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HYPERBOLIC HEAT CONDUCTION IN TWO SEMI-INFINITE BODIES IN CONTACT*

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Abstract. We find, under the viewpoint of the hyperbolic model of heat conduction, the exact analytical solution for the temperature distribution in all points of two semi-infinite homogeneous isotropic bodies that initially are at uniform temperatures T_0^1 and T_0^2, respectively, suddenly placed together at time $t = 0$ and assuming that the contact between the bodies is perfect. We make graphics of the obtained temperature profiles of two bodies at different times and points. And finally, we compare the temperature solution obtained from hyperbolic model to the parabolic or classical solution, for the same problem of heat conduction.

Keywords: hyperbolic heat conduction, relaxation time

MSC 2000: 80A20

1. Introduction

In many technological applications great amounts of heat are applied to materials in very short times (for instance pulsed-laser processing of metals and semiconductors, film applications, laser surgery, etc.). In such cases the results predicted by the use of the classical parabolic Fourier heat equation are far from the experimental results (see for instance the survey papers [1] and [2]). Then a more accurate heat transfer theory is needed. The easier alternative is to use the modified Fourier law

$$q(x,t) + \tau \frac{\partial q(x,t)}{\partial t} = -k \nabla T(x,t) \tag{1}$$

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which states that heat flux does not begin at the instant t when the temperature gradient is calculated, but at time $t+\tau$, where τ is an assumed constant material characteristic which is called the thermal relaxation time. This assumption gives rise (for homogeneous materials) to the hyperbolic differential heat conduction equation

$$\frac{\partial T}{\partial t}+\tau\frac{\partial^2 T}{\partial t^2}=\alpha\frac{\partial^2 T}{\partial x^2} \tag{2}$$

(see [1]) where $\alpha = k/\varrho c$ is the thermal diffusivity of the medium and we assume the thermal conductivity k, the specific heat c and the density ϱ of the material are constant. The parameter τ is called the relaxation time and represents the time lag for the beginning of heat flow after a gradient temperature has been imposed. We assume τ is constant, too.

Many physical situations have been considered in the perspective of this hyperbolic heat transmission equation (see bibliography in the above quoted articles). In this paper we study the problem of two semi-infinite bodies at uniform temperatures T_0^1 and T_0^2, respectively, assuming that they are suddenly placed together and at the instant $t=0$ they are in perfect contact, this is to say, there is no contact thermal resistance. This question has been considered with some physical variants from the viewpoint of classical Fourier theory, for instance in [3] and in the just described setting by Kazimi and Erdman in [4] with the hyperbolic model. However, they only give an approximate solution for the temperature on the interface of the two bodies valid for very small or very large times. In this paper we provide a complete exact solution for the temperature problem at all points of the bodies involved under the same hypothesis.

2. Analytical development

Consider two semi-infinite isotropic bodies with different but constant physical properties ϱ_i, c_i, k_i and τ_i, where the subscript $i=1,2$ refers to every one of the two bodies which are initially held at two uniform temperatures, T_0^1 and T_0^2, respectively. The bodies are placed together, and at time zero the heat conduction process begins. We assume that there is no thermal contact resistance on the interface. Then, the governing equations for the temperatures of the two bodies are

$$\alpha_i\frac{\partial^2 T_i}{\partial x^2}=\tau_i\frac{\partial^2 T_i}{\partial t^2}+\frac{\partial T_i}{\partial t}, \quad (i=1,2), \tag{3}$$

and the initial and boundary conditions are

$$\forall x < 0 \quad T_1(x,0) = T_0^1, \tag{4}$$

$$\forall x > 0 \quad T_2(x,0) = T_0^2, \tag{5}$$

$$\forall x < 0 \quad \frac{\partial T_1}{\partial t}(x,0) = 0, \tag{6}$$

$$\forall x > 0 \quad \frac{\partial T_2}{\partial t}(x,0) = 0, \tag{7}$$

$$\forall t > 0 \quad q_1(0,t) = q_2(0,t), \tag{8}$$

$$\forall t > 0 \quad T_1(0,t) = T_2(0,t), \tag{9}$$

$$\forall t > 0 \quad T_1(-\infty,t) = \lim_{x\to-\infty} T_1(x,t) = T_0^1, \tag{10}$$

$$\forall t > 0 \quad T_2(\infty,t) = \lim_{x\to\infty} T_2(x,t) = T_0^2. \tag{11}$$

We solve this problem by the Laplace transform with respect to the variable t. From conditions (4), (5), (6) and (7), the Laplace transform of equation (3) takes the form

$$\alpha_i \frac{\partial^2 \widehat{T}_i}{\partial x^2}(x,s) - (\tau_i s^2 + s)\widehat{T}_i(x,s) = -(1+\tau_i s)T_0^i,$$

which is a second order ordinary differential equation with respect to the spatial variable x with a solution

$$\widehat{T}_i(x,s) = A_i \mathrm{e}^{\beta_i x} + B_i \mathrm{e}^{-\beta_i x} + \frac{T_0^i}{s}, \quad \text{where } \beta_i = \sqrt{\frac{s+\tau_i s^2}{\alpha_i}}. \tag{12}$$

By condition (10) we have

$$\lim_{x\to-\infty} \widehat{T}_1(x,s) = \int_0^\infty \mathrm{e}^{-st} \lim_{x\to-\infty} T_1(x,t)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st} T_0^1\,\mathrm{d}t = \frac{T_0^1}{s},$$

and from (12) we obtain

$$\lim_{x\to-\infty} \widehat{T}_1(x,s) = \lim_{x\to-\infty} A_1 \mathrm{e}^{\beta_1 x} + B_1 \mathrm{e}^{-\beta_1 x} + \frac{T_0^1}{s} = \frac{T_0^1}{s}.$$

Since $\beta_i > 0$ we must take $B_1 = 0$. Analogously, using condition (11) we obtain $A_2 = 0$.

To use condition (8), we have to express the flux depending on temperature. It is known (see for instance [1]) that

$$q_i(0,t) = \frac{-k_i}{\tau_i}\,\mathrm{e}^{\frac{-t}{\tau_i}} \int_0^t \mathrm{e}^{\frac{\eta}{\tau_i}} \frac{\partial T_i}{\partial x}(0,\eta)\,\mathrm{d}\eta, \tag{13}$$

and taking the Laplace transform yields

$$\mathcal{L}\left[\mathrm{e}^{\frac{-t}{\tau_i}} \int_0^t \mathrm{e}^{\frac{\eta}{\tau_i}} \frac{\partial T_i}{\partial x}(0,\eta)\,\mathrm{d}\eta\right] = \frac{\tau_i}{\tau_i s+1}\frac{\partial \widehat{T}_i}{\partial x}(0,s).$$

Now by condition (8)

$$\frac{-k_1}{\tau_1}\frac{\tau_1}{(\tau_1 s+1)}\frac{\partial \widehat{T}_1}{\partial x}(0,s) = \frac{-k_2}{\tau_2}\frac{\tau_2}{(\tau_2 s+1)}\frac{\partial \widehat{T}_2}{\partial x}(0,s).$$

Computing $\frac{\partial \widehat{T}_1}{\partial x}(0,s)$ and $\frac{\partial \widehat{T}_2}{\partial x}(0,s)$ from equation (12) we get

$$\frac{-k_1}{\sqrt{\alpha_1}}\,A_1\frac{1}{\sqrt{\tau_1 s+1}} = \frac{k_2}{\sqrt{\alpha_2}}\,B_2\frac{1}{\sqrt{\tau_2 s+1}}. \tag{14}$$

Finally, by condition (9) we obtain

$$\widehat{T}_1(0,s) = \widehat{T}_2(0,s) \tag{15}$$

and hence

$$\widehat{T}_1(x,s) = \frac{(T_0^1 - T_0^2)}{s\left(\frac{-k_1\sqrt{\alpha_2}\sqrt{\tau_2 s+1}}{k_2\sqrt{\alpha_1}\sqrt{\tau_1 s+1}} - 1\right)}\mathrm{e}^{\sqrt{\frac{s+\tau_1 s^2}{\alpha_1}}x} + \frac{T_0^1}{s}, \tag{16}$$

$$\widehat{T}_2(x,s) = \frac{T_0^2}{s} + (T_0^1 - T_0^2)\frac{\sqrt{\tau_2 s+1}}{s\left(\sqrt{\tau_2 s+1} + \frac{k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2}}\sqrt{\tau_1 s+1}\right)}\mathrm{e}^{-x\sqrt{\frac{s+\tau_2 s^2}{\alpha_2}}}. \tag{17}$$

The hard problem is to find the inverse Laplace transform of these functions. We start with $\widehat{T}_2(x,s)$. Put

$$g_1(s) := \frac{\sqrt{\tau_2 s+1}}{s\left(\sqrt{\tau_2 s+1} + \frac{k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2}}\sqrt{\tau_1 s+1}\right)}$$

and

$$g_2(s) := \mathrm{e}^{-x\sqrt{s+\tau_2 s^2/\alpha_2}}.$$

Then by the convolution theorem

$$\mathcal{L}^{-1}[g(s)] = \int_0^t G_1(t-u)G_2(u)\,\mathrm{d}u,$$

where $G_1(u)$ and $G_2(u)$ are the Laplace inverses of $g_1(s)$ and $g_2(s)$, respectively. We start by finding $\mathcal{L}^{-1}[g_2(s)]$. By a well known property of the inverse Laplace transform (see for instance [5, p. 507])

$$\mathcal{L}^{-1}[g_2(s)] = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}^{-1}\Big[\frac{g_2(s)}{s}\Big]. \tag{18}$$

The inverse of $\frac{g_2(s)}{s}$ has been calculated in [6] as

$$\begin{aligned}\mathcal{L}^{-1}&\Big[\frac{g_2(s)}{s}\Big](t)\\ &= H\Big(t-\frac{x}{v_2}\Big)\left(\mathrm{e}^{-\frac{x}{2\sqrt{\alpha_2\tau_2}}} + \frac{xv_2}{4\alpha_2\tau_2}\int_{\frac{x}{v_2}}^{t}\mathrm{e}^{-\frac{u}{2\tau_2}}\frac{I_1\Big(\sqrt{\big(\frac{u}{2\tau_2}\big)^2-\frac{x^2}{4\alpha_2\tau_2}}\Big)}{\sqrt{\big(\frac{u}{2\tau_2}\big)^2-\frac{x^2}{4\alpha_2\tau_2}}}\,\mathrm{d}u\right),\end{aligned}$$

where $H(u)$ is the Heaviside function and

$$v_i := \sqrt{\frac{\alpha_i}{\tau_i}}, \quad i=1,2$$

is the speed of thermal propagation in every body. Hence by (18)

$$\begin{aligned}G_2(t) &:= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{L}^{-1}\Big[\frac{g_2(s)}{s}\Big](t)\Big)\\ &= \delta\Big(t-\frac{x}{v_2}\Big)\left(\mathrm{e}^{-\frac{x}{2\sqrt{\alpha_2\tau_2}}} + \frac{v_2x}{4\alpha_2\tau_2}\int_{\frac{x}{v_2}}^{t}\mathrm{e}^{\frac{-u}{2\tau_2}}\frac{I_1\Big(\sqrt{\big(\frac{u}{2\tau_2}\big)^2-\frac{x^2}{4\alpha_2\tau_2}}\Big)}{\sqrt{\big(\frac{u}{2\tau_2}\big)^2-\frac{x^2}{4\alpha_2\tau_2}}}\,\mathrm{d}u\right)\\ &\qquad + H\Big(t-\frac{x}{v_2}\Big)\frac{v_2x}{4\alpha_2\tau_2}\mathrm{e}^{\frac{-t}{2\tau_2}}\frac{I_1\Big(\sqrt{\big(\frac{t}{2\tau_2}\big)^2-\frac{x^2}{4\alpha_2\tau_2}}\Big)}{\sqrt{\big(\frac{t}{2\tau_2}\big)^2-\frac{x^2}{4\alpha_2\tau_2}}}\end{aligned}$$

where $\delta(u)$ is Dirac's δ distribution.

In order to obtain $\mathcal{L}^{-1}[g_1(s)]$ we want to use Bromwich's formula. To get a convergent integral in this formula we need to consider

$$\frac{g_1(s)}{s} = \frac{\sqrt{\tau_2 s+1}}{s^2\Big(\sqrt{\tau_2 s+1}+\frac{k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2}}\sqrt{\tau_1 s+1}\Big)}.$$

Then we use (18) again replacing $g_2(s)$ by $g_1(s)$. By Bromwich's formula

$$\mathcal{L}^{-1}\Big[\frac{g_1(s)}{s}\Big](t) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{e}^{st}\frac{\sqrt{\tau_2 s+1}}{s^2\Big(\sqrt{\tau_2 s+1}+\frac{k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2}}\sqrt{\tau_1 s+1}\Big)}\,\mathrm{d}s \tag{19}$$

where we take any $\gamma > 0$. From now on we shall always assume that $\tau_1 \leqslant \tau_2$. Bearing in mind that $s=-1/\tau_1$ and $s=-1/\tau_2$ are branch points we use the contour of Fig. 1.

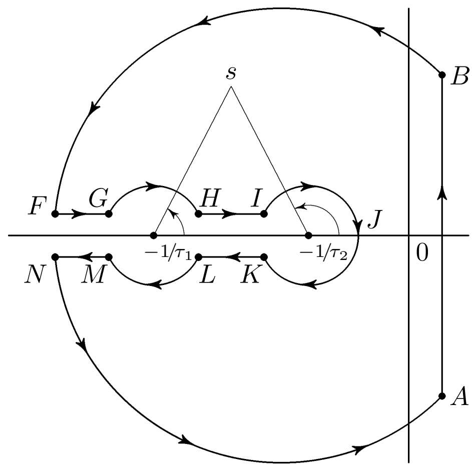

Figure 1. Bromwich contour.

Remark that there is only a (double) pole at $s = 0$ within the above contour. In fact, another possible pole will be located at the real number

$$s = \frac{\alpha_1 k_2^2 - \alpha_2 k_1^2}{\tau_2 \alpha_2 k_1^2 - \tau_1 \alpha_1 k_2^2}$$

but a careful verification shows that, under the selected branch of every square root, it is a strange value for which the denominator of (19) vanishes. To simplify, define

$$B = \frac{k_2 \sqrt{\alpha_1 \tau_1}}{k_1 \sqrt{\alpha_2 \tau_2}}. \tag{20}$$

Then the residue at $s = 0$ is

$$\begin{aligned} R(0) &= \lim_{s \to 0} \frac{\mathrm{d}}{\mathrm{d}s} \left(s^2 \mathrm{e}^{st} \frac{\sqrt{\tau_2 s + 1}}{s^2 \left(\sqrt{\tau_2 s + 1} + \frac{k_2 \sqrt{\alpha_1}}{k_1 \sqrt{\alpha_2}} \sqrt{\tau_1 s + 1} \right)} \right) \\ &= \frac{t}{1 + B\sqrt{\frac{\tau_2}{\tau_1}}} + \frac{B\left(\sqrt{\frac{\tau_2}{\tau_1}} - \sqrt{\frac{\tau_1}{\tau_2}}\right)}{2\left(\sqrt{\frac{1}{\tau_1}} + B\sqrt{\frac{1}{\tau_2}}\right)^2}. \end{aligned} \tag{21}$$

Hence, by (19),

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{g_1(s)}{s}\right] &= \frac{t}{1 + B\sqrt{\frac{\tau_2}{\tau_1}}} + \frac{B\left(\sqrt{\frac{\tau_2}{\tau_1}} - \sqrt{\frac{\tau_1}{\tau_2}}\right)}{2\left(\sqrt{\frac{1}{\tau_1}} + B\sqrt{\frac{1}{\tau_2}}\right)^2} \\ &\quad - \frac{1}{\pi} \int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{\mathrm{e}^{yt}}{y^2} \frac{B\sqrt{-y - \frac{1}{\tau_2}}\sqrt{y + \frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}} \, \mathrm{d}y. \end{aligned}$$

According to (18), $\mathcal{L}^{-1}[g_1(s)]$ is obtained by differentiating

$$G_1(t) = \frac{1}{1+B\sqrt{\frac{\tau_2}{\tau_1}}} - \frac{1}{\pi}\int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{e^{yt}}{y} \frac{B\sqrt{-y-\frac{1}{\tau_2}}\sqrt{y+\frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}}\,dy.$$

Finally, $\mathcal{L}^{-1}[g(s)]$ is obtained by the convolution theorem and hence, after simplifications, for the temperature in body 2 we get

$$\begin{aligned}
(22)\quad T_2(x,t) = T_0^2 + (T_0^1 - T_0^2)\Bigg[& H\Big(t-\frac{x}{v_2}\Big) e^{-\frac{x}{2\sqrt{\alpha_2\tau_2}}} \\
& \times \left(\frac{1}{1+B\sqrt{\frac{\tau_2}{\tau_1}}} - \frac{1}{\pi}\int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{e^{y\left(t-\frac{x}{v_2}\right)}}{y} \frac{B\sqrt{-y-\frac{1}{\tau_2}}\sqrt{y+\frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}}\,dy \right) \\
& + H\Big(t-\frac{x}{v_2}\Big) \int_{\frac{x}{v_2}}^{t} \frac{x v_2}{4\alpha_2\tau_2}\, e^{\frac{-q}{2\tau_2}}\, \frac{I_1\left(\sqrt{\left(\frac{q}{2\tau_2}\right)^2 - \frac{x^2}{4\alpha_2\tau_2}}\right)}{\sqrt{\left(\frac{q}{2\tau_2}\right)^2 - \frac{x^2}{4\alpha_2\tau_2}}} \\
& \times \left(\frac{1}{1+B\sqrt{\frac{\tau_2}{\tau_1}}} - \frac{1}{\pi}\int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{e^{y(t-q)}}{y} \frac{B\sqrt{-y-\frac{1}{\tau_2}}\sqrt{y+\frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}}\,dy \right) dq \Bigg] \\
& \forall\, x \geqslant 0, \ \ \forall\, t \geqslant 0.
\end{aligned}$$

A similar analysis is applied to the body 1 to obtain the temperature profile. We arrive at

$$\begin{aligned}
(23)\quad T_1(x,t) = T_0^1 + (T_0^2 - T_0^1)\Bigg[& H\Big(t+\frac{x}{v_1}\Big) e^{\frac{x}{2\sqrt{\alpha_1\tau_1}}} \\
& \times \left(\frac{B}{B+\sqrt{\frac{\tau_1}{\tau_2}}} + \frac{1}{\pi}\int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{e^{y\left(t+\frac{x}{v_1}\right)}}{y} \frac{B\sqrt{-y-\frac{1}{\tau_2}}\sqrt{y+\frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}}\,dy \right) \\
& - H\Big(t+\frac{x}{v_1}\Big) \int_{\frac{-x}{v_1}}^{t} \frac{x v_1}{4\alpha_1\tau_1} e^{\frac{-q}{2\tau_1}}\, \frac{I_1\left(\sqrt{\left(\frac{q}{2\tau_1}\right)^2 - \frac{x^2}{4\alpha_1\tau_1}}\right)}{\sqrt{\left(\frac{q}{2\tau_1}\right)^2 - \frac{x^2}{4\alpha_1\tau_1}}} \\
& \times \left(\frac{B}{B+\sqrt{\frac{\tau_1}{\tau_2}}} + \frac{1}{\pi}\int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{e^{y(t-q)}}{y} \frac{B\sqrt{-y-\frac{1}{\tau_2}}\sqrt{y+\frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}}\,dy \right) dq \Bigg] \\
& \forall\, x \leqslant 0, \ \ \forall\, t \geqslant 0,
\end{aligned}$$

which solves completely our problem. The integrals in equations (23) and (22) cannot be expressed in terms of tabulated functions and must therefore be evaluated numerically.

2.1. Discussion: The interface temperature

To illustrate the following general theoretical discussion and make graphics of the temperature profiles in a concrete case, we choose uranium dioxide UO_2 for body 1 and liquid sodium Na for body 2, since they offer a good model to study a hypothetical accident condition in nuclear reactors (see [7]). The values of physical parameters used are taken from this paper.

- UO_2: $\alpha_1 = 4.89 \cdot 10^{-7}\,\frac{\mathrm{m}^2}{\mathrm{s}}$, $k_1 = 0.5\,\frac{\mathrm{cal}}{\mathrm{m}\,{}^\circ\mathrm{C}}$, $T_0^1 = 3000\,{}^\circ\mathrm{C}$, $\tau_1 = 1.69 \cdot 10^{-13}\,\mathrm{s}$
- Na: $\alpha_2 = 3.55 \cdot 10^{-5}\,\frac{\mathrm{m}^2}{\mathrm{s}}$, $k_2 = 9.15\,\frac{\mathrm{cal}}{\mathrm{m}\,{}^\circ\mathrm{C}}$, $T_0^2 = 800\,{}^\circ\mathrm{C}$, $\tau_2 = 6.72 \cdot 10^{-12}\,\mathrm{s}$.

All graphics have been made with version 4 of program *Mathematica*.

In [4] and [7] approximations of the interface temperature valid for very short or very large times are given. However, putting $x = 0$ in (22) we obtain the exact value of the interface temperature:

$$
\begin{aligned}
(24)\quad T(t) &= T_0^2 + (T_0^1 - T_0^2) \\
&\quad \times \left(\frac{1}{1 + B\sqrt{\frac{\tau_2}{\tau_1}}} - \frac{1}{\pi} \int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{\mathrm{e}^{yt}}{y} \, \frac{B\sqrt{-y - \frac{1}{\tau_2}}\sqrt{y + \frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}} \, \mathrm{d}y \right) \qquad \forall\, t > 0
\end{aligned}
$$

which is readily seen to be a monotone decreasing function. By the initial and final value theorems on Laplace transforms and (17) we get

$$
\begin{aligned}
(25)\qquad \lim_{t \to 0} T(t) &= \lim_{s \to \infty} s\mathcal{L}^{-1}[T_2(0,t)](s) \\
&= T_0^2 + (T_0^1 - T_0^2) \frac{k_1\sqrt{\alpha_2 \tau_2}}{k_1\sqrt{\alpha_2\tau_2} + k_2\sqrt{\alpha_1\tau_1}}
\end{aligned}
$$

for the initial temperature on the interface and

$$
\begin{aligned}
(26)\qquad \lim_{t \to \infty} T(t) &= \lim_{s \to 0} s\mathcal{L}^{-1}[T_2(0,t)](s) \\
&= T_0^2 + (T_0^1 - T_0^2) \frac{k_1\sqrt{\alpha_2}}{k_1\sqrt{\alpha_2} + k_2\sqrt{\alpha_1}}
\end{aligned}
$$

for the limit temperature on the interface, which turns out to be independent of the relaxation parameters τ_1 and τ_2.

Three main conclusions can be obtained from these formulae. The first is not unexpected from the setting (9) and gives us an instantaneous variation of the temperature on $t = 0$, i.e. condition (9) implies the temperatures $T_1(x,t)$ and $T_2(x,t)$

to be discontinuous functions at the point $x = 0$, $t = 0$. The second is that, if the relaxation time parameters τ_1 and τ_2 are equal, we have

$$T(t) = T_0^2 + (T_0^1 - T_0^2)\,\frac{k_1\sqrt{\alpha_2}}{k_1\sqrt{\alpha_2} + k_2\sqrt{\alpha_1}} \tag{27}$$

since $T(t)$ is decreasing and (25) and (26) give now the same value, i.e. when $\tau_1 = \tau_2$ then the interface temperature is constant. It can be seen that this value is the same as that predicted for the instantaneous interface temperature by the classical Fourier model (see [8] and [3]).

Another consequence is the following: If the two bodies are made of different materials, in spite of the relaxation parameters τ_1 and τ_2 being close, the approximation $\tau_1 \approx \tau_2$ is unacceptable because the integral in (24) can be extended over a large interval and differences between the constant value given by (27) and the actual value (24) of the interface temperature can be considerable. This is clear in Fig. 2 where $T(t)$ is plotted in the general case and the constant value is obtained assuming $\tau_1 = \tau_2$.

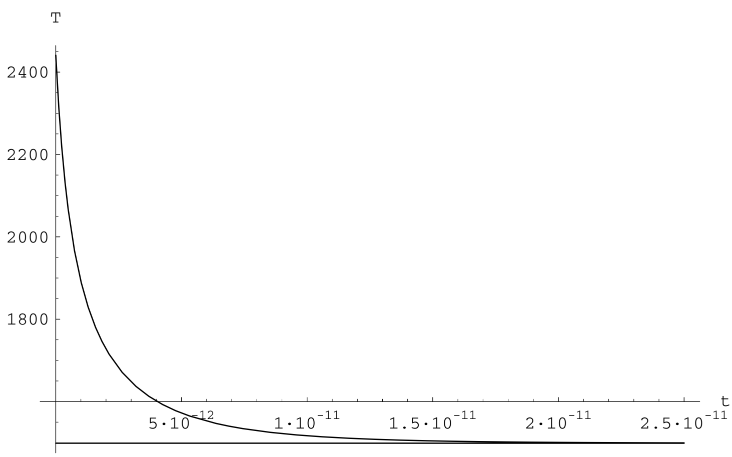

Figure 2. Comparison between interface temperature in cases $\tau_1 \neq \tau_2$ and $\tau_1 = \tau_2$.

According to (25) and (26) the initial interface temperature is 2441.05 °C and the limit interface temperature is 1498.9 °C. However, assuming $\tau_1 = \tau_2$ (27) gives a constant interface temperature of 1498.9 °C. In spite of the very small time elapsed to get practically the limit value of the interface temperature, the great difference in temperatures between the two different assumptions may have dramatic consequences.

It is also important to compare the exact formula (24) with the approximate formula given by Kazimi and Erdmann in [4] for small times. Fig. 3 shows the exact interface temperature (continuous line), the approximate interface temperature of Kazimi and Erdmann (dashed line) and the constant interface temperature taking $\tau_1 = \tau_2$. It is clear that for times in the interval from $5 \cdot 10^{-12}$ s to 10^{-11} s the above approximation produces considerable errors.

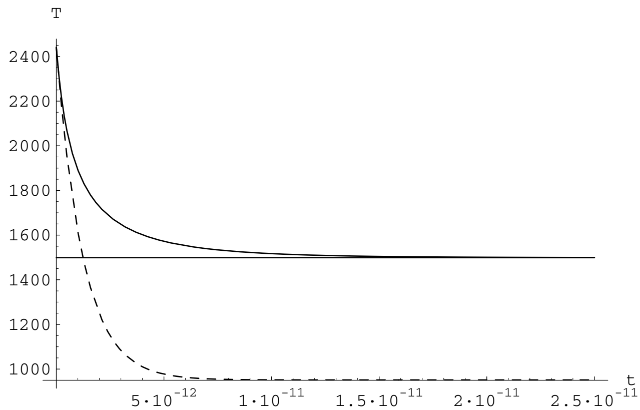

Figure 3. Interface temperature: exact (continuous line) and approximate (dashed line) for $\tau_1 \neq \tau_2$ and in the case $\tau_1 = \tau_2$ (horizontal line).

2.2. Discussion: The general temperature

The presence of the Heaviside function in (23) and (22) shows the existence of a discontinuity in temperatures $T_1(x,t)$ and $T_2(x,t)$ along the right line $x = -v_1 t$ and $x = v_2 t$, respectively, due to the ondulatory character of the hyperbolic heat equation produced by relaxation times τ_1 and τ_2. However, this effect is appreciated only for small times and abscisa values due to the small absolute values of the slopes $-v_1$ and v_2. We see in Fig. 4 the global aspect of spatial temporal evolution of temperatures in both bodies.

For a better appreciation of the discontinuity effect of the Heaviside function in body 1, see Fig. 5.

At every point x of body 1 at the instant $t = -x/v_1$, there is a jump in the temperature profile of magnitude

$$S_1(x) = (T_0^2 - T_0^1)\mathrm{e}^{\frac{x}{2\sqrt{\alpha_1 \tau_1}}} \left(\frac{B}{B + \sqrt{\frac{\tau_1}{\tau_2}}} + \frac{1}{\pi} \int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{B}{y} \frac{\sqrt{-y - \frac{1}{\tau_2}}\sqrt{y + \frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}}\, \mathrm{d}y \right)$$

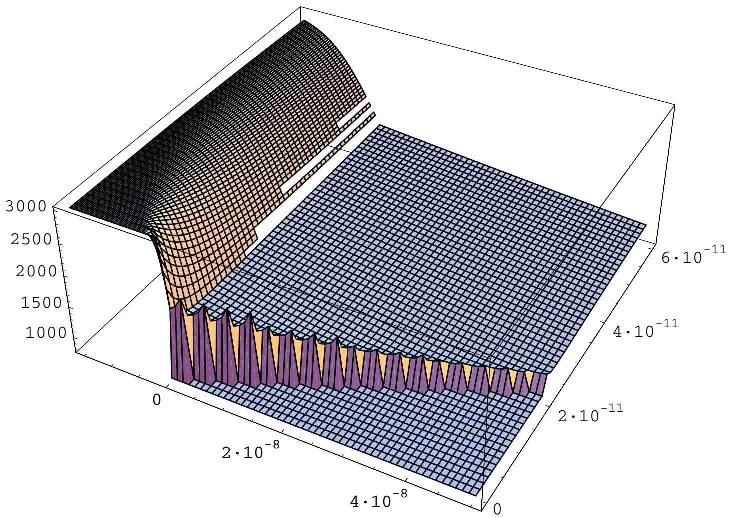

Figure 4. Plot of temperatures $T_1(x,t)$ and $T_2(x,t), \tau_1 \neq \tau_2$.

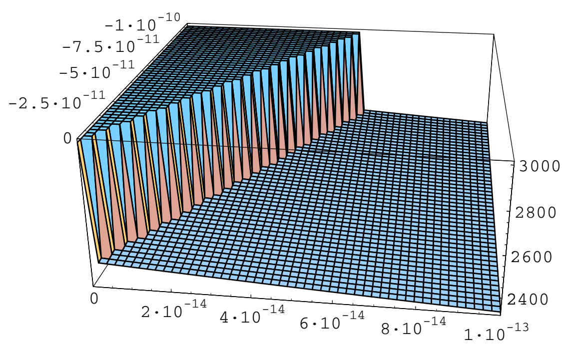

Figure 5. Temperature in body 1, $\tau_1 \neq \tau_2$, $x \in [-10^{10}, 0]$, $t \in [0, 10^{-13}]$.

and at every point x of body 2 at the instant $t = x/v_2$, there is a jump in the temperature profile of magnitude

$$S_2(x) = (T_0^2 - T_1^0)\mathrm{e}^{-\frac{x}{2\sqrt{\alpha_2\tau_2}}} \times \left(\frac{1}{1 + B\sqrt{\frac{\tau_2}{\tau_1}}} - \frac{1}{\pi} \int_{-\frac{1}{\tau_1}}^{-\frac{1}{\tau_2}} \frac{B}{y} \frac{\sqrt{-y - \frac{1}{\tau_2}}\sqrt{y + \frac{1}{\tau_1}}}{B^2 y + \frac{B^2}{\tau_1} - y - \frac{1}{\tau_2}} \, \mathrm{d}y \right),$$

which clearly decreases quickly when x increases.

A closer look reveals a surprising fact. It is remarkable that, for every fixed time, the temperature along body 2 is not monotone decreasing in the spatial interval $[0, v_2 t]$. Fig. 6 shows this phenomenon with the effect of the Heaviside function. We have plotted the temperature $T_2(x,t)$ in the spatial interval $[0, 5 \cdot 10^{-8}]$ at fixed instants $t = 0.5 \cdot 10^{-11}$ s, 10^{-11} s and $t = 1.5 \cdot 10^{-11}$ s.

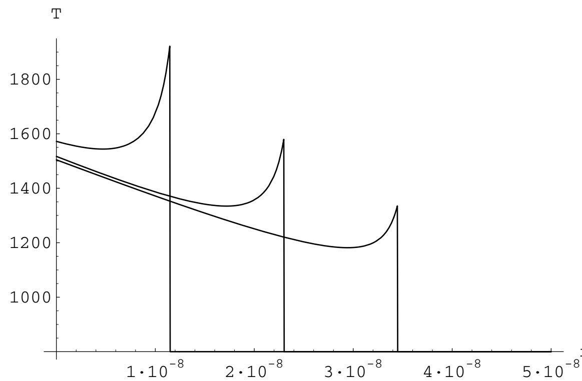

Figure 6. Temperature in body 2 at times $t = 0.5 \cdot 10^{-11}$ s, $t = 10^{-11}$ s, $t = 1.5 \cdot 10^{-11}$ s if $\tau_1 \neq \tau_2$.

This pattern in the temperature evolution is due to the different speed of heat transmission in the two materials. In fact, if the relaxation parameter is the same in both bodies, temperature becomes a decreasing function of x for every fixed t. This is clear from (22) putting $\tau_1 = \tau_2$. In Fig. 7 we present a sketch of the situation choosing $\tau_2 = 1.69 \cdot 10^{-13}$ s (the relaxation time of body 1) at times $t = 0.2 \cdot 10^{-12}$ s, $t = 0.5 \cdot 10^{-12}$ s and $t = 10^{-12}$ s, which shows the relevant influence of non equal relaxation times.

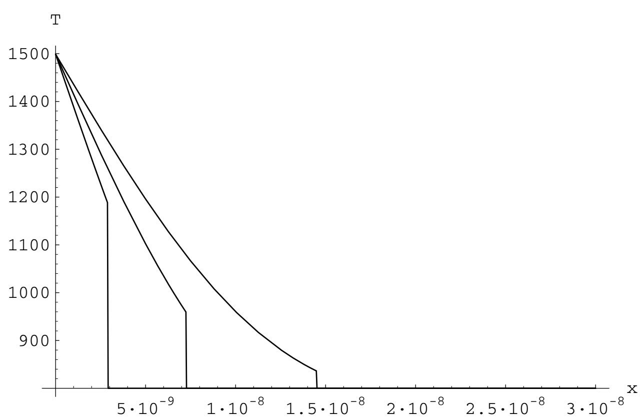

Figure 7. Temperature in body 2 at times $t = 0.2 \cdot 10^{-12}$ s, $t = 0.5 \cdot 10^{-12}$ s, $t = 10^{-12}$ s when $\tau_1 = \tau_2$.

Analogously, the temporal evolution of the temperature at fixed points in body 2 is illustrated in Fig. 8 if $\tau_1 \neq \tau_2$ and in Fig. 9 if $\tau_1 = \tau_2$. The same is done for body 1 in Fig. 10 in the case $\tau_1 \neq \tau_2$. We omit a figure in the case $\tau_1 = \tau_2$ because it is similar.

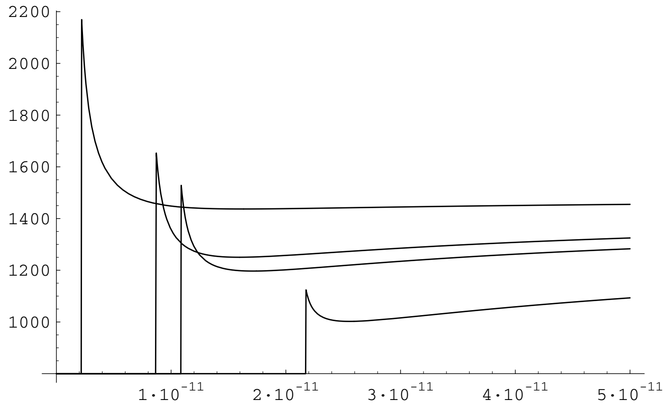

Figure 8. Evolution of temperature at points $x = 10^{-8}$ m, $x = 2.5 \cdot 10^{-8}$ m and $x = 5 \cdot 10^{-8}$ m if $\tau_1 \neq \tau_2$.

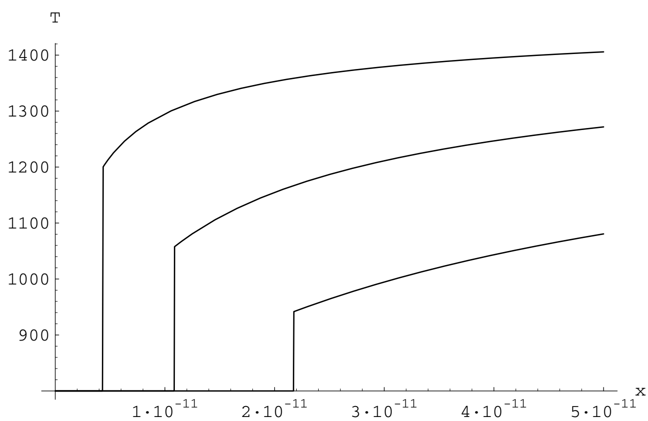

Figure 9. Evolution of temperature in points $x = 10^{-8}$ m, $x = 2.5{\cdot}10^{-8}$ m and $x = 5{\cdot}10^{-8}$ m if $\tau_1 = \tau_2$.

2.3. Comparison with parabolic model

Finally, we make a brief comparison with the classical solution by the Fourier method. The solution of our problem from the point of view of the parabolic model is known since the first half of the 20th century and can be found in the modern

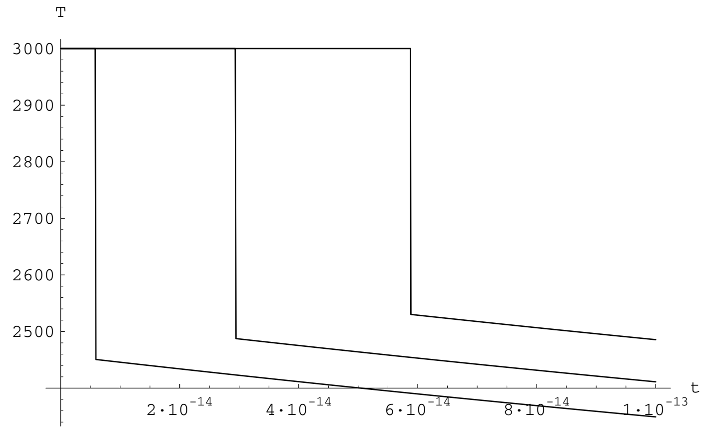

Figure 10. Evolution of temperature at points $x = -10^{-11}$ m, $x = -5 \cdot 10^{-11}$ m and $x = -10^{-10}$ m if $\tau_1 \neq \tau_2$.

edition [8] of the classical book of Carslaw and Hager. In this case the temperature profiles are

$$T_1(x,t) = T_0^1 + \frac{(T_0^2 - T_0^1)k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2} + k_2\sqrt{\alpha_1}}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{-x}{2\sqrt{t\alpha_1}}} \mathrm{e}^{-u^2}\,\mathrm{d}u\right), \tag{28}$$

$$T_2(x,t) = T_0^2 + \frac{(T_0^1 - T_0^2)k_1\sqrt{\alpha_2}}{k_1\sqrt{\alpha_2} + k_2\sqrt{\alpha_1}}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{t\alpha_2}}} \mathrm{e}^{-u^2}\,\mathrm{d}u\right). \tag{29}$$

The parabolic interface temperature is constant with respect to time with value

$$\frac{T_0^1 k_1\sqrt{\alpha_2} + T_0^2 k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2} + k_2\sqrt{\alpha_1}}$$

which coincides with the limit hyperbolic interface temperature (26). Fig. 12 shows the interface temperature in the parabolic (dashed line) and hyperbolic model (continuous line). We can see that for this concrete case, the necessary time, from the point of view of the hyperbolic model, to reach at the interface the equilibrium temperature is more or less $2 \cdot 10^{-11}$ s. However, in the parabolic model the equilibrium temperature at interface is reached as soon as the process of heat conduction begins. In these processes in which great amounts of heat are applied to materials in very short times, times of nano and picoseconds are important, therefore, the parabolic model gives rise to very important errors because of the great difference of temperatures at these times.

Parabolic solutions (28) and (29) show that for all $t > 0$, although t is very small, and for all $x > 0$, although x is very big, temperature is always different

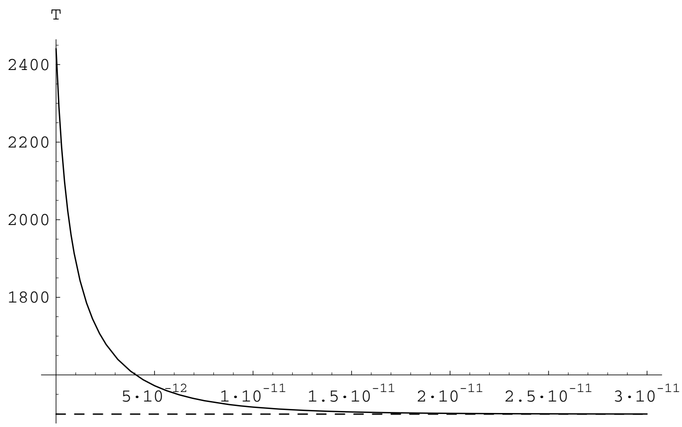

Figure 11. Parabolic (dashed line) and hyperbolic (continuous line) temperature at interface.

from the initial one. Initially, all points are at initial temperature and if a change in temperature is due to a contribution of heat, then the speed of heat propagation is infinite. This fact is showed in Fig. 11. In the parabolic model the temperature is always different from the initial one (although this change amounts only to two or three degrees). We can also see in Fig. 11 that the hyperbolic temperature is divided in two zones due to the Heaviside function: a zone where the body is at the initial temperature because heat has not yet arrived, and another zone where heat has arrived and the temperature has changed; this fact shows that in this case the speed of the heat conduction is finite.

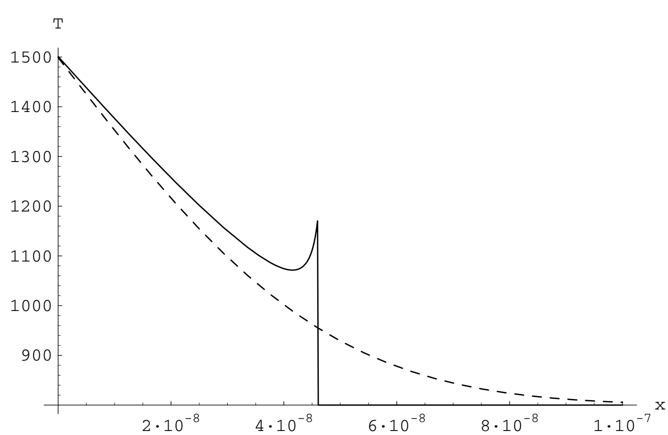

Figure 12. Parabolic (dashed line) and hyperbolic (continuous line) temperature at time $2 \cdot 10^{-11}$ s in body 2.

3. Conclusions

We have considered the distribution of temperatures of two semi-infinite bodies with constant initial temperature placed together at time $t = 0$, assuming perfect thermal contact from the viewpoint of the hyperbolic equation of heat transmission. We have obtained an exact analytical solution for the temperature at all points and times for every body. If the materials in contact are different, it is necessary to work with different relaxation times τ_1, τ_2 since the approximation $\tau_1 \approx \tau_2$ gives values of temperatures very different from the correct situation $\tau_1 \neq \tau_2$ at the initial moments, precisely when the hyperbolic theory of heat must replace the ordinary Fourier theory. Moreover, with the exact solution at our disposal, we see that the previous approximate solutions for the interface temperature give significant errors and must be disregarded. Future research work must consider the more realistic case of the existence of thermal contact resistance on the interface and the introduction of a model which would avoid the instantaneous change of temperature on the interface at $t = 0$.

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