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A GENERALIZATION OF THE HOLDITCH THEOREM FOR THE PLANAR HOMOTHETIC MOTIONS

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Abstract. In this paper, under the one-parameter closed planar homothetic motion, a generalization of Holditch Theorem is obtained by using two different line segments (with fixed lengths) whose endpoints move along two different closed curves.

Keywords: Steiner formula, Holditch Theorem, homothetic motion

MSC 2000: 53A17

1. Introduction

H. Holditch [4] gave the following noteworthy classical theorem in 1858: if the endpoints A, B of a fixed segment $\overline{AB}$ with length $a+b$ are rotated once along an oval (Eilinie) k in the Euclidean plane, then a given fixed point X ($\overline{AX}=a$, $\overline{XB}=b$) of $\overline{AB}$ describes a closed, not necesarilly convex, curve $k(X)$. The area F of the *Holditch-Ring* bounded by the curves k and $k(X)$ is $F=\pi ab$. Later, this classical result was generalized by different methods [1]–[9], [11]–[13].

Let E and E' be the moving and fixed Euclidean planes and $\{O;\boldsymbol{e}_1,\boldsymbol{e}_2\}$ and $\{O';\boldsymbol{e}'_1,\boldsymbol{e}'_2\}$ their coordinate systems, respectively. By taking $\boldsymbol{OO'}=\boldsymbol{u}=u_1\boldsymbol{e}_1+u_1\boldsymbol{e}_2$ for $u_1,u_2\in\mathbb{R}$, the motion defined by the transformation

$$\boldsymbol{x}'=h\boldsymbol{x}-\boldsymbol{u} \tag{1}$$

is called a *one-parameter planar homothetic motion* and denoted by E/E', where h is a homothetic scale of the motion E/E' and $\boldsymbol{x}$ and $\boldsymbol{x}'$ are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X\in E$, respectively. The homothetic scale h and the vectors $\boldsymbol{x}$, $\boldsymbol{x}'$ and $\boldsymbol{u}$ are continuously differentiable functions of a real parameter t. Furthermore, at the initial time $t=0$

the coordinate systems coincide. Taking $\varphi = \varphi(t)$ as the rotation angle between $\boldsymbol{e}_1$ and $\boldsymbol{e}'_1$, equations

$$\begin{aligned} \boldsymbol{e}_1 &= \cos\varphi \boldsymbol{e}'_1 + \sin\varphi \boldsymbol{e}'_2, \\ \boldsymbol{e}_2 &= -\sin\varphi \boldsymbol{e}'_1 + \cos\varphi \boldsymbol{e}'_2 \end{aligned} \tag{2}$$

can be written. If

$$\begin{aligned} u_j(t+T) &= u_j(t), \quad j = 1, 2, \\ \varphi(t+T) &= \varphi(t) + 2\pi\nu, \quad \forall t \in [0, T] \end{aligned} \tag{3}$$

then the motion E/E' is called a *one-parameter closed planar homothetic motion* with the period $T > 0$ and the rotation number $\nu \in \mathbb{Z}$. During a closed motion, each of the points can pass through several times on its orbit in period $[0, T]$. We call the oriented surface area of the orbit curve multiplied by the number of passing through (Durchlaufzahl) the *orbit surface area.* To avoid the cases of pure translation and pure rotation we assume that

$$\dot{\varphi}(t) = \mathrm{d}\varphi/\mathrm{d}t \neq 0.$$

Under the one-parameter closed planar homothetic motions, if $P = (p_1, p_2)$ is the pole point of the motion at a time t, then the sliding velocity of a fixed point $X = (x_1, x_2) \in E$ with respect to E' is

$$\mathrm{d}\boldsymbol{x}' = \{(x_1 - p_1)\,\mathrm{d}h - (x_2 - p_2)h\,\mathrm{d}\varphi\}\boldsymbol{e}_1 + \{(x_1 - p_1)h\,\mathrm{d}\varphi + (x_2 - p_2)\,\mathrm{d}h\}\boldsymbol{e}_2. \tag{4}$$

Furthermore, the area F_X described by the fixed point X, given by the Gauss area formula [10], is

$$F_X = \frac{1}{2}\oint (x'_1\,\mathrm{d}x'_2 - x'_2\,\mathrm{d}x'_1), \tag{5}$$

where the integration is taken along the closed orbit curve of X. Then we obtain

$$\begin{aligned} 2F_X = (x_1^2 + x_2^2)\int_0^T h^2(t)\,\mathrm{d}\varphi(t) - 2x_1\int_0^T p_1(t)h^2(t)\,\mathrm{d}\varphi(t) \\ - 2x_2\int_0^T p_2(t)h^2(t)\,\mathrm{d}\varphi(t) \end{aligned} \tag{6}$$

$$
\begin{aligned}
&+ \int_0^T \{u_1(t)p_1(t)h(t)\,\mathrm{d}\varphi(t) + u_2(t)p_2(t)h(t)\,\mathrm{d}\varphi(t) + u_1(t)p_2(t)\,\mathrm{d}h(t) \\
&\qquad\qquad - u_2(t)p_1(t)\,\mathrm{d}h(t)\} \\
&+ x_1 \int_0^T \{u_2(t)\,\mathrm{d}h(t) - 2p_2(t)h(t)\,\mathrm{d}h(t) + h(t)\,\mathrm{d}u_2(t)\} \\
&+ x_2 \int_0^T \{-u_1(t)\,\mathrm{d}h(t) + 2p_1(t)h(t)\,\mathrm{d}h(t) - h(t)\,\mathrm{d}u_1(t)\}.
\end{aligned}
$$

Moreover, using the mean value theorem of integral calculus for the closed interval $0 \leqslant t \leqslant T$, there exists at least one point $t_0 \in [0, T]$ such that

$$
\int_0^T h^2(t)\,\mathrm{d}\varphi(t) = \int_0^T h^2(t)\dot{\varphi}(t)\,\mathrm{d}t = 2h^2(t_0)\pi\nu. \tag{7}
$$

By taking $\nu \neq 0$, the Steiner point $S = (s_1, s_2)$ for the closed planar homothetic motion can be written as

$$
s_j = \frac{\int_0^t h^2(t)p_j(t)\,\mathrm{d}\varphi(t)}{\int_0^T h^2(t)\,\mathrm{d}\varphi(t)}, \quad j = 1, 2. \tag{8}
$$

Thus, from Eqs. (6), (7) and (8) we get [3]

$$
F_X = F_O + h^2(t_0)\pi\nu(x_1^2 + x_2^2 - 2x_1s_1 - 2x_2s_2) + \mu_1x_1 + \mu_2x_2, \tag{9}
$$

where F_O is the area of the origin point of the moving coordinate system and

$$
\begin{aligned}
\mu_1 &= \frac{1}{2}\int_0^T \{-2h(t)p_2(t)\,\mathrm{d}h(t) + h(t)\,\mathrm{d}u_2(t) + u_2(t)\,\mathrm{d}h(t)\}, \\
\mu_2 &= \frac{1}{2}\int_0^T \{2h(t)p_1(t)\,\mathrm{d}h(t) - h(t)\,\mathrm{d}u_1(t) - u_1(t)\,\mathrm{d}h(t)\}.
\end{aligned} \tag{10}
$$

Eq. (9) is called the *Steiner area formula* for the one-parameter closed planar homothetic motion E/E'.

2. A generalization of Holditch theorem to two different closed curves

Theorem 1. *Let E_i/E' be one-parameter closed planar homothetic motions with the same orientation, the same number of passing through (Durchlaufzahl) and the rotation numbers ν_i $(i = 1, 2)$. Under closed homothetic motions E_i/E', if the endpoints A_i and B_i of lines $g_i = A_iB_i$ draw orbit curves $k_A, k_B \in E'$, respectively, then the points $X_i \in g_i$ $(\overline{A_iX_i} = \lambda_i a$, $\overline{X_iB_i} = \lambda_i b)$ describe orbit curves k_i with orbit surface areas F_i. The difference $F = F_1 - F_2$ of the orbit surface areas F_i depends on the parameters a, b, λ_i, ν_i, h, while it is independent of k_A and k_B.*

P r o o f. Let A_iB_i $(i = 1, 2)$ have the directions of the real axes of the moving planes E_i. In this case we have $A_i = (0, 0)$, $B_i = (\lambda_i(a + b), 0)$, $X_i = (\lambda_i a, 0)$. Using the generalization of the Holditch Theorem given by Kuruoğlu and Yüce [11], for the orbit surface areas F_{A_i}, F_{B_i}, F_i of points A_i, B_i, X_i we get

$$F_1 = \frac{aF_{B_1} + bF_{A_1}}{a + b} - h^2(t_0)\pi\nu_1\lambda_1^2 ab \tag{11}$$

and

$$F_2 = \frac{aF_{B_2} + bF_{A_2}}{a + b} - h^2(t_0)\pi\nu_2\lambda_2^2 ab. \tag{12}$$

Hence, we obtain

$$F = F_1 - F_2 = \frac{a(F_{B_1} - F_{B_2}) + b(F_{A_1} - F_{A_2})}{a + b} + (\nu_2\lambda_2^2 - \nu_1\lambda_1^2)h^2(t_0)\pi ab. \tag{13}$$

Since the points A_i (B_i) draw the closed curves k_A (k_B) with the same orientation under the homothetic motions E_i/E', we can write $F_{A_1} = F_{A_2}$ $(F_{B_1} = F_{B_2})$. Then we get

$$F = F_1 - F_2 = (\nu_2\lambda_2^2 - \nu_1\lambda_1^2)h^2(t_0)\pi\nu ab. \tag{14}$$

Special case 1: In the special case of the homothetic scale $h \equiv 1$, Eq. (14) yields

$$F = F_1 - F_2 = (\nu_2\lambda_2^2 - \nu_1\lambda_1^2)\pi\nu ab \tag{15}$$

which was given by Pottman [7].

Special case 2: If $k_A = k_B$, $\nu_2 = 1$ and $\lambda_1 = 0$ then Eq. (14) yields the Holditch theorem for the closed planar homothetic motion which was given by Tutar and Kuruoğlu [3].

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