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David Buhagiar; Emmanuel Chetcuti

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ON COMPLETE-COCOMPLETE SUBSPACES OF AN INNER PRODUCT SPACE

David Buhagiar, Emanuel Chetcuti, Msida

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Abstract. In this note we give a measure-theoretic criterion for the completeness of an inner product space. We show that an inner product space S is complete if and only if there exists a σ-additive state on $C(S)$, the orthomodular poset of complete-cocomplete subspaces of S. We then consider the problem of whether every state on $E(S)$, the class of splitting subspaces of S, can be extended to a Hilbertian state on $E(\overline{S})$; we show that for the dense hyperplane S (of a separable Hilbert space) constructed by P. Pták and H. Weber in Proc. Am. Math. Soc. *129* (2001), 2111–2117, every state on $E(S)$ is a restriction of a state on $E(\overline{S})$.

Keywords: Hilbert space, inner product space, orthogonally closed subspace, complete and cocomplete subspaces, finitely and σ-additive state

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1. Introduction

If H is a Hilbert space, then we denote by $L(H)$ the system of all closed subspaces of H. In 1936, G. Birkhoff and J. von Neumann showed [2] that the set of assertions of a quantum system has the structure of projective geometry. Presently, apart from $L(H)$, other families of closed subspaces of incomplete inner product spaces are being used as axiomatical quantum mechanical models. It is therefore of great importance to characterize Hilbert spaces in the family of inner product spaces. There are various algebraic and measure-theoretic completeness criteria which can be found in [3].

Let S be an *inner product space* (real or complex) and let $\langle\cdot,\cdot\rangle$ be the inner product on S. Let $A^{\perp} = \{b \in S\colon\ \langle a,b\rangle = 0 \text{ for all } a \in A\}$ for each subset $A \subset S$. Recall that M is said to be cocomplete if $M = A^{\perp}$ for some complete subspace A of S.

In what follows we will consider the following families of closed subspaces of S:

$$F(S) = \{M \subset S\colon\ M^{\perp\perp} = M\},$$
$$E(S) = \{M \subset S\colon\ M \oplus M^{\perp} = S\},$$
$$C(S) = \{M \subset S\colon\ M \text{ is complete or cocomplete}\}.$$

The main results of the paper are Theorem 2.6, where we characterize complete inner product spaces as those for which the orthomodular poset of complete-cocomplete subspaces admit a σ-additive state, and Theorem 3.7, where we show that whenever $C(S)$ consists of the finite/co-finite dimensional subspaces, every state on $C(S)$ extends to a state on $E(\overline{S})$.

2. σ-additive states on $C(S)$

The main measure-theoretic completeness criterion is due to J. Hamhalter and P. Pták [6]. It states that for a separable incomplete inner product space S, $F(S)$ admits no σ-additive states. In the proof, use is made of Gleason's theorem which states that for a separable Hilbert space H, $\dim H \geqslant 3$, every σ-additive state on $L(H)$ is described via a von Neumann density operator [5]. In this section we show that, for a separable inner product space S, any σ-additive state on $C(S)$ induces a state on $F(S)$ and therefore for an incomplete separable inner product space, $C(S)$ admits no σ-additive states.

A *σ-additive state* on an orthomodular poset L is defined as follows [7]:

Definition 2.1. A σ-additive state on an orthomodular poset L is a mapping $s\colon L \to [0,1]$ such that

I. $s(1) = 1$;

II. if $\{A_i\colon\ i \in I\}$ is a countable orthogonal collection in L and $\bigvee_{i\in I} A_i$ exists in L, then $s\Big(\bigvee_{i\in I} A_i\Big) = \sum_{i\in I} s(A_i)$.

Finitely additive states on L are similarly defined, but with (II) holding only for finitely many mutually orthogonal elements of L.

It can be verified (see [9]) that for any inner product space S, $E(S)$ and $C(S)$ form orthomodular posets with respect to set inclusion $\subset$ and orthogonal complementation $\perp$. We also recall that $C(S) \subset E(S) \subset F(S)$.

Lemma 2.1. *Let $A \in C(S)$ and suppose that $\{u_i\colon i \in I\}$ is a maximal orthonormal system in A. Let $[u_i]$ stand for the linear span of u_i. Then $\bigvee_{i\in I}[u_i]$ exists in $C(S)$ and, moreover, $\bigvee_{i\in I}[u_i] = A$.*

P r o o f. Evidently, $[u_i] \subset A$ for all $i \in I$. If $M \in E(S)$ satisfies $M \supset [u_i]$ for all $i \in I$ and $M \subset A$, then, since $S = M \oplus M^\perp$, $a = m + m'$ for all $a \in A$, where $m \in M$ and $m' \in M^\perp$. This implies that $m' = a - m \in A \cap M^\perp$; that is, $m' \perp u_i$ for all $i \in I$. Since $m' \in A$, it follows that $m' = 0$ and therefore $a \in M$. Hence $M = A$. $\square$

Lemma 2.2. *Let s be a σ-additive state on $C(S)$ where S is a separable inner product space. Then for any orthonormal system $\{x_i\colon i \in I\} \subset S$ we have*

$$\sum_{i\in I} s([x_i]) \leqslant 1.$$

P r o o f. Extend $\{x_i\colon i \in I\}$ to $\{x_i\colon i \in I\} \cup \{y_j\colon j \in J\}$, a maximal orthonormal system of S. By Lemma 2.1, it follows that

$$\bigvee_{i\in I}[x_i] \vee \bigvee_{i\in J}[y_j] = S.$$

Moreover, since S is separable, $|I \cup J| \leqslant \aleph_0$ and therefore, by the σ-additivity of s we have

$$\sum_{i\in I} s([x_i]) + \sum_{j\in J} s([y_j]) = 1.$$

Hence, $\sum_{i\in I} s([x_i]) \leqslant 1$. $\square$

We shall say that a subspace $M \subset S$ (S separable) is *consistent with respect to a σ-additive state s* on $C(S)$ when for any two maximal orthonormal systems $\{a_i\colon i \in I\}$ and $\{b_i\colon i \in I\}$ in M we have

$$\sum_{i\in I} s([a_i]) = \sum_{i\in I} s([b_i]).$$

For any σ-additive state s on $C(S)$, define

$$L_s(S) = \{M \in F(S)\colon M \text{ is } \textit{consistent with respect to } s\}.$$

R e m a r k 2.1. For a separable inner product space S we have $C(S) \subset L_s(S)$ for any σ-additive state s on $C(S)$.

Lemma 2.3. *Let S be a separable inner product space. Let $\{x_i \colon i \in I\}$ be an orthonormal system in S and $M = \{x_i \colon i \in I\}^{\perp\perp}$.*

(1) *If $\{y_j \colon j \in J\}$ is a maximal orthonormal system in $M^\perp$, then $\{x_i \colon i \in I\} \cup \{y_j \colon j \in J\}$ is a maximal orthonormal system in S.*

(2) *$M^\perp \in L_s(S)$ for any σ-additive state s on $C(S)$.*

Proof. (1) Let $v \in S$ satisfy $v \perp x_i$ for all $i \in I$ and $v \perp y_j$ for all $j \in J$. This implies that $v \in \{x_i \colon i \in I\}^\perp = M^\perp$. Hence $v = 0$ because $\{y_j \colon j \in J\}$ is a maximal orthonormal system in $M^\perp$.

(2) Suppose that $\{y_j \colon j \in J\}$ and $\{z_j \colon j \in J\}$ are two maximal orthonormal systems in $M^\perp$. Then by part (1), $\{x_i \colon i \in I\} \cup \{y_j \colon j \in J\}$ and $\{x_i \colon i \in I\} \cup \{z_j \colon j \in J\}$ are two maximal orthonormal systems in S. By Lemma 2.1 we have

$$1 = s(S) = \sum_{i \in I} s([x_i]) + \sum_{j \in J} s([y_j]) = \sum_{i \in I} s([x_i]) + \sum_{j \in J} s([z_j])$$

and therefore

$$\sum_{j \in J} s([y_j]) = \sum_{j \in J} s([z_j]).$$

□

Lemma 2.4. *Let S be a separable inner product space. If $M \in F(S)$, then $M^\perp \in L_s(S)$ for any σ-additive state s on $C(S)$.*

Proof. Starting with a countable dense subset in M, we can, by the Gram-Schmidt process, convert its maximal linearly independent subset into an orthonormal countable collection $\{e_i \colon i \in I\}$ such that

$$M = \overline{\operatorname{span}\{e_i \colon i \in I\}}^S = \bigvee_{i \in I} [e_i] \quad \text{in } F(S).$$

Thus, $M = \operatorname{span}\{e_i \colon i \in I\}^{\perp\perp}$ and hence the result follows from Lemma 2.3 part (2). □

Corollary 2.5. *For a separable inner product space S we have $F(S) = L_s(S)$ for any σ-additive state s on $C(S)$.*

Proof. Let $M \in F(S)$. Since $M^\perp \in F(S)$, Lemma 2.4 implies that for any σ-additive state s on $C(S)$, $M = M^{\perp\perp} \in L_s(S)$. □

Finally, we formulate the main result of this section.

Theorem 2.6. *If S is an incomplete inner product space, then $C(S)$ possesses no σ-additive states. Thus, S is complete if and only if $C(S)$ possesses a σ-additive state.*

P r o o f. Assume to the contrary that s is a σ-additive state on $C(S)$, where S is a separable incomplete inner product space. We show that the map

$$\hat{s}\colon\ F(S) \to [0,1],\ \ M \mapsto \sum_{i\in I} s([x_i])$$

where $\{x_i\colon\ i \in I\}$ is a maximal orthonormal system of M, defines a state on $F(S)$. This would contradict the Hamhalter-Pták result [6]. First we note that by Corollary 2.5, $\hat{s}$ is well defined. We also have $\hat{s}(S) = 1$ and $\hat{s}(\{0\}) = 0$. Let $\{A_i\colon\ i \in I\}$ be a countable orthogonal collection in $F(S)$. For each A_i we have

$$A_i = (\operatorname{span}\{x_{ij}\colon\ j \in J_i\})^{\perp\perp},$$

where $\{x_{ij}\colon\ j \in J_i\}$ is an orthonormal basis in A_i. We now show that $\bigcup\limits_i \{x_{ij}\colon\ j \in J_i\}$ is a maximal orthonormal sequence in $\Big(\operatorname{span}\Big(\bigcup\limits_{i\in I} A_i\Big)\Big)^{\perp\perp} = \bigvee\limits_{i\in I} A_i$ in $F(S)$. Clearly, $\bigcup\limits_i \{x_{ij}\colon\ j \in J_i\}$ is an orthonormal system in $\Big(\operatorname{span}\Big(\bigcup\limits_{i\in I} A_i\Big)\Big)^{\perp\perp}$. If $v \perp x_{ij}$ for all $i \in I$, $j \in J_i$, then $v \in A_i^{\perp}$ for all $i \in I$. This implies that $v \in \bigcap\limits_{i\in I} A_i^{\perp} = \Big(\operatorname{span}\Big(\bigcup\limits_{i\in I} A_i\Big)\Big)^{\perp}$. Hence, either $v = 0$ or $v \notin \Big(\operatorname{span}\Big(\bigcup\limits_{i\in I} A_i\Big)\Big)^{\perp\perp}$. Therefore we have

$$\hat{s}\Big(\bigvee_{i\in I} A_i\Big) = \sum_{i,j} s([x_{ij}]) = \sum_{i\in I}\sum_{j\in J_i} s([x_{ij}]) = \sum_{i\in I} \hat{s}(A_i).$$

$\square$

3. Finitely additive states on $C(S)$

In this section we consider finitely additive states on $C(S)$. A question considered in [9] is whether there is always a state on $E(S)$ which is not a restriction of a Hilbertian state on $E(\overline{S})$. To show that there exists an inner product space S for which every state on $E(S)$ is a restriction of a state on $E(\overline{S})$, we take a separable inner product space S which is a dense hyperplane and so in this particular case we have $E(S) = C(S)$. Moreover, in this example we have that $C(S)$ consists of the finite and cofinite dimensional subspaces of S. First we show that such an inner product space does exist. The following construction is due to P. Pták and H. Weber [9].

Theorem 3.1. *Let H be a separable Hilbert space. Then H possesses a dense hyperplane, S, such that $C(S)$ consists of finite or cofinite dimensional subspaces of S. Thus, $C(S)$ is a modular lattice.*

Proof. We make use of the following lemma. $\square$

Lemma 3.2. *Let H be a separable Hilbert space. Then we have*

(1) $\operatorname{card}(H) = 2^{\aleph_0}$;

(2) $\operatorname{card}\{\{v_i\}\colon$ *sequence in* $H\} = 2^{\aleph_0}$;

(3) $\operatorname{card}\{A \subset H\colon A$ *is a complete infinite dimensional subspace of* $H\} = 2^{\aleph_0}$.

Proof. (1) Since H is separable, the linear dimension of H is $2^{\aleph_0}$. Let U be a linear basis for H; thus $\operatorname{card}(U) = 2^{\aleph_0}$. Moreover, if D is the field (real or complex) of H, we have

$$\operatorname{card}(D \times U) = 2^{\aleph_0} \cdot 2^{\aleph_0}.$$

Define $S_n = \{v \in H\colon$ the decomposition of v in terms of U consists of n terms$\}$. Then $H = \bigcup\limits_{n\in\mathbb{N}} S_n$ and $S_n \cap S_m = \emptyset$ for $n \neq m$. We show that $\operatorname{card}(S_n) = 2^{\aleph_0}$. By the well-ordering principle, we can well-order U. For every $v \in S_n$ we have $v = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$ where $\{u_i\colon i \leqslant n\} \subset U$ and $\{\alpha_i\colon i \in \mathbb{N}\} \subset D$. Moreover, we can express the linear expansion of v in such a way that $u_i \leqslant u_j$ for $i \leqslant j$ $(i, j \leqslant n)$. Define a map

$$\psi_n\colon S_n \to (D \times U)^n, \;\; v \mapsto ((\alpha_1, u_1), (\alpha_2, u_2), \ldots, (\alpha_n, u_n)).$$

Clearly ψ_n is injective. Moreover, $\operatorname{card}(D \times U)^n = (2^{\aleph_0})^n = 2^{\aleph_0}$. Hence $\operatorname{card}(S_n) \leqslant 2^{\aleph_0}$. Therefore $\operatorname{card}(H) \leqslant 2^{\aleph_0}$. Surely $\operatorname{card}(H) \geqslant 2^{\aleph_0}$, which completes the proof of (1).

(2) The cardinality of the set $L = \{\{v_i\}\colon$ sequence in $H\}$ is equal to $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$.

(3) Let

$$\mathfrak{A} = \{A \subset H\colon A \text{ is a complete infinite dimensional subspace of } H\}.$$

Since H is separable, there is an orthonormal basis (ONB) $E = \{e_i\colon i \in \mathbb{N}\}$ in H. Every infinite subset $C \subset E$ defines a complete infinite dimensional subspace of H (consider $\overline{\operatorname{span}}\{e_i \in C\}$) and moreover, distinct subsets define distinct subspaces. Thus we have

$$\operatorname{card}(\mathfrak{A}) \geqslant 2^{\aleph_0}.$$

For any $A \in \mathfrak{A}$, define $F_A = \{\{u_i\}\colon$ ONB of $A\}$. For all $A \in \mathfrak{A}$ we have $F_A \neq \emptyset$ and therefore by the axiom of choice we can choose a representative from F_A. Denote it by $\varphi(A)$. Surely we have that if $A, B \in \mathfrak{A}$ and $A \neq B$, then $\varphi(A) \neq \varphi(B)$. Thus φ is an injective map from A into L defined in part (2) above. Thus $\operatorname{card}(\mathfrak{A}) \leqslant 2^{\aleph_0}$. $\square$

P r o o f of Theorem 3.1. By Lemma 3.2 we know that the collection

$$\mathfrak{A} = \{A \subset H\colon\ A \text{ is a complete infinite dimesional subspace of } H\}$$

has cardinality $2^{\aleph_0}$. By the well-ordering principle, we can well-order the family $\mathfrak{A}$. Let ω be the initial ordering of cardinality $2^{\aleph_0}$. Then $\mathfrak{A} = \{A_\alpha\colon\ \alpha < \omega\}$. We will construct a linearly independent family of vectors in H, $\{v_\alpha\colon\ \alpha < \omega\}$, such that $v_\alpha \in A_\alpha$ for each α, $\alpha < \omega$. We proceed by transfinite induction. Let $v_1 \in A_1$ be an arbitrary non-zero vector. Suppose that the collection $\{v_\beta\colon\ \beta < \alpha\}$ has been constructed. Since the linear dimension of A_α is $2^{\aleph_0}$ and since $\operatorname{card}\{\beta\colon\ \beta < \alpha\} < 2^{\aleph_0}$, there must be a vector, v_α, in A_α so that the family $\{v_\beta\colon\ \beta \leqslant \alpha\}$ is linearly independent. Thus we can construct a family $V = \{v_\beta\colon\ \beta < \omega\}$ of linearly independent vectors of H such that $v_\beta \in A_\beta$. Extend this to a Hamel basis U for H. We can well-order the unbounded interval $[1, \infty)$ in $\mathbb{R}$ to get $\{p_\alpha\colon\ \alpha < \omega\}$. Then define a linear map f on H into $\mathbb{R}$ by $f(v_\alpha) = p_\alpha$ for all $v_\alpha \in V$ and $f(u) = 0$ for all $u \in U \setminus V$. Then f is an unbounded linear functional on H. Then $S = \mathcal{N}(f)$ is a dense hyperplane in H. By the construction, S does not contain a complete subspace of infinite dimension. □

It seems that in the reasoning applied in [9] to deduce that every state on $E(S)$ (S being the inner product space considered in Theorem 3.1) is a restriction of a state on $E(\overline{S})$, it was assumed that every state on $E(S)$ can be expressed as a convex sum of a state which lives on a finite dimensional subspace of S and the free state. This is however not true as we show below. Yet, the claim put by Pták and Weber in [9] is true as we will also prove.

First we construct a state on $E(S)$ which cannot be expressed as a convex sum of a state which lives on a finite dimensional subspace and the free state (i.e. the state which vanishes on all finite dimensional subspaces).

Proposition 3.3. *Every state s on $E(\overline{S})$ defines a finitely additive state on $E(S)$.*

P r o o f. Define $\hat{s}\colon\ E(S) \to [0,1]$ by $\hat{s}(M) = s(\overline{M})$. Certainly, $\hat{s}(\{0\}) = 0$. If $A \subset B^{\perp_S}$ in $E(S)$, then

$$\hat{s}(A \oplus B) = s(\overline{A \oplus B}) = s(\overline{A} \oplus \overline{B}) = s(\overline{A}) + s(\overline{B}) = \hat{s}(A) + \hat{s}(B).$$

□

Proposition 3.4. *Let S be the inner product space considered in Theorem* 3.1. *There are states on $E(S)$ which cannot be expressed as a convex sum $\alpha s_1 + (1-\alpha)s_0$ with s_1 being a state living on a finite dimensional subspace and s_0 being the free state.*

P r o o f will be divided in three parts:

Claim 1. *Let $x \in \overline{S}$, $\|x\| = 1$. The map*

$$\hat{s}_x \colon E(\overline{S}) \to [0,1], \quad M \mapsto \langle P_M x, x\rangle$$

defines a state on $E(\overline{S})$. Hence by Proposition 3.3, *$\hat{s}_x$ defines a state s_x on $E(S)$ determined by $s_x(A) = \hat{s}_x(\overline{A})$.*

Claim 2. *Let $M \in E(S)$ be finite dimensional ($M \neq \{0\}$). There is a state s on $E(S)$ which does not live on any finite dimensional subspace of S and satisfy $s(M) > 0$.*

P r o o f. Since M is complete, $\overline{S} = M \oplus M^{\perp_{\overline{S}}}$. Let $y \in M^{\perp_{\overline{S}}} \setminus M^{\perp_S}$. (Note that $M^{\perp_{\overline{S}}} \setminus M^{\perp_S}$ is not empty because S is not complete.) Let $x \in M$, $x \neq 0$. Put $\tilde{z} = x + y$ and let $z = \tilde{z}/\|\tilde{z}\|$. Consider the state s_z on $E(S)$. Then

$$s_z(M) = \langle P_M z, z\rangle \neq 0.$$

Suppose to the contrary, that $s_z(N) = 1$ for some finite dimensional subspace of S. Then $\langle P_N z, z\rangle = \|P_N z\|^2 = 1$. But since $z = P_N z + P_{N^{\perp_{\overline{S}}}} z$, we have

$$1 = \|z\|^2 = \|P_N z\|^2 + \|P_{N^{\perp_{\overline{S}}}} z\|^2 = 1 + \|P_{N^{\perp_{\overline{S}}}} z\|^2.$$

This implies that $P_{N^{\perp_{\overline{S}}}} z = 0$ and therefore $z \in N$. This is a contradiction because $z \notin S$. □

Claim 3. *The state s_z cannot be expressed as a convex sum $\alpha s_1 + (1-\alpha)s_0$ with s_1 being a state living on a finite dimensional subspace and s_0 being the free state.*

P r o o f. Suppose that $s_z = \alpha s_1 + (1-\alpha)s_0$ where $s_1(N) = 1$, $N \in E(S)$, N is finite dimensional and s_0 is the free state on $E(S)$. Let $v \in N^{\perp_S}$. Then $s_1([v]) = 0$ and therefore

$$s_z([v]) = \langle P_{[v]} z, z\rangle = 0.$$

This implies that $v \perp z$. This holds for all $v \in N^{\perp_S}$, and therefore we have

$$N^{\perp_S} \subset [z]^{\perp_{\overline{S}}}.$$

This implies that

$$N = N^{\perp_S \perp_{\overline{S}}} \supset z^{\perp_{\overline{S}} \perp_{\overline{S}}} = [z],$$

which is a contradiction since $z \notin S$. □

Thus Proposition 3.4 is proved. □

However, next we show that the claim put in [9] is true, that is, every state on $E(S) = C(S)$ is a restriction of a state on $E(\overline{S}) = C(\overline{S}) = L(\overline{S})$, where S is the inner product space constructed in 3.1. We shall need the following notion.

Definition 3.1 (Trace operator)**.** A bounded operator T on a Hilbert space H is called a *trace operator* if $\sum\limits_{i \in I} \langle Tx_i, x_i \rangle$ exists for any orthonormal basis $\{x_i \colon i \in I\}$ in H. By $\mathrm{Tr}(H)$ we shall denote the class of all trace operators on H.

Let us recall a classical result.

Lemma 3.5. *Let T be a positive Hermitian operator on a Hilbert space H. If $T \in \mathrm{Tr}(H)$, then there exists a constant $k \in [0, \infty)$ such that, for any orthonormal basis $\{x_i \colon i \in I\}$ of H, we have*

$$\sum_{i \in I} \langle Tx_i, x_i \rangle = k.$$

Proof. Let $\{x_i \colon i \in I\}$ be an orthonormal basis of H. If $T \in \mathrm{Tr}(H)$, then $\sum\limits_{i \in I} \langle Tx_i, x_i \rangle$ converges; say

$$\sum_{i \in I} \langle Tx_i, x_i \rangle = k.$$

Since T is Hermitian and positive, $k \in [0, \infty)$. Let $\{y_i \colon i \in I\}$ be an orthonormal basis of H. Then

$$\begin{aligned} \sum_{i \in I} \langle Ty_i, y_i \rangle &= \sum_{i \in I} \langle T^{\frac{1}{2}} y_i, T^{\frac{1}{2}} y_i \rangle = \sum_{i \in I} \|T^{\frac{1}{2}} y_i\|^2 \\ &= \sum_{i \in I} \sum_{j \in I} |\langle T^{\frac{1}{2}} y_i, x_j \rangle|^2 = \sum_{j \in I} \sum_{i \in I} |\langle y_i, T^{\frac{1}{2}} x_j \rangle|^2 = \sum_{j \in I} \|T^{\frac{1}{2}} x_j\|^2 = k. \end{aligned}$$

□

Let $T \in \mathrm{Tr}(H)$ be a positive (and non-zero) Hermitian operator on a Hilbert space H. Consider the map

$$\hat{s}_T\colon\ L(H) \to [0,1],\quad M \mapsto \begin{cases} \dfrac{1}{k}\sum\limits_{i\in J}\langle Tx_i, x_i\rangle, \\ 0 \quad \text{if } M = \{0\}, \end{cases}$$

where $\{x_i\colon\ i \in J\}$ is an orthonormal basis of M and $k = \sum\limits_{i\in I}\langle Ty_i, y_i\rangle$ for some orthonormal basis $\{y_i\colon\ i \in I\}$ of H. We show that $\hat{s}_T$ defines a σ-additive state on $L(H)$.

By Lemma 3.5 it is clear, that $\hat{s}_T$ is well-defined on $L(H)$, that is, the value of $\hat{s}_T$ on any $M \in L(H)$ is in $[0,1]$ and, moreover, is independent of the orthonormal basis of M chosen to compute $\hat{s}_T(M)$. Also, $\hat{s}_T(\{0\}) = 0$ and $\hat{s}_T(H) = 1$. Let us verify that for any sequence $\{M_k\colon\ k \in \mathbb{N}\}$ of mutually orthogonal complete subspaces of H, we have

$$\hat{s}_T\Bigl(\bigvee_{k\in\mathbb{N}} M_k\Bigr) = \sum_{k\in\mathbb{N}} \hat{s}_T(M_k);$$

that is, we verify that $\hat{s}_T$ is σ-additive. Indeed, let $\{u_{ki}\colon\ i \in J_k\}$ be an orthonormal basis in M_k. It suffices to show that $\bigcup\limits_{k\in\mathbb{N}}\{u_{ki}\colon\ i \in J_k\}$ is a maximal orthonormal system in $\bigvee\limits_{k\in\mathbb{N}} M_k$. Certainly, $\bigcup\limits_{k\in\mathbb{N}}\{u_{ki}\colon\ i \in J_k\} \subset \bigvee\limits_{k\in\mathbb{N}} M_k$. Let $v \in \bigvee\limits_{k\in\mathbb{N}} M_k$ satisfy $v \perp u_{ki}$ for all $k \in \mathbb{N}$ and $i \in J_k$. Then $v \in \bigcap\limits_{k\in\mathbb{N}} M_k^{\perp} = \Bigl(\bigvee\limits_{k\in\mathbb{N}} M_k\Bigr)^{\perp}$, and therefore $v = 0$.

Definition 3.2 (Frame function). Let H be a real or complex Hilbert space. We say that a mapping $f\colon\ \mathcal{S}(H) \to \mathbb{R}$ defined on the collection of unit vectors of H is a *frame function* on H if there is a constant W, called the *weight* of f, such that for any orthonormal basis $\{x_i\colon\ i \in I\}$ of H we have

$$\sum_{i\in I} f(x_i) = W.$$

For an inner product space S, a map $f\colon\ \mathcal{S}(S) \to \mathbb{R}$ is called a *frame-type function* on $\overline{S}$ if

(i) for any orthonormal system $\{x_i\colon\ i \in J\} \subset S$, $\{f(x_i)\colon\ i \in J\}$ is summable;

(ii) for any finite dimensional subspace K of S, $f|_{\mathcal{S}(K)}$ is a frame function on K.

We shall make use of the following theorem. For the proof, the reader is referred to [3, Theorem 3.2.21].

Theorem 3.6. *Let S be an infinite dimensional inner product space. Let $f\colon \mathcal{S}(S) \to \mathbb{R}$ be a frame-type function on $\overline{S}$. Then there is a unique Hermitian trace operator T on $\overline{S}$ such that*

$$f(x) = \langle Tx, x\rangle \quad \textit{for all } x \in \mathcal{S}(S).$$

We say that a finitely additive state s on $C(S)$ is a *restriction* of a state on $L(\overline{S})$ if there exists a state $\hat{s}$ on $L(\overline{S})$ such that $s(M) = \hat{s}(M)$ holds for all complete subspaces of S.

Theorem 3.7. *Let S be an inner product space such that $C(S)$ consists either of the finite or of the cofinite dimensional subspaces of S. Then every finitely additive state on $C(S)$ is a restriction of a state on $L(\overline{S})$.*

Proof. Let S be such an inner product space and let s be a finitely additive state on $C(S)$. Consider the mapping

$$f\colon \mathcal{S}(S) \to \mathbb{R}, \quad x \mapsto s([x]).$$

Suppose that $\{x_i\colon i \in J\}$ is an orthonormal system in S. For any finite subset $J_0 \subset J$ we have

$$\sum_{i\in J_0} f(x_i) = \sum_{i\in J_0} s([x_i]) = s(\operatorname{span}\{x_i\colon i \in J_0\}) < 1.$$

Therefore, $\sum\limits_{i\in J} f(x_i)$ converges. One can easily verify that for any finite dimensional subspace K of S, $f|_{\mathcal{S}(K)}$ is a frame function on K and therefore f is a frame-type function on $\overline{S}$. By Theorem 3.6, it follows that there is a unique Hermitian trace operator T on $\overline{S}$ such that

$$f(x) = \langle Tx, x\rangle \quad \text{for all } x \in \mathcal{S}(S).$$

It follows from the definition of f and from the continuity of T that T is positive. (Any Hermitian operator is bounded.) Indeed, let $y \in \overline{S}$. Then there is a sequence $\{x_i\} \subset S$ such that $x_i \to y$. Then

$$\begin{aligned}\langle Ty, y\rangle &= \langle T(\lim_{i\to\infty} x_i), \lim_{i\to\infty} x_i\rangle = \langle \lim_{i\to\infty} T(x_i), \lim_{i\to\infty} x_i\rangle \\ &= \lim_{i\to\infty} \langle Tx_i, x_i\rangle = \lim_{i\to\infty} s([x_i]) \geqslant 0.\end{aligned}$$

If T is non-zero, then we can define a $\sigma-$additive state $\hat{s}_T$ on $L(\overline{S})$. It is not difficult to check that $\hat{s}_T$ coincides with s on all finite dimensional subspaces of S.

Indeed, let M be a finite dimensional subspace of S and $\{x_i\colon i \leqslant n\}$ an orthonormal basis of M. Then

$$\hat{s}_T(M) = \sum_{i\leqslant n} \langle Tx_i, x_i\rangle = \sum_{i\leqslant n} f(x_i) = \sum_{i\leqslant n} s([x_i]) = s(M).$$

This implies that s is the restriction of $\hat{s}_T$.

If T is the zero operator on $\overline{S}$, then

$$\langle Tx, x\rangle = f(x) = s([x]) = 0,$$

for all $x \in \mathcal{S}(S)$. This implies that s is zero on all finite dimensional subspaces of S and one on all cofinite dimensional subspaces, that is, s is the 'free state' on $C(S)$. But there is a state, $\hat{s}$, on $L(\overline{S})$ which vanishes on all finite dimensional subspaces of $\overline{S}$ (see [7]). It is clear that s is a restriction of $\hat{s}$. This completes the proof. □

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Authors' address: *D. Buhagiar*, *E. Chetcuti*, Department of Mathematics, University of Malta, Msida MSD.06, Malta, e-mail: `david.buhagiar@um.edu.mt`, `emanria@maltanet.net`.