\sim -1\colon && -\operatorname{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim 0\colon && \partial u/\partial t - \operatorname{div}_{x}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.
\end{aligned}$$

$k = 3$

$$\begin{aligned}
&\sim -3\colon && \partial u/\partial \tau = 0;\\
&\sim -2\colon && \partial u_1/\partial \tau - \operatorname{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim -1\colon && \partial u_2/\partial \tau - \operatorname{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim 0\colon && \partial u/\partial t - \operatorname{div}_{x}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.
\end{aligned}$$

$3 < k < 4$

$$\begin{aligned}
&\sim -k\colon && \partial u/\partial \tau = 0;\\
&\sim -k+1\colon && \partial u_1/\partial \tau = 0;\\
&\sim -2\colon && -\operatorname{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim -k+2\colon && \partial u_2/\partial \tau = 0;\\
&\sim -1\colon && \operatorname{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim 0\colon && \partial u/\partial t - \operatorname{div}_{x}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.
\end{aligned}$$

$k = 4$

$$\begin{aligned}
&\sim -4\colon && \partial u/\partial \tau = 0;\\
&\sim -3\colon && \partial u_1/\partial \tau = 0;\\
&\sim -2\colon && \partial u_2/\partial \tau - \operatorname{div}_{y_2}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim -1\colon && -\operatorname{div}_{y_1}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim 0\colon && \partial u/\partial t - \operatorname{div}_{x}(a(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f.
\end{aligned}$$

$k > 4$

$$\begin{aligned}
&\sim -k\colon && \partial u/\partial \tau = 0;\\
&\sim -k+1\colon && \partial u_1/\partial \tau = 0;\\
&\sim -k+2\colon && \partial u_2/\partial \tau = 0;\\
&\sim -2\colon && \operatorname{div}_{y_2}(\tilde{a}(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim -1\colon && -\operatorname{div}_{y_1}(\tilde{a}(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = 0;\\
&\sim 0\colon && \partial u/\partial t - \operatorname{div}_{x}(\tilde{a}(\nabla_x u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2)) = f,
\end{aligned}$$

where

$$\tilde{a}(x,t) = \int_0^1 a(x, y_1, y_2, t, \tau)\, d\tau.$$

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