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MODELING, MATHEMATICAL AND NUMERICAL ANALYSIS OF ELECTRORHEOLOGICAL FLUIDS*

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Abstract. Many electrorheological fluids are suspensions consisting of solid particles and a carrier oil. If such a suspension is exposed to a strong electric field the effective viscosity increases dramatically. In this paper we first derive a model which captures this behaviour. For the resulting system of equations we then prove local in time existence of strong solutions for large data. For these solutions we finally derive error estimates for a fully implicit time-discretization.

Keywords: Maxwell's equations, electrorheological fluids, constitutive relations, Galerkin approximation

MSC 2000: 35Q35, 76W05, 65M60, 65M15

0. INTRODUCTION

Many *electrorheological fluids* (abbreviated: ERFs) are *suspensions* consisting of particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field. The observed *increase* of the measured shear stresses (or the measured viscosity) is essentially due to the existence of particle structures forming in the presence of an electric field hindering the flow and resulting in a higher, apparent viscosity. For an overview especially of microscopic models and explanations in electrorheology we refer the reader to Parthasarathy/Klingenberg [36].

In the first section we develop a model which captures the above described features. There are many ways to model ERFs and we refer the reader to the discussion in [39], [43], [19]. Here we model the ERF in a homogenized sense within the framework of continuum mechanics and follow the procedure from Rajagopal/Růžička [39], (cf. [44], [19]). In particular we take into account the complex interaction of the

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electro-magnetic fields and the moving liquid, thus treating the electric field as a variable that is determined by Maxwell's equations. The final system describing the motion of ERFs is derived from the general balance laws of thermodynamics and electrodynamics by a non-dimensionalization and a subsequent approximation.

In the second section we show the existence of strong solutions for the mechanical part of the system describing the flow of ERFs, i.e. the balance of mass and momentum. The constitutive relation for the extra stress tensor implies that the system possesses p-structure, where however $p = p(|\mathbf{E}|^2)$ is a *material function* and not a constant. Thus the natural functional setting are *generalized Lebesgue and Sobolev spaces*. The basic properties of these spaces can be found in Kováčik/Rákosník [28] (cf. Diening [10], [11], Diening/Růžička [15], [16], [17] for more recent results and the web-page [40] for up-to-date information). The method presented here is based on ideas developed in [31], [32], [6], [30], [33] (cf. [21], [13] for an overview of recent results for generalized Newtonian fluids) to handle situations when the elliptic operator is monotone, but due to the properties of the convective term the *theory of monotone operators* is not applicable. Our presentation follows the treatment in Diening [12], Diening/Růžička [14].

In the third section we prove error estimates for the difference between a *strong* solution of the continuous system and a *weak* solution of the fully implicit time-discretization of this system under the additional assumption that $p = \text{const}$. In contrast to the mathematical analysis there are only few numerical results for such a system (cf. [5], [4], [37], [13]). Here we generalize the treatment of Diening/Prohl/Růžička [13] to the case that the extra stress tensor is not derived from a potential.

1. Modeling

We start by stating Maxwell's equations. Here we use the so-called "statistical formulation", which is based on a "dipole-current-loop" model (cf. Eringen/Maugin [20], Hutter/van de Ven [27], Grot [25], Pao [35]):

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.1}$$

$$\operatorname{curl}\mathbf{H} = \frac{\partial \mathbf{D}^e}{\partial t} + \mathbf{J}, \tag{1.2}$$

$$\operatorname{div}\mathbf{D}^e = q^e, \tag{1.3}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{1.4}$$

where $\mathbf{E}$ is the *electric field*, $\mathbf{B}$ the *magnetic flux density*, $\mathbf{H}$ is the *magnetic field* given by $\mathbf{H} = {\mu_0}^{-1}\mathbf{B} - \mathbf{M}$ with the *magnetization* $\mathbf{M}$, $\mathbf{D}^e$ is the *dielectric displacement* given by $\mathbf{D}^e = \mathbf{P} + \varepsilon_0 \mathbf{E}$ with the *electric polarization* $\mathbf{P}$, $\mathbf{J}$ the *current density*, q^e the *density of the free electric charges* and ε_0 and μ_0 denote the dielectric constant and the permeability in vacuo, respectively.

Now we state the thermo-mechanical balance laws. The balance of mass and momentum are[1]

$$\dot{\varrho} + \varrho \operatorname{div} \mathbf{v} = 0, \tag{1.5}$$

$$\varrho \dot{\mathbf{v}} - \operatorname{div} \mathbf{T} = \mathbf{f} + \mathbf{f}^e, \tag{1.6}$$

respectively, where ϱ is the *mass density*, $\mathbf{T}$ the *Cauchy stress tensor*[2], $\mathbf{f}$ the *mechanical force density* and $\mathbf{f}^e$ is the *electro-magnetic force density* which is given by (cf. pages 284–285 of [35])[3]

$$\mathbf{f}^e = \mathbf{q}^e \boldsymbol{\mathcal{E}} + [\boldsymbol{\mathcal{J}} + \dot{\mathbf{P}} - [\nabla \mathbf{v}]\mathbf{P} + (\operatorname{div} \mathbf{v})\mathbf{P}] \times \mathbf{B} + [\nabla \mathbf{B}]^\top \boldsymbol{\mathcal{M}} + [\nabla \boldsymbol{\mathcal{E}}]\mathbf{P} \tag{1.7}$$

where $\boldsymbol{\mathcal{E}}$ is the *effective electric field strength* defined as

$$\boldsymbol{\mathcal{E}} = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{1.8}$$

$\boldsymbol{\mathcal{J}}$ the *conductive current density* given by

$$\boldsymbol{\mathcal{J}} = \mathbf{J} - \mathbf{q}^e \mathbf{v} \tag{1.9}$$

and $\boldsymbol{\mathcal{M}}$ the *effective magnetization* defined through

$$\boldsymbol{\mathcal{M}} = \mathbf{M} + \mathbf{v} \times \mathbf{P}. \tag{1.10}$$

The balance of angular momentum takes the form

$$\mathbf{x} \times \varrho \dot{\mathbf{v}} - \operatorname{div}(\mathbf{x} \times \mathbf{T}) = \mathbf{x} \times \mathbf{f} + \mathbf{l}^e, \tag{1.11}$$

in which $\mathbf{l}^e$ denotes the electro-magnetic torque density (cf. p. 284–285 of [35]) given by

$$\mathbf{l}^e = \mathbf{x} \times \mathbf{f}^e + \mathbf{P} \times \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{M}} \times \mathbf{B}. \tag{1.12}$$

[1] The material time derivative is denoted by a superposed dot or by $\mathrm{d}/\mathrm{d}t$.

[2] $\mathbf{T}$ is introduced via $\mathbf{t} = \mathbf{T} \cdot \mathbf{n}$, where $\mathbf{t}$ is the *Cauchy stress vector* and $\mathbf{n}$ the *outer unit normal vector*.

[3] Here and in the following we use the notation $[\nabla \mathbf{v}]\mathbf{w} = \left(w_j \partial v_i / \partial x_j\right)_{i=1,2,3}$, where the summation convention over repeated indices is used. We will use that convention throughout this paper.

The balance of total energy takes the form

$$\varrho \frac{\mathrm{d}}{\mathrm{d}t}\Big(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\Big) = \operatorname{div}(\mathbf{T}^{\top}\mathbf{v} - \mathbf{q}) + (\mathbf{f} + \mathbf{f}^{e})\cdot\mathbf{v} + w + w^{e}, \tag{1.13}$$

where e denotes the *specific internal energy*, $\mathbf{q}$ the *heat flux*, w the *mechanical energy production density* and w^e the *electro-magnetic energy supply density* which is given as (cf. p. 284–285 of [35])

$$w_e = \mathcal{J}\cdot\mathcal{E} + \mathcal{E}\cdot\dot{\mathbf{P}} - \mathcal{M}\cdot\dot{\mathbf{B}} + \mathcal{E}\cdot\mathbf{P}\operatorname{div}\mathbf{v}. \tag{1.14}$$

Using (1.6) together with (1.14), we obtain from (1.13) the balance of internal energy according to

$$\varrho\dot{e} + \operatorname{div}\mathbf{q} = \mathbf{T}\cdot\mathbf{L} + \mathcal{J}\cdot\mathcal{E} + \mathcal{E}\cdot\dot{\mathbf{P}} - \mathcal{M}\cdot\dot{\mathbf{B}} + \mathbf{P}\cdot\mathcal{E}\operatorname{div}\mathbf{v} + w, \tag{1.15}$$

where $\mathbf{L} = \nabla\mathbf{v}$ is the *velocity gradient.* We interpret the second law of thermodynamics in the form of the Clausius-Duhem inequality

$$\varrho\dot{\eta} + \operatorname{div}\frac{\mathbf{q}}{\theta} - \frac{w}{\theta} \geqslant 0, \tag{1.16}$$

where η is the *specific entropy* and θ the absolute temperature.

The system (1.1)–(1.4), (1.5), (1.6), (1.15) and (1.16) which describes the motion of the liquid has far more unknowns than equations. It is rendered determinate by providing appropriate *constitutive relations* reflecting the material properties. Towards this end, we will assume that

$$\varrho,\ \theta,\ \nabla\theta,\ \mathbf{v},\ \mathbf{D},\ \mathbf{E},\ \mathbf{B}, \tag{1.17}$$

where $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\top})$ is the *symmetric velocity gradient*, are the independent variables and thus we provide constitutive relations for

$$e,\ \eta,\ \mathbf{T},\ \mathbf{q},\ \mathbf{P},\ \mathcal{M},\ \mathcal{J} \tag{1.18}$$

of the form

$$f = \hat{f}(\varrho, \theta, \nabla\theta, \mathbf{v}, \mathbf{D}, \mathbf{E}, \mathbf{B}), \tag{1.19}$$

where f stands for any of the quantities in (1.18).

Both the material and the balance equations are subject to invariance requirements. It is well known that the mechanical balance laws (1.5), (1.6) and (1.15) are form-invariant under Galilean transformations given by

$$\mathbf{x}^{*} = \mathbf{Q}\mathbf{x} + \mathbf{v}_0 t + \mathbf{b}_0, \quad t^{*} = t, \tag{1.20}$$

where $\mathbf{v}_0$, $\mathbf{b}_0$ are constant vectors and $\mathbf{Q}$ is a time independent orthogonal tensor, while Maxwell's equations (1.1)–(1.4) are form-invariant under Lorentz transformations. We are interested in non-relativistic effects and it is well-known that there are problems with consistent invariance requirements for all thermo-mechanical and electro-magnetic balance laws and constitutive equations in a non-relativistic situation (cf. [25], [38], [44]). To avoid these difficulties we shall make the following *invariance requirements*: We assume that the quantities (1.18), describing the material properties, are invariant under Galilean transformations (1.20)[4]. Moreover we require that all balance laws (1.5), (1.6), (1.15), (1.16) and (1.1)–(1.4) are form-invariant under Galilean transformations (1.20). These two requirements imply consistent transformation formulæ for all necessary quantities (cf. [44]). In particular, we obtain from the invariance requirements that the constitutive relations (1.19) are isotropic functions of their arguments and that (1.19) has to be replaced by (cf. Grot [25])

$$f = \hat{f}(\varrho, \theta, \nabla\theta, \mathbf{D}, \boldsymbol{\mathcal{E}}, \mathbf{B}), \tag{1.21}$$

where f stands for any of the quantities in (1.18).

In addition to restrictions placed on the constitutive response functions by the invariance requirements we have additional strictures due to the requirement of the second law of thermodynamics. We shall now determine the restrictions imposed by requiring that all admissible processes of the body, i.e. processes compatible with the balance laws and the constitutive response functions, meet the Clausius-Duhem inequality (1.16). Introducing the specific Helmholtz free energy ψ through

$$\psi = e - \eta\theta - \frac{1}{\varrho}\boldsymbol{\mathcal{E}} \cdot \mathbf{P}, \tag{1.22}$$

and substituting it into (1.16) we obtain, with the help of the energy balance (1.15) and the balance of mass (1.5), the *dissipation inequality*

$$-\varrho(\dot{\psi} + \eta\,\dot{\theta}) + \mathbf{T} \cdot \mathbf{L} - \frac{\mathbf{q} \cdot \nabla\theta}{\theta} - \dot{\boldsymbol{\mathcal{E}}} \cdot \mathbf{P} - \boldsymbol{\mathcal{M}} \cdot \dot{\mathbf{B}} + \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}} \geqslant 0. \tag{1.23}$$

[4] Note that one usually assumes that the constitutive relations depend on $\mathbf{L}$ instead of $\mathbf{D}$, and then one deduces from the principle of material frame indifference, i.e. $(1.20)_1$ is replaced by $\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t)$, that the dependence on $\mathbf{L}$ has to reduce to a dependence on $\mathbf{D}$ only. In fact, this is the only relevant consequence of the stronger requirement of material frame indifference for us which cannot be obtained from the requirement that the material properties are invariant under Galilean transformations (1.20) only.

From (1.21) and (1.22) we get that $\psi = \psi(\varrho, \theta, \nabla\theta, \mathbf{D}, \boldsymbol{\mathcal{E}}, \mathbf{B})$. If we now compute $\dot{\psi}$ explicitely we can re-write (1.23), also using (1.5), as

$$(1.24)\quad -\varrho\Big(\frac{\partial\psi}{\partial\theta} + \eta\Big)\dot{\theta} - \varrho\frac{\partial\psi}{\partial\mathbf{D}}\cdot\dot{\mathbf{D}} - \Big(\boldsymbol{\mathcal{M}} + \varrho\frac{\partial\psi}{\partial\mathbf{B}}\Big)\cdot\dot{\mathbf{B}} + \Big(\mathbf{T} + \varrho^2\frac{\partial\psi}{\partial\varrho}\mathbf{I}\Big)\cdot\mathbf{D}$$
$$+ \mathbf{T}\cdot\mathbf{W} - \varrho\frac{\partial\psi}{\partial\nabla\theta}(\nabla\theta)^{\cdot} - \frac{\mathbf{q}\cdot\nabla\theta}{\theta} - \Big(\varrho\frac{\partial\psi}{\partial\boldsymbol{\mathcal{E}}} + \mathbf{P}\Big)\cdot\dot{\boldsymbol{\mathcal{E}}} + \boldsymbol{\mathcal{J}}\cdot\boldsymbol{\mathcal{E}} \geqslant 0.$$

Using the linearity of (1.24) with respect to the dotted quantities and $\mathbf{W}$ and their independence on the arguments appearing in the constitutive relations (1.21) one easily deduces (cf. Coleman, Noll [9], Truesdell/Noll [45], Grot [25])

$$(1.25)\quad \eta = -\frac{\partial\psi}{\partial\theta},\quad \frac{\partial\psi}{\partial\nabla\theta} = \mathbf{0},\quad \frac{\partial\psi}{\partial\mathbf{D}} = \mathbf{0},$$
$$\mathbf{P} = -\varrho\frac{\partial\psi}{\partial\boldsymbol{\mathcal{E}}},\quad \boldsymbol{\mathcal{M}} = -\varrho\frac{\partial\psi}{\partial\mathbf{B}},\quad \mathbf{T}^{\top} = \mathbf{T},$$

and the *reduced dissipation inequality*

$$(1.26)\quad \Big(\mathbf{T} + \varrho^2\frac{\partial\psi}{\partial\varrho}\mathbf{I}\Big)\cdot\mathbf{D} - \frac{\mathbf{q}\cdot\nabla\theta}{\theta} + \boldsymbol{\mathcal{J}}\cdot\boldsymbol{\mathcal{E}} \geqslant 0,$$

where ψ, η, $\mathbf{P}$ and $\boldsymbol{\mathcal{M}}$ are functions of ϱ, θ, $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$ only.

1.1. Electrorheological approximation

The equations derived in the last section may be simplified in view of electrorheological applications. Towards this end it is recommendable to carry out an appropriate *non-dimensionalization* with a subsequent *approximation.* All assumptions made in this section are based upon our understanding of the behaviour of ERFs, both from the theoretical and experimental point of view (cf. [7], [8], [18], [44], [46]).

Firstly, we shall assume that the Cauchy stress tensor $\mathbf{T}$ does not depend on the electric flux density $\mathbf{B}$, i.e.

$$(1.27)\quad \mathbf{T} = \hat{\mathbf{T}}(\varrho, \theta, \nabla\theta, \mathbf{D}, \boldsymbol{\mathcal{E}}).$$

This assumption reflects the observation that the material properties of an ERF do not change if a magnetic field is applied, because surely the particles in an ERF bear no magnetic properties.

Secondly, we shall assume that we are dealing with a dielectricum, i.e.

$$(1.28)\quad \boldsymbol{\mathcal{M}} \equiv \mathbf{0} \quad \text{where} \quad \boldsymbol{\mathcal{M}} = \mathbf{M} + \mathrm{v}\times\mathbf{P}.$$

Note that this assumption ensures that an apparent magnetization can only be generated by a moving polarized fluid (cf. [25]). This common assumption is a crucial point for deriving the so-called "quasi-electrostatic equations". In view of (1.25) the assumption (1.28) also implies that the Helmholtz free energy ψ, and thus also the polarization $\mathbf{P}$ and the entropy η, are only functions of ϱ, θ and $\boldsymbol{\mathcal{E}}$.

Thirdly, we shall assume that the fluid is electrically non-conducting, i.e.

$$\mathcal{J} \equiv \mathbf{0}. \tag{1.29}$$

This assumption may not be fully justified in general, because some ERFs exhibit a certain electrical conductivity which is often due to the content of water. However, many of them are free of water and have very low electrical conductivity (for example the polyurethane dispersions described in detail in [7], [8]), and thus we may restrict ourselves to such a class.

In order to reach the final *electrorheological approximation* and to determine and retain terms that are dominant and discard others that are insignificant, we will carry out a dimensional analysis which follows closely the one in [38], [44]. Towards this end we may introduce the following *dimensionless* quantities[5]:

$$\begin{aligned} &\bar{\mathbf{E}} = \frac{\mathbf{E}}{E_0}, \quad \bar{\mathbf{B}} = \frac{\mathbf{B}}{B_0}, \quad \bar{q}^e = \frac{q^e}{q_0}, \quad \bar{\mathbf{T}} = \frac{\mathbf{T}}{T_0}, \quad \bar{\mathbf{v}} = \frac{\mathbf{v}}{V_0}, \quad \bar{\mathbf{x}} = \frac{\mathbf{x}}{L_0}, \\ &\bar{t} = \frac{t}{t_0}, \quad \bar{\mathbf{P}} = \frac{\mathbf{P}}{\varepsilon_0 E_0}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_0}, \quad \bar{\mathbf{f}} = \frac{\mathbf{f}}{f_0}, \quad \bar{\theta} = \frac{\theta}{\theta_0}, \end{aligned} \tag{1.30}$$

where the quantities with the subscript "0" are appropriate characteristic quantities of the problem in question. In typical problems and for many ERFs (cf. [7], [8]), we envisage that

$$\begin{aligned} &E_0 \sim 3 \cdot (10^4 - 10^6)\,\mathrm{V\,m^{-1}}, && V_0 \sim (10^{-3} - 1)\,\mathrm{m\,s^{-1}}, \\ &L_0 \sim 5 \cdot (10^{-4} - 10^{-3})\,\mathrm{m}, && \eta_0 \sim (10^{-2} - 10^{-1})\,\mathrm{kg\,(m\ s)^{-1}}, \\ &t_0 \sim (10^{-3} - 1)\,\mathrm{s}, && \varrho_0 \sim 10^3\,\mathrm{kg\,m^{-3}}. \end{aligned} \tag{1.31}$$

The time t_0 may be either a characteristic electric or hydrodynamic time, depending on the specific problem. Moreover, ϱ_0 and η_0 are the density and the dynamic viscosity of the fluid in the *absence* of an electric field, respectively. Using (1.31), the Reynolds number $\mathrm{Re} = (\varrho_0\, L_0\, V_0)/\eta_0$ and the Strouhal number $\mathrm{Str} = L_0/(V_0 t_0)$ lie in the range

$$5 \cdot 10^{-3} \leqslant \mathrm{Re} \leqslant 5 \cdot 10^2 \quad \text{and} \quad 5 \cdot 10^{-4} \leqslant \mathrm{Str} \leqslant 5 \cdot 10^3, \tag{1.32}$$

[5] In this section, dimensionless quantities and operators are denoted by a superposed bar.

respectively. Magnetic quantities are missing in (1.31). No experimental observation is known to us that shows that the magnetic field plays a significant role in electrorheological applications. Usually, no external magnetic field is applied and thus $\mathbf{B}$ is only induced due to the electric field. We interpret the *secondary* role of $\mathbf{B}$ in ERFs through the assumptions that

$$\frac{E_0}{B_0}\frac{L_0}{c^2 t_0} = O(1), \tag{1.33}$$

resulting in

$$B_0 \sim (10^{-16} - 10^{-10})\,\mathrm{Vs/m^2}. \tag{1.34}$$

Recall that $c \approx 3 \cdot 10^8\,\mathrm{m\,s^{-1}}$ denotes the speed of electro-magnetic waves in vacuo. (1.33) is consistent with the assumption that the magnetic flux density is only induced by oscillations of the electric field and/or the motion of a polarized body (cf. (1.42)). Let us introduce a small non-dimensional number ε through

$$\varepsilon \equiv 10^{-3}, \tag{1.35}$$

which measures the importance of the terms. The situation described above—together with an assumption that there are only few free charges in the fluid—can thus be summarized as

$$\begin{aligned}
\frac{L_0}{c\,t_0} &= O(\varepsilon^3) - O(\varepsilon^4), & \frac{V_0}{c} &= O(\varepsilon^3) - O(\varepsilon^4),\\
\frac{V_0\,t_0}{L_0} &= O(\varepsilon^{-1}) - O(\varepsilon), & \frac{q_0\,L_0}{\varepsilon_0 E_0} &= O(\varepsilon^3),\\
\frac{B_0\,L_0}{E_0\,t_0} &= O(\varepsilon^5) - O(\varepsilon^8), & \frac{E_0\,V_0}{B_0\,c^2} &= O(1).
\end{aligned} \tag{1.36}$$

The non-dimensionalized system of balance laws may then be approximated by retaining terms up to order ε^2, while neglecting terms of higher order.

Firstly, let us discuss the role of $\boldsymbol{\mathcal{E}}$ in the constitutive relations. It follows from the definition of $\boldsymbol{\mathcal{E}}$ that

$$\bar{\boldsymbol{\mathcal{E}}} = \frac{\boldsymbol{\mathcal{E}}}{E_0} = \bar{\mathbf{E}} + \frac{V_0\,B_0}{E_0}\,\bar{\mathbf{v}} \times \bar{\mathbf{B}} = \bar{\mathbf{E}} + O(\varepsilon^5), \tag{1.37}$$

where we used that

$$\frac{V_0\,B_0}{E_0} = O(\varepsilon^5) - O(\varepsilon^7). \tag{1.38}$$

Thus, we can replace $\bar{\boldsymbol{\mathcal{E}}}$ by $\bar{\mathbf{E}}$ in all non-dimensionalized constitutive relations.

The dimensionless form of Maxwell's equations (1.1)–(1.4) may be obtained upon using the definitions of $\mathbf{H}$, $\mathbf{D}^e$, (1.28), (1.36) and (1.37) as

$$\overline{\mathrm{div}}\overline{\mathbf{E}} + \overline{\mathrm{div}}\overline{\mathbf{P}} = \underbrace{\frac{q_0\, L_0}{\varepsilon_0 E_0}}_{O(\varepsilon^3)}\, \bar{q}^e + O(\varepsilon^5), \qquad \overline{\mathrm{curl}}\overline{\mathbf{E}} + \underbrace{\frac{B_0\, L_0}{E_0\, t_0}}_{O(\varepsilon^5)}\, \frac{\partial \overline{\mathbf{B}}}{\partial \bar{t}} = \mathbf{0}, \qquad \overline{\mathrm{div}}\overline{\mathbf{B}} = 0,$$

$$\overline{\mathrm{curl}}\overline{\mathbf{B}} + \underbrace{\frac{E_0\, V_0}{B_0\, c^2}}_{O(1)}\, \overline{\mathrm{curl}}(\bar{\mathbf{v}} \times \overline{\mathbf{P}}) = \underbrace{\frac{E_0}{B_0}\, \frac{L_0}{c^2\, t_0}}_{O(1)}\, \frac{\partial}{\partial \bar{t}}(\overline{\mathbf{E}} + \overline{\mathbf{P}}) - \underbrace{\frac{q_0\, L_0}{\varepsilon_0 E_0}\, \frac{E_0\, V_0}{B_0\, c^2}}_{O(\varepsilon^3)}\, \bar{q}^e \bar{\mathbf{v}} + O(\varepsilon^5),$$

where in $O(\varepsilon^5)$ only terms coming from (1.37) are included and where we also used the relation $\varepsilon_0 \mu_0 = c^{-2}$. Neglecting terms of $O(\varepsilon^3)$, we obtain the *electrorheological approximation of Maxwell's equations* according to[6]

$$\mathrm{div}(\varepsilon_0 \mathbf{E} + \mathbf{P}) = 0, \tag{1.39}$$

$$\mathrm{curl}\, \mathbf{E} = \mathbf{0}, \tag{1.40}$$

$$\mathrm{div}\, \mathbf{B} = 0, \tag{1.41}$$

$$\frac{1}{\mu_0}\, \mathrm{curl}\, \mathbf{B} + \mathrm{curl}(\mathbf{v} \times \mathbf{P}) = \frac{\partial(\varepsilon_0 \mathbf{E} + \mathbf{P})}{\partial t}, \tag{1.42}$$

where $\mathbf{P} = \mathbf{P}(\varrho, \theta, \mathbf{E})$.

Now we turn to the approximation of the thermo-mechanical balance laws. The conservation of mass (1.5) remains unaffected. In the momentum equation (1.6) we re-write the electro-magnetic force $\mathbf{f}^e$ using (1.8), (1.28), (1.29) and then use (1.36) and (1.37), which leads to

$$\begin{aligned}
&\frac{\varrho_0\, V_0\, L_0}{\varepsilon_0\, E_0^2\, t_0}\, \bar{\varrho} \frac{\partial \bar{\mathbf{v}}}{\partial \bar{t}} + \frac{\varrho_0\, V_0^2}{\varepsilon_0\, E_0^2}\, \bar{\varrho}[\overline{\nabla} \bar{\mathbf{v}}]\bar{\mathbf{v}} - \frac{T_0}{\varepsilon_0\, E_0^2}\, \overline{\mathrm{div}}\overline{\mathbf{T}} \\
&= f_0 \frac{L_0}{\varepsilon_0\, E_0^2} \bar{\mathbf{f}} + \underbrace{\frac{q_0\, L_0}{\varepsilon_0\, E_0}}_{O(\varepsilon^3)} \Big(\bar{q}_e \overline{\mathbf{E}} + \underbrace{\frac{V_0\, B_0}{E_0}}_{O(\varepsilon^5)}\, \bar{q}_e \bar{\mathbf{v}} \times \overline{\mathbf{B}} \Big) + \underbrace{\frac{B_0\, L_0}{E_0\, t_0}}_{O(\varepsilon^5)}\, \frac{\partial \overline{\mathbf{P}}}{\partial \bar{t}} \times \overline{\mathbf{B}} \\
&\quad + \underbrace{\frac{V_0\, B_0}{E_0}}_{O(\varepsilon^5)} ([\overline{\nabla} \overline{\mathbf{P}}]\bar{\mathbf{v}} + (\overline{\mathrm{div}} \bar{\mathbf{v}})\overline{\mathbf{P}} \times \overline{\mathbf{B}} + \bar{\mathbf{v}} \times ([\overline{\nabla} \overline{\mathbf{B}}]\overline{\mathbf{P}})) + [\overline{\nabla} \overline{\mathbf{E}}]\overline{\mathbf{P}} + O(\varepsilon^5),
\end{aligned} \tag{1.43}$$

where in $O(\varepsilon^5)$ only terms coming from (1.37) are included. We see that all underbraced terms on the right-hand side of (1.43) have to be neglected. We shall retain

[6] Since $\boldsymbol{\mathcal{M}} = \mathbf{0}$, we can rewrite (1.39)–(1.42) in terms of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{D}^e$ only.

the mechanical force term and the term with the Cauchy stress. Furthermore, one easily computes that

$$
\frac{\varrho_0\, V_0\, L_0}{\varepsilon_0\, E_0^2\, t_0} = \begin{cases} O(1) - O(\varepsilon^2) & \text{if } E_0^2 \sim 9 \cdot 10^{12}\,\mathrm{V}^2\,\mathrm{m}^{-2}, \\ O(\varepsilon^{-1}) - O(\varepsilon^1) & \text{if } E_0^2 \sim 9 \cdot 10^{10}\,\mathrm{V}^2\,\mathrm{m}^{-2}, \\ O(\varepsilon^{-2}) - O(1) & \text{if } E_0^2 \sim 9 \cdot 10^{8}\,\mathrm{V}^2\,\mathrm{m}^{-2}, \end{cases} \tag{1.44}
$$

$$
\frac{\varrho_0\, V_0^2}{\varepsilon_0\, E_0^2} = \begin{cases} O(1) - O(\varepsilon^2) & \text{if } E_0^2 \sim 9 \cdot 10^{12}\,\mathrm{V}^2\,\mathrm{m}^{-2}, \\ O(\varepsilon^{-1}) - O(\varepsilon^1) & \text{if } E_0^2 \sim 9 \cdot 10^{10}\,\mathrm{V}^2\,\mathrm{m}^{-2}, \\ O(\varepsilon^{-2}) - O(1) & \text{if } E_0^2 \sim 9 \cdot 10^{8}\,\mathrm{V}^2\,\mathrm{m}^{-2}. \end{cases} \tag{1.45}
$$

Therefore also the first and the second term on the left-hand side of (1.43) have to be kept. With regard to the approximation of the other thermo-mechanical non-dimensionalized equations, we only replace $\bar{\boldsymbol{\mathcal{E}}}$ by $\bar{\mathbf{E}}$ since we have no indication of the behaviour of the other quantities.

Therefore, the *electrorheological approximation of the thermo-mechanical balance laws* is given by

$$\dot{\varrho} + \varrho \operatorname{div} \mathbf{v} = 0, \tag{1.46}$$

$$\varrho \dot{\mathbf{v}} - \operatorname{div} \mathbf{T} = \mathbf{f} + [\nabla \mathbf{E}]\mathbf{P}, \tag{1.47}$$

$$c_v \varrho \dot{\theta} - k \Delta \theta - \Big(\frac{\partial \mathbf{P}}{\partial \theta} \cdot \dot{\mathbf{E}} + \frac{\partial \pi}{\partial \theta} \operatorname{tr} \mathbf{D} \Big) \theta = (\mathbf{T} - \pi \mathbf{I}) \cdot \mathbf{D} + w, \tag{1.48}$$

$$(\mathbf{T} - \pi \mathbf{I}) \cdot \mathbf{D} - \frac{(\nabla \theta) \cdot \mathbf{q}}{\theta} \geqslant 0, \tag{1.49}$$

where we used the definition of the *specific heat* c_v and of the *thermodynamic pressure* π according to

$$c_v = -\theta \frac{\partial^2 \psi}{\partial \theta^2}, \qquad \pi = -\varrho^2 \frac{\partial \psi}{\partial \varrho}.$$

Moreover c_v, $\mathbf{P}$, π and ψ are functions of ϱ, θ and $\mathbf{E}$; while we have for the Cauchy stress $\mathbf{T} = \mathbf{T}(\varrho, \theta, \nabla\theta, \mathbf{D}, \mathbf{E})$.

1.2. Constitutive relations

Now we will develop a constitutive theory for ERFs. In order to keep the already very long and complicated formulæ as simple as possible we keep the dependence on $\nabla\theta$ only in the constitutive relation for the heat flux $\mathbf{q}$ and assume that

$$\mathbf{q} = -k \nabla \theta, \tag{1.50}$$

where the *thermal conductivity* k is a positive constant. In all other constitutive relations we drop the dependence on $\nabla\theta$. We also restrict ourselves to the case of an incompressible ERF, i.e.

$$\operatorname{tr}\mathbf{D} = 0, \tag{1.51}$$

and consequently we also drop the dependence on ϱ in all constitutive relations. Moreover we assume a linear dependence of the polarization $\mathbf{P}$ on the electric field $\mathbf{E}$, i.e.

$$\mathbf{P} = \chi^E(\theta)\mathbf{E}, \tag{1.52}$$

where χ^E is the *dielectric susceptibility.* The Cauchy stress can be splited according to $\mathbf{T} = -\pi\mathbf{I} + \mathbf{S}$. From the above assumptions and (1.27) we get that the *extra stress tensor* $\mathbf{S}$ is of the form

$$\mathbf{S} = \mathbf{S}(\theta, \mathbf{D}, \mathbf{E}). \tag{1.53}$$

From representation theorems (cf. the appendix of [20] and the references stated there) it follows that the most general form for $\mathbf{S}$ is given by

$$\begin{aligned} \mathbf{S} = \alpha_2 \mathbf{E}\otimes\mathbf{E} + \alpha_3\mathbf{D} + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{DE}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{DE}) \\ + \alpha_6(\mathbf{D}^2\mathbf{E}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{D}^2\mathbf{E}), \end{aligned} \tag{1.54}$$

where α_i, $i = 2, \dots 6$ may be functions of the invariants

$$\theta,\ |\mathbf{E}|^2,\ \operatorname{tr}\mathbf{D}^2,\ \operatorname{tr}\mathbf{D}^3,\ \operatorname{tr}(\mathbf{DE}\otimes\mathbf{E}),\ \operatorname{tr}(\mathbf{D}^2\mathbf{E}\otimes\mathbf{E}). \tag{1.55}$$

In view of certain peculiarities in the behaviour of the normal stress differences in the case $\alpha_4 \neq 0$ even in the absence of an electric field (cf. [33]) and due to previous mathematical investigations for shear dependent viscous fluids, which suggests that terms involving $\mathbf{D}^2$ can be treated as a perturbation (cf. [31], [33]), we assume that

$$\alpha_4 \equiv 0, \qquad \alpha_6 \equiv 0. \tag{1.56}$$

Based on experimental data (cf. [26], [3], [2], [1], [47]) we assume that in the presence and the absence of an electric field the ERF behaves like a *generalized Newtonian fluid* with power p, where the power p can depend on the magnitude of the electric field $|\mathbf{E}|^2$. Moreover, we restrict ourselves to the case that the material functions α_2, α_3 and α_5 depend only on the invariants θ, $|\mathbf{D}|^2$ and $|\mathbf{E}|^2$ and that all terms have the same *growth behaviour.* Thus we deal with the following model for the extra stress tensor $\mathbf{S}$

$$\begin{aligned} \mathbf{S} = \alpha_{21}((1 + |\mathbf{D}|^2)^{(p-1)/2} - 1)\mathbf{E}\otimes\mathbf{E} + (\alpha_{31} + \alpha_{33}|\mathbf{E}|^2)(1 + |\mathbf{D}|^2)^{(p-2)/2}\mathbf{D} \\ + \alpha_{51}(1 + |\mathbf{D}|^2)^{(p-2)/2}(\mathbf{DE}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{DE}), \end{aligned} \tag{1.57}$$

where α_{ij} are constants and $p = p(|\mathbf{E}|^2)$ is a C^1-function such that

$$1 < p_\infty \leqslant p(|\mathbf{E}|^2) \leqslant p_0. \tag{1.58}$$

To ensure the validity of the Clausius-Duhem inequality we further require that the constant coefficients α_{ij} and the function p are such that (cf. [44, Lemma 1.4.46])

$$\alpha_{31} > 0, \qquad \alpha_{33} > 0, \qquad \alpha_{33} + \frac{4}{3}\alpha_{51} > 0, \tag{1.59}$$

$$k(p_0)|\alpha_{21}| < \begin{cases} 2\sqrt{\alpha_{33}}\sqrt{2\alpha_{51}} & \text{if } \alpha_{33} \leqslant \frac{4}{3}\alpha_{51}, \\ \sqrt{\frac{3}{2}}(\alpha_{33} + \frac{4}{3}\alpha_{51}) & \text{if } \frac{4}{3}|\alpha_{51}| \leqslant \alpha_{33}, \end{cases} \tag{1.60}$$

where $k(p_0) = 1$ if $p_0 \leqslant 3$ and $k(p_0) > 1$ is a computable constant for $p_0 > 3$. Note that these requirements ensure that the operator induced by $-\operatorname{div}\mathbf{S}(\mathbf{D}, \mathbf{E})$ is *coercive.*

2. Flows of shear dependent electrorheological fluids

In the previous section we have shown that the isothermal flow of an incompressible shear dependent ERF is governed by the following system[7]

$$\begin{aligned} \operatorname{div}\mathbf{E} &= 0, \\ \operatorname{curl}\mathbf{E} &= \mathbf{0}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \partial_t\mathbf{v} - \operatorname{div}\mathbf{S} + [\nabla\mathbf{v}]\mathbf{v} + \nabla\pi &= \mathbf{f} + \chi^E[\nabla\mathbf{E}]\mathbf{E}, \\ \operatorname{div}\mathbf{v} &= 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \operatorname{div}\mathbf{B} &= 0, \\ \mu_0^{-1}\operatorname{curl}\mathbf{B} + \chi^E\operatorname{curl}(\mathbf{v}\times\mathbf{E}) &= (\varepsilon_0 + \chi^E)\partial_t\mathbf{E}, \end{aligned} \tag{2.3}$$

$$\mathbf{S}\cdot\mathbf{D} + w = 0, \tag{2.4}$$

where the extra stress tensor $\mathbf{S}$ is given by (1.57), (1.58).

The system (2.1)–(2.4) is separated. We first solve the quasi-static Maxwell's equations (2.1) for the electric field and then seek for the velocity field by solving (2.2). Knowing $\mathbf{E}$ and $\mathbf{v}$ we can solve (3.2) and (2.4). Note that the equation (2.4) has to be interpreted as an equation for the mechanical energy supply density w. It was already pointed out in the previous section that the magnetic induction $\mathbf{B}$ is of

[7] We have divided equation (1.47) by the constant density ϱ_0 and adapted the notation appropriately.

secondary importance, which is reflected by the structure of the above system. Moreover, the quasi-static Maxwell's equations (2.1) are widely studied in the literature (cf. the overview article Milani/Picard [34]). Since in this investigation of ERFs we are mainly interested in the velocity field $\mathbf{v}$, we shall only consider the system (2.2), in which $\mathbf{E}$ is assumed to be any given vector field, having certain regularity properties. Moreover, for simplicity we shall complete (2.2) by *space periodic boundary conditions* and an *initial condition* $\mathbf{v}_0$.

In order to prove existence results for the system (2.2) we need some structure conditions for the extra stress tensor $\mathbf{S}$, which unfortunately are stronger than the conditions we have to assume for the validity of the Clausius-Duhem inequality, which is a physical requirement. In the following we assume that the constant coefficients α_{ij} and the function p are such that the operator induced by $-\operatorname{div}\mathbf{S}(\mathbf{D},\mathbf{E})$ is *uniformly monotone*, i.e.

$$\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}} B_{ij}B_{kl} \geqslant \gamma_1(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p(|\mathbf{E}|^2)-2)/2}|\mathbf{B}|^2 \tag{2.5}$$

is satisfied for all $\mathbf{B},\mathbf{D} \in X := \{\mathbf{D} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}},\ \operatorname{tr}\mathbf{D} = 0\}$, and that the following *growth conditions* are satisfied for $i,j,k,l,n = 1,2,3$,

$$\Big|\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}}\Big| \leqslant \gamma_2(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p(|\mathbf{E}|^2)-2)/2}, \tag{2.6}$$

$$\Big|\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial E_n}\Big| \leqslant \gamma_3|\mathbf{E}|(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p(|\mathbf{E}|^2)-1)/2}(1+\ln(1+|\mathbf{D}|^2)). \tag{2.7}$$

Conditions for α_{ij} and p that ensure the validity of (2.5) can be found in [44, Chapter 1]. We will show that the *coercivity*, i.e. that

$$\mathbf{S}(\mathbf{D},\mathbf{E})\cdot\mathbf{D} \geqslant c(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p(|\mathbf{E}|^2)-2)/2}|\mathbf{D}|^2 \tag{2.8}$$

holds for all $\mathbf{D} \in X$, is a consequence of (2.5).[8]

Before formulating the main result of this section, we introduce some notation. Let $\Omega = (0,L)^3$, $L \in (0,\infty)$ be a cube in $\mathbb{R}^3$ and denote $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$ and $\Gamma_{j+3} = \partial\Omega \cap \{x_j = L\}$, for $j = 1,2,3$. For $T \in (0,\infty)$, we denote by Q_T the time-space cylinder $I \times \Omega$, where $I = [0,T]$ is a time interval. By $\mathcal{D}(\Omega)$ we denote the space of smooth periodic functions with mean value zero. Let further $q > 1$ and $k > 0$. Then $(L^q(\Omega), \|\cdot\|_q)$ and $(W^{k,q}(\Omega), \|\cdot\|_{k,q})$, respectively, is used for the usual Lebesgue and Sobolev spaces, of periodic functions with mean value zero. By

[8] As was already pointed out the coercivity and the Clausius-Duhem inequality are almost equivalent. In fact, if p is independent of $|\mathbf{E}|^2$ than these two requirements are the same.

$\langle f, g\rangle := \int_\Omega fg\,\mathrm{d}x$ we denote the scalar product with respect to space. We also need Lebesgue and Sobolev spaces with variable exponents, which are denoted by $L^{p(\cdot)}(G)$ and $W^{k,p(\cdot)}(G)$, respectively, where $G = \Omega$ or $G = Q_T$. For a given $p(\cdot) \in L^\infty(G)$, $1 < p_\infty \leqslant p(x) \leqslant p_0 < \infty$, we define the *modular*

$$\varrho_p(f) = \varrho_{p,G}(f) := \int_G |f(y)|^{p(y)}\,\mathrm{d}y.$$

Similarly to the Luxemburg norm in Orlicz spaces we define

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 \mid \varrho_p(\lambda^{-1}f) < 1\},$$

which is a norm on the *generalized Lebesgue space*

$$L^{p(\cdot)}(G) := \{f \in L^1(G) \mid \varrho_p(\lambda^{-1}f) < \infty \text{ for some } \lambda > 0\}.$$

Generalized Sobolev spaces are defined analogously. We refer to Kováčik/Rákosník [28] for a detailed treatment of these spaces. Moreover, we denote by $L^q(I;X)$ the Bochner spaces which are equipped with the norm $\left(\int_I \|\cdot\|_X^q\,\mathrm{d}s\right)^{1/q}$. In the following we use for the partial derivative with respect to time the symbol ∂_t. We shall further make frequent use of spaces of divergence free functions defined by

$$\begin{aligned}
&\mathcal{V} := \{\psi \in \mathcal{D}(\Omega)\colon\ \operatorname{div}\psi = 0\},\\
&V_p := \text{the closure of } \mathcal{V} \text{ with respect to the } \|\nabla\cdot\|_p\text{-norm},
\end{aligned}$$

and use the following expressions, for functions $\mathbf{v}$ and $\mathbf{E}$ defined on the space-time cylinder Q_T,

$$\mathcal{I}(t,\mathbf{v}) := \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}(t),\mathbf{E}(t))}{\partial D_{kl}} D_{ij}(\nabla\mathbf{v})(t)D_{kl}(\nabla\mathbf{v})(t)\,\mathrm{d}x, \tag{2.9}$$

$$\mathcal{J}(t,\mathbf{v}) := \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}(t),\mathbf{E}(t))}{\partial D_{kl}} D_{ij}(\partial_t\mathbf{v})(t)D_{kl}(\partial_t\mathbf{v})(t)\,\mathrm{d}x, \tag{2.10}$$

which are related to the extra stress tensor $\mathbf{S}$.

We are seeking solutions $\mathbf{v}$ of the system (2.2) completed with the initial condition

$$\mathbf{v}(0) = \mathbf{v}_0, \tag{2.11}$$

and with space-periodic boundary conditions

$$\mathbf{v}\big|_{\Gamma_j} = \mathbf{v}\big|_{\Gamma_{j+3}}, \qquad \nabla\mathbf{v}\big|_{\Gamma_j} = \nabla\mathbf{v}\big|_{\Gamma_{j+3}}, \qquad \pi\big|_{\Gamma_j} = \pi\big|_{\Gamma_{j+3}}, \tag{2.12}$$

for $j = 1,2,3$. Now we can formulate the main result of this section.

Theorem 2.13. *Assume that the extra stress tensor* $\mathbf{S}$ *satisfies* (2.5)–(2.7) *and* $\mathbf{S}(\mathbf{0},\mathbf{E}) = \mathbf{0}$. *Let* $\mathbf{v}_0 \in W^{2,2}(\Omega) \cap V_p$ *be a given initial velocity,* $\mathbf{f} \in C(I; W^{1,2}(\Omega))$, $\partial_t \mathbf{f} \in C(I; L^2(\Omega))$ *be a given force,* $\mathbf{E} \in W^{1,\infty}(I; W^{1,\infty}(\Omega))$ *be a given electric field and let* $p = p(|\mathbf{E}|^2)$ *be a* C^1*-function with* $p_\infty \leqslant p(|\mathbf{E}|^2) \leqslant p_0$. *If*

$$\frac{3}{2} < p_\infty \leqslant p_0 \leqslant 2$$

then there exists a time $T^* > 0$, *such that a strong solution* $\mathbf{v}$ *of the system* (2.2) *exists on* $I' := [0, T^*]$. *This solution satisfies*

$$\operatorname*{ess\,sup}_{s \in I'} \|\partial_t \mathbf{v}(s)\|_2^2 + \int_0^{T^*} \mathcal{I}(t,\mathbf{v})^{\frac{5p_\infty - 6}{2-p_\infty}} + \mathcal{J}(t,\mathbf{v})\,\mathrm{d}t \leqslant C(\mathbf{f}, \mathbf{v}_0, \mathbf{E}). \tag{2.14}$$

In particular we have that for $1 < r < 6(p_\infty - 1)$

$$\begin{aligned} &\mathbf{v} \in L^{p_\infty \frac{5p_\infty - 6}{2-p_\infty}}\Big(I'; W^{2,\frac{3p_\infty}{p_\infty+1}}(\Omega)\Big) \cap C(I'; V_r), \\ &\partial_t \mathbf{v} \in L^{\frac{p_\infty(5p_\infty-6)}{(3p_\infty-2)(p_\infty-1)}}\Big(I'; W^{1,\frac{3p_\infty}{p_\infty+1}}(\Omega)\Big) \cap L^\infty(I'; L^2(\Omega)), \\ &\partial_t^2 \mathbf{v} \in L^2(I'; (V_2)^*). \end{aligned} \tag{2.15}$$

Remark 2.16. With a more refined technique one can show that the statement of the theorem is valid for $\frac{7}{5} < p_\infty \leqslant p_0 \leqslant 2$ (cf. [14, Theorem 21]).

The main problem in the proof of the previous theorem consists in the identification of the limit

$$\lim_{N\to\infty} \int_0^T \int_\Omega \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}) \cdot \mathbf{D}(\varphi)\,\mathrm{d}x\,\mathrm{d}t$$

where $\mathbf{v}^N$ is some approximate solution of (2.2). The method used here is based on Vitali's convergence theorem and the almost everywhere convergence of $\mathbf{D}\mathbf{v}^N$. This method was developed in [31], [32], [6], [30], [14] to handle situations when the theory of monotone operators fails to identify the above limit. It is worth noticing that unsteady problems for ERFs cannot be treated with the help of monotonicity methods even for large p_∞ due to the *non-standard growth* of the governing system, i.e. within the classical Sobolev spaces our assumptions (2.5)–(2.7) imply

$$C(1 + |\mathbf{D}|)^{p_\infty - 2}|\mathbf{D}|^2 \leqslant \mathbf{S}(\mathbf{D}, \mathbf{E}) \cdot \mathbf{D} \leqslant \tilde{C}(1 + |\mathbf{D}|)^{p_0 - 2}|\mathbf{D}|^2.$$

Before we start with the proof of the above theorem we need some preliminary results related to the extra stress tensor $\mathbf{S}$. Let us start with an algebraic lemma.

We write $f \cong g$ iff there exist constants $C_0, C_1 > 0$ such that

$$C_0 f \leqslant g \leqslant C_1 f,$$

where we always indicate on which quantities the constants may depend.

Lemma 2.17. *For all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ and all $q > 1$ there holds*

$$\int_0^1 (1 + |\mathbf{B} + s(\mathbf{A} - \mathbf{B})|)^{q-2}\,\mathrm{d}s \cong (1 + |\mathbf{B}| + |\mathbf{A}|)^{q-2},$$

with constants depending on q only.

Proof. The proof can be found in [24, Lemma 8.3]. □

Remark 2.18. Since $|\mathbf{A}| + |\mathbf{A} - \mathbf{B}| \leqslant 2(|\mathbf{A}| + |\mathbf{B}|) \leqslant 4(|\mathbf{A}| + |\mathbf{A} - \mathbf{B}|)$ we immediately obtain from Lemma 2.17 that for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ and all $q > 1$ there holds

$$\int_0^1 (1 + |\mathbf{B} + s(\mathbf{A} - \mathbf{B})|)^{q-2}\,\mathrm{d}s \cong (1 + |\mathbf{B}| + |\mathbf{A} - \mathbf{B}|)^{q-2},$$

with constants depending on q only.

Lemma 2.19. *Suppose that $\mathbf{S}$ satisfies* (2.5) *and* (2.6) *and $\mathbf{S}(\mathbf{0}, \mathbf{E}) = \mathbf{0}$. Then there holds for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and all $\mathbf{E} \in \mathbb{R}^3$*

$$\text{(a)} \qquad \mathbf{S}(\mathbf{A}, \mathbf{E}) \cdot \mathbf{A} \cong |\mathbf{A}|^2 (1 + |\mathbf{A}|)^{p(|\mathbf{E}|^2)-2},$$

$$\text{(b)} \qquad (\mathbf{S}(\mathbf{A}, \mathbf{E}) - \mathbf{S}(\mathbf{B}, \mathbf{E})) \cdot (\mathbf{A} - \mathbf{B}) \cong |\mathbf{A} - \mathbf{B}|^2 (1 + |\mathbf{B}| + |\mathbf{A}|)^{p(|\mathbf{E}|^2)-2},$$

$$\text{(c)} \qquad |\mathbf{S}(\mathbf{A}, \mathbf{E}) - \mathbf{S}(\mathbf{B}, \mathbf{E})| \cong |\mathbf{A} - \mathbf{B}| (1 + |\mathbf{B}| + |\mathbf{A}|)^{p(|\mathbf{E}|^2)-2},$$

$$\text{(d)} \qquad |\mathbf{S}(\mathbf{A}, \mathbf{E})| \cong |\mathbf{A}| (1 + |\mathbf{A}|)^{p(|\mathbf{E}|^2)-2},$$

with constants depending on p_∞, p_0 (cf. (1.58)*) and $1 + |\mathbf{E}|^2$ only.*

Proof. Note that the statement (a) is a special case of (b) by choosing $\mathbf{B} = \mathbf{0}$ and using $\mathbf{S}(\mathbf{0}, \mathbf{E}) = \mathbf{0}$. In the same way (d) follows from (c). In order to prove (b) one notices that (2.5), (2.6) and Lemma 2.17 yield

$$\begin{aligned}
&(\mathbf{S}(\mathbf{A}, \mathbf{E}) - \mathbf{S}(\mathbf{B}, \mathbf{E})) \cdot (\mathbf{A} - \mathbf{B}) \\
&\quad = \int_0^1 \frac{\partial S_{ij}(\mathbf{B} + s(\mathbf{A} - \mathbf{B}), \mathbf{E})}{\partial D_{kl}} (A - B)_{kl} (A - B)_{ij}\,\mathrm{d}s \\
&\quad \cong |\mathbf{A} - \mathbf{B}|^2 \int_0^1 (1 + |\mathbf{B} + s(\mathbf{A} - \mathbf{B})|)^{p-2}\,\mathrm{d}s \\
&\quad \cong |\mathbf{A} - \mathbf{B}|^2 (1 + |\mathbf{B}| + |\mathbf{A}|)^{p-2},
\end{aligned}$$

where we used $(1+y^2)^{\frac{1}{2}} \cong (1+|y|)$. From this we immediately obtain

$$\begin{aligned}|\mathbf{A}-\mathbf{B}|^2(1+|\mathbf{B}|+|\mathbf{A}|)^{p-2} &\leqslant c(\mathbf{S}(\mathbf{A},\mathbf{E})-\mathbf{S}(\mathbf{B},\mathbf{E}))\cdot(\mathbf{A}-\mathbf{B})\\ &\leqslant c|\mathbf{S}(\mathbf{A},\mathbf{E})-\mathbf{S}(\mathbf{B},\mathbf{E})|\,|\mathbf{A}-\mathbf{B}|,\end{aligned}$$

which delivers the first inequality in (c). For the other inequality we use (2.6) and Lemma 2.17 to obtain

$$\begin{aligned}|\mathbf{S}(\mathbf{A},\mathbf{E})-\mathbf{S}(\mathbf{B},\mathbf{E})| &= \left|\int_0^1 \frac{\partial^2 S_{ij}(\mathbf{B}+s(\mathbf{A}-\mathbf{B}),\mathbf{E})}{\partial D_{kl}}\,\mathrm{d}s(A-B)_{kl}\right|\\ &\leqslant c|\mathbf{A}-\mathbf{B}|(1+|\mathbf{B}|+|\mathbf{A}|)^{p-2},\end{aligned}$$

which finishes the proof. □

Remark 2.20. Note that in the right-hand sides in Lemma 2.19 one can replace $1+|\mathbf{B}|+|\mathbf{A}|$ by $1+|\mathbf{B}|+|\mathbf{A}-\mathbf{B}|$.

Now we derive *lower bounds* for the expressions $\mathcal{I}(t,\mathbf{v})$ and $\mathcal{J}(t,\mathbf{v})$, defined in (2.9) and (2.10), for which we will often simply write $\mathcal{I}(\mathbf{v})$ and $\mathcal{J}(\mathbf{v})$. They arise from testing (2.2) with $-\Delta\mathbf{v}$ and "$\partial_t^2\mathbf{v}$", respectively. The expression $(1+|\mathbf{D}\mathbf{v}|^2)^{1/2}$ will appear quite often, so it is very useful to introduce the abbreviation

$$\tilde{D}\mathbf{v} := (1+|\mathbf{D}\mathbf{v}|^2)^{1/2}. \tag{2.21}$$

As a consequence of (2.5) we have

$$\mathcal{I}(t,\mathbf{v}) \geqslant \gamma_1 \int_\Omega (\tilde{D}\mathbf{v}(t))^{p(|\mathbf{E}(t)|^2)-2}|\mathbf{D}(\nabla\mathbf{v})(t)|^2\,\mathrm{d}x, \tag{2.22}$$

$$\mathcal{J}(t,\mathbf{v}) \geqslant \gamma_1 \int_\Omega (\tilde{D}\mathbf{v}(t))^{p(|\mathbf{E}(t)|^2)-2}|\mathbf{D}(\partial_t\mathbf{v})(t)|^2\,\mathrm{d}x. \tag{2.23}$$

Note that $\partial_j\partial_k v_m = \partial_j D_{km}\mathbf{v} + \partial_k D_{mj}\mathbf{v} - \partial_m D_{jk}\mathbf{v}$, which implies

$$|\boldsymbol{\nabla}^2\mathbf{v}| \leqslant 3|\mathbf{D}(\nabla\mathbf{v})| \leqslant 3|\boldsymbol{\nabla}^2\mathbf{v}|. \tag{2.24}$$

Thus, $|\mathbf{D}(\nabla\mathbf{v})|$ can always be replaced by $|\boldsymbol{\nabla}^2\mathbf{v}|$ (and vice versa) by increasing the multiplicative constant.

Lemma 2.25. *Let* $\mathbf{S}$ *satisfy* (2.5) *and* (2.6). *Then for all (sufficiently smooth)* $\mathbf{v}$, *for all* $1 \leqslant r \leqslant 2$, *and almost every* $t \in I$ *there holds:*

$$\|\mathbf{D}(\nabla\mathbf{v})(t)\|_r \leqslant C(\mathcal{I}(t,\mathbf{v}))^{1/2}\big\|(\tilde{D}\mathbf{v}(t))^{\frac{2-p(|\mathbf{E}(t)|^2)}{2}}\big\|_{2/(2-r)}, \tag{2.26}$$

$$\|\mathbf{D}(\partial_t\mathbf{v})(t)\|_r \leqslant C(\mathcal{J}(t,\mathbf{v}))^{1/2}\big\|(\tilde{D}\mathbf{v}(t))^{\frac{2-p(|\mathbf{E}(t)|^2)}{2}}\big\|_{2/(2-r)}, \tag{2.27}$$

where $2r/(2-r) = \infty$ *for* $r = 2$.

Proof. Observe that $1 \leqslant 2/r < \infty$ and $1 < (2/r)' = 2/(2-r) \leqslant \infty$. Further for $1 \leqslant r < 2$ we have

$$\begin{aligned}\|\mathbf{D}\mathbf{w}\|_r^r &= \int_\Omega ((\tilde{D}\mathbf{v})^{p-2}|\mathbf{D}\mathbf{w}|^2)^{r/2}(\tilde{D}\mathbf{v})^{(2-p)r/2}\,\mathrm{d}x\\ &\leqslant \Big(\int_\Omega (\tilde{D}\mathbf{v})^{p-2}|\mathbf{D}\mathbf{w}|^2\,\mathrm{d}x\Big)^{r/2}\|(\tilde{D}\mathbf{v})^{(2-p)r/2}\|_{2/(2-r)}\\ &= \Big(\int_\Omega (\tilde{D}\mathbf{v})^{p-2}|\mathbf{D}\mathbf{w}|^2\,\mathrm{d}x\Big)^{r/2}\|(\tilde{D}\mathbf{v})^{(2-p)/2}\|_{2r/(2-r)}^r.\end{aligned}$$

Choosing now $\mathbf{w} = \nabla\mathbf{v}$ and $\mathbf{w} = \partial_t\mathbf{v}$ and using (2.22) and (2.23), respectively, we obtain the assertions of the lemma for $r < 2$. The case $r = 2$ is treated similarly. □

Lemma 2.28. *Let* $\mathbf{S}$ *satisfy* (2.5) *and* (2.6). *For all (sufficiently smooth)* $\mathbf{v}$ *with* $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$ *and almost every* $t \in I$ *there holds*

$$\|\nabla\mathbf{v}(t)\|_{1,\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \leqslant C(\mathcal{I}(t,\mathbf{v})+1), \tag{2.29}$$

$$\|\partial_t\mathbf{v}(t)\|_{1,\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \leqslant C\mathcal{J}(t,\mathbf{v})^{p_\infty/2}(\mathcal{I}(t,\mathbf{v})+1)^{(2-p_\infty)/2} \tag{2.30}$$

$$\leqslant C(\mathcal{J}(t,\mathbf{v})+\mathcal{I}(t,\mathbf{v})+1). \tag{2.31}$$

Proof. From Lemma 2.25 ($r \mapsto \frac{3p_\infty}{p_\infty+1}$) we deduce, also using $2-p \leqslant 2-p_\infty$,

$$\begin{aligned}\|\mathbf{D}(\nabla\mathbf{v})\|_{\frac{3p_\infty}{p_\infty+1}} &\leqslant C\mathcal{I}(\mathbf{v})^{1/2}\|(\tilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{6p_\infty}{2-p_\infty}}\\ &\leqslant C\mathcal{I}(\mathbf{v})^{1/2}\|(\tilde{D}\mathbf{v})^{\frac{2-p_\infty}{2}}\|_{\frac{6p_\infty}{2-p_\infty}}\\ &\leqslant C\mathcal{I}(\mathbf{v})^{1/2}(1+\|\mathbf{D}\mathbf{v}\|_{3p_\infty})^{\frac{2-p_\infty}{2}}\\ &\leqslant C\mathcal{I}(\mathbf{v})^{1/2}\big(1+C\|\nabla\mathbf{D}\mathbf{v}\|_{\frac{3p_\infty}{p_\infty+1}}\big)^{\frac{2-p_\infty}{2}},\end{aligned}$$

since $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$. Due to $\nabla\mathbf{D}\mathbf{v} = \mathbf{D}(\nabla\mathbf{v})$, this implies

$$\|\mathbf{D}(\nabla\mathbf{v})\|_{\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \leqslant C(\mathcal{I}(\mathbf{v})+1).$$

From (2.24) and $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$ we get

$$\|\nabla \mathbf{v}\|^{p_\infty}_{1,\frac{3p_\infty}{p_\infty+1}} \leqslant C(\mathcal{I}(\mathbf{v})+1).$$

Analogously we can use Lemma 2.5 to get

$$\begin{aligned}
\|\mathbf{D}(\partial_t \mathbf{v})\|_{\frac{3p_\infty}{p_\infty+1}} &\leqslant C\mathcal{J}(\mathbf{v})\|^{1/2}\|(\tilde{D}\mathbf{v})^{\frac{2-p_\infty}{2}}\|_{\frac{6p_\infty}{2-p_\infty}} \\
&\leqslant C\mathcal{J}(\mathbf{v})^{1/2}\big(1+C\|\nabla \mathbf{D}\mathbf{v}\|_{\frac{3p_\infty}{p_\infty+1}}\big)^{\frac{2-p_\infty}{2}} \\
&\overset{(2.29)}{\leqslant} C\mathcal{J}(\mathbf{v})^{1/2}\big(1+C(\mathcal{I}(\mathbf{v})+1)^{\frac{1}{p_\infty}}\big)^{\frac{2-p_\infty}{2}} \\
&\leqslant C\mathcal{J}(\mathbf{v})^{1/2}(1+\mathcal{I}(\mathbf{v}))^{\frac{2-p_\infty}{2p_\infty}}.
\end{aligned}$$

Again $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$ and Korn's inequality imply

$$\|\partial_t \mathbf{v}\|_{1,\frac{3p_\infty}{p_\infty+1}} \leqslant C\|\mathbf{D}(\partial_t \mathbf{v})\|_{\frac{3p_\infty}{p_\infty+1}} \leqslant C\mathcal{J}(\mathbf{v})^{1/2}(1+\mathcal{I}(\mathbf{v}))^{\frac{2-p_\infty}{2p_\infty}},$$

which proves (2.30). The last inequality follows from Young's inequality. □

2.1. A priori estimates

Now we use a Galerkin approximation to derive *a priori* estimates for approximate solutions $\mathbf{v}^N$ of the system (2.2). These estimates allow the limiting process $N \to \infty$ showing the existence of a solution $\mathbf{v}$ of the system (2.2).

Let $\{\boldsymbol{\omega}^r\}$ denote the set consisting of the eigenvectors of the Stokes operator denoted by A. Let λ_r be the corresponding eigenvalues and $X_N := \operatorname{span}\{\boldsymbol{\omega}^1, \dots, \boldsymbol{\omega}^N\}$. Note that $\langle \boldsymbol{\omega}^r, 1\rangle = 0$. Define $P^N\mathbf{v} := \sum_{r=1}^N \langle \mathbf{v}, \boldsymbol{\omega}^r\rangle\, \boldsymbol{\omega}^r$. Then we have

$$\lambda_r \left\langle \boldsymbol{\omega}^r, \mathbf{v}^N\right\rangle = \left\langle A\boldsymbol{\omega}^r, \mathbf{v}^N\right\rangle = \left\langle \nabla\boldsymbol{\omega}^r, \nabla\mathbf{v}^N\right\rangle \tag{2.32}$$

and $P^N\colon W^{s,2} \to (X_N, \|\cdot\|_{s,2})$ are uniformly continuous for all $s \in [0,3]$ (cf. [42], [30]).

Setting $\mathbf{f}^N = P^N\mathbf{f}$ we seek the approximate solution $\mathbf{v}^N(t,x) = \sum_{r=1}^N c_r^N(t)\boldsymbol{\omega}^r(x)$, where the coefficients $c_r^N(t)$ solve the Galerkin system (for all $1 \leqslant r \leqslant N$)

$$\begin{aligned}
&\left\langle \partial_t \mathbf{v}^N, \boldsymbol{\omega}^r\right\rangle + \left\langle \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E})\mathbf{D}\boldsymbol{\omega}^r\right\rangle + \left\langle [\nabla \mathbf{v}^N]\mathbf{v}^N, \boldsymbol{\omega}^r\right\rangle \\
&\qquad = \left\langle \mathbf{f}^N, \boldsymbol{\omega}^r\right\rangle - \chi^E \left\langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}\boldsymbol{\omega}^r\right\rangle, \\
&\mathbf{v}^N(0) = P^N\mathbf{v}_0.
\end{aligned} \tag{2.33}$$

Since the matrix $\langle \omega_j, \omega_k \rangle$ with $j,k = 1,\dots,N$ is positive definite, the Galerkin system (2.33) can be re-written as a system of ordinary differential equations. This in turn fulfills the Carathéodory conditions and is therefore solvable locally in time, i.e. on a small time interval $I^* = [0,T^*)$. From the assumptions on $\mathbf{f}$ in Theorem 2.13 it follows that $\mathbf{f}^N = P^N\mathbf{f} \in L^\infty(I; W^{1,2}(\Omega))$ and $\partial_t \mathbf{f}^N = P^N(\partial_t \mathbf{f}) \in L^2(I; L^2(\Omega))$. This implies $c_r^N, \partial_t c_r^N, \partial_t^2 c_r^N \in L^2(I^*)$. Thus $\mathbf{v}^N, \partial_t \mathbf{v}^N, \partial_t^2 \mathbf{v}^N \in L^2(I^*; X_N)$. (Note that the norms may depend on N). To ensure solvability for large times at least for this finite dimensional problem we have to establish a first *a priori* estimate.

Since $\mathbf{v}^N \in L^2(I^*; X_N)$, we can test (2.33) with $\mathbf{v}^N$ and get

$$\text{(2.34)} \qquad \frac{1}{2} d_t \|\mathbf{v}^N\|_2^2 + \left\langle \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}), \mathbf{D}\mathbf{v}^N \right\rangle = \left\langle \mathbf{f}^N, \mathbf{v}^N \right\rangle - \chi^E \left\langle \mathbf{E} \otimes \mathbf{E}, \mathbf{D}\mathbf{v}^N \right\rangle.$$

Note that $\left\langle [\nabla \mathbf{v}^N]\mathbf{v}^N, \mathbf{v}^N \right\rangle = 0$ due to $\operatorname{div} \mathbf{v}^N = 0$. From the coercivity of $\mathbf{S}$ (cf. Lemma 2.19 (a)) and the pointwise inequalities

$$(1+y^2)^{\frac{q-2}{2}} y^2 \geqslant C(q)(y^q - 1), \qquad (1+y^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}} \geqslant (1+y^2)^{\frac{p_\infty - 2}{2}}$$

we deduce that the second term on the left-hand side of (2.34) is bounded from below by

$$C_2 \int_\Omega (1+|\mathbf{E}|^2)(|\mathbf{D}\mathbf{v}^N|^{p(|\mathbf{E}|^2)} + |\mathbf{D}\mathbf{v}^N|^{p_\infty})\, dx - C \int_\Omega 1 + |\mathbf{E}|^2\, dx.$$

The terms on the right-hand side of (2.34) are bounded from above by

$$\frac{C_2}{2} \int_\Omega (1+|\mathbf{E}|^2)|\mathbf{D}\mathbf{v}^N|^{p_\infty}\, dx + C\|\mathbf{E}\|_2^2 + C\|\mathbf{f}\|_2^{p'_\infty}.$$

Integration over time and Gronwall's inequality thus imply

$$\max_{[0,T^*]} \|\mathbf{v}^N\|_2^2 + \int_0^{T^*} \int_\Omega |\mathbf{D}\mathbf{v}^N|^{p(|\mathbf{E}|^2)} + |\mathbf{D}\mathbf{v}^N|^{p_\infty}\, dx\, dt \leqslant C(T, \mathbf{f}, \mathbf{v}_0, \mathbf{E}).$$

In particular we get

$$\|c_r^N\|_{L^\infty(I^*)} \leqslant C(T, \mathbf{f}, \mathbf{v}_0, \mathbf{E}), \qquad 1 \leqslant r \leqslant N.$$

As a consequence we can iterate Carathéodory's theorem to push the solvability of the Galerkin system (2.33) up to any fixed time interval $I = [0,T]$. Hence, independently of N

$$\text{(2.35)} \qquad \|\mathbf{v}^N\|_{L^\infty(I;L^2(\Omega))}^2 + \varrho_{p(|\mathbf{E}|^2),Q_T}(\mathbf{D}\mathbf{v}^N) + \|\nabla \mathbf{v}^N\|_{L^{p_\infty}(Q_T)}^{p_\infty} \leqslant C,$$

where we have also used Korn's inequality in $L^{p_\infty}(\Omega)$.

We got the first *a priori* estimate by using $\mathbf{v}^N$ as a test function. To derive our second *a priori* estimate we want to use $A\mathbf{v}^N$ as a test function. The special choice of base functions $\boldsymbol{\omega}^r$ ensures that we do not leave X_N, the space of admissible test functions. More explicitly we multiply the rth equation of the Galerkin system (2.33) by $\lambda_r c_r^N$ and use (2.32) to obtain

$$
\begin{aligned}
(2.36)\qquad & \langle \partial_t \mathbf{v}^N, A\mathbf{v}^N\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}^N,\mathbf{E}), \mathbf{D}(A\mathbf{v}^N)\rangle + \langle [\nabla \mathbf{v}^N]\mathbf{v}^N, A\mathbf{v}^N\rangle \\
&= \langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{v}^N\rangle - \chi^E \langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}(A\mathbf{v}^N)\rangle .
\end{aligned}
$$

Due to the periodicity we have $A = -\Delta$, and thus

$$
(2.37)\qquad \int_\Omega [\nabla \mathbf{v}^N]\mathbf{v}^N \cdot A\mathbf{v}^N \, dx = \int_\Omega \frac{\partial v_j^N}{\partial x_k}\frac{\partial v_i^N}{\partial x_j}\frac{\partial v_i^N}{\partial x_k}\, dx \leqslant \|\nabla \mathbf{v}^N\|_3^3,
$$

$$
\begin{aligned}
(2.38)\quad -\chi^{\mathbf{E}} \int_\Omega \mathbf{E}\otimes\mathbf{E}\cdot \mathbf{D}(A\mathbf{v}^N)\, dx &= 2\chi^E \int_\Omega E_i \frac{\partial E_j}{\partial x_k} D_{ij}\Big(\frac{\partial \mathbf{v}^N}{\partial x_k}\Big)\, dx \\
&\leqslant \frac{\gamma_1}{8}\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla \mathbf{v}^N)|^2\, dx \\
&\quad + C(\gamma_1, \mathbf{E}, \nabla \mathbf{E}) \int_\Omega (\tilde{D}\mathbf{v}^N)^{2-p(|\mathbf{E}|^2)}\, dx,
\end{aligned}
$$

$$
\begin{aligned}
(2.39)\quad \int_\Omega \mathbf{S}(\mathbf{D}\mathbf{v}^N,\mathbf{E})\cdot \mathbf{D}(A\mathbf{v}^N)\, dx &= \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}^N,\mathbf{E})}{\partial D_{kl}} D_{kl}(\nabla \mathbf{v}^N) D_{ij}(\nabla \mathbf{v}^N)\, dx \\
&\quad + \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}^N,\mathbf{E})}{\partial E_k}\nabla E_k D_{ij}(\nabla \mathbf{v}^N)\, dx.
\end{aligned}
$$

The right-hand side of (2.39) is bounded from below by

$$
\begin{aligned}
&\frac{1}{2}\mathcal{I}(\mathbf{v}^N) + \frac{\gamma_1}{2}\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla \mathbf{v}^N)|^2\, dx \\
&\quad - \frac{\gamma_1}{8}\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla \mathbf{v}^N)|^2\, dx \\
&\quad - C(\gamma_1, \nabla \mathbf{E})\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)}(1+\ln(\tilde{D}\mathbf{v}^N)^2)^2\, dx,
\end{aligned}
$$

where we used the definition of $\mathcal{I}$, (2.22) and Young's inequality. Thus we have

$$
\begin{aligned}
(2.40)\qquad & d_t\|\boldsymbol{\nabla}\mathbf{v}^N\|_2^2 + \mathcal{I}(\mathbf{v}^N) + \frac{\gamma_1}{2}\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla \mathbf{v}^N)|^2\, dx \\
&\leqslant C(1 + \|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3 + |\langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{v}^N\rangle| + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N)),
\end{aligned}
$$

where we also used the estimate $\ln(1+y^2) \leqslant c(1+y^2)^{\frac{1}{4}}$ and $p(|\mathbf{E}|^2) \leqslant p_0 \leqslant 2$, $2 - p(|\mathbf{E}|^2) \leqslant p(|\mathbf{E}|^2)$.

If $p > \frac{11}{5}$ one can show that $\|\nabla \mathbf{v}^N\|_3^3 \leqslant C_\varepsilon \|\nabla \mathbf{v}^N\|_p^p \|\nabla \mathbf{v}^N\|_2^2 + \varepsilon \mathcal{I}(\mathbf{v}^N)$ (see [30]), which enables us to apply Gronwall's inequality after absorbing $\varepsilon\mathcal{I}(\mathbf{v}^N)$ on the left-hand side. This would give us a global estimate. If $p > \frac{5}{3}$ we can show that $\|\nabla \mathbf{v}^N\|_3^3 \leqslant C_\varepsilon \|\nabla \mathbf{v}^N\|_p^p \|\nabla \mathbf{v}^N\|_2^R + \varepsilon \mathcal{I}(\mathbf{v}^N)$ for some constant $1 < R < \infty$ and thereafter absorb $\varepsilon\mathcal{I}(\mathbf{v}^N)$ on the left-hand side and apply a local version of Gronwall's inequality (cf. Lemma 2.52). This would give us an estimate for small times. Nevertheless we will not make use of these facts, since we are also interested in smaller values of p than $\frac{5}{3}$.

We will test immediately with "$\partial_t \mathbf{v}^N \partial_t$" to get in addition to (2.40) another estimate. Then we will use the resulting *two estimates at the same time* to derive quite strong *a priori* estimates for $\mathbf{v}^N$ for values up to $p > \frac{3}{2}$. Let us take the time derivative of the Galerkin system (2.33):

$$\begin{aligned}(2.41)\qquad &\left\langle \partial_t^2 \mathbf{v}^N, \boldsymbol{\omega}^r \right\rangle + \left\langle \partial_t \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}), \mathbf{D}\boldsymbol{\omega}^r \right\rangle + \left\langle \partial_t([\nabla \mathbf{v}^N]\mathbf{v}^N), \boldsymbol{\omega}^r \right\rangle \\ &\quad = \left\langle \partial_t \mathbf{f}^N, \boldsymbol{\omega}^r \right\rangle - \chi^E \left\langle \partial_t(\mathbf{E}\otimes\mathbf{E}), \mathbf{D}\boldsymbol{\omega}^r \right\rangle,\end{aligned}$$

for $1 \leqslant r \leqslant N$. Since $\mathbf{v}^N \in W^{2,2}(I; X_n)$, this makes sense and we can even test with $\partial_t \mathbf{v}^N \in W^{1,2}(I; X_n)$ resulting in

$$\begin{aligned}&\frac{1}{2} d_t \|\partial_t \mathbf{v}^N\|_2^2 + \left\langle \partial_t \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}), \mathbf{D}(\partial_t \mathbf{v}^N) \right\rangle + \left\langle \partial_t([\nabla \mathbf{v}^N]\mathbf{v}^N), \partial_t \mathbf{v}^N \right\rangle \\ &\quad = \left\langle \partial_t \mathbf{f}^N, \partial_t \mathbf{v}^N \right\rangle - \chi^E \left\langle \partial_t(\mathbf{E}\otimes\mathbf{E}), \mathbf{D}(\partial_t \mathbf{v}^N) \right\rangle.\end{aligned}$$

Similarly as in (2.38) and (2.39) we get

$$\begin{aligned}-\chi^{\mathbf{E}} \int_\Omega \partial_t(\mathbf{E}\otimes\mathbf{E}) \cdot \mathbf{D}(\partial_t \mathbf{v}^N)\,\mathrm{d}x &\leqslant \frac{\gamma_1}{8} \int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2} |\mathbf{D}(\partial_t \mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad + C(\gamma_1, \mathbf{E}, \partial_t \mathbf{E}) \int_\Omega (\tilde{D}\mathbf{v}^N)^{2-p(|\mathbf{E}|^2)}\,\mathrm{d}x, \\ \int_\Omega \partial_t \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}) \cdot \mathbf{D}(\partial_t \mathbf{v}^N)\,\mathrm{d}x &= \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}^N, \mathbf{E})}{\partial D_{kl}} D_{kl}(\partial_t \mathbf{v}^N) D_{ij}(\partial_t \mathbf{v}^N)\,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}^N, \mathbf{E})}{\partial E_k} \partial_t E_k D_{ij}(\partial_t \mathbf{v}^N)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\mathcal{J}(\mathbf{v}^N) + \frac{\gamma_1}{2} \int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2} |\mathbf{D}(\partial_t \mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad - \frac{\gamma_1}{8} \int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2} |\mathbf{D}(\partial_t \mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad - C(\gamma_1, \partial_t \mathbf{E}) \int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)} (1 + \ln(\tilde{D}\mathbf{v}^N)^2)^2\,\mathrm{d}x,\end{aligned}$$

where we used the definition of $\mathcal{J}$, (2.23) and Young's inequality. This yields (cf. (2.40)), also using $\operatorname{div}\mathbf{v}^N = 0$ in the convective term,

$$(2.42)\qquad \begin{aligned} d_t\|\partial_t\mathbf{v}^N\|_2^2 + \mathcal{J}(\mathbf{v}^N) + \frac{\gamma_1}{2}\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\partial_t\mathbf{v}^N)|^2\,\mathrm{d}x \\ \leqslant C\big(1 + \big|\big\langle[\nabla\mathbf{v}^N]\partial_t\mathbf{v}^N, \partial_t\mathbf{v}^N\big\rangle\big| + \big|\big\langle\partial_t\mathbf{f}^N, \partial_t\mathbf{v}^N\big\rangle\big| + \|\nabla\mathbf{v}^N\|_3^3 \\ + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N)\big). \end{aligned}$$

Recall that

$$(2.43)\qquad \begin{aligned} d_t\|\boldsymbol{\nabla}\mathbf{v}^N\|_2^2 + \mathcal{I}(\mathbf{v}^N) + \frac{\gamma_1}{2}\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla\mathbf{v}^N)|^2\,\mathrm{d}x \\ \leqslant C\big(1 + \|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3 + \big|\big\langle\boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{v}^N\big\rangle\big| + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N)\big). \end{aligned}$$

At first sight, we have gained nothing. We have to control one more bad term, namely $|\langle[\nabla\mathbf{v}^N]\partial_t\mathbf{v}^N, \partial_t\mathbf{v}^N\rangle|$, but we only got more information about the time derivative of $\mathbf{v}^N$. But the critical term $\|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3$, which gave the lower bound for p, has no time derivatives. The next lemma shows that $\mathcal{J}(\mathbf{v}^N)$ reveals indeed more information.

Lemma 2.44. *Let $1 < q < \infty$, then for almost every $t \in I$*

$$(2.45)\qquad \begin{aligned} d_t(\|\tilde{D}\mathbf{v}(t)\|_q^q) &\leqslant C\mathcal{J}(t,\mathbf{v})^{\frac{1}{2}}(\varrho_{2q-p(|\mathbf{E}(t)|^2),\Omega}(\tilde{D}\mathbf{v}(t)))^{1/2} \\ &\leqslant \varepsilon\mathcal{J}(t,\mathbf{v}) + C_\varepsilon\varrho_{2q-p(|\mathbf{E}(t)|^2),\Omega}(\tilde{D}\mathbf{v}(t)), \end{aligned}$$

where $\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}) = \int_\Omega(\tilde{D}\mathbf{v})^{2q-p(|\mathbf{E}|^2)}\,\mathrm{d}x$ even if $2q - p(|\mathbf{E}|^2) < 1$.

Proof. Note that

$$\partial_t((\tilde{D}\mathbf{v})^q) = q(\tilde{D}\mathbf{v})^{q-2}(D_{jk}\mathbf{v})(\partial_t D_{jk}\mathbf{v}).$$

Hence

$$\begin{aligned} d_t(\|\tilde{D}\mathbf{v}\|_q^q) &\leqslant q\int_\Omega(\tilde{D}\mathbf{v})^{q-1}\|\partial_t\mathbf{D}\mathbf{v}\|\,\mathrm{d}x \\ &= q\int_\Omega(\tilde{D}\mathbf{v})^{\frac{p-2}{2}}|\mathbf{D}(\partial_t\mathbf{v})|(\tilde{D}\mathbf{v})^{q-\frac{1}{2}p}\,\mathrm{d}x \\ &\leqslant qC\mathcal{J}(\mathbf{v})^{\frac{1}{2}}(\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}))^{\frac{1}{2}}, \end{aligned}$$

by Hölder's inequality, which proves the first assertion. The second follows from Young's inequality. □

This lemma enables us to produce $d_t(\|\tilde{D}\mathbf{v}^N\|_q^q)$ on the left-hand side of (2.42) if we add $C\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v})$ to the right-hand side. Thus we have three terms to control:

$$\|\nabla\mathbf{v}^N\|_3^3, \quad |\langle[\nabla\mathbf{v}^N]\partial_t\mathbf{v}^N, \partial_t\mathbf{v}^N\rangle|, \quad \varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}^N). \tag{2.46}$$

The first and the second one will be easier to estimate for large q, but the third one for small q. The problem now is to find the *optimal* choice for q. We start by examining which values of q are needed for the first and the second term. In view of local Gronwall's inequality (cf. Lemma 2.52), we will be able to control arbitrary powers of $\|\tilde{D}\mathbf{v}^N\|_q^q$ and $\|\partial_t\mathbf{v}^N\|_2^2$.

Lemma 2.47. *Let $q > \frac{9-3p_\infty}{2}$, then there exists a constant $R_1 = R_1(p_\infty) > q$, such that*

$$\|\nabla\mathbf{v}\|_3^3 \leqslant C_\varepsilon\|\tilde{D}\mathbf{v}\|_q^{R_1} + \varepsilon\mathcal{I}(\mathbf{v}) + \varepsilon.$$

Proof. If $q \geqslant 3$, then there is nothing to prove, so assume $q < 3$. We interpolate $L^3(\Omega) = [L^q(\Omega), L^{3p_\infty}(\Omega)]_\theta$ with $\theta = \frac{(3-q)p_\infty}{3p_\infty - q}$, $1-\theta = \frac{q(p_\infty-1)}{3p_\infty-q}$ and obtain

$$\|\nabla\mathbf{v}\|_3^3 \leqslant \|\nabla\mathbf{v}\|_q^{3(1-\theta)}\|\nabla\mathbf{v}\|_{3p_\infty}^{3\theta}.$$

If $3\theta < p_\infty$, there exists an $\delta > 1$ such that

$$\begin{aligned}
\|\nabla\mathbf{v}\|_3^3 &\leqslant C_\varepsilon\|\nabla\mathbf{v}\|_q^{3(1-\theta)\delta'} + \varepsilon\|\nabla\mathbf{v}\|_{3p_\infty}^{p_\infty} \\
&\leqslant C_\varepsilon\|\nabla\mathbf{v}\|_q^{3(1-\theta)\delta'} + \varepsilon C\|\nabla\mathbf{v}\|_{1,\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \\
&\leqslant C_\varepsilon\|\nabla\mathbf{v}\|_q^{3(1-\theta)\delta'} + \varepsilon C(\mathcal{I}(\mathbf{v}) + 1),
\end{aligned}$$

where we used Lemma 2.28. So by Korn's inequality

$$\|\nabla\mathbf{v}\|_3^3 \leqslant C_{\varepsilon_2}\|\tilde{D}\mathbf{v}\|_q^{3(1-\theta)\delta'} + \varepsilon_2\mathcal{I}(\mathbf{v}) + \varepsilon_2.$$

We still have to verify $3\theta < p_\infty$, but this is equivalent to

$$\frac{3(3-q)p_\infty}{3p_\infty - q} < p_\infty \Longleftrightarrow \frac{9-3p_\infty}{2} < q,$$

which holds due to the assumption on q. $\square$

Lemma 2.48. *Let $q > \frac{9-3p_\infty}{2}$, then there exist constants $R_2 = R_2(p_\infty) > 2$ and $R_3 = R_3(p_\infty) > q$ such that*

$$|\langle[\nabla \mathbf{v}]\partial_t \mathbf{v}, \partial_t \mathbf{v}\rangle| \leqslant \varepsilon \mathcal{J}(\mathbf{v}) + C_\varepsilon(\|\partial_t \mathbf{v}\|_2^{R_2} + \|\tilde{D}\mathbf{v}\|_q^{R_3} + 1).$$

Proof. Note that Lemma 2.25 ($r \mapsto \frac{2q}{2-p_\infty+q}$) implies

$$\begin{aligned} \|\mathbf{D}(\partial_t \mathbf{v})\|_{\frac{2q}{2-p_\infty+q}} &\leqslant C\mathcal{J}(\mathbf{v})^{\frac{1}{2}} \|(\tilde{D}\mathbf{v})\|_{\frac{2q}{2-p_\infty}}^{\frac{2-p}{2}} \\ &\leqslant C\mathcal{J}(\mathbf{v})^{\frac{1}{2}} \|\tilde{D}\mathbf{v}\|_q^{\frac{2-p_\infty}{2}}, \end{aligned} \tag{2.49}$$

where we used that $(1+y^2)^{(2-p)/4} \leqslant (1+y^2)^{(2-p_\infty)/4}$. Furthermore we have the embedding $W^{1,\frac{2q}{2-p_\infty+q}}(\Omega) \hookrightarrow L^{\frac{6q}{6-3p_\infty+q}}(\Omega)$. Since $\frac{9-3p_\infty}{2} < q$ is equivalent to $\frac{2q}{q-1} < \frac{6q}{6-3p_\infty+q}$, we can interpolate $L^{\frac{2q}{q-1}}(\Omega) = [L^2(\Omega), L^{\frac{6q}{6-3p_\infty+q}}(\Omega)]_\theta$. This and Korn's and Young's inequalities imply

$$\begin{aligned} |\langle[\nabla \mathbf{v}]\partial_t \mathbf{v}, \partial_t \mathbf{v}\rangle| &\leqslant \|\partial_t \mathbf{v}\|_{\frac{2q}{q-1}}^2 \|\nabla \mathbf{v}\|_q \\ &\leqslant C\|\partial_t \mathbf{v}\|_2^{2(1-\theta)} \|\partial_t \mathbf{v}\|_{\frac{6q}{6-3p_\infty+q}}^{2\theta} \|\nabla \mathbf{v}\|_q \\ &\leqslant C\|\partial_t \mathbf{v}\|_2^{2(1-\theta)} \|\partial_t \nabla \mathbf{v}\|_{\frac{2q}{2-p_\infty+q}}^{2\theta} \|\nabla \mathbf{v}\|_q \\ &\overset{(2.49)}{\leqslant} C\|\partial_t \mathbf{v}\|_2^{2(1-\theta)} (\mathcal{J}(\mathbf{v})^{\frac{1}{2}} \|\tilde{D}\mathbf{v}\|_q^{\frac{2-p_\infty}{2}})^{2\theta} \|\nabla \mathbf{v}\|_q \\ &\leqslant \varepsilon \mathcal{J}(\mathbf{v}) + C_\varepsilon(\|\partial_t \mathbf{v}\|_2^{R_2} + \|\tilde{D}\mathbf{v}\|_q^{R_3} + 1). \end{aligned}$$

□

It is indeed interesting that both terms $|\langle[\nabla \mathbf{v}]\partial_t \mathbf{v}, \partial_t \mathbf{v}\rangle|$ and $\|\nabla \mathbf{v}^N\|_3^3$ require the same bound for q, which is $q > \frac{1}{2}(9-3p_\infty)$. Now we have to find the upper bound for q, in order to control $\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}^N)$. For that we require $q \leqslant \frac{1}{2}(3+p_\infty)$ and obtain

$$\int_\Omega |\tilde{D}\mathbf{v}^N|^{2q-p(|\mathbf{E}|^2)}\,\mathrm{d}x \leqslant \int_\Omega |\tilde{D}\mathbf{v}^N|^{2q-p_\infty}\,\mathrm{d}x = \|\tilde{D}\mathbf{v}^N\|_{2q-p_\infty}^{2q-p_\infty} \leqslant C(\|\nabla \mathbf{v}^N\|_3^3 + 1),$$

since $2q - p_\infty \leqslant 3$. That means that $\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}^N)$ can be controlled if $\|\nabla \mathbf{v}^N\|_3^3$ can be controlled. But for $p_\infty > \frac{3}{2}$ we can always find q such that

$$\frac{9-3p_\infty}{2} < q \leqslant \frac{3+p_\infty}{2}.$$

Thus all terms in (2.46) can be controlled under this condition. It remains to control the terms involving $\mathbf{f}^N$ in (2.42) and (2.43), which is easily established by

$$\begin{aligned}|\langle\nabla\mathbf{f}^N,\nabla\mathbf{v}^N\rangle| &\leqslant \|P^N\mathbf{f}\|_{1,2}\,\|\nabla\mathbf{v}^N\|_2 \leqslant C\|\mathbf{f}\|_{1,2}\,\|\nabla\mathbf{v}^N\|_2\\ &\leqslant C\|\mathbf{f}\|_{1,2}^2 + C\|\tilde{D}\mathbf{v}^N\|_q^2,\\ |\langle\partial_t\mathbf{f}^N,\partial_t\mathbf{v}^N\rangle| &\leqslant \|P^N(\partial_t\mathbf{f})\|_2\,\|\partial_t\mathbf{v}^N\|_2 \leqslant C\|\partial_t\mathbf{f}\|_2\,\|\partial_t\mathbf{v}^N\|_2\\ &\leqslant C\|\partial_t\mathbf{f}\|_2^2 + C\|\partial_t\mathbf{v}^N\|_2^2.\end{aligned}$$

Finally we have, since $p(|\mathbf{E}|^2) \leqslant p_0 \leqslant 2 \leqslant q$,

$$\varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N) \leqslant \|\tilde{D}\mathbf{v}^N\|_{p_0}^{p_0} \leqslant C\|\tilde{D}\mathbf{v}^N\|_q^2. \tag{2.50}$$

Hence by Lemma 2.47, Lemma 2.48, Korn's inequality, and the above calculations we get, for $\max(2, \frac{9-3p_\infty}{2}) < q \leqslant \frac{3+p_\infty}{2}$,

$$\begin{aligned}&d_t\|\partial_t\mathbf{v}^N\|_2^2 + d_t(\|\tilde{D}\mathbf{v}(t)\|_q^q) + d_t\|\nabla\mathbf{v}^N\|_2^2 + \mathcal{I}(\mathbf{v}^N) + \frac{\gamma_1}{2}\mathcal{J}(\mathbf{v}^N)\\ &\quad\leqslant C(1 + |\langle[\nabla\mathbf{v}]\partial_t\mathbf{v}^N,\partial_t\mathbf{v}^N\rangle| + |\langle\partial_t\mathbf{f}^N,\partial_t\mathbf{v}^N\rangle| + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N)\\ &\qquad + |\langle\nabla\mathbf{f}^N,\nabla\mathbf{v}^N\rangle| + \|\nabla\mathbf{v}^N\|_3^3 + \varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}^N))\\ &\quad\leqslant C(1 + \|\tilde{D}\mathbf{v}^N\|_q^{\max(R_1,R_3,2)} + \|\partial_t\mathbf{v}^N\|_2^{\max(R_2,2)} + \|\mathbf{f}\|_{1,2}^2 + \|\partial_t\mathbf{f}\|_2^2).\end{aligned}$$

The following lemma ensures that for small times T' we get boundedness (uniformly with respect to N) of the following expressions, for $\max(2, \frac{9-3p_\infty}{2}) < q \leqslant \frac{3+p_\infty}{2}$:

$$\begin{aligned}&\|\partial_t\mathbf{v}^N\|_{L^\infty(I';L^2(\Omega))}^2, &&\|\nabla\mathbf{v}^N\|_{L^\infty(I';L^q(\Omega))}^q,\\ &\|\mathcal{I}(\mathbf{v}^N)\|_{L^1(I')}, &&\|\mathcal{J}(\mathbf{v}^N)\|_{L^1(I')},\end{aligned} \tag{2.51}$$

where $I' = [0,T']$. These *a priori* estimates in turn are sufficient to pass to the limit $N \to \infty$ to get a solution $\mathbf{v}$ of our original problem (2.2).

Lemma 2.52 (local version of Gronwall's lemma). *Let $T, \alpha, c_0 > 0$ be given constants and let $0 < h \in C([0,T])$, $0 \leqslant f \in C^1([0,T])$ satisfy*

$$f'(t) \leqslant h(t) + c_0 f(t)^{1+\alpha}. \tag{2.53}$$

Then

$$f(t) \leqslant H(t) + H(t_0)\big((1 - \alpha c_0 H(t_0)^\alpha\, t)^{-\frac{1}{\alpha}} - 1\big)$$

for all $t \in [0, t_0)$, where

$$H(t) := f(0) + \int_0^t h(s)\,\mathrm{d}s,$$

and where t_0 is defined by the condition $\alpha c_0 H(t_0)^\alpha\, t_0 = 1$.

Proof. Define $a\colon [0,t_0) \to \mathbb{R}^{\geqslant 0}$ by

$$a(t) := H(t_0)\big((1-\alpha c_0 H(t_0)^{\alpha}\, t)^{-\frac{1}{\alpha}} - 1\big).$$

Then a solves

$$\begin{aligned} a'(t) &= c_0(H(t_0)+a(t))^{1+\alpha},\\ a(0) &= 0. \end{aligned}$$

Setting $z(t) := H(t) + a(t)$ we see that for all $t \in [0,t_0)$ holds

$$\begin{aligned} z'(t) = h(t) + a'(t) &= h(t) + c_0(H(t_0)+a(t))^{1+\alpha}\\ &> h(t) + c_0(H(t)+a(t))^{1+\alpha} = h(t) + c_0 z(t)^{1+\alpha}. \end{aligned}$$

Since $z(0) \geqslant f(0)$ we get from this and (2.53) that $f'(0) < z'(0)$. Consequently, there exists $t' > 0$ such that for all $t \in [0,t']$ holds $f(t) \leqslant z(t)$. Iterating this argument we obtain the assertion of the lemma. □

In order to derive the last estimate from Theorem 2.13 we go once more into (2.36) and move the term with the time derivative to the right-hand side. This gives

$$\mathcal{I}(\mathbf{v}^N) \leqslant C\big(1 + \|\nabla \mathbf{v}^N\|_3^3 + |\langle \nabla \mathbf{f}^N, \nabla \mathbf{v}^N\rangle| + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N) + |\langle \partial_t \mathbf{v}^N, -\Delta \mathbf{v}^N\rangle|\big).$$

Using

$$\|\mathbf{f}^N\|_{L^\infty(I;W^{1,2}(\Omega))} = \|P^N \mathbf{f}\|_{L^\infty(I;W^{1,2}(\Omega))} \leqslant C\|\mathbf{f}\|_{L^\infty(I;W^{1,2}(\Omega))} \leqslant C,$$

together with (2.50), (2.51) and Lemma 2.47, for $q > \max\big(2, \frac{1}{2}(9-3p_\infty)\big)$, we get

$$\mathcal{I}(\mathbf{v}^N) \leqslant C\big(1 + |\langle \partial_t \mathbf{v}^N, -\Delta \mathbf{v}^N\rangle|\big). \tag{2.54}$$

The following lemma gives control of the remaining term $|\langle \partial_t \mathbf{v}^N, -\Delta \mathbf{v}^N\rangle|$.

Lemma 2.55. *For $1 < p_\infty \leqslant 2$ there holds*

$$|\langle \partial_t \mathbf{v}, \Delta \mathbf{v}\rangle| \leqslant C\|\partial_t \mathbf{v}\|_2^{\frac{4(p_\infty-1)}{3p_\infty-2}} \mathcal{J}(\mathbf{v})^{\frac{2-p_\infty}{2(3p_\infty-2)}} (\mathcal{I}(\mathbf{v})+1)^{\frac{p_\infty+2}{2(3p_\infty-2)}}.$$

Proof. With the help of Lemma 2.28 we conclude

$$\begin{aligned} |\langle \partial_t \mathbf{v}, \Delta \mathbf{v}\rangle| &\leqslant \|\partial_t \mathbf{v}\|_{\frac{3p_\infty}{2p_\infty-1}} \|\mathbf{v}\|_{2,\frac{3p_\infty}{p_\infty+1}}\\ &\leqslant C\|\partial_t \mathbf{v}\|_{\frac{3p_\infty}{2p_\infty-1}} (\mathcal{I}(\mathbf{v})+1)^{1/p_\infty}\\ &\leqslant C\|\partial_t \mathbf{v}\|_2^{1-\theta} \|\partial_t \mathbf{v}\|_{1,\frac{3p_\infty}{1+p_\infty}}^{\theta} (\mathcal{I}(\mathbf{v})+1)^{1/p_\infty}\\ &\leqslant C\|\partial_t \mathbf{v}\|_2^{1-\theta} \big(\mathcal{J}(\mathbf{v})^{\frac{1}{2}} (\mathcal{I}(\mathbf{v})+1)^{\frac{2-p_\infty}{2p_\infty}}\big)^{\theta} (\mathcal{I}(\mathbf{v})+1)^{1/p_\infty}, \end{aligned}$$

where we used the interpolation $L^{\frac{3p_\infty}{2p_\infty-1}}(\Omega) = [L^2(\Omega), L^{3p_\infty}(\Omega)]_\theta$ with $\theta = \frac{2-p_\infty}{3p_\infty-2}$, $1-\theta = \frac{4p_\infty-4}{3p_\infty-2}$. Consequently $\frac{2-p_\infty}{2p_\infty}\theta + \frac{1}{p_\infty} = \frac{p_\infty+2}{2(3p_\infty-2)}$. This proves the lemma. □

This lemma, (2.54) and (2.51) imply

$$1+\mathcal{I}(\mathbf{v}^N) \leqslant C\big(1+\mathcal{J}(\mathbf{v}^N)^{\frac{2-p_\infty}{2(3p_\infty-2)}}(\mathcal{I}(\mathbf{v}^N)+1)^{\frac{p_\infty+2}{2(3p_\infty-2)}}\big).$$

Thus by Young's inequality, which is applicable for $p_\infty > \frac{6}{5}$, we get

$$\mathcal{I}(\mathbf{v}^N) \leqslant C\big(1+\mathcal{J}(\mathbf{v}^N)^{\frac{2-p_\infty}{5p_\infty-6}}\big), \tag{2.56}$$

which raised to the power $\frac{5p_\infty-6}{2-p_\infty}$ gives, in view of (2.51),

$$\mathcal{I}(\mathbf{v}^N)^{\frac{5p_\infty-6}{2-p_\infty}} \leqslant C\big(1+\mathcal{J}(\mathbf{v}^N)\big) \leqslant C.$$

This and (2.51) implies that the following expressions are bounded independently on N, for $\max\big(2, \frac{9-3p_\infty}{2}\big) < q \leqslant \frac{3+p_\infty}{2}$,

$$\begin{aligned} &\|\partial_t \mathbf{v}^N\|^2_{L^\infty(I';L^2(\Omega))}, && \|\nabla \mathbf{v}^N\|^q_{L^\infty(I';L^q(\Omega))}, \\ &\|\mathcal{I}(\mathbf{v}^N)\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I')}, && \|\mathcal{J}(\mathbf{v}^N)\|_{L^1(I')}. \end{aligned} \tag{2.57}$$

2.2. Passage to the limit

From (2.57) and Lemma 2.28 it follows that

$$\|\mathbf{v}^N\|_{L^{p_\infty\frac{5p_\infty-6}{2-p_\infty}}(I';W^{2,p_\infty}(\Omega))} \leqslant C, \tag{2.58}$$

$$\|\partial_t \mathbf{v}^N\|_{L^\infty(I';L^2(\Omega))} + \|\partial_t \mathbf{v}^N\|_{L^{p_\infty}(I';W^{1,\frac{3p_\infty}{p_\infty+1}}(\Omega))} \leqslant C, \tag{2.59}$$

since $\langle \mathbf{v}^N, 1\rangle = 0$. Thus we can pick a subsequence (still denoted by $\mathbf{v}^N$) with

$$\mathbf{v}^N \rightharpoonup \mathbf{v} \quad \text{in } L^{p_\infty\frac{5p_\infty-6}{2-p_\infty}}(I';W^{2,p_\infty}(\Omega)), \tag{2.60}$$

$$\partial_t\mathbf{v}^N \rightharpoonup \partial_t\mathbf{v} \quad \text{in } L^\infty(I';L^2(\Omega)) \cap L^{p_\infty}(I';W^{1,\frac{3p_\infty}{p_\infty+1}}(\Omega)), \tag{2.61}$$

where we have used the fact that the weak limit of distributions on $I\times\Omega$ is unique. Since $W^{2,p}(\Omega) \hookrightarrow\hookrightarrow W^{1,2}(\Omega)$ for $p > \frac{6}{5}$, the lemma of Aubin-Lions implies the existence of a subsequence such that

$$\nabla \mathbf{v}^N \to \nabla \mathbf{v} \quad \text{in } L^2(I'\times\Omega). \tag{2.62}$$

As a consequence we get convergence of the convective term

$$[\nabla \mathbf{v}^N]\mathbf{v}^N \to [\nabla \mathbf{v}]\mathbf{v} \quad \text{in } L^1(I'\times\Omega). \tag{2.63}$$

Observe that we have due to Lemma 2.19 (c) (with $\mathbf{B} = \mathbf{0}$) and $p(|\mathbf{E}|^2) \leqslant p_0 \leqslant 2$

$$\|\mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E})\|_{L^2(I' \times \Omega)} \leqslant C(\mathbf{E}) \|(\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-1}\|_{L^2(I' \times \Omega)} \leqslant C(1 + \|\nabla \mathbf{v}^N\|_{L^2(I' \times \Omega)}) \leqslant C. \tag{2.64}$$

On the other hand by (2.62) $\mathbf{D}\mathbf{v}^N \to \mathbf{D}\mathbf{v}$ a.e. in $I' \times \Omega$, so

$$\mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}) \to \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}) \quad \text{a.e. in } I' \times \Omega \tag{2.65}$$

due to the continuity properties of $\mathbf{S}$. Now Vitali's convergence theorem, (2.64) and (2.65) imply

$$\mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}) \to \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}) \quad \text{in } L^1(I' \times \Omega). \tag{2.66}$$

Now we can easily pass to the limit in the Galerkin system (2.33). Indeed, choose $\boldsymbol{\omega}^r$ and $\varphi \in C_0^\infty(I')$, then we can conclude from (2.33), (2.61), (2.63), and (2.66) that

$$\int_{I'} \varphi(\langle \partial_t \mathbf{v}, \boldsymbol{\omega}^r \rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}), \mathbf{D}\boldsymbol{\omega}^r \rangle + \langle [\nabla \mathbf{v}]\mathbf{v}, \boldsymbol{\omega}^r \rangle) \, \mathrm{d}t = \int_{I'} \varphi(\langle \mathbf{f}, \boldsymbol{\omega}^r \rangle - \chi^E \langle \mathbf{E} \otimes \mathbf{E}, \mathbf{D}\boldsymbol{\omega}^r \rangle) \, \mathrm{d}t.$$

Furthermore $\mathbf{v}$ fulfills

$$\|\partial_t \mathbf{v}\|_{L^2(I' \times \Omega)} + \|\mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E})\|_{L^1(I' \times \Omega)} + \|[\nabla \mathbf{v}]\mathbf{v}\|_{L^{\frac{4}{3}}(I' \times \Omega)} \leqslant C.$$

Since $\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \ldots\}$ is dense in $W^{s,2}(\Omega) \cap V_{p_\infty}$ and $W^{s,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for $s > \frac{5}{2}$, we deduce that

$$\int_{I'} \varphi(\langle \partial_t \mathbf{v}, \boldsymbol{\omega} \rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}), \mathbf{D}\boldsymbol{\omega} \rangle + \langle [\nabla \mathbf{v}], \mathbf{v}, \boldsymbol{\omega} \rangle) \, \mathrm{d}t = \int_{I'} \varphi(\langle \mathbf{f}, \boldsymbol{\omega} \rangle - \chi^E \langle \mathbf{E} \otimes \mathbf{E}, \mathbf{D}\boldsymbol{\omega} \rangle) \, \mathrm{d}t$$

is fulfilled for all $\boldsymbol{\omega} \in W^{s,2}(\Omega) \cap V_{p_\infty}$, especially for all $\boldsymbol{\omega} \in \mathcal{V}$. Note that

$$\langle \partial_t \mathbf{v}, \boldsymbol{\omega} \rangle, \ \langle \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}), \mathbf{D}\boldsymbol{\omega} \rangle, \ \langle [\nabla \mathbf{v}]\mathbf{v}, \boldsymbol{\omega} \rangle, \ \langle \mathbf{f}, \boldsymbol{\omega} \rangle, \ \langle \mathbf{E} \otimes \mathbf{E}, \mathbf{D}\boldsymbol{\omega} \rangle \in L^1(I')$$

and thus we obtain for all $\boldsymbol{\omega} \in \mathcal{V}$ and a.e. $t \in I'$

$$\langle \partial_t \mathbf{v}, \boldsymbol{\omega} \rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}), \mathbf{D}\boldsymbol{\omega} \rangle + \langle [\nabla \mathbf{v}]\mathbf{v}, \boldsymbol{\omega} \rangle = \langle \mathbf{f}, \boldsymbol{\omega} \rangle - \chi^E \langle \mathbf{E} \otimes \mathbf{E}, \mathbf{D}\boldsymbol{\omega} \rangle. \tag{2.67}$$

It remains to show that $\mathbf{v}(0) = \mathbf{v}_0$. The embedding $W^{1,2}(I') \hookrightarrow C^{\frac{1}{2}}(\overline{I'})$ and the interpolation $L^\infty(I') = [L^2(I'), W^{1,2}(I')]_{\frac{1}{2}}$ imply

$$\begin{aligned}(2.68)\quad \|P^N\mathbf{v}_0 - \mathbf{v}(0)\|_2 &= \|\mathbf{v}^N(0) - \mathbf{v}(0)\|_2 \\ &\leqslant C\,\underbrace{\|\mathbf{v}^N - \mathbf{v}\|^{\frac{1}{2}}_{L^2(I';L^2(\Omega))}}_{\to 0}\,\underbrace{\|\partial_t\mathbf{v}^N - \partial_t\mathbf{v}\|^{\frac{1}{2}}_{L^2(I';L^2(\Omega))}}_{\leqslant C} \to 0.\end{aligned}$$

Since $P^N\mathbf{v}_0 \to \mathbf{v}_0$ in $L^2(\Omega)$ we get $\mathbf{v}(0) = \mathbf{v}_0$. Overall we have shown by (2.67) and (2.68) that $\mathbf{v}$ satisfies (2.2) in the weak sense. It remains to prove the estimates for $\mathbf{v}$, $\mathcal{I}(\mathbf{v})$ and $\mathcal{J}(\mathbf{v})$. First of all, from (2.60) and (2.61) it follows that

$$(2.69)\qquad \|\partial_t\mathbf{v}\|_{L^\infty(I';L^2(\Omega))} + \|\mathbf{v}\|_{L^{p_\infty\frac{5p_\infty-6}{2-p_\infty}}(I';W^{2,p_\infty}(\Omega))} \leqslant C.$$

The passage to the limit in the expressions $\|\mathcal{I}(\mathbf{v}^N)\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I')}$ and $\|\mathcal{J}(\mathbf{v}^N)\|_{L^1(I')}$ is possible, since due to (2.62), (2.58), (2.59) and the convexity of $\mathcal{I}$ and $\mathcal{J}$ in $\mathbf{D}(\nabla\mathbf{v})$ and $\mathbf{D}(\partial_t\mathbf{v})$, respectively, we can use De Giorgi's semicontinuity theorem (cf. [23], p. 132) and a version of it (cf. [12]) to obtain

$$(2.70)\qquad \int_0^{T'} \mathcal{I}(t,\mathbf{v})^{\frac{5p_\infty-6}{2-p_\infty}} + \mathcal{J}(t,\mathbf{v})\,dt \leqslant C.$$

Moreover from this, (2.30) and Young's inequality we get

$$(2.71)\qquad \int_0^{T'} \|\partial_t\mathbf{v}\|_{1,\frac{3p_\infty}{p_\infty+1}}^{\frac{p_\infty(5p_\infty-6)}{(3p_\infty-2)(p_\infty-1)}}\,dt \leqslant C\int_0^{T'} \mathcal{I}(t,\mathbf{v})^{\frac{5p_\infty-6}{2-p_\infty}} + \mathcal{J}(t,\mathbf{v})\,dt \leqslant C.$$

In order to obtain the estimate for $\partial_t^2\mathbf{v}$ we differentiate (2.67) with respect to time in the sense of distributions, which yields for all $\boldsymbol{\omega} \in \mathcal{V}$ and all $\varphi \in C_0^\infty(I')$

$$\begin{aligned}(2.72)\quad \int_0^{T'} \langle\partial_t^2\mathbf{v},\boldsymbol{\omega}\rangle\varphi\,dt = \int_0^{T'} &-\langle\partial_t\mathbf{S}(\mathbf{Dv},\mathbf{E}),\mathbf{D}\boldsymbol{\omega}\rangle\varphi + \langle 2\mathbf{v}\otimes\partial_t\mathbf{v},\mathbf{D}\boldsymbol{\omega}\rangle\varphi \\ &+ \langle\partial_t\mathbf{f},\boldsymbol{\omega}\rangle\varphi - 2\chi^E\langle\mathbf{E}\otimes\partial_t\mathbf{E},\mathbf{D}\boldsymbol{\omega}\rangle\varphi\,dt.\end{aligned}$$

From (2.6), (2.7) and $p(|\mathbf{E}|^2) \leqslant p_0 \leqslant 2$ we get

$$\|\partial_t\mathbf{S}(\mathbf{Dv},\mathbf{E})\|_2^2 \leqslant C\big(1 + \mathcal{J}(\mathbf{v}) + \|\nabla\mathbf{v}|_2^2 + \|\nabla\mathbf{v}\|_3^3\big),$$

which due to (2.69) and (2.70) belongs to $L^1(I')$. From (2.69) and the assumptions on the data we easily see that also the other three terms on the right-hand side of (2.72) belong to $L^1(I')$ if $\boldsymbol{\omega} \in L^2(I';V_2)$. This implies $\partial_t^2\mathbf{v} \in L^2(I';(V_2)^*)$. From (2.69), (2.71) and the parabolic embedding (cf. [14]) we finally get $\mathbf{v} \in C(I';V_r)$, $1 \leqslant r < 6(p_\infty - 1)$. This finishes the proof of Theorem 2.13.

Now we discuss a time discretization of the system (2.2) under the additional assumption that

$$p = \text{const.}$$

and consequently we have to modify our basic assumptions on $\mathbf{S}$. We assume that the following *monotonicity condition*

$$\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}} B_{ij} B_{kl} \geqslant \gamma_1 (1 + |\mathbf{E}|^2)(1 + |\mathbf{D}|^2)^{\frac{p-2}{2}} |\mathbf{B}|^2, \tag{3.1}$$

is satisfied for all $\mathbf{B}, \mathbf{D} \in X := \{\mathbf{D} \in \mathbb{R}^{3\times 3}_{\text{sym}}, \operatorname{tr} \mathbf{D} = 0\}$, and that the following *growth conditions* are satisfied for $i, j, k, l, n = 1, 2, 3$,

$$\left| \frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}} \right| \leqslant \gamma_2 (1 + |\mathbf{E}|^2)(1 + |\mathbf{D}|^2)^{\frac{p-2}{2}}, \tag{3.2}$$

$$\left| \frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial E_n} \right| \leqslant \gamma_3 |\mathbf{E}| (1 + |\mathbf{E}|^2)(1 + |\mathbf{D}|^2)^{\frac{p-1}{2}}. \tag{3.3}$$

For the numerical analysis we need some additional notation. Let $I_k = \{t_m\}_{m=0}^M$ be a given net in an interval $I = [0, t_M]$ with a constant *time-step* size $k := t_m - t_{m-1}$. We denote by $d_t \mathbf{v}^m := k^{-1}(\mathbf{v}^m - \mathbf{v}^{m-1})$ the divided difference in time. By $l^q(I_k; X)$ we denote the space of functions $\{\varphi^m\}_{m=0}^M$ with finite norm $\left(k \sum\limits_{m=0}^{M} \|\varphi^m\|_X^q\right)^{1/p}$. For $q = \infty$, functions $\{\varphi^m\}_{m=0}^M$ need to satisfy the bound $\max\limits_{0 \leqslant m \leqslant M} \|\varphi^m\|_X < \infty$.

The problem (2.2) is approximated by a time discretization by means of the *implicit Euler scheme*:

Algorithm 3.4. *Let there be given a time-step size $k > 0$ and the corresponding net $I_k = \{t_m\}_{m=0}^M$. For $m \geqslant 1$ and $\mathbf{v}^{m-1}$ given from the previous step, compute an iterate $\mathbf{v}^m$ that solves*

$$\begin{aligned} d_t \mathbf{v}^m - \operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{v}^m, \mathbf{E}(t_m)) + [\nabla \mathbf{v}^m]\mathbf{v}^m + \nabla \pi^m & \\ = \mathbf{f}(t_m) + \chi^E [\nabla \mathbf{E}(t_m)] \mathbf{E}(t_m)&, \\ \operatorname{div} \mathbf{v}^m = 0&, \\ \mathbf{v}^0 = \mathbf{v}_0&, \end{aligned} \tag{3.5}$$

endowed with space-periodic boundary conditions (2.6).

The main result of this section is:

Theorem 3.6. *Assume that the extra stress tensor* $\mathbf{S}$ *satisfies* (3.1)–(3.3) *and* $\mathbf{S}(\mathbf{0}, \mathbf{E}) = \mathbf{0}$. *Let* $\mathbf{v}_0 \in W^{2,2}(\Omega) \cap V_p$ *be a given initial velocity,* $\mathbf{f} \in C(I; W^{1,2}(\Omega))$, $\partial_t \mathbf{f} \in C(I; L^2(\Omega))$ *be a given force,* $\mathbf{E} \in C^1(\bar{I}; C^1(\Omega))$ *be a given electric field. Let* $\mathbf{v}$ *be a strong solution of the problem* (2.2) *on the interval* $I' = [0, T']$ *for* $p \in \left[\frac{5}{3}, 2\right]$ *satisfying* (2.14) *and* (2.15). *Suppose that* $\mathbf{v}^m$ *is a weak solution of the problem* (3.5) *satisfying* (3.19) *and* $t_M \leqslant T'$. *Then for all*

$$\alpha < \alpha_0(p) := \frac{5p-6}{4(p-1)} \tag{3.7}$$

there exists a constant C *that only depends on* $\mathbf{v}_0$, $\mathbf{f}$, Ω, T' *and* α *but not on the time-step size* k, *such that the following error estimate is valid, provided that the time-step size is chosen sufficiently small, i.e.* $k \leqslant k_0(p, T')$,

$$\max_{1 \leqslant m \leqslant M} \|\mathbf{v}(t_m) - \mathbf{v}^m\|_2^2 + k \sum_{m=1}^{M} \|\mathbf{D}(\mathbf{v}(t_m) - \mathbf{v}^m)\|_p^2 \leqslant C k^{2\alpha}. \tag{3.8}$$

Remark 3.9. With a more refined technique (cf. [13]) one can show that the assertion of the theorem holds for $p \in \left(\frac{11+\sqrt{21}}{10}, 2\right] \approx (1.5583, 2]$.

Before we start with the proof of Theorem 3.6 we need some additional properties of quantities related to $\mathbf{S}$. Due to (3.1)–(3.3) we get that $\mathcal{I}(t, \mathbf{v})$ and $\mathcal{J}(t, \mathbf{v})$ defined in (2.9) and (2.10) satisfy the analogue of (2.22) and (2.23), i.e.

$$\mathcal{I}(t, \mathbf{v}) \geqslant \gamma_1 \int_\Omega (\tilde{D}\mathbf{v}(t))^{p-2} |\mathbf{D}(\nabla \mathbf{v})(t)\|^2 \, dx,$$

$$\mathcal{J}(t, \mathbf{v}) \geqslant \gamma_1 \int_\Omega (\tilde{D}\mathbf{v}(t))^{p-2} |\mathbf{D}(\partial_t \mathbf{v})(t)|^2 \, dx.$$

The discrete analogue for $\mathcal{J}(\mathbf{v})$ for a function defined on a net I_k reads as follows

$$\mathcal{K}(\mathbf{v}^m) := \int_\Omega \int_0^1 \frac{\partial S_{ij}(\mathbf{D}(s\mathbf{v}^m + (1-s)\mathbf{v}^{m-1}), \mathbf{E}(t_m))}{\partial D_{kl}} \, ds \, D_{ij}(d_t \mathbf{v}^m) D_{kl}(d_t \mathbf{v}^m) \, dx,$$

which due to (3.1) and Lemma 2.17 satisfies

$$\mathcal{K}(\mathbf{v}^m) \geqslant C_3 \int_\Omega (1 + |\mathbf{D}\mathbf{v}^m|^2 + |\mathbf{D}\mathbf{v}^{m-1}|^2)^{\frac{p-2}{2}} |\mathbf{D}(d_t \mathbf{v}^m)|^2 \, dx. \tag{3.10}$$

Lemma 3.11. *Let* $\mathbf{S}$ *satisfy* (3.1) *and* (3.2). *Then for all (sufficiently smooth)* $\mathbf{v}$ *with* $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$, *for all* $1 \leqslant q < \infty$, *and almost every* $t \in I$ *there holds:*

$$\|\nabla\mathbf{v}(t)\|^2_{\frac{6q}{6-3p+q}} + \|\mathbf{D}(\nabla\mathbf{v})(t)\|^2_{\frac{2q}{2-p+q}} \leqslant C\mathcal{I}(t,\mathbf{v})\|\tilde{D}\mathbf{v}(t)\|_q^{2-p}, \tag{3.12}$$

$$\|\partial_t\mathbf{v}(t)\|^2_{\frac{6q}{6-3p+q}} + \|\mathbf{D}(\partial_t\mathbf{v})(t)\|^2_{\frac{2q}{2-p+q}} \leqslant C\mathcal{J}(t,\mathbf{v})\|\tilde{D}\mathbf{v}(t)\|_q^{2-p}. \tag{3.13}$$

Proof. Lemma 2.25 $\left(r \mapsto \frac{2q}{2-p+q}\right)$ and $p = \text{const.}$ imply

$$\begin{aligned}\|\mathbf{D}(\nabla\mathbf{v})\|_{\frac{2q}{2-p+q}} &\leqslant C\mathcal{I}(\mathbf{v})^{\frac12}\|(\tilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{2q}{2-p}} \\ &\leqslant C\mathcal{I}(\mathbf{v})^{\frac12}\|\tilde{D}\mathbf{v}\|_q^{\frac{2-p}{2}},\end{aligned}$$

which together with the embedding $W^{2,\frac{2q}{2-p+q}}(\Omega) \hookrightarrow W^{1,\frac{6q}{6-3p+q}}(\Omega)$ proves the first assertion. The second assertion follows analogously. □

Since $\mathcal{K}(\mathbf{v}^m)$ is the discrete version of $\mathcal{J}(\mathbf{v})$ we immediately obtain in the same way as in Lemma 3.11 and Lemma 2.28:

Lemma 3.14. *Let* $\mathbf{S}$ *satisfy* (3.1) *and* (3.2). *For all (sufficiently smooth)* $\mathbf{v}^m$ *with* $\int_\Omega \mathbf{v}^m\,\mathrm{d}x = 0$ *there holds for all* $q \in [1,\infty)$:

$$\begin{aligned}&\|d_t\mathbf{v}^m\|^2_{\frac{6q}{6-3p+q}} + \|\mathbf{D}(d_t\mathbf{v}^m)\|^2_{\frac{2q}{2-p+q}} \\ &\qquad\leqslant C\mathcal{K}(\mathbf{v}^m)(\|\tilde{D}\mathbf{v}^m\|_q + \|\tilde{D}\mathbf{v}^{m-1}\|_q)^{2-p},\end{aligned} \tag{3.15}$$

$$\|d_t\mathbf{v}^m\|_{3p}^p + \|d_t\nabla\mathbf{v}^m\|^p_{\frac{3p}{p+1}} \leqslant C(1+\mathcal{I}(\mathbf{v}^m)+\mathcal{I}(\mathbf{v}^{m-1}))^{\frac{2-p}{2}}\mathcal{K}(\mathbf{v}^m)^{p/2}, \tag{3.16}$$

$$\leqslant C(1+\mathcal{I}(\mathbf{v}^m)+\mathcal{I}(\mathbf{v}^{m-1})+\mathcal{K}(\mathbf{v}^m)). \tag{3.17}$$

The following lemma ensures the solvability of the problem (3.5).

Lemma 3.18. *Let* $\mathbf{S}$, $\mathbf{v}_0$, $\mathbf{f}$ *and* $\mathbf{E}$ *satisfy the assumptions of Theorem* 3.6. *Then there exists a weak solution* $\mathbf{v}^m$ *of* (3.5) *satisfying*

$$\max_{1\leqslant m\leqslant M}\|\mathbf{v}^m\|_2^2 + k\sum_{m=1}^M \|\mathbf{D}\mathbf{v}^m\|_p^p \leqslant C(\mathbf{f},\mathbf{v}_0,\mathbf{E}), \tag{3.19}$$

whenever $p > \frac32$.

Proof. First of all note that the strategy employed in the proof of Theorem 2.13 to ensure the existence of strong solutions is not applicable in the *discrete case*, since there is no discrete version of the local Gronwall's inequality. For $p > \frac{9}{5}$ the estimate (3.19) is sufficient to ensure the existence of weak solutions using the theory of monotone operators (cf. [29]). For this we must view (3.5), with k and m fixed, as a steady system with the discrete time derivative as the right-hand side. In order to prove the lemma for $p > \frac{3}{2}$ we proceed as follows (cf. [22], [41]). We approximate (3.5) by the mollified system

$$
\begin{aligned}
(3.20) \qquad d_t \mathbf{v}_n^m - \operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{v}_n^m, \mathbf{E}(t_m)) + [\nabla \mathbf{v}^m](\mathbf{v}_n^m)_{1/n} + \nabla \pi_n^m \\
= \mathbf{f}_n(t_m) + \chi^E [\nabla \mathbf{E}_n(t_m)] \mathbf{E}_n(t_m), \\
\operatorname{div} \mathbf{v}_n^m = 0,
\end{aligned}
$$

where $(\mathbf{v}_n^m)_{1/n} = w_{1/n} * \mathbf{v}_n^m$ is the usual mollification. Now we fix m and k and move the discrete time derivative to the right-hand side and view (3.20) as a steady system. Using the Galerkin method and the theory of monotone operators[9] it is easy to show that there exists a weak solution to (3.20) satisfying the estimate (3.19). The *key* observation is that

$$[\nabla \mathbf{v}^m](\mathbf{v}_n^m)_{1/n} \quad \text{is bounded in} \quad L^{\frac{3p}{6-p}}(\Omega)$$

uniformly with respect to n. To take advantage of this property we must use L^∞-test functions which ensure the almost everywhere convergence of $\mathbf{D}\mathbf{v}_n^m$. This argument is elaborated in detail in [41] and one can follow exactly the argumentation there. As a result one obtains that $\mathbf{D}\mathbf{v}_n^m$ converges a.e. in Ω to $\mathbf{D}\mathbf{v}^m$, which together with Vitali's convergence theorem enables the limiting process in the weak formulation of (3.20). □

In order to verify Theorem 3.6 we have to deal with two problems. Namely that the discrete solution $\mathbf{v}^m$ of the problem (3.5) is only weak and secondly that the information about $\partial_t^2 \mathbf{v}$ is also weak. Thus we introduce an auxiliary problem to split these problems subsequently. We follow the procedure introduced in [37] and consider the following auxiliary problem:

Algorithm 3.21. *Suppose that* $\mathbf{v}$ *is a strong solution to the problem* 2.2 *with the properties stated in Theorem* 2.13. *Then determine* $\mathbf{V}^m$, $m = 1, \ldots, M$, *that*

[9] Note that the mollified convective term maps the space V_p into $W^{-1,p}(\Omega)$ for $p > \frac{3}{2}$.

solves

$$
\begin{aligned}
(3.22)\qquad d_t\mathbf{V}^m - \operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{V}^m,\mathbf{E}(t_m)) + [\nabla\mathbf{V}^m]\mathbf{v}(t_m) + \nabla\Pi^m &\\
= \mathbf{f}(t_m) + \chi^E[\nabla\mathbf{E}(t_m)]\mathbf{E}(t_m),&\\
\operatorname{div}\mathbf{V}^m = 0,&\\
\mathbf{V}^0 = \mathbf{v}_0,&
\end{aligned}
$$

endowed with space-periodic boundary conditions (2.12).

We have linearized the convective term with respect to the continuous solution $\mathbf{v}(t_m)$, for which we have good regularity properties. The hope is that $\mathbf{V}^m$ *inherits the regularity* from $\mathbf{v}$. In fact this is the case at the expense of restricting ourselves to a smaller range of p's.

Proposition 3.23. *Let* $\mathbf{S}$, $\mathbf{v}_0$, $\mathbf{f}$ *and* $\mathbf{E}$ *satisfy the assumptions of Theorem* 3.6. *Let* $\mathbf{v}$ *defined on* $I=[0,T']$ *be the strong solution ensured by this theorem and let* $t_M<T'$. *Then there exists a strong solution* $\mathbf{V}^m$ *of the problem* (3.22) *whenever* $p\in\left[\frac{5}{3},2\right]$. *This solution satisfies*

$$
(3.24)\qquad \max_{1\leqslant m\leqslant M}\|d_t\mathbf{V}^m\|_2^2 + k\sum_{m=1}^{M}\left(\mathcal{I}(\mathbf{V}^m)^{\frac{5p-6}{2-p}} + \mathcal{K}(\mathbf{V}^m)\right) \leqslant C(\mathbf{f},\mathbf{v}_0,\mathbf{E}).
$$

In particular we have that for all $1<r<6(p-1)$ *it holds*

$$
\begin{aligned}
(3.25)\qquad \mathbf{V}^m &\in l^{p\frac{5p-6}{2-p}}\left(I_k;W^{2,\frac{3p}{p+1}}(\Omega)\right)\cap l^\infty(I_k;V_r),\\
d_t\mathbf{V}^m &\in l^{\frac{p(5p-6)}{(3p-2)(p-1)}}\left(I_k;W^{1,\frac{3p}{p+1}}(\Omega)\right)\cap l^\infty(I_k;L^2(\Omega)).
\end{aligned}
$$

P r o o f. The existence of a strong solution $\mathbf{V}^m$ of (3.22) follows from the regularity in (3.25) using the Galerkin approach with eigenfunctions of the Stokes operator as a basis. The regularity (3.25) follows in the same way as in the proof of Theorem 2.13 from (3.24) using also Lemma 3.14. Thus we shall only derive these estimates. For all missing details in the following computations we refer to [30, Section 5.3].

First of all we test the weak formulation of (3.22), which reads for all $\varphi\in V_p$

$$
\begin{aligned}
(3.26)\qquad \langle d_t\mathbf{V}^m,\varphi\rangle + \langle\mathbf{S}(\mathbf{D}\mathbf{V}^m,\mathbf{E}(t_m)),\mathbf{D}\varphi\rangle + \langle[\nabla\mathbf{V}^m]\mathbf{v}(t_m),\varphi\rangle&\\
= \langle\mathbf{f}(t_m),\varphi\rangle - \chi^E\langle\mathbf{E}(t_m)\otimes\mathbf{E}(t_m),\mathbf{D}\varphi\rangle,&
\end{aligned}
$$

with $\mathbf{V}^m$ and sum up over all iteration steps to obtain the first *a priori* estimate

$$
(3.27)\qquad \max_{1\leqslant m\leqslant M}\|\mathbf{V}^m\|_2^2 + k\sum_{m=1}^{M}\|\mathbf{D}\mathbf{V}^m\|_p^p \leqslant C,
$$

where we used that $\langle[\nabla\mathbf{V}^m]\mathbf{v}(t_m),\mathbf{V}^m\rangle = 0$.

The next step is to use in (3.26) $-\Delta \mathbf{V}^m$ as a test function. Again we use that $\operatorname{div} \mathbf{v}(t_m) = 0$ in the linearized convective term, the properties of $\mathbf{S}$ (cf. (3.1)–(3.3)), the definition of $\mathcal{I}(\mathbf{V}^m)$ and obtain, after summation up to level $N \in \{1, \dots, M\}$,

$$\tag{3.28} \begin{aligned} &\|\nabla \mathbf{V}^N\|_2^2 + k \sum_{m=1}^{N} \mathcal{I}(\mathbf{V}^m) \\ &\leqslant C\Big(1 + k \sum_{m=1}^{N} \int_\Omega |\nabla \mathbf{v}(t_m)|\, |\nabla \mathbf{V}^m|^2 \,\mathrm{d}x \\ &\quad + k \sum_{m=1}^{N} \int_\Omega \Big| \frac{\partial S_{ij}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m))}{\partial E_n} \nabla E_n \cdot D_{ij}(\nabla \mathbf{V}^m) \Big| \,\mathrm{d}x \Big). \end{aligned}$$

The last term on the right-hand side can be bounded by (cf. (3.3))

$$\tag{3.29} \varepsilon k \sum_{m=1}^{N} \mathcal{I}(\mathbf{V}^m) + Ck \sum_{m=1}^{N} \|\tilde{D}\mathbf{V}^m\|_p^p,$$

where the first term is absorbed in the left-hand side of (3.28). The second term on the right-hand side in (3.28) can, for $1 < r < 6(p-1)$, $\alpha \in (0,1)$, be estimated by

$$\tag{3.30} \|\nabla \mathbf{v}(t_m)\|_r \, \|\nabla \mathbf{V}^m\|_{2r'}^2 \leqslant C \|\nabla \mathbf{V}^m\|_{2r'}^2 = C \|\nabla \mathbf{V}^m\|_{2r'}^{2(\alpha + 1 - \alpha)},$$

where r' is the dual exponent to r and where we used $\mathbf{v} \in C(I; V_r)$. Now, for $p > \frac{4}{3}$ and $\frac{3p}{3p-2} < r < 6(p-1)$ we interpolate $L^{2r'}(\Omega)$ both between $L^2(\Omega)$ and $L^{3p}(\Omega)$ and between $L^p(\Omega)$ and $L^{3p}(\Omega)$, which gives

$$\tag{3.31} \begin{aligned} \|\nabla \mathbf{V}^m\|_{2r'} &\leqslant \|\nabla \mathbf{V}^m\|_2^{\frac{r(3p-2)-3p}{r(3p-2)}} \|\nabla \mathbf{V}^m\|_{3p}^{\frac{3p}{r(3p-2)}}, \\ \|\nabla \mathbf{V}^m\|_{2r'} &\leqslant \|\nabla \mathbf{V}^m\|_p^{\frac{1}{4}\frac{r(3p-2)-3p}{r}} \|\nabla \mathbf{V}^m\|_{3p}^{\frac{3}{4}\frac{r(2-p)+p}{r}}. \end{aligned}$$

Using also (2.29) the right-hand side of (3.30) can be estimated by

$$\tag{3.32} C(1 + \|\nabla \mathbf{V}^m\|_2^2)^{Q_1} \, \|\nabla \mathbf{V}^m\|_p^{pQ_2} (1 + \mathcal{I}(\mathbf{V}^m))^{Q_3},$$

where

$$\begin{gathered} Q_1 = (1-\alpha) \frac{r(3p-2) - 3p}{r(3p-2)}, \qquad Q_2 = \alpha \frac{1}{2p} \frac{r(3p-2) - 3p}{r}, \\ Q_3 = (1-\alpha) \frac{2}{p} \frac{3p}{r(3p-2)} + \alpha \frac{3}{2p} \frac{r(2-p)+p}{r}. \end{gathered}$$

Young's inequality together with the requirements

$$Q_2 \cdot \delta = \frac{1}{1+\varepsilon}, \quad Q_3 \cdot \delta' = 1, \quad \frac{1}{\delta} + \frac{1}{\delta'} = 1$$

for any prescribed $\varepsilon > 0$ yields

$$1 + \|\nabla \mathbf{V}^N\|_2^2 + k \sum_{m=1}^{N} \mathcal{I}(\mathbf{V}^m) \leqslant C\Big(1 + k \sum_{m=1}^{N} \|\nabla \mathbf{V}^m\|_p^{\frac{p}{1+\varepsilon}} (1 + \|\nabla \mathbf{V}^m\|_2^2)^{\lambda_\varepsilon(r)}\Big),$$

where

$$\lambda_\varepsilon(r) \searrow \lambda = \frac{2(p-1)(2-p)}{3p^2 - 5p + 1} \quad \text{for } \varepsilon \searrow 0, \ r \nearrow 6(p-1).$$

In view of (3.27) we have to check whether $\lambda < 1$, which holds for $p \in \big(\frac{11+\sqrt{21}}{10}, 2\big]$. Therefore we can employ discrete Gronwall's lemma and obtain our second *a priori* estimate

$$\max_{1 \leqslant m \leqslant M} \|\nabla \mathbf{V}^m\|_2^2 + k \sum_{m=1}^{M} \mathcal{I}(\mathbf{V}^m) \leqslant C. \tag{3.33}$$

Now we want to use "$d_t^2 \mathbf{V}^m$" as a test function in (3.26). This in fact will give us the lower bound $p \geqslant \frac{5}{3}$. Firstly, we have to introduce $\mathbf{V}^{-1}$. For that we set for all $\varphi \in V_p$

$$\begin{aligned} \frac{1}{k}\langle \mathbf{V}^0 - \mathbf{V}^{-1}, \varphi\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{V}^0, \mathbf{E}(0)), \mathbf{D}\varphi\rangle + \langle [\nabla \mathbf{V}^0]\mathbf{V}^0, \varphi\rangle \\ = \langle \mathbf{f}(0), \varphi\rangle - \chi^E \langle \mathbf{E}(0) \otimes \mathbf{E}(0), \mathbf{D}\varphi\rangle. \end{aligned}$$

Using $\mathbf{V}^0 = \mathbf{v}_0$, $p \leqslant 2$ and the assumption on $\mathbf{v}_0$ and $\mathbf{E}$ we obtain

$$\begin{aligned} \|d_t \mathbf{V}^0\|_2^2 \leqslant C(\|\mathbf{f}(0)\|_2^2 + \|[\nabla \mathbf{v}_0]\mathbf{v}_0\|_2^2 + \|\operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{v}_0, \mathbf{E}(0))\|_2^2 \\ + \|\mathbf{E}(0) \otimes \mathbf{D}\mathbf{E}(0)\|_2^2) \leqslant C. \end{aligned} \tag{3.34}$$

Now we can take the discrete time derivative of the weak formulation (3.26), use $d_t \mathbf{V}^m$ as a test function, and sum up to level $N \in \{1, \ldots, M\}$, to obtain

$$\begin{aligned} &\|d_t \mathbf{V}^N\|_2^2 \\ &\quad + \frac{1}{k} \sum_{m=1}^{N} \int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_{m-1})))\mathbf{D}(\mathbf{V}^m - \mathbf{V}^{m-1})\, \mathrm{d}x \\ &\quad \leqslant C\Big(1 + k \sum_{m=1}^{N} \Big| \int_\Omega [\nabla \mathbf{V}^m] d_t \mathbf{v}(t_{m-1}) \cdot d_t \mathbf{V}^m \, \mathrm{d}x \Big| \Big), \end{aligned} \tag{3.35}$$

where we used (3.34). From the formula $d_t\mathbf{v}(t_m) = k^{-1}\int_{t_{m-1}}^{t_m} \partial_t \mathbf{v}(s)\,\mathrm{d}s$ and $(2.15)_2$ we deduce

$$\|d_t\mathbf{v}(t_m)\|_2 \leqslant \operatorname*{ess\,sup}_I \|\partial_t\mathbf{v}\|_2 \leqslant C, \tag{3.36}$$

and thus we can bound the last term in (3.35) by

$$\begin{aligned}\|d_t\mathbf{v}(t_{m-1})\|_2 \,\|\, |\nabla\mathbf{v}^m|\, |d_t\mathbf{V}^m|\, \|_2 &\leqslant C\|\nabla\mathbf{V}^m\|_4 \,\|d_t\mathbf{V}^m\|_4 \\ &\leqslant \varepsilon\mathcal{K}(\mathbf{V}^m) + C\mathcal{I}(\mathbf{V}^m),\end{aligned} \tag{3.37}$$

where we used (3.15), (3.12) with $q = 2$, (3.33) and Young's inequality. However we have to check whether

$$\frac{12}{8-3p} \geqslant 4 \Longleftrightarrow p \geqslant \frac{5}{3},$$

which is the lower bound from the proposition. Furthermore we have for the second term on the left-hand side of (3.35)

$$\begin{aligned}
&k^{-1}\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_{m-1}))) \cdot \mathbf{D}(\mathbf{V}^m - \mathbf{V}^{m-1})\,\mathrm{d}x \\
&= k^{-1}\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_m))) \cdot \mathbf{D}(\mathbf{V}^m - \mathbf{V}^{m-1})\,\mathrm{d}x \\
&\quad + k^{-1}\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_{m-1}))) \cdot \mathbf{D}(\mathbf{V}^m - \mathbf{V}^{m-1})\,\mathrm{d}x \\
&= k\mathcal{K}(\mathbf{V}^m) \\
&\quad + k\int_\Omega\int_0^1 \frac{S_{ij}(\mathbf{D}\mathbf{V}^{m-1}, (1-\tau)\mathbf{E}(t_{m-1}) + \tau\mathbf{E}(t_m))}{\partial E_n}\,\mathrm{d}\tau\, d_t E_n(t_m) D_{ij}(d_t\mathbf{V}^m)\,\mathrm{d}x.
\end{aligned}$$

The last term is moved to the right-hand side and there estimated by

$$\varepsilon k\mathcal{K}(\mathbf{V}^m) + Ck(\|\tilde{D}\mathbf{V}^m\|_p^p + \|\tilde{D}\mathbf{V}^{m-1}\|_p^p), \tag{3.38}$$

where we used (3.3) and (3.10). Note that the last term is finite after summation over m, due to (3.27). Alltogether, we have therefore derived our third *a priori* estimate

$$\max_{1\leqslant m\leqslant M} \|d_t\mathbf{V}^m\|_2^2 + k\sum_{m=1}^{M} \mathcal{K}(\mathbf{V}^m) \leqslant C. \tag{3.39}$$

Using $-\Delta \mathbf{V}^m$ as a test function in (3.26), where also the term with the discrete time derivative is estimated, yields for $p > \frac{3}{2}$ and $\frac{6}{3p-2} < r < 6(p-1)$ (cf. (3.28)–(3.30))

$$
\begin{aligned}
(3.40) \qquad 1 + \mathcal{I}(\mathbf{V}^m) &\leqslant C\big(1 + \varepsilon \mathcal{I}(\mathbf{V}^m) + \|\tilde{D}\mathbf{V}^m\|_p^p + \|\nabla \mathbf{V}^m\|_{2r'}^2 \\
&\qquad + \|d_t \mathbf{V}^m\|_{\frac{3p}{2p-1}} \|\nabla^2 \mathbf{V}^m\|_{\frac{3p}{p+1}}\big) \\
&\leqslant C\big(1 + C_\varepsilon \|\nabla \mathbf{V}^m\|_2^2 + \varepsilon \mathcal{I}(\mathbf{V}^m)(1 + \|\tilde{D}\mathbf{V}^m\|_2^{2-p}) \\
&\qquad + \|d_t \mathbf{V}^m\|_{\frac{3p}{2p-1}} \|\nabla^2 \mathbf{V}^m\|_{\frac{3p}{p+1}}\big) \\
&\leqslant C\big(C_\varepsilon + \varepsilon \mathcal{I}(\mathbf{V}^m) + \|d_t \mathbf{V}^m\|_{\frac{3p}{2p-1}} (1 + \mathcal{I}(\mathbf{V}^m))^{1/p}\big),
\end{aligned}
$$

where we used $\mathbf{V}^m \in l^\infty(I_k; W^{1,2}\Omega)$ and $p \leqslant 2$; the interpolation of $L^{2r'}(\Omega)$ between $L^2(\Omega)$ and $L^{\frac{12}{8-3p}}(\Omega)$, which is possible for $p > \frac{3}{2}$, and (3.12) with $q = 2$; again $\mathbf{V}^m \in l^\infty(I_k; W^{1,2}(\Omega))$ and finally (2.29). For ε sufficiently small we can absorb the term $c\varepsilon \mathcal{I}(\mathbf{V}^m)$ into the left-hand side of (3.40). Thus we get

$$
(3.41) \qquad (1 + \mathcal{I}(\mathbf{V}^m))^{\frac{p-1}{p}} \leqslant C\big(1 + \|d_t \mathbf{V}^m\|_{\frac{3p}{2p-1}}\big).
$$

Now we interpolate $L^{\frac{3p}{2p-1}}(\Omega)$ between $L^2(\Omega)$ and $L^{3p}(\Omega)$, and use that $d_t \mathbf{V}^m \in l^\infty(I_k; L^2(\Omega))$ and (3.16), to arrive at

$$
(3.42) \qquad (1 + \mathcal{I}(\mathbf{V}^m))^{\frac{p-1}{p}} \leqslant C\big(1 + \mathcal{K}(\mathbf{V}^m)^{\lambda/2}(1 + \mathcal{I}(\mathbf{V}^m) + \mathcal{I}(\mathbf{V}^{m-1}))^{\lambda \frac{2-p}{2p}}\big),
$$

with $\lambda = \frac{2-p}{3p-2}$. We raise this inequality to the power γ and apply Young's inequality to get

$$
\begin{aligned}
(3.43) \qquad & (1 + \mathcal{I}(\mathbf{V}^m))^{\gamma \frac{p-1}{p}} \\
&\leqslant C\big(1 + \mathcal{K}(\mathbf{V}^m)^{\gamma \frac{\lambda}{2}}(1 + \mathcal{I}(\mathbf{V}^m) + \mathcal{I}(\mathbf{V}^{m-1}))^{\gamma \lambda \frac{2-p}{2p}}\big) \\
&\leqslant C\big(1 + C_\varepsilon \mathcal{K}(\mathbf{V}^m) + \varepsilon(1 + \mathcal{I}(\mathbf{V}^m) + \mathcal{I}(\mathbf{V}^{m-1}))^{\frac{2\gamma}{2-\gamma\lambda} \lambda \frac{2-p}{2p}}\big).
\end{aligned}
$$

We now require $\gamma \frac{p-1}{p} = \frac{2\gamma}{2-\gamma\lambda} \lambda \frac{2-p}{2p}$, which gives $\gamma = \frac{p}{p-1} \frac{5p-6}{2-p}$. With this γ and ε sufficiently small we can absorb the last term in (3.43) into the left-hand side after summation over all time steps. Thus we have derived

$$
(3.44) \qquad k \sum_{m=0}^{M} \mathcal{I}(\mathbf{V}^m)^{\frac{5p-6}{2-p}} \leqslant C\Big(1 + k \sum_{m=0}^{M} \mathcal{K}(\mathbf{V}^m)\Big) \leqslant C.
$$

The proof is complete. □

Proposition 3.23 shows that the solution $\mathbf{V}^m$ of (3.22) has the same regularity properties as the solution $\mathbf{v}$ of the problem (2.2). Thus we can *split* the error into two parts, namely

$$\mathbf{v}(t_m) - \mathbf{v}^m = (\mathbf{v}(t_m) - \mathbf{V}^m) + (\mathbf{V}^m - \mathbf{v}^m) =: \boldsymbol{\eta}^m + \mathbf{e}^m. \tag{3.45}$$

Before we discuss these errors we need one more property of $\mathbf{S}$.

Lemma 3.46. *Let* $\mathbf{S}$ *satisfy* (3.1) *and* (3.2). *Then for all (sufficiently smooth)* $\mathbf{v}$, $\mathbf{w}$, *for all* $1 \leqslant r < \infty$, *and almost every* $t \in I'$ *there holds*

$$\begin{aligned}\|\mathbf{D}(\mathbf{v}(t) - \mathbf{w}(t))\|^2_{\frac{2r}{2-p+r}} &\leqslant C\langle \mathbf{S}(\mathbf{D}\mathbf{v}(t), \mathbf{E}(t)) - \mathbf{S}(\mathbf{D}\mathbf{w}(t), \mathbf{E}(t)), \mathbf{D}(\mathbf{v}(t) - \mathbf{w}(t))\rangle \\ &\quad \times (1 + \|\mathbf{D}\mathbf{v}(t)\|_r + \|\mathbf{D}(\mathbf{v}(t) - \mathbf{w}(t))\|_r)^{2-p}.\end{aligned}$$

Proof. We have using Lemma 2.19

$$\begin{aligned}\|\mathbf{D}(\mathbf{v} - \mathbf{w})\|^{\frac{2r}{2-p+r}}_{\frac{2r}{2-p+r}} &= \int_\Omega ((1 + |\mathbf{D}\mathbf{v}| + |\mathbf{D}(\mathbf{v} - \mathbf{w})|)^{p-2} |\mathbf{D}(\mathbf{v} - \mathbf{w})|^2)^{\frac{r}{2-p+r}} \\ &\quad \times (1 + |\mathbf{D}\mathbf{v}| + |\mathbf{D}(\mathbf{v} - \mathbf{w})|)^{\frac{(2-p)r}{2-p+r}} \, \mathrm{d}x \\ &\leqslant \left(\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}) - \mathbf{S}(\mathbf{D}\mathbf{w}, \mathbf{E})) \cdot \mathbf{D}(\mathbf{v} - \mathbf{w}) \, \mathrm{d}x \right)^{\frac{r}{2-p+r}} \\ &\quad \times \left(\int_\Omega (1 + |\mathbf{D}\mathbf{v}| + |\mathbf{D}(\mathbf{v} - \mathbf{w})|)^r \, \mathrm{d}x \right)^{\frac{2-p}{2-p+r}},\end{aligned}$$

which immediately gives the assertion. □

Let us first discuss the error $\boldsymbol{\eta}^m$, where we can take advantage of the regularity properties for $\mathbf{v}$ and $\mathbf{V}^m$. The error $\boldsymbol{\eta}^m$ is governed by the following system, which holds for all $\boldsymbol{\varphi} \in V_p$,

$$\begin{aligned}&\langle d_t \boldsymbol{\eta}^m, \boldsymbol{\varphi}\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}(t_m), \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)), \mathbf{D}\boldsymbol{\varphi}\rangle \\ &\qquad + \langle [\nabla \boldsymbol{\eta}^m] \mathbf{v}(t_m), \boldsymbol{\varphi}\rangle = \langle \mathbf{R}^m, \boldsymbol{\varphi}\rangle,\end{aligned} \tag{3.47}$$

supplemented with

$$\mathbf{R}^m := d_t \mathbf{v}(t_m) - \partial_t \mathbf{v}(t_m) = -\frac{1}{k} \int_{t_{m-1}}^{t_m} (s - t_{m-1}) \partial_t^2 \mathbf{v}(s) \, \mathrm{d}s. \tag{3.48}$$

From (3.48) and (2.15) we compute that

$$\|\mathbf{R}^m\|_2^2 \leqslant C \sup_{s \in [t_{m-1}, t_m]} \|\partial_t \mathbf{v}(s)\|_2^2, \tag{3.49}$$

$$\|\mathbf{R}^m\|^2_{(\mathbf{V}_2)^*} \leqslant Ck \int_{t_{m-1}}^{t_m} \|\partial_t^2 \mathbf{v}(s)\|^2_{(\mathbf{V}_2)^*} \, \mathrm{d}s. \tag{3.50}$$

If we use $\boldsymbol{\eta}^m$ as a test function in (3.47) and sum over all iteration steps, we obtain, for $1 < r < 6(p-1)$,

$$\begin{aligned}(3.51) \qquad \max_{1\leqslant m\leqslant M} \|\boldsymbol{\eta}^m\|_2^2 + k\sum_{m=1}^{M}\left(\|\mathbf{D}\boldsymbol{\eta}^m\|^2_{\frac{2r}{2-p+r}} + \|\mathbf{D}\boldsymbol{\eta}^m\|_p^2\right) \\ \leqslant C(r)k\sum_{m=1}^{M}\langle \mathbf{R}^m, \boldsymbol{\eta}^m\rangle,\end{aligned}$$

where we have used Lemma 3.46 and $\mathbf{v}(t_m), \mathbf{V}^m \in l^\infty(I_k; V_r)$. We can bound the term on the right-hand side with the help of the embedding $W^{1,\frac{2r}{2-p+r}}(\Omega) \hookrightarrow W^{\frac{2r-6+3p}{2r},2}(\Omega)$ and the interpolation of $W^{\frac{2r-6+3p}{2r},2}(\Omega)$ between $W^{1,2}(\Omega)$ and $L^2(\Omega)$ as follows

$$\begin{aligned}(3.52) \qquad \langle \mathbf{R}^m, \boldsymbol{\eta}^m\rangle &\leqslant \|\mathbf{R}^m\|_2^{1-\frac{2r-6+3p}{2r}} \|\mathbf{R}^m\|_{(V_2)^*}^{\frac{2r-6+3p}{2r}} \|\boldsymbol{\eta}^m\|_{V_{\frac{2r}{2-p+r}}} \\ &\leqslant C(\mathbf{f}, \mathbf{v}_0)\|\mathbf{R}^m\|_{(V_2)^*}^{\frac{2r-6+3p}{r}} + \frac{1}{2}\|\mathbf{D}\boldsymbol{\eta}^m\|^2_{\frac{2r}{2-p+r}},\end{aligned}$$

where we also used Korn's and Young's inequalities and (3.49). Now, we move the last term in (3.52) to the left-hand side of (3.51) and it remains to bound the first term in (3.52). Note that

$$(3.53) \qquad \frac{2r-6+3p}{2r} =: \tilde{\alpha}(p,r) \nearrow \alpha_0(p) := \frac{5p-6}{4(p-1)}, \quad \text{for} \quad r \nearrow 6(p-1).$$

From (3.50) and $(2.15)_3$ we derive

$$\begin{aligned}k\sum_{m=1}^{M}\|\mathbf{R}^m\|_{(V_2)^*}^{2\tilde{\alpha}(p,r)} &\leqslant Ck^{2\tilde{\alpha}(p,r)}\left(\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m}\|\partial_t^2\mathbf{v}(s)\|^2_{(V_2)^*}\,\mathrm{d}s\right)^{\tilde{\alpha}(p,r)} \\ &\leqslant Ck^{2\tilde{\alpha}(p,r)},\end{aligned}$$

which together with (3.51), (3.52) yields

$$(3.54) \qquad \max_{1\leqslant m\leqslant M}\|\boldsymbol{\eta}^m\|_2^2 + k\sum_{m=1}^{M}\|\mathbf{D}\boldsymbol{\eta}^m\|_p^2 \leqslant C(r)k^{2\tilde{\alpha}(p,r)},$$

with $\tilde{\alpha}(p,r)$ defined in (3.53).

We still have to deal with the error $\mathbf{e}^m$, which is governed by the system

$$(3.55) \qquad \langle d_t\mathbf{e}^m, \boldsymbol{\varphi}\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{v}^m, \mathbf{E}(t_m)), \mathbf{D}\boldsymbol{\varphi}\rangle = \langle \mathbf{r}^m, \boldsymbol{\varphi}\rangle,$$

which holds for all $\varphi \in V_p$, and where

$$
(3.56) \qquad \begin{aligned} -\mathbf{r}^m &= [\nabla \mathbf{V}^m]\mathbf{v}(t_m) - [\nabla \mathbf{v}^m]\mathbf{v}^m \\ &= [\nabla \mathbf{V}^m]\boldsymbol{\eta}^m + [\nabla \mathbf{V}^m]\mathbf{e}^m + [\nabla \mathbf{e}^m]\mathbf{v}^m. \end{aligned}
$$

If we use in (3.55) the test function $\mathbf{e}^m$ and sum over all iteration steps, we get

$$
(3.57) \qquad \begin{aligned} \max_{1 \leqslant m \leqslant M} \|\mathbf{e}^m\|_2^2 + k \sum_{m=1}^{M} \frac{\|\mathbf{D}\mathbf{e}^m\|_p^2}{C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p}} &\leqslant Ck \sum_{m=1}^{M} \int_\Omega |\boldsymbol{\eta}^m|\,|\mathbf{e}^m|\,|\nabla \mathbf{V}^m| \, \mathrm{d}x + Ck \sum_{m=1}^{M} \int_\Omega |\mathbf{e}^m|^2 \, |\nabla \mathbf{V}^m| \, \mathrm{d}x \\ &=: Ck \sum_{m=1}^{M} (I_1^m + I_2^m). \end{aligned}
$$

For the lower bound of the elliptic term we used Lemma 3.46 with $r = p$ and the uniform bound for $\nabla \mathbf{V}^m \in l^\infty(I_k; L^p(\Omega))$. With the help of Hölder's inequality, the interpolation inequality

$$
\|\mathbf{v}\|_{2r'} \leqslant \|\mathbf{v}\|_2^{1-\lambda} \|\nabla \mathbf{v}\|_p^\lambda
$$

with $\lambda = \frac{3p}{r(5p-6)}$ and $\nabla \mathbf{V}^m \in l^\infty(I_k; L^r(\Omega))$, $\frac{3p}{5p-6} < r < 6(p-1)$, we find that the term I_1^m is bounded by

$$
(3.58) \qquad \begin{aligned} &\|\nabla \mathbf{V}^m\|_r \, \|\mathbf{e}^m\|_{2r'} \|\boldsymbol{\eta}^m\|_{2r'} \\ &\leqslant C \|\boldsymbol{\eta}^m\|_2^{1-\lambda} \|\nabla \boldsymbol{\eta}^m\|_p^\lambda \, \|\mathbf{e}^m\|_2^{1-\lambda} \frac{\|\mathbf{D}\mathbf{e}^m\|_p^\lambda}{(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^{\lambda/2}} (C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^{\lambda/2} \\ &\leqslant C \|\mathbf{e}^m\|_2 \, \|\boldsymbol{\eta}^m\|_2 (C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^{\frac{\lambda}{2(1-\lambda)}} + \frac{\frac{1}{2}\|\mathbf{D}\mathbf{e}^m\|_p}{(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^{1/2}} \|\mathbf{D}\boldsymbol{\eta}^m\|_p \\ &\leqslant C \|\boldsymbol{\eta}^m\|_2^2 + C(1 + \|\mathbf{D}\mathbf{e}^m\|_p^p)^{\frac{(2-p)\lambda}{p(1-\lambda)}} \|\mathbf{e}^m\|_2^2 + C \|\mathbf{D}\boldsymbol{\eta}^m\|_p^2 + \frac{\frac{1}{2}\|\mathbf{D}\mathbf{e}^m\|_p^2}{C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p}}. \end{aligned}
$$

The last term on the right-hand side is absorbed into the left-hand side of (3.57). For the first term and the third term in the last line of (3.58) we use estimate (3.54). The term I_2^m is easier. We get

$$
(3.59) \qquad \begin{aligned} \|\nabla \mathbf{V}^m\|_r \|\mathbf{e}^m\|_{2r'}^2 &\leqslant C \|\mathbf{e}^m\|_2^{2(1-\lambda)} \frac{\|\mathbf{D}\mathbf{e}^m\|_p^{2\lambda}}{(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^\lambda} (C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^\lambda \\ &\leqslant C(1 + \|\mathbf{D}\mathbf{e}^m\|_p^p)^{\frac{(2-p)\lambda}{p(1-\lambda)}} \|\mathbf{e}^m\|_2^2 + \frac{1}{2} \frac{\|\mathbf{D}\mathbf{e}^m\|_p^2}{C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p}}. \end{aligned}
$$

Thus we arrive at

$$\max_{1\leqslant m\leqslant M}\|\mathbf{e}^m\|_2^2 + k\sum_{m=1}^{M}\frac{\|\mathbf{D}\mathbf{e}^m\|_p^2}{C+\|\mathbf{D}\mathbf{e}^m\|_p^{2-p}} \tag{3.60}$$

$$\leqslant Ck^{2\tilde{\alpha}(p,r)} + k\sum_{m=1}^{M}(C+\|\mathbf{D}\mathbf{e}^m\|_p^p)^{\frac{2-p}{p}\frac{\lambda}{1-\lambda}}\|\mathbf{e}^m\|_2^2$$

and we can use the discrete Gronwall's lemma whenever $\frac{2-p}{p}\frac{\lambda}{1-\lambda} < 1$, where $\lambda = \frac{3p}{r(5p-6)}$, $1 < r < 6(p-1)$. One easily computes that this requirement is equivalent to $p > \frac{11+\sqrt{21}}{10}$. After the application of the discrete Gronwall's lemma we obtain that the left-hand side of (3.60) is bounded by $Ck^{2\tilde{\alpha}(p,r)}$, with $\tilde{\alpha}(p,r)$ given by (3.53). We can always choose r such that $2\tilde{\alpha}(p,r) > 1$ and we readily obtain that

$$\max_{1\leqslant m\leqslant M}\|\mathbf{D}\mathbf{e}^m\|_p^2 \leqslant C$$

and in turn we derive

$$\max_{1\leqslant m\leqslant M}\|\mathbf{e}^m\|_2^2 + k\sum_{m=1}^{M}\|\mathbf{D}\mathbf{e}^m\|_p^2 \leqslant C(r)k^{\tilde{\alpha}(p,r)}. \tag{3.61}$$

Since the same estimates hold for η^m we have furnished the proof of Theorem 3.6.

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References

[1] *B. Abu-Jdayil, P. O. Brunn*: Effects of nonuniform electric field on slit flow of an electrorheological fluid. J. Rheol. *39* (1995), 1327–1341.
[2] *B. Abu-Jdayil, P. O. Brunn*: Effects of electrode morphology on the slit flow of an electrorheological fluid. J. Non-Newtonian Fluid Mech. *63* (1966), 45–61.
[3] *B. Abu-Jdayil, P. O. Brunn*: Study of the flow behaviour of electrorheological fluids at shear- and flow- mode. Chem. Eng. and Proc. *36* (1997), 281–289.
[4] *W. Bao, J. W. Barrett*: A priori and a posteriori error bounds for a nonconforming linear finite element approximation of a non-Newtonian flow. RAIRO Modél. Math. Anal. Numér. *32* (1998), 843–858.
[5] *J. Baranger, K. Najib, and D. Sandri*: Numerical analysis of a three-fields model for a quasi-Newtonian flow. Comput. Methods Appl. Mech. Engrg. *109* (1993), 281–292.
[6] *H. Bellout, F. Bloom, and J. Nečas*: Young measure-valued solutions for non-Newtonian incompressible fluids. Comm. Partial Differential Equations *19* (1994), 1763–1803.

[7] *R. Bloodworth*: Electrorgeological fluids based on polyurethane dispersions. In: Electrorheological Fluids (R. Tao, G. D. Roy, eds.). World Scientific, 1994, pp. 67–83.

[8] *R. Bloodworth, E. Wendt*: Materials for ER-fluids. Int. J. Mod. Phys. B *23/24* (1996), 2951–2964.

[9] *B. D. Coleman, W. Noll*: The thermodynamics of elastic materials with heat conduction and viscosity. Arch. Rational Mech. Anal. *13* (1963), 167–178.

[10] *L. Diening*: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. Math. Inequ. Appl. *7* (2004), 245–253. Preprint 2002-02, University Freiburg.

[11] *L. Diening*: Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. Math. Nachr. *268* (2004), 31–43.

[12] *L. Diening*: Theoretical and numerical results for electrorheological fluids. PhD. thesis. University Freiburg, 2002.

[13] *L. Diening, A. Prohl, and M. Růžička*: On time discretizations for generalized Newtonian fluids. In: Nonlinear Problems in Mathematical Physics and Related Topics II. In honour of Professor O. A. Ladyzhenskaya (M. Sh. Birman, S. Hildebrandt, V. Solonnikov, and N. N. Uraltseva, eds.). Kluwer/Plenum, New York, 2002, pp. 89–118.

[14] *L. Diening, M. Růžička*: Strong solutions for generalized Newtonian fluids. J. Math. Fluid. Mech. Accepted. Preprint 2003-8, University Freiburg.

[15] *L. Diening, M. Růžička*: Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. J. Reine Angew. Math. *563* (2003), 197–220.

[16] *L. Diening, M. Růžička*: Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, Part I. J. Math. Anal. Appl. (2004), 559–571.

[17] *L. Diening, M. Růžička*: Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, Part II. J. Math. Anal. Appl. (2004), 572–588.

[18] *W. Eckart*: Theoretische Untersuchungen von elektrorheologischen Flüssigkeiten bei homogenen und inhomogenen elektrischen Feldern. Shaker Verlag, Aachen, 2000.

[19] *W. Eckart, M. Růžička*: Modeling micropolar electrorheological fluids. Accepted. Preprint 2003-11, University Freiburg.

[20] *A. C. Eringen, G. Maugin*: Electrodynamics of Continua, Vol. I and II. Springer-Verlag, New York, 1989.

[21] *J. Frehse, J. Málek*: Problems due to the no-slip boundary in incompressible fluid dynamics. In: Geometric Analysis and Nonlinear Partial Differential Equations. Springer-Verlag, Berlin, 2003, pp. 559–571.

[22] *J. Frehse, J. Málek, and M. Steinhauer*: An existence result for fluids with shear dependent viscosity—steady flows. Nonlinear Anal. *30* (1997), 3041–3049.

[23] *M. Giaquinta, G. Modica, and J. Souček*: Cartesian currents in the calculus of variations. II. Variational integrals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Vol. 38. Springer-Verlag, Berlin, 1998.

[24] *E. Giusti*: Direct Methods in the Calculus of Variations. Unione Matematica Italiana, Bologna, 1994. (In Italian.)

[25] *R. A. Grot*: Relativistic continuum physics: Electromagnetic interactions. In: Continuum Physics (A. C. Eringen, ed.). Academic Press, 1976, pp. 130–221.

[26] *T. C. Halsey, J. E. Martin, and D. Adolf*: Rheology of Electrorheological Fluids. Phys. Rev. Letters *68* (1992), 1519–1522.

[27] *K. Hutter, A. A. F. van de Ven*: Field Matter Interactions in Thermoelastic Solids. Lecture Notes in Physics, Vol. 88. Springer-Verlag, Berlin, 1978.

[28] *O. Kováčik, J. Rákosník*: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math. J. *41* (1991), 592–618.

[29] *J. L. Lions*: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Dunod, Paris, 1969. (In French.)
[30] *J. Málek, J. Nečas, M. Rokyta, and M. Růžička*: Weak and Measure-Valued Solutions to Evolutionary PDEs. Applied Mathematics and Mathematical Computations, Vol. 13. Chapman & Hall, London, 1996.
[31] *J. Málek, J. Nečas, and M. Růžička*: On the non-Newtonian incompressible fluids. Math. Models Methods Appl. Sci. *3* (1993), 35–63.
[32] *J. Málek, J. Nečas, and M. Růžička*: On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains. The case $p \geqslant 2$. Adv. Differential Equations *6* (2001), 257–302.
[33] *J. Málek, K. R. Rajagopal, and M. Růžička*: Existence and regularity of solutions and the stability of the rest state for fluids with shear dependent viscosity. Math. Models Methods Appl. Sci. *5* (1995), 789–812.
[34] *A. Milani, R. Picard*: Decomposition theorems and their application to non-linear electro- and magneto-static boundary value problems. Lecture Notes in Math., Vol. 1357. Springer-Verlag, 1988, pp. 317–340.
[35] *Y. H. Pao*: Electromagnetic forces in deformable continua. Mechanics Today, Vol. 4 (S. Nemat-Nasser, ed.). Pergamon Press, 1978, pp. 209–306.
[36] *M. Parthasarathy, D. J. Klingenberg*: Mechanism and models. Materials, Sciences and Engineering *R17* (1966), 57–103.
[37] *A. Prohl, M. Růžička*: On fully implicit space-time discretization for motions of incompressible fluids with shear dependent viscosities: The case $p \leqslant 2$. SIAM J. Numer. Anal. *39* (2001), 241–249.
[38] *K. R. Rajagopal, M. Růžička*: On the modeling of electrorheological materials. Mech. Research Comm. *23* (1996), 401–407.
[39] *K. R. Rajagopal, M. Růžička*: Mathematical modelling of electrorheological materials. Cont. Mech. and Thermodynamics *13* (2001), 59–78.
[40] Helsinki research group on variable exponent Lebesgue and Sobolev spaces. http://www.math.helsinki.fi/analysis/varsobgroup/.
[41] *M. Růžička*: A note on steady flow of fluids with shear dependent viscosity. Proceedings of the Second World Congress of Nonlinear Analysts (Athens, 1996). Nonlinear Anal. *30* (1997), 3029–3039.
[42] *M. Růžička*: Flow of shear dependent electrorheological fluids: Unsteady space periodic case. In: Applied Nonlinear Analysis (A. Sequeira, ed.). Kluwer/Plenum, New York, 1999, pp. 485–504.
[43] *M. Růžička*: Electrorheological fluids: Modeling and mathematical theory. RIMS Kokyuroku *1146* (2000), 16–38.
[44] *M. Růžička*: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, Vol. 1748. Springer-Verlag, Berlin, 2000.
[45] *C. Truesdell, W. Noll*: The Non-Linear Field Theories of Mechanics. Handbuch der Physik, Vol. III/3. Springer-Verlag, New York, 1965.
[46] *T. Wunderlich*: Der Einfluß der Elektrodenoberfläche und der Strömungsform auf den elektrorheologischen Effekt. PhD. thesis. University Erlangen-Nürnberg, 2000.
[47] *T. Wunderlich, P. O. Brunn*: Pressure drop measurements inside a flat channel—with flush mounted and protruding electrodes of varable length—using an electrorheological fluid. Experiments in Fluids *28* (2000), 455–461.

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MODELING, MATHEMATICAL AND NUMERICAL ANALYSIS OF ELECTRORHEOLOGICAL FLUIDS*

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Abstract. Many electrorheological fluids are suspensions consisting of solid particles and a carrier oil. If such a suspension is exposed to a strong electric field the effective viscosity increases dramatically. In this paper we first derive a model which captures this behaviour. For the resulting system of equations we then prove local in time existence of strong solutions for large data. For these solutions we finally derive error estimates for a fully implicit time-discretization.

Keywords: Maxwell's equations, electrorheological fluids, constitutive relations, Galerkin approximation

MSC 2000: 35Q35, 76W05, 65M60, 65M15

0. Introduction

Many *electrorheological fluids* (abbreviated: ERFs) are *suspensions* consisting of particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field. The observed *increase* of the measured shear stresses (or the measured viscosity) is essentially due to the existence of particle structures forming in the presence of an electric field hindering the flow and resulting in a higher, apparent viscosity. For an overview especially of microscopic models and explanations in electrorheology we refer the reader to Parthasarathy/Klingenberg [36].

In the first section we develop a model which captures the above described features. There are many ways to model ERFs and we refer the reader to the discussion in [39], [43], [19]. Here we model the ERF in a homogenized sense within the framework of continuum mechanics and follow the procedure from Rajagopal/Růžička [39], (cf. [44], [19]). In particular we take into account the complex interaction of the

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electro-magnetic fields and the moving liquid, thus treating the electric field as a variable that is determined by Maxwell's equations. The final system describing the motion of ERFs is derived from the general balance laws of thermodynamics and electrodynamics by a non-dimensionalization and a subsequent approximation.

In the second section we show the existence of strong solutions for the mechanical part of the system describing the flow of ERFs, i.e. the balance of mass and momentum. The constitutive relation for the extra stress tensor implies that the system possesses p-structure, where however $p = p(|\mathbf{E}|^2)$ is a *material function* and not a constant. Thus the natural functional setting are *generalized Lebesgue and Sobolev spaces.* The basic properties of these spaces can be found in Kováčik/Rákosník [28] (cf. Diening [10], [11], Diening/Růžička [15], [16], [17] for more recent results and the web-page [40] for up-to-date information). The method presented here is based on ideas developed in [31], [32], [6], [30], [33] (cf. [21], [13] for an overview of recent results for generalized Newtonian fluids) to handle situations when the elliptic operator is monotone, but due to the properties of the convective term the *theory of monotone operators* is not applicable. Our presentation follows the treatment in Diening [12], Diening/Růžička [14].

In the third section we prove error estimates for the difference between a *strong* solution of the continuous system and a *weak* solution of the fully implicit time-discretization of this system under the additional assumption that $p = \text{const}$. In contrast to the mathematical analysis there are only few numerical results for such a system (cf. [5], [4], [37], [13]). Here we generalize the treatment of Diening/Prohl/Růžička [13] to the case that the extra stress tensor is not derived from a potential.

1. Modeling

We start by stating Maxwell's equations. Here we use the so-called "statistical formulation", which is based on a "dipole-current-loop" model (cf. Eringen/Maugin [20], Hutter/van de Ven [27], Grot [25], Pao [35]):

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.1}$$

$$\operatorname{curl} \mathbf{H} = \frac{\partial \mathbf{D}^e}{\partial t} + \mathbf{J}, \tag{1.2}$$

$$\operatorname{div} \mathbf{D}^e = q^e, \tag{1.3}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{1.4}$$

where $\mathbf{E}$ is the *electric field*, $\mathbf{B}$ the *magnetic flux density*, $\mathbf{H}$ is the *magnetic field* given by $\mathbf{H} = {\mu_0}^{-1}\mathbf{B} - \mathbf{M}$ with the *magnetization* $\mathbf{M}$, $\mathbf{D}^e$ is the *dielectric displacement* given by $\mathbf{D}^e = \mathbf{P} + \varepsilon_0 \mathbf{E}$ with the *electric polarization* $\mathbf{P}$, $\mathbf{J}$ the *current density*, q^e the *density of the free electric charges* and ε_0 and μ_0 denote the dielectric constant and the permeability in vacuo, respectively.

Now we state the thermo-mechanical balance laws. The balance of mass and momentum are[1]

$$\dot{\varrho} + \varrho \operatorname{div} \mathbf{v} = 0, \tag{1.5}$$

$$\varrho \dot{\mathbf{v}} - \operatorname{div} \mathbf{T} = \mathbf{f} + \mathbf{f}^e, \tag{1.6}$$

respectively, where ϱ is the *mass density*, $\mathbf{T}$ the *Cauchy stress tensor*[2], $\mathbf{f}$ the *mechanical force density* and $\mathbf{f}^e$ is the *electro-magnetic force density* which is given by (cf. pages 284–285 of [35])[3]

$$\mathbf{f}^e = \mathbf{q}^e \boldsymbol{\mathcal{E}} + [\boldsymbol{\mathcal{J}} + \dot{\mathbf{P}} - [\nabla \mathbf{v}]\mathbf{P} + (\operatorname{div} \mathbf{v})\mathbf{P}] \times \mathbf{B} + [\nabla \mathbf{B}]^\top \boldsymbol{\mathcal{M}} + [\nabla \boldsymbol{\mathcal{E}}]\mathbf{P} \tag{1.7}$$

where $\boldsymbol{\mathcal{E}}$ is the *effective electric field strength* defined as

$$\boldsymbol{\mathcal{E}} = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{1.8}$$

$\boldsymbol{\mathcal{J}}$ the *conductive current density* given by

$$\boldsymbol{\mathcal{J}} = \mathbf{J} - \mathbf{q}^e \mathbf{v} \tag{1.9}$$

and $\boldsymbol{\mathcal{M}}$ the *effective magnetization* defined through

$$\boldsymbol{\mathcal{M}} = \mathbf{M} + \mathbf{v} \times \mathbf{P}. \tag{1.10}$$

The balance of angular momentum takes the form

$$\mathbf{x} \times \varrho \dot{\mathbf{v}} - \operatorname{div}(\mathbf{x} \times \mathbf{T}) = \mathbf{x} \times \mathbf{f} + \mathbf{l}^e, \tag{1.11}$$

in which $\mathbf{l}^e$ denotes the electro-magnetic torque density (cf. p. 284–285 of [35]) given by

$$\mathbf{l}^e = \mathbf{x} \times \mathbf{f}^e + \mathbf{P} \times \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{M}} \times \mathbf{B}. \tag{1.12}$$

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[1] The material time derivative is denoted by a superposed dot or by $\mathrm{d}/\mathrm{d}t$.

[2] $\mathbf{T}$ is introduced via $\mathbf{t} = \mathbf{T} \cdot \mathbf{n}$, where $\mathbf{t}$ is the *Cauchy stress vector* and $\mathbf{n}$ the *outer unit normal vector.*

[3] Here and in the following we use the notation $[\nabla \mathbf{v}]\mathbf{w} = \left(w_j \partial v_i / \partial x_j\right)_{i=1,2,3}$, where the summation convention over repeated indices is used. We will use that convention throughout this paper.

The balance of total energy takes the form

$$\varrho \frac{\mathrm{d}}{\mathrm{d}t}\Big(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\Big) = \operatorname{div}(\mathbf{T}^\top \mathbf{v} - \mathbf{q}) + (\mathbf{f} + \mathbf{f}^e)\cdot\mathbf{v} + w + w^e, \tag{1.13}$$

where e denotes the *specific internal energy*, $\mathbf{q}$ the *heat flux*, w the *mechanical energy production density* and w^e the *electro-magnetic energy supply density* which is given as (cf. p. 284–285 of [35])

$$w_e = \boldsymbol{\mathcal{J}}\cdot\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}\cdot\dot{\mathbf{P}} - \boldsymbol{\mathcal{M}}\cdot\dot{\mathbf{B}} + \boldsymbol{\mathcal{E}}\cdot\mathbf{P}\operatorname{div}\mathbf{v}. \tag{1.14}$$

Using (1.6) together with (1.14), we obtain from (1.13) the balance of internal energy according to

$$\varrho\dot{e} + \operatorname{div}\mathbf{q} = \mathbf{T}\cdot\mathbf{L} + \boldsymbol{\mathcal{J}}\cdot\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}\cdot\dot{\mathbf{P}} - \boldsymbol{\mathcal{M}}\cdot\dot{\mathbf{B}} + \mathbf{P}\cdot\boldsymbol{\mathcal{E}}\operatorname{div}\mathbf{v} + w, \tag{1.15}$$

where $\mathbf{L} = \nabla\mathbf{v}$ is the *velocity gradient*. We interpret the second law of thermodynamics in the form of the Clausius-Duhem inequality

$$\varrho\dot{\eta} + \operatorname{div}\frac{\mathbf{q}}{\theta} - \frac{w}{\theta} \geqslant 0, \tag{1.16}$$

where η is the *specific entropy* and θ the absolute temperature.

The system (1.1)–(1.4), (1.5), (1.6), (1.15) and (1.16) which describes the motion of the liquid has far more unknowns than equations. It is rendered determinate by providing appropriate *constitutive relations* reflecting the material properties. Towards this end, we will assume that

$$\varrho,\ \theta,\ \nabla\theta,\ \mathbf{v},\ \mathbf{D},\ \mathbf{E},\ \mathbf{B}, \tag{1.17}$$

where $\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^\top)$ is the *symmetric velocity gradient*, are the independent variables and thus we provide constitutive relations for

$$e,\ \eta,\ \mathbf{T},\ \mathbf{q},\ \mathbf{P},\ \boldsymbol{\mathcal{M}},\ \boldsymbol{\mathcal{J}} \tag{1.18}$$

of the form

$$f = \hat{f}(\varrho, \theta, \nabla\theta, \mathbf{v}, \mathbf{D}, \mathbf{E}, \mathbf{B}), \tag{1.19}$$

where f stands for any of the quantities in (1.18).

Both the material and the balance equations are subject to invariance requirements. It is well known that the mechanical balance laws (1.5), (1.6) and (1.15) are form-invariant under Galilean transformations given by

$$\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{v}_0 t + \mathbf{b}_0, \quad t^* = t, \tag{1.20}$$

where $\mathbf{v}_0$, $\mathbf{b}_0$ are constant vectors and $\mathbf{Q}$ is a time independent orthogonal tensor, while Maxwell's equations (1.1)–(1.4) are form-invariant under Lorentz transformations. We are interested in non-relativistic effects and it is well-known that there are problems with consistent invariance requirements for all thermo-mechanical and electro-magnetic balance laws and constitutive equations in a non-relativistic situation (cf. [25], [38], [44]). To avoid these difficulties we shall make the following *invariance requirements*: We assume that the quantities (1.18), describing the material properties, are invariant under Galilean transformations (1.20)[4]. Moreover we require that all balance laws (1.5), (1.6), (1.15), (1.16) and (1.1)–(1.4) are form-invariant under Galilean transformations (1.20). These two requirements imply consistent transformation formulæ for all necessary quantities (cf. [44]). In particular, we obtain from the invariance requirements that the constitutive relations (1.19) are isotropic functions of their arguments and that (1.19) has to be replaced by (cf. Grot [25])

$$f = \hat{f}(\varrho, \theta, \nabla\theta, \mathbf{D}, \boldsymbol{\mathcal{E}}, \mathbf{B}), \tag{1.21}$$

where f stands for any of the quantities in (1.18).

In addition to restrictions placed on the constitutive response functions by the invariance requirements we have additional strictures due to the requirement of the second law of thermodynamics. We shall now determine the restrictions imposed by requiring that all admissible processes of the body, i.e. processes compatible with the balance laws and the constitutive response functions, meet the Clausius-Duhem inequality (1.16). Introducing the specific Helmholtz free energy ψ through

$$\psi = e - \eta\theta - \frac{1}{\varrho}\boldsymbol{\mathcal{E}} \cdot \mathbf{P}, \tag{1.22}$$

and substituting it into (1.16) we obtain, with the help of the energy balance (1.15) and the balance of mass (1.5), the *dissipation inequality*

$$-\varrho(\dot{\psi} + \eta\,\dot{\theta}) + \mathbf{T} \cdot \mathbf{L} - \frac{\mathbf{q} \cdot \nabla\theta}{\theta} - \dot{\boldsymbol{\mathcal{E}}} \cdot \mathbf{P} - \boldsymbol{\mathcal{M}} \cdot \dot{\mathbf{B}} + \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}} \geqslant 0. \tag{1.23}$$

[4] Note that one usually assumes that the constitutive relations depend on $\mathbf{L}$ instead of $\mathbf{D}$, and then one deduces from the principle of material frame indifference, i.e. $(1.20)_1$ is replaced by $\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t)$, that the dependence on $\mathbf{L}$ has to reduce to a dependence on $\mathbf{D}$ only. In fact, this is the only relevant consequence of the stronger requirement of material frame indifference for us which cannot be obtained from the requirement that the material properties are invariant under Galilean transformations (1.20) only.

From (1.21) and (1.22) we get that $\psi = \psi(\varrho, \theta, \nabla\theta, \mathbf{D}, \boldsymbol{\mathcal{E}}, \mathbf{B})$. If we now compute $\dot{\psi}$ explicitely we can re-write (1.23), also using (1.5), as

$$(1.24)\quad -\varrho\Big(\frac{\partial\psi}{\partial\theta} + \eta\Big)\dot{\theta} - \varrho\frac{\partial\psi}{\partial\mathbf{D}} \cdot \dot{\mathbf{D}} - \Big(\boldsymbol{\mathcal{M}} + \varrho\frac{\partial\psi}{\partial\mathbf{B}}\Big) \cdot \dot{\mathbf{B}} + \Big(\mathbf{T} + \varrho^2\frac{\partial\psi}{\partial\varrho}\mathbf{I}\Big) \cdot \mathbf{D}$$
$$+ \mathbf{T} \cdot \mathbf{W} - \varrho\frac{\partial\psi}{\partial\nabla\theta}(\nabla\theta)^{\cdot} - \frac{\mathbf{q} \cdot \nabla\theta}{\theta} - \Big(\varrho\frac{\partial\psi}{\partial\boldsymbol{\mathcal{E}}} + \mathbf{P}\Big) \cdot \dot{\boldsymbol{\mathcal{E}}} + \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}} \geqslant 0.$$

Using the linearity of (1.24) with respect to the dotted quantities and $\mathbf{W}$ and their independence on the arguments appearing in the constitutive relations (1.21) one easily deduces (cf. Coleman, Noll [9], Truesdell/Noll [45], Grot [25])

$$(1.25)\quad \eta = -\frac{\partial\psi}{\partial\theta}, \quad \frac{\partial\psi}{\partial\nabla\theta} = \mathbf{0}, \quad \frac{\partial\psi}{\partial\mathbf{D}} = \mathbf{0},$$
$$\mathbf{P} = -\varrho\frac{\partial\psi}{\partial\boldsymbol{\mathcal{E}}}, \quad \boldsymbol{\mathcal{M}} = -\varrho\frac{\partial\psi}{\partial\mathbf{B}}, \quad \mathbf{T}^\top = \mathbf{T},$$

and the *reduced dissipation inequality*

$$(1.26)\quad \Big(\mathbf{T} + \varrho^2\frac{\partial\psi}{\partial\varrho}\mathbf{I}\Big) \cdot \mathbf{D} - \frac{\mathbf{q} \cdot \nabla\theta}{\theta} + \boldsymbol{\mathcal{J}} \cdot \boldsymbol{\mathcal{E}} \geqslant 0,$$

where ψ, η, $\mathbf{P}$ and $\boldsymbol{\mathcal{M}}$ are functions of ϱ, θ, $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$ only.

1.1. Electrorheological approximation

The equations derived in the last section may be simplified in view of electrorheological applications. Towards this end it is recommendable to carry out an appropriate *non-dimensionalization* with a subsequent *approximation.* All assumptions made in this section are based upon our understanding of the behaviour of ERFs, both from the theoretical and experimental point of view (cf. [7], [8], [18], [44], [46]).

Firstly, we shall assume that the Cauchy stress tensor $\mathbf{T}$ does not depend on the electric flux density $\mathbf{B}$, i.e.

$$(1.27)\quad \mathbf{T} = \hat{\mathbf{T}}(\varrho, \theta, \nabla\theta, \mathbf{D}, \boldsymbol{\mathcal{E}}).$$

This assumption reflects the observation that the material properties of an ERF do not change if a magnetic field is applied, because surely the particles in an ERF bear no magnetic properties.

Secondly, we shall assume that we are dealing with a dielectricum, i.e.

$$(1.28)\quad \boldsymbol{\mathcal{M}} \equiv \mathbf{0} \quad \text{where} \quad \boldsymbol{\mathcal{M}} = \mathbf{M} + \mathbf{v} \times \mathbf{P}.$$

Note that this assumption ensures that an apparent magnetization can only be generated by a moving polarized fluid (cf. [25]). This common assumption is a crucial point for deriving the so-called "quasi-electrostatic equations". In view of (1.25) the assumption (1.28) also implies that the Helmholtz free energy ψ, and thus also the polarization $\mathbf{P}$ and the entropy η, are only functions of ϱ, θ and $\boldsymbol{\mathcal{E}}$.

Thirdly, we shall assume that the fluid is electrically non-conducting, i.e.

$$\boldsymbol{\mathcal{J}} \equiv \mathbf{0}. \tag{1.29}$$

This assumption may not be fully justified in general, because some ERFs exhibit a certain electrical conductivity which is often due to the content of water. However, many of them are free of water and have very low electrical conductivity (for example the polyurethane dispersions described in detail in [7], [8]), and thus we may restrict ourselves to such a class.

In order to reach the final *electrorheological approximation* and to determine and retain terms that are dominant and discard others that are insignificant, we will carry out a dimensional analysis which follows closely the one in [38], [44]. Towards this end we may introduce the following *dimensionless* quantities[5]:

$$\begin{gathered} \overline{\mathbf{E}} = \frac{\mathbf{E}}{E_0}, \quad \overline{\mathbf{B}} = \frac{\mathbf{B}}{B_0}, \quad \bar{q}^e = \frac{q^e}{q_0}, \quad \overline{\mathbf{T}} = \frac{\mathbf{T}}{T_0}, \quad \bar{\mathbf{v}} = \frac{\mathbf{v}}{V_0}, \quad \bar{\mathbf{x}} = \frac{\mathbf{x}}{L_0}, \\ \bar{t} = \frac{t}{t_0}, \quad \overline{\mathbf{P}} = \frac{\mathbf{P}}{\varepsilon_0 E_0}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_0}, \quad \bar{\mathbf{f}} = \frac{\mathbf{f}}{f_0}, \quad \bar{\theta} = \frac{\theta}{\theta_0}, \end{gathered} \tag{1.30}$$

where the quantities with the subscript "0" are appropriate characteristic quantities of the problem in question. In typical problems and for many ERFs (cf. [7], [8]), we envisage that

$$\begin{aligned} &E_0 \sim 3 \cdot (10^4 - 10^6)\,\mathrm{V\,m^{-1}}, && V_0 \sim (10^{-3} - 1)\,\mathrm{m\,s^{-1}}, \\ &L_0 \sim 5 \cdot (10^{-4} - 10^{-3})\,\mathrm{m}, && \eta_0 \sim (10^{-2} - 10^{-1})\,\mathrm{kg\,(m\ s)^{-1}}, \\ &t_0 \sim (10^{-3} - 1)\,\mathrm{s}, && \varrho_0 \sim 10^3\,\mathrm{kg\,m^{-3}}. \end{aligned} \tag{1.31}$$

The time t_0 may be either a characteristic electric or hydrodynamic time, depending on the specific problem. Moreover, ϱ_0 and η_0 are the density and the dynamic viscosity of the fluid in the *absence* of an electric field, respectively. Using (1.31), the Reynolds number $\mathrm{Re} = (\varrho_0\, L_0\, V_0)/\eta_0$ and the Strouhal number $\mathrm{Str} = L_0/(V_0 t_0)$ lie in the range

$$5 \cdot 10^{-3} \leqslant \mathrm{Re} \leqslant 5 \cdot 10^2 \quad \text{and} \quad 5 \cdot 10^{-4} \leqslant \mathrm{Str} \leqslant 5 \cdot 10^3, \tag{1.32}$$

[5] In this section, dimensionless quantities and operators are denoted by a superposed bar.

respectively. Magnetic quantities are missing in (1.31). No experimental observation is known to us that shows that the magnetic field plays a significant role in electrorheological applications. Usually, no external magnetic field is applied and thus $\mathbf{B}$ is only induced due to the electric field. We interpret the *secondary* role of $\mathbf{B}$ in ERFs through the assumptions that

$$\frac{E_0}{B_0}\,\frac{L_0}{c^2 t_0} = O(1), \tag{1.33}$$

resulting in

$$B_0 \sim (10^{-16} - 10^{-10})\,\mathrm{Vs/m^2}. \tag{1.34}$$

Recall that $c \approx 3 \cdot 10^8\,\mathrm{m\,s^{-1}}$ denotes the speed of electro-magnetic waves in vacuo. (1.33) is consistent with the assumption that the magnetic flux density is only induced by oscillations of the electric field and/or the motion of a polarized body (cf. (1.42)). Let us introduce a small non-dimensional number ε through

$$\varepsilon \equiv 10^{-3}, \tag{1.35}$$

which measures the importance of the terms. The situation described above—together with an assumption that there are only few free charges in the fluid—can thus be summarized as

$$\begin{aligned} \frac{L_0}{c\,t_0} &= O(\varepsilon^3) - O(\varepsilon^4), & \frac{V_0}{c} &= O(\varepsilon^3) - O(\varepsilon^4), \\ \frac{V_0\,t_0}{L_0} &= O(\varepsilon^{-1}) - O(\varepsilon), & \frac{q_0\,L_0}{\varepsilon_0 E_0} &= O(\varepsilon^3), \\ \frac{B_0\,L_0}{E_0\,t_0} &= O(\varepsilon^5) - O(\varepsilon^8), & \frac{E_0\,V_0}{B_0\,c^2} &= O(1). \end{aligned} \tag{1.36}$$

The non-dimensionalized system of balance laws may then be approximated by retaining terms up to order ε^2, while neglecting terms of higher order.

Firstly, let us discuss the role of $\boldsymbol{\mathcal{E}}$ in the constitutive relations. It follows from the definition of $\boldsymbol{\mathcal{E}}$ that

$$\bar{\boldsymbol{\mathcal{E}}} = \frac{\boldsymbol{\mathcal{E}}}{E_0} = \bar{\mathbf{E}} + \frac{V_0\,B_0}{E_0}\,\bar{\mathbf{v}} \times \bar{\mathbf{B}} = \bar{\mathbf{E}} + O(\varepsilon^5), \tag{1.37}$$

where we used that

$$\frac{V_0\,B_0}{E_0} = O(\varepsilon^5) - O(\varepsilon^7). \tag{1.38}$$

Thus, we can replace $\bar{\boldsymbol{\mathcal{E}}}$ by $\bar{\mathbf{E}}$ in all non-dimensionalized constitutive relations.

The dimensionless form of Maxwell's equations (1.1)–(1.4) may be obtained upon using the definitions of $\mathbf{H}$, $\mathbf{D}^e$, (1.28), (1.36) and (1.37) as

$$\overline{\operatorname{div}}\overline{\mathbf{E}} + \overline{\operatorname{div}}\overline{\mathbf{P}} = \underbrace{\frac{q_0\,L_0}{\varepsilon_0 E_0}}_{O(\varepsilon^3)}\,\bar{q}^e + O(\varepsilon^5), \qquad \overline{\operatorname{curl}}\overline{\mathbf{E}} + \underbrace{\frac{B_0\,L_0}{E_0\,t_0}}_{O(\varepsilon^5)}\,\frac{\partial\overline{\mathbf{B}}}{\partial\bar{t}} = \mathbf{0}, \qquad \overline{\operatorname{div}}\overline{\mathbf{B}} = 0,$$

$$\overline{\operatorname{curl}}\overline{\mathbf{B}} + \underbrace{\frac{E_0\,V_0}{B_0\,c^2}}_{O(1)}\,\overline{\operatorname{curl}}(\bar{\mathbf{v}} \times \overline{\mathbf{P}}) = \underbrace{\frac{E_0}{B_0}\,\frac{L_0}{c^2\,t_0}}_{O(1)}\,\frac{\partial}{\partial\bar{t}}(\overline{\mathbf{E}} + \overline{\mathbf{P}}) - \underbrace{\frac{q_0\,L_0}{\varepsilon_0 E_0}\,\frac{E_0\,V_0}{B_0\,c^2}}_{O(\varepsilon^3)}\,\bar{q}^e\bar{\mathbf{v}} + O(\varepsilon^5),$$

where in $O(\varepsilon^5)$ only terms coming from (1.37) are included and where we also used the relation $\varepsilon_0\mu_0 = c^{-2}$. Neglecting terms of $O(\varepsilon^3)$, we obtain the *electrorheological approximation of Maxwell's equations* according to[6]

$$\operatorname{div}(\varepsilon_0\mathbf{E} + \mathbf{P}) = 0, \tag{1.39}$$

$$\operatorname{curl}\mathbf{E} = \mathbf{0}, \tag{1.40}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{1.41}$$

$$\frac{1}{\mu_0}\operatorname{curl}\mathbf{B} + \operatorname{curl}(\mathbf{v} \times \mathbf{P}) = \frac{\partial(\varepsilon_0\mathbf{E} + \mathbf{P})}{\partial t}, \tag{1.42}$$

where $\mathbf{P} = \mathbf{P}(\varrho, \theta, \mathbf{E})$.

Now we turn to the approximation of the thermo-mechanical balance laws. The conservation of mass (1.5) remains unaffected. In the momentum equation (1.6) we re-write the electro-magnetic force $\mathbf{f}^e$ using (1.8), (1.28), (1.29) and then use (1.36) and (1.37), which leads to

$$\begin{aligned}
&\frac{\varrho_0\,V_0\,L_0}{\varepsilon_0\,E_0^2\,t_0}\,\bar{\varrho}\frac{\partial\bar{\mathbf{v}}}{\partial\bar{t}} + \frac{\varrho_0\,V_0^2}{\varepsilon_0\,E_0^2}\,\bar{\varrho}[\overline{\nabla}\bar{\mathbf{v}}]\bar{\mathbf{v}} - \frac{T_0}{\varepsilon_0\,E_0^2}\,\overline{\operatorname{div}}\overline{\mathbf{T}} \\
&\quad = f_0\frac{L_0}{\varepsilon_0\,E_0^2}\bar{\mathbf{f}} + \underbrace{\frac{q_0\,L_0}{\varepsilon_0\,E_0}}_{O(\varepsilon^3)}\Big(\overline{q_e}\overline{\mathbf{E}} + \underbrace{\frac{V_0\,B_0}{E_0}}_{O(\varepsilon^5)}\,\bar{q}_e\bar{\mathbf{v}} \times \overline{\mathbf{B}}\Big) + \underbrace{\frac{B_0\,L_0}{E_0\,t_0}}_{O(\varepsilon^5)}\,\frac{\partial\overline{\mathbf{P}}}{\partial\bar{t}} \times \overline{\mathbf{B}} \\
&\qquad + \underbrace{\frac{V_0\,B_0}{E_0}}_{O(\varepsilon^5)}([\overline{\nabla}\overline{\mathbf{P}}]\bar{\mathbf{v}} + (\overline{\operatorname{div}}\bar{\mathbf{v}})\overline{\mathbf{P}} \times \overline{\mathbf{B}} + \bar{\mathbf{v}} \times ([\overline{\nabla}\overline{\mathbf{B}}]\overline{\mathbf{P}})) + [\overline{\nabla}\overline{\mathbf{E}}]\overline{\mathbf{P}} + O(\varepsilon^5),
\end{aligned} \tag{1.43}$$

where in $O(\varepsilon^5)$ only terms coming from (1.37) are included. We see that all underbraced terms on the right-hand side of (1.43) have to be neglected. We shall retain

[6] Since $\boldsymbol{\mathcal{M}} = \mathbf{0}$, we can rewrite (1.39)–(1.42) in terms of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{D}^e$ only.

the mechanical force term and the term with the Cauchy stress. Furthermore, one easily computes that

$$\frac{\varrho_0\, V_0\, L_0}{\varepsilon_0\, E_0^2\, t_0} = \begin{cases} O(1) - O(\varepsilon^2) & \text{if } E_0^2 \sim 9 \cdot 10^{12}\, \mathrm{V}^2\, \mathrm{m}^{-2}, \\ O(\varepsilon^{-1}) - O(\varepsilon^1) & \text{if } E_0^2 \sim 9 \cdot 10^{10}\, \mathrm{V}^2\, \mathrm{m}^{-2}, \\ O(\varepsilon^{-2}) - O(1) & \text{if } E_0^2 \sim 9 \cdot 10^{8}\, \mathrm{V}^2\, \mathrm{m}^{-2}\,, \end{cases} \tag{1.44}$$

$$\frac{\varrho_0\, V_0^2}{\varepsilon_0\, E_0^2} = \begin{cases} O(1) - O(\varepsilon^2) & \text{if } E_0^2 \sim 9 \cdot 10^{12}\, \mathrm{V}^2\, \mathrm{m}^{-2}, \\ O(\varepsilon^{-1}) - O(\varepsilon^1) & \text{if } E_0^2 \sim 9 \cdot 10^{10}\, \mathrm{V}^2\, \mathrm{m}^{-2}, \\ O(\varepsilon^{-2}) - O(1) & \text{if } E_0^2 \sim 9 \cdot 10^{8}\, \mathrm{V}^2\, \mathrm{m}^{-2}. \end{cases} \tag{1.45}$$

Therefore also the first and the second term on the left-hand side of (1.43) have to be kept. With regard to the approximation of the other thermo-mechanical non-dimensionalized equations, we only replace $\overline{\boldsymbol{\mathcal{E}}}$ by $\overline{\mathbf{E}}$ since we have no indication of the behaviour of the other quantities.

Therefore, the *electrorheological approximation of the thermo-mechanical balance laws* is given by

$$\dot{\varrho} + \varrho \operatorname{div} \mathbf{v} = 0, \tag{1.46}$$

$$\varrho\dot{\mathbf{v}} - \operatorname{div} \mathbf{T} = \mathbf{f} + [\nabla\mathbf{E}]\mathbf{P}, \tag{1.47}$$

$$c_v \varrho\, \dot{\theta} - k\Delta\theta - \Big(\frac{\partial \mathbf{P}}{\partial \theta} \cdot \dot{\mathbf{E}} + \frac{\partial \pi}{\partial \theta} \operatorname{tr} \mathbf{D}\Big)\theta = (\mathbf{T} - \pi\mathbf{I}) \cdot \mathbf{D} + w, \tag{1.48}$$

$$(\mathbf{T} - \pi\mathbf{I}) \cdot \mathbf{D} - \frac{(\nabla\theta) \cdot \mathbf{q}}{\theta} \geqslant 0, \tag{1.49}$$

where we used the definition of the *specific heat* c_v and of the *thermodynamic pressure* π according to

$$c_v = -\theta \frac{\partial^2 \psi}{\partial \theta^2}, \qquad \pi = -\varrho^2 \frac{\partial \psi}{\partial \varrho}.$$

Moreover c_v, $\mathbf{P}$, π and ψ are functions of ϱ, θ and $\mathbf{E}$; while we have for the Cauchy stress $\mathbf{T} = \mathbf{T}(\varrho, \theta, \nabla\theta, \mathbf{D}, \mathbf{E})$.

1.2. Constitutive relations

Now we will develop a constitutive theory for ERFs. In order to keep the already very long and complicated formulæ as simple as possible we keep the dependence on $\nabla\theta$ only in the constitutive relation for the heat flux $\mathbf{q}$ and assume that

$$\mathbf{q} = -k\nabla\theta, \tag{1.50}$$

where the *thermal conductivity* k is a positive constant. In all other constitutive relations we drop the dependence on $\nabla\theta$. We also restrict ourselves to the case of an incompressible ERF, i.e.

$$\operatorname{tr}\mathbf{D} = 0, \tag{1.51}$$

and consequently we also drop the dependence on ϱ in all constitutive relations. Moreover we assume a linear dependence of the polarization $\mathbf{P}$ on the electric field $\mathbf{E}$, i.e.

$$\mathbf{P} = \chi^E(\theta)\mathbf{E}, \tag{1.52}$$

where χ^E is the *dielectric susceptibility.* The Cauchy stress can be splited according to $\mathbf{T} = -\pi\mathbf{I} + \mathbf{S}$. From the above assumptions and (1.27) we get that the *extra stress tensor* $\mathbf{S}$ is of the form

$$\mathbf{S} = \mathbf{S}(\theta, \mathbf{D}, \mathbf{E}). \tag{1.53}$$

From representation theorems (cf. the appendix of [20] and the references stated there) it follows that the most general form for $\mathbf{S}$ is given by

$$\begin{aligned} \mathbf{S} = {} & \alpha_2 \mathbf{E}\otimes\mathbf{E} + \alpha_3\mathbf{D} + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{DE}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{DE}) \\ & + \alpha_6(\mathbf{D}^2\mathbf{E}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{D}^2\mathbf{E}), \end{aligned} \tag{1.54}$$

where α_i, $i = 2, \ldots 6$ may be functions of the invariants

$$\theta,\ |\mathbf{E}|^2,\ \operatorname{tr}\mathbf{D}^2,\ \operatorname{tr}\mathbf{D}^3,\ \operatorname{tr}(\mathbf{DE}\otimes\mathbf{E}),\ \operatorname{tr}(\mathbf{D}^2\mathbf{E}\otimes\mathbf{E}). \tag{1.55}$$

In view of certain peculiarities in the behaviour of the normal stress differences in the case $\alpha_4 \neq 0$ even in the absence of an electric field (cf. [33]) and due to previous mathematical investigations for shear dependent viscous fluids, which suggests that terms involving $\mathbf{D}^2$ can be treated as a perturbation (cf. [31], [33]), we assume that

$$\alpha_4 \equiv 0, \qquad \alpha_6 \equiv 0. \tag{1.56}$$

Based on experimental data (cf. [26], [3], [2], [1], [47]) we assume that in the presence and the absence of an electric field the ERF behaves like a *generalized Newtonian fluid* with power p, where the power p can depend on the magnitude of the electric field $|\mathbf{E}|^2$. Moreover, we restrict ourselves to the case that the material functions α_2, α_3 and α_5 depend only on the invariants θ, $|\mathbf{D}|^2$ and $|\mathbf{E}|^2$ and that all terms have the same *growth behaviour.* Thus we deal with the following model for the extra stress tensor $\mathbf{S}$

$$\begin{aligned} \mathbf{S} = {} & \alpha_{21}((1+|\mathbf{D}|^2)^{(p-1)/2} - 1)\mathbf{E}\otimes\mathbf{E} + (\alpha_{31} + \alpha_{33}|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p-2)/2}\mathbf{D} \\ & + \alpha_{51}(1+|\mathbf{D}|^2)^{(p-2)/2}(\mathbf{DE}\otimes\mathbf{E} + \mathbf{E}\otimes\mathbf{DE}), \end{aligned} \tag{1.57}$$

where α_{ij} are constants and $p = p(|\mathbf{E}|^2)$ is a C^1-function such that

$$1 < p_\infty \leqslant p(|\mathbf{E}|^2) \leqslant p_0. \tag{1.58}$$

To ensure the validity of the Clausius-Duhem inequality we further require that the constant coefficients α_{ij} and the function p are such that (cf. [44, Lemma 1.4.46])

$$\alpha_{31} > 0, \qquad \alpha_{33} > 0, \qquad \alpha_{33} + \frac{4}{3}\alpha_{51} > 0, \tag{1.59}$$

$$k(p_0)|\alpha_{21}| < \begin{cases} 2\sqrt{\alpha_{33}}\sqrt{2\alpha_{51}} & \text{if } \alpha_{33} \leqslant \frac{4}{3}\alpha_{51}, \\ \sqrt{\frac{3}{2}}(\alpha_{33} + \frac{4}{3}\alpha_{51}) & \text{if } \frac{4}{3}|\alpha_{51}| \leqslant \alpha_{33}, \end{cases} \tag{1.60}$$

where $k(p_0) = 1$ if $p_0 \leqslant 3$ and $k(p_0) > 1$ is a computable constant for $p_0 > 3$. Note that these requirements ensure that the operator induced by $-\operatorname{div}\mathbf{S}(\mathbf{D}, \mathbf{E})$ is *coercive.*

2. Flows of shear dependent electrorheological fluids

In the previous section we have shown that the isothermal flow of an incompressible shear dependent ERF is governed by the following system[7]

$$\begin{aligned} \operatorname{div}\mathbf{E} &= 0, \\ \operatorname{curl}\mathbf{E} &= \mathbf{0}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} \partial_t\mathbf{v} - \operatorname{div}\mathbf{S} + [\nabla\mathbf{v}]\mathbf{v} + \nabla\pi &= \mathbf{f} + \chi^E[\nabla\mathbf{E}]\mathbf{E}, \\ \operatorname{div}\mathbf{v} &= 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \operatorname{div}\mathbf{B} &= 0, \\ \mu_0^{-1}\operatorname{curl}\mathbf{B} + \chi^E\operatorname{curl}(\mathbf{v} \times \mathbf{E}) &= (\varepsilon_0 + \chi^E)\partial_t\mathbf{E}, \end{aligned} \tag{2.3}$$

$$\mathbf{S} \cdot \mathbf{D} + w = 0, \tag{2.4}$$

where the extra stress tensor $\mathbf{S}$ is given by (1.57), (1.58).

The system (2.1)–(2.4) is separated. We first solve the quasi-static Maxwell's equations (2.1) for the electric field and then seek for the velocity field by solving (2.2). Knowing $\mathbf{E}$ and $\mathbf{v}$ we can solve (3.2) and (2.4). Note that the equation (2.4) has to be interpreted as an equation for the mechanical energy supply density w. It was already pointed out in the previous section that the magnetic induction $\mathbf{B}$ is of

[7] We have divided equation (1.47) by the constant density ϱ_0 and adapted the notation appropriately.

secondary importance, which is reflected by the structure of the above system. Moreover, the quasi-static Maxwell's equations (2.1) are widely studied in the literature (cf. the overview article Milani/Picard [34]). Since in this investigation of ERFs we are mainly interested in the velocity field $\mathbf{v}$, we shall only consider the system (2.2), in which $\mathbf{E}$ is assumed to be any given vector field, having certain regularity properties. Moreover, for simplicity we shall complete (2.2) by *space periodic boundary conditions* and an *initial condition* $\mathbf{v}_0$.

In order to prove existence results for the system (2.2) we need some structure conditions for the extra stress tensor $\mathbf{S}$, which unfortunately are stronger than the conditions we have to assume for the validity of the Clausius-Duhem inequality, which is a physical requirement. In the following we assume that the constant coefficients α_{ij} and the function p are such that the operator induced by $-\operatorname{div}\mathbf{S}(\mathbf{D},\mathbf{E})$ is *uniformly monotone*, i.e.

$$\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}}\, B_{ij}B_{kl} \geqslant \gamma_1(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p(|\mathbf{E}|^2)-2)/2}|\mathbf{B}|^2 \tag{2.5}$$

is satisfied for all $\mathbf{B},\mathbf{D} \in X := \{\mathbf{D} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}},\ \operatorname{tr}\mathbf{D} = 0\}$, and that the following *growth conditions* are satisfied for $i,j,k,l,n = 1,2,3$,

$$\Big|\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}}\Big| \leqslant \gamma_2(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p(|\mathbf{E}|^2)-2)/2}, \tag{2.6}$$

$$\Big|\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial E_n}\Big| \leqslant \gamma_3|\mathbf{E}|(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p(|\mathbf{E}|^2)-1)/2}(1+\ln(1+|\mathbf{D}|^2)). \tag{2.7}$$

Conditions for α_{ij} and p that ensure the validity of (2.5) can be found in [44, Chapter 1]. We will show that the *coercivity*, i.e. that

$$\mathbf{S}(\mathbf{D},\mathbf{E})\cdot\mathbf{D} \geqslant c(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{(p(|\mathbf{E}|^2)-2)/2}|\mathbf{D}|^2 \tag{2.8}$$

holds for all $\mathbf{D} \in X$, is a consequence of (2.5).[8]

Before formulating the main result of this section, we introduce some notation. Let $\Omega = (0,L)^3$, $L \in (0,\infty)$ be a cube in $\mathbb{R}^3$ and denote $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$ and $\Gamma_{j+3} = \partial\Omega \cap \{x_j = L\}$, for $j = 1,2,3$. For $T \in (0,\infty)$, we denote by Q_T the time-space cylinder $I \times \Omega$, where $I = [0,T]$ is a time interval. By $\mathcal{D}(\Omega)$ we denote the space of smooth periodic functions with mean value zero. Let further $q > 1$ and $k > 0$. Then $(L^q(\Omega), \|\cdot\|_q)$ and $(W^{k,q}(\Omega), \|\cdot\|_{k,q})$, respectively, is used for the usual Lebesgue and Sobolev spaces, of periodic functions with mean value zero. By

[8] As was already pointed out the coercivity and the Clausius-Duhem inequality are almost equivalent. In fact, if p is independent of $|\mathbf{E}|^2$ than these two requirements are the same.

$\langle f, g\rangle := \int_\Omega fg\,\mathrm{d}x$ we denote the scalar product with respect to space. We also need Lebesgue and Sobolev spaces with variable exponents, which are denoted by $L^{p(\cdot)}(G)$ and $W^{k,p(\cdot)}(G)$, respectively, where $G = \Omega$ or $G = Q_T$. For a given $p(\cdot) \in L^\infty(G)$, $1 < p_\infty \leqslant p(x) \leqslant p_0 < \infty$, we define the *modular*

$$\varrho_p(f) = \varrho_{p,G}(f) := \int_G |f(y)|^{p(y)}\,\mathrm{d}y.$$

Similarly to the Luxemburg norm in Orlicz spaces we define

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 \mid \varrho_p(\lambda^{-1}f) < 1\},$$

which is a norm on the *generalized Lebesgue space*

$$L^{p(\cdot)}(G) := \{f \in L^1(G) \mid \varrho_p(\lambda^{-1}f) < \infty \text{ for some } \lambda > 0\}.$$

Generalized Sobolev spaces are defined analogously. We refer to Kováčik/Rákosník [28] for a detailed treatment of these spaces. Moreover, we denote by $L^q(I;X)$ the Bochner spaces which are equipped with the norm $\bigl(\int_I \|\cdot\|_X^q\,\mathrm{d}s\bigr)^{1/q}$. In the following we use for the partial derivative with respect to time the symbol ∂_t. We shall further make frequent use of spaces of divergence free functions defined by

$$\mathcal{V} := \{\boldsymbol{\psi} \in \mathcal{D}(\Omega) \colon \operatorname{div}\boldsymbol{\psi} = 0\},$$
$$V_p := \text{the closure of } \mathcal{V} \text{ with respect to the } \|\nabla\cdot\|_p\text{-norm},$$

and use the following expressions, for functions $\mathbf{v}$ and $\mathbf{E}$ defined on the space-time cylinder Q_T,

$$\mathcal{I}(t,\mathbf{v}) := \int_\Omega \frac{\partial S_{ij}(\mathbf{Dv}(t),\mathbf{E}(t))}{\partial D_{kl}} D_{ij}(\nabla\mathbf{v})(t) D_{kl}(\nabla\mathbf{v})(t)\,\mathrm{d}x, \tag{2.9}$$

$$\mathcal{J}(t,\mathbf{v}) := \int_\Omega \frac{\partial S_{ij}(\mathbf{Dv}(t),\mathbf{E}(t))}{\partial D_{kl}} D_{ij}(\partial_t\mathbf{v})(t) D_{kl}(\partial_t\mathbf{v})(t)\,\mathrm{d}x, \tag{2.10}$$

which are related to the extra stress tensor $\mathbf{S}$.

We are seeking solutions $\mathbf{v}$ of the system (2.2) completed with the initial condition

$$\mathbf{v}(0) = \mathbf{v}_0, \tag{2.11}$$

and with space-periodic boundary conditions

$$\mathbf{v}\big|_{\Gamma_j} = \mathbf{v}\big|_{\Gamma_{j+3}}, \qquad \nabla\mathbf{v}\big|_{\Gamma_j} = \nabla\mathbf{v}\big|_{\Gamma_{j+3}}, \qquad \pi\big|_{\Gamma_j} = \pi\big|_{\Gamma_{j+3}}, \tag{2.12}$$

for $j = 1, 2, 3$. Now we can formulate the main result of this section.

Theorem 2.13. *Assume that the extra stress tensor* $\mathbf{S}$ *satisfies* (2.5)–(2.7) *and* $\mathbf{S}(\mathbf{0},\mathbf{E})=\mathbf{0}$. *Let* $\mathbf{v}_0\in W^{2,2}(\Omega)\cap V_p$ *be a given initial velocity,* $\mathbf{f}\in C(I;W^{1,2}(\Omega))$, $\partial_t\mathbf{f}\in C(I;L^2(\Omega))$ *be a given force,* $\mathbf{E}\in W^{1,\infty}(I;W^{1,\infty}(\Omega))$ *be a given electric field and let* $p=p(|\mathbf{E}|^2)$ *be a* C^1*-function with* $p_\infty\leqslant p(|\mathbf{E}|^2)\leqslant p_0$. *If*

$$\frac{3}{2}<p_\infty\leqslant p_0\leqslant 2$$

then there exists a time $T^*>0$, *such that a strong solution* $\mathbf{v}$ *of the system* (2.2) *exists on* $I':=[0,T^*]$. *This solution satisfies*

$$\operatorname*{ess\,sup}_{s\in I'}\|\partial_t\mathbf{v}(s)\|_2^2+\int_0^{T^*}\mathcal{I}(t,\mathbf{v})^{\frac{5p_\infty-6}{2-p_\infty}}+\mathcal{J}(t,\mathbf{v})\,\mathrm{d}t\leqslant C(\mathbf{f},\mathbf{v}_0,\mathbf{E}). \tag{2.14}$$

In particular we have that for $1<r<6(p_\infty-1)$

$$\begin{aligned}\mathbf{v}&\in L^{p_\infty\frac{5p_\infty-6}{2-p_\infty}}\Big(I';W^{2,\frac{3p_\infty}{p_\infty+1}}(\Omega)\Big)\cap C(I';V_r),\\ \partial_t\mathbf{v}&\in L^{\frac{p_\infty(5p_\infty-6)}{(3p_\infty-2)(p_\infty-1)}}\Big(I';W^{1,\frac{3p_\infty}{p_\infty+1}}(\Omega)\Big)\cap L^\infty(I';L^2(\Omega)),\\ \partial_t^2\mathbf{v}&\in L^2(I';(V_2)^*).\end{aligned} \tag{2.15}$$

Remark 2.16. With a more refined technique one can show that the statement of the theorem is valid for $\frac{7}{5}<p_\infty\leqslant p_0\leqslant 2$ (cf. [14, Theorem 21]).

The main problem in the proof of the previous theorem consists in the identification of the limit

$$\lim_{N\to\infty}\int_0^T\int_\Omega \mathbf{S}(\mathbf{D}\mathbf{v}^N,\mathbf{E})\cdot\mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}x\,\mathrm{d}t$$

where $\mathbf{v}^N$ is some approximate solution of (2.2). The method used here is based on Vitali's convergence theorem and the almost everywhere convergence of $\mathbf{D}\mathbf{v}^N$. This method was developed in [31], [32], [6], [30], [14] to handle situations when the theory of monotone operators fails to identify the above limit. It is worth noticing that unsteady problems for ERFs cannot be treated with the help of monotonicity methods even for large p_∞ due to the *non-standard growth* of the governing system, i.e. within the classical Sobolev spaces our assumptions (2.5)–(2.7) imply

$$C(1+|\mathbf{D}|)^{p_\infty-2}|\mathbf{D}|^2\leqslant\mathbf{S}(\mathbf{D},\mathbf{E})\cdot\mathbf{D}\leqslant\tilde{C}(1+|\mathbf{D}|)^{p_0-2}|\mathbf{D}|^2.$$

Before we start with the proof of the above theorem we need some preliminary results related to the extra stress tensor $\mathbf{S}$. Let us start with an algebraic lemma.

We write $f \cong g$ iff there exist constants $C_0, C_1 > 0$ such that

$$C_0 f \leqslant g \leqslant C_1 f,$$

where we always indicate on which quantities the constants may depend.

Lemma 2.17. *For all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ and all $q > 1$ there holds*

$$\int_0^1 (1 + |\mathbf{B} + s(\mathbf{A} - \mathbf{B})|)^{q-2} \, \mathrm{d}s \cong (1 + |\mathbf{B}| + |\mathbf{A}|)^{q-2},$$

with constants depending on q only.

Proof. The proof can be found in [24, Lemma 8.3]. □

Remark 2.18. Since $|\mathbf{A}| + |\mathbf{A} - \mathbf{B}| \leqslant 2(|\mathbf{A}| + |\mathbf{B}|) \leqslant 4(|\mathbf{A}| + |\mathbf{A} - \mathbf{B}|)$ we immediately obtain from Lemma 2.17 that for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$ and all $q > 1$ there holds

$$\int_0^1 (1 + |\mathbf{B} + s(\mathbf{A} - \mathbf{B})|)^{q-2} \, \mathrm{d}s \cong (1 + |\mathbf{B}| + |\mathbf{A} - \mathbf{B}|)^{q-2},$$

with constants depending on q only.

Lemma 2.19. *Suppose that $\mathbf{S}$ satisfies* (2.5) *and* (2.6) *and $\mathbf{S}(\mathbf{0}, \mathbf{E}) = \mathbf{0}$. Then there holds for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and all $\mathbf{E} \in \mathbb{R}^3$*

(a) $$\mathbf{S}(\mathbf{A}, \mathbf{E}) \cdot \mathbf{A} \cong |\mathbf{A}|^2 (1 + |\mathbf{A}|)^{p(|\mathbf{E}|^2)-2},$$

(b) $$(\mathbf{S}(\mathbf{A}, \mathbf{E}) - \mathbf{S}(\mathbf{B}, \mathbf{E})) \cdot (\mathbf{A} - \mathbf{B}) \cong |\mathbf{A} - \mathbf{B}|^2 (1 + |\mathbf{B}| + |\mathbf{A}|)^{p(|\mathbf{E}|^2)-2},$$

(c) $$|\mathbf{S}(\mathbf{A}, \mathbf{E}) - \mathbf{S}(\mathbf{B}, \mathbf{E})| \cong |\mathbf{A} - \mathbf{B}| (1 + |\mathbf{B}| + |\mathbf{A}|)^{p(|\mathbf{E}|^2)-2},$$

(d) $$|\mathbf{S}(\mathbf{A}, \mathbf{E})| \cong |\mathbf{A}| (1 + |\mathbf{A}|)^{p(|\mathbf{E}|^2)-2},$$

with constants depending on p_∞, p_0 (cf. (1.58)*) and $1 + |\mathbf{E}|^2$ only.*

Proof. Note that the statement (a) is a special case of (b) by choosing $\mathbf{B} = \mathbf{0}$ and using $\mathbf{S}(\mathbf{0}, \mathbf{E}) = \mathbf{0}$. In the same way (d) follows from (c). In order to prove (b) one notices that (2.5), (2.6) and Lemma 2.17 yield

$$\begin{aligned}
&(\mathbf{S}(\mathbf{A}, \mathbf{E}) - \mathbf{S}(\mathbf{B}, \mathbf{E})) \cdot (\mathbf{A} - \mathbf{B}) \\
&\qquad = \int_0^1 \frac{\partial S_{ij}(\mathbf{B} + s(\mathbf{A} - \mathbf{B}), \mathbf{E})}{\partial D_{kl}} (A - B)_{kl} (A - B)_{ij} \, \mathrm{d}s \\
&\qquad \cong |\mathbf{A} - \mathbf{B}|^2 \int_0^1 (1 + |\mathbf{B} + s(\mathbf{A} - \mathbf{B})|)^{p-2} \, \mathrm{d}s \\
&\qquad \cong |\mathbf{A} - \mathbf{B}|^2 (1 + |\mathbf{B}| + |\mathbf{A}|)^{p-2},
\end{aligned}$$

where we used $(1+y^2)^{\frac{1}{2}} \cong (1+|y|)$. From this we immediately obtain

$$\begin{aligned}|\mathbf{A}-\mathbf{B}|^2(1+|\mathbf{B}|+|\mathbf{A}|)^{p-2} &\leqslant c(\mathbf{S}(\mathbf{A},\mathbf{E})-\mathbf{S}(\mathbf{B},\mathbf{E}))\cdot(\mathbf{A}-\mathbf{B})\\ &\leqslant c|\mathbf{S}(\mathbf{A},\mathbf{E})-\mathbf{S}(\mathbf{B},\mathbf{E})|\,|\mathbf{A}-\mathbf{B}|,\end{aligned}$$

which delivers the first inequality in (c). For the other inequality we use (2.6) and Lemma 2.17 to obtain

$$\begin{aligned}|\mathbf{S}(\mathbf{A},\mathbf{E})-\mathbf{S}(\mathbf{B},\mathbf{E})| &= \left|\int_0^1 \frac{\partial^2 S_{ij}(\mathbf{B}+s(\mathbf{A}-\mathbf{B}),\mathbf{E})}{\partial D_{kl}}\,\mathrm{d}s(A-B)_{kl}\right|\\ &\leqslant c|\mathbf{A}-\mathbf{B}|(1+|\mathbf{B}|+|\mathbf{A}|)^{p-2},\end{aligned}$$

which finishes the proof. □

Remark 2.20. Note that in the right-hand sides in Lemma 2.19 one can replace $1+|\mathbf{B}|+|\mathbf{A}|$ by $1+|\mathbf{B}|+|\mathbf{A}-\mathbf{B}|$.

Now we derive *lower bounds* for the expressions $\mathcal{I}(t,\mathbf{v})$ and $\mathcal{J}(t,\mathbf{v})$, defined in (2.9) and (2.10), for which we will often simply write $\mathcal{I}(\mathbf{v})$ and $\mathcal{J}(\mathbf{v})$. They arise from testing (2.2) with $-\Delta\mathbf{v}$ and "$\partial_t^2\mathbf{v}$", respectively. The expression $(1+|\mathbf{D}\mathbf{v}|^2)^{1/2}$ will appear quite often, so it is very useful to introduce the abbreviation

$$\tilde{D}\mathbf{v} := (1+|\mathbf{D}\mathbf{v}|^2)^{1/2}. \tag{2.21}$$

As a consequence of (2.5) we have

$$\mathcal{I}(t,\mathbf{v}) \geqslant \gamma_1 \int_\Omega (\tilde{D}\mathbf{v}(t))^{p(|\mathbf{E}(t)|^2)-2}|\mathbf{D}(\nabla\mathbf{v})(t)|^2\,\mathrm{d}x, \tag{2.22}$$

$$\mathcal{J}(t,\mathbf{v}) \geqslant \gamma_1 \int_\Omega (\tilde{D}\mathbf{v}(t))^{p(|\mathbf{E}(t)|^2)-2}|\mathbf{D}(\partial_t\mathbf{v})(t)|^2\,\mathrm{d}x. \tag{2.23}$$

Note that $\partial_j\partial_k v_m = \partial_j D_{km}\mathbf{v} + \partial_k D_{mj}\mathbf{v} - \partial_m D_{jk}\mathbf{v}$, which implies

$$|\boldsymbol{\nabla}^2\mathbf{v}| \leqslant 3|\mathbf{D}(\nabla\mathbf{v})| \leqslant 3|\boldsymbol{\nabla}^2\mathbf{v}|. \tag{2.24}$$

Thus, $|\mathbf{D}(\nabla\mathbf{v})|$ can always be replaced by $|\boldsymbol{\nabla}^2\mathbf{v}|$ (and vice versa) by increasing the multiplicative constant.

Lemma 2.25. *Let* $\mathbf{S}$ *satisfy* (2.5) *and* (2.6). *Then for all (sufficiently smooth)* $\mathbf{v}$, *for all* $1 \leqslant r \leqslant 2$, *and almost every* $t \in I$ *there holds:*

$$\|\mathbf{D}(\nabla\mathbf{v})(t)\|_r \leqslant C(\mathcal{I}(t,\mathbf{v}))^{1/2}\big\|(\tilde{D}\mathbf{v}(t))^{\frac{2-p(|\mathbf{E}(t)|^2)}{2}}\big\|_{2/(2-r)}, \tag{2.26}$$

$$\|\mathbf{D}(\partial_t\mathbf{v})(t)\|_r \leqslant C(\mathcal{J}(t,\mathbf{v}))^{1/2}\big\|(\tilde{D}\mathbf{v}(t))^{\frac{2-p(|\mathbf{E}(t)|^2)}{2}}\big\|_{2/(2-r)}, \tag{2.27}$$

where $2r/(2-r) = \infty$ *for* $r = 2$.

P r o o f. Observe that $1 \leqslant 2/r < \infty$ and $1 < (2/r)' = 2/(2-r) \leqslant \infty$. Further for $1 \leqslant r < 2$ we have

$$\begin{aligned}
\|\mathbf{D}\mathbf{w}\|_r^r &= \int_\Omega ((\tilde{D}\mathbf{v})^{p-2}|\mathbf{D}\mathbf{w}|^2)^{r/2}(\tilde{D}\mathbf{v})^{(2-p)r/2}\,\mathrm{d}x \\
&\leqslant \left(\int_\Omega (\tilde{D}\mathbf{v})^{p-2}|\mathbf{D}\mathbf{w}|^2\,\mathrm{d}x\right)^{r/2}\big\|(\tilde{D}\mathbf{v})^{(2-p)r/2}\big\|_{2/(2-r)} \\
&= \left(\int_\Omega (\tilde{D}\mathbf{v})^{p-2}|\mathbf{D}\mathbf{w}|^2\,\mathrm{d}x\right)^{r/2}\big\|(\tilde{D}\mathbf{v})^{(2-p)/2}\big\|_{2r/(2-r)}^r.
\end{aligned}$$

Choosing now $\mathbf{w} = \nabla\mathbf{v}$ and $\mathbf{w} = \partial_t\mathbf{v}$ and using (2.22) and (2.23), respectively, we obtain the assertions of the lemma for $r < 2$. The case $r = 2$ is treated similarly. $\square$

Lemma 2.28. *Let* $\mathbf{S}$ *satisfy* (2.5) *and* (2.6). *For all (sufficiently smooth)* $\mathbf{v}$ *with* $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$ *and almost every* $t \in I$ *there holds*

$$\|\nabla\mathbf{v}(t)\|_{1,\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \leqslant C(\mathcal{I}(t,\mathbf{v}) + 1), \tag{2.29}$$

$$\|\partial_t\mathbf{v}(t)\|_{1,\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \leqslant C\mathcal{J}(t,\mathbf{v})^{p_\infty/2}(\mathcal{I}(t,\mathbf{v}) + 1)^{(2-p_\infty)/2} \tag{2.30}$$

$$\leqslant C(\mathcal{J}(t,\mathbf{v}) + \mathcal{I}(t,\mathbf{v}) + 1). \tag{2.31}$$

P r o o f. From Lemma 2.25 ($r \mapsto \frac{3p_\infty}{p_\infty+1}$) we deduce, also using $2 - p \leqslant 2 - p_\infty$,

$$\begin{aligned}
\|\mathbf{D}(\nabla\mathbf{v})\|_{\frac{3p_\infty}{p_\infty+1}} &\leqslant C\mathcal{I}(\mathbf{v})^{1/2}\|(\tilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{6p_\infty}{2-p_\infty}} \\
&\leqslant C\mathcal{I}(\mathbf{v})^{1/2}\|(\tilde{D}\mathbf{v})^{\frac{2-p_\infty}{2}}\|_{\frac{6p_\infty}{2-p_\infty}} \\
&\leqslant C\mathcal{I}(\mathbf{v})^{1/2}(1 + \|\mathbf{D}\mathbf{v}\|_{3p_\infty})^{\frac{2-p_\infty}{2}} \\
&\leqslant C\mathcal{I}(\mathbf{v})^{1/2}\big(1 + C\|\nabla\mathbf{D}\mathbf{v}\|_{\frac{3p_\infty}{p_\infty+1}}\big)^{\frac{2-p_\infty}{2}},
\end{aligned}$$

since $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$. Due to $\boldsymbol{\nabla}\mathbf{D}\mathbf{v} = \mathbf{D}(\nabla\mathbf{v})$, this implies

$$\|\mathbf{D}(\nabla\mathbf{v})\|_{\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \leqslant C(\mathcal{I}(\mathbf{v}) + 1).$$

From (2.24) and $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$ we get

$$\|\nabla \mathbf{v}\|^{p_\infty}_{1,\frac{3p_\infty}{p_\infty+1}} \leqslant C(\mathcal{I}(\mathbf{v})+1).$$

Analogously we can use Lemma 2.5 to get

$$\begin{aligned}
\|\mathbf{D}(\partial_t \mathbf{v})\|_{\frac{3p_\infty}{p_\infty+1}} &\leqslant C\mathcal{J}(\mathbf{v})\|^{1/2}\|(\tilde{D}\mathbf{v})^{\frac{2-p_\infty}{2}}\|_{\frac{6p_\infty}{2-p_\infty}} \\
&\leqslant C\mathcal{J}(\mathbf{v})^{1/2}\big(1+C\|\boldsymbol{\nabla}\mathbf{D}\mathbf{v}\|_{\frac{3p_\infty}{p_\infty+1}}\big)^{\frac{2-p_\infty}{2}} \\
&\overset{(2.29)}{\leqslant} C\mathcal{J}(\mathbf{v})^{1/2}\big(1+C(\mathcal{I}(\mathbf{v})+1)^{\frac{1}{p_\infty}}\big)^{\frac{2-p_\infty}{2}} \\
&\leqslant C\mathcal{J}(\mathbf{v})^{1/2}(1+\mathcal{I}(\mathbf{v}))^{\frac{2-p_\infty}{2p_\infty}}.
\end{aligned}$$

Again $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$ and Korn's inequality imply

$$\|\partial_t \mathbf{v}\|_{1,\frac{3p_\infty}{p_\infty+1}} \leqslant C\|\mathbf{D}(\partial_t\mathbf{v})\|_{\frac{3p_\infty}{p_\infty+1}} \leqslant C\mathcal{J}(\mathbf{v})^{1/2}(1+\mathcal{I}(\mathbf{v}))^{\frac{2-p_\infty}{2p_\infty}},$$

which proves (2.30). The last inequality follows from Young's inequality. $\square$

2.1. A priori estimates

Now we use a Galerkin approximation to derive *a priori* estimates for approximate solutions $\mathbf{v}^N$ of the system (2.2). These estimates allow the limiting process $N \to \infty$ showing the existence of a solution $\mathbf{v}$ of the system (2.2).

Let $\{\boldsymbol{\omega}^r\}$ denote the set consisting of the eigenvectors of the Stokes operator denoted by A. Let λ_r be the corresponding eigenvalues and $X_N := \operatorname{span}\{\boldsymbol{\omega}^1,\dots,\boldsymbol{\omega}^N\}$. Note that $\langle \boldsymbol{\omega}^r, 1\rangle = 0$. Define $P^N\mathbf{v} := \sum\limits_{r=1}^{N} \langle \mathbf{v}, \boldsymbol{\omega}^r\rangle\, \boldsymbol{\omega}^r$. Then we have

$$\lambda_r \left\langle \boldsymbol{\omega}^r, \mathbf{v}^N\right\rangle = \left\langle A\boldsymbol{\omega}^r, \mathbf{v}^N\right\rangle = \left\langle \boldsymbol{\nabla}\boldsymbol{\omega}^r, \boldsymbol{\nabla}\mathbf{v}^N\right\rangle \tag{2.32}$$

and $P^N\colon W^{s,2} \to (X_N, \|\cdot\|_{s,2})$ are uniformly continuous for all $s \in [0,3]$ (cf. [42], [30]).

Setting $\mathbf{f}^N = P^N\mathbf{f}$ we seek the approximate solution $\mathbf{v}^N(t,x) = \sum\limits_{r=1}^{N} c_r^N(t)\boldsymbol{\omega}^r(x)$, where the coefficients $c_r^N(t)$ solve the Galerkin system (for all $1 \leqslant r \leqslant N$)

$$\begin{aligned}
\left\langle \partial_t \mathbf{v}^N, \boldsymbol{\omega}^r\right\rangle + \left\langle \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E})\mathbf{D}\boldsymbol{\omega}^r\right\rangle + \left\langle [\nabla \mathbf{v}^N]\mathbf{v}^N, \boldsymbol{\omega}^r\right\rangle \\
= \left\langle \mathbf{f}^N, \boldsymbol{\omega}^r\right\rangle - \chi^E \left\langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}\boldsymbol{\omega}^r\right\rangle, \\
\mathbf{v}^N(0) = P^N\mathbf{v}_0.
\end{aligned} \tag{2.33}$$

Since the matrix $\langle \boldsymbol{\omega}_j, \boldsymbol{\omega}_k \rangle$ with $j,k = 1,\dots,N$ is positive definite, the Galerkin system (2.33) can be re-written as a system of ordinary differential equations. This in turn fulfills the Carathéodory conditions and is therefore solvable locally in time, i.e. on a small time interval $I^* = [0,T^*)$. From the assumptions on $\mathbf{f}$ in Theorem 2.13 it follows that $\mathbf{f}^N = P^N\mathbf{f} \in L^\infty(I;W^{1,2}(\Omega))$ and $\partial_t\mathbf{f}^N = P^N(\partial_t\mathbf{f}) \in L^2(I;L^2(\Omega))$. This implies $c_r^N, \partial_t c_r^N, \partial_t^2 c_r^N \in L^2(I^*)$. Thus $\mathbf{v}^N, \partial_t\mathbf{v}^N, \partial_t^2\mathbf{v}^N \in L^2(I^*;X_N)$. (Note that the norms may depend on N). To ensure solvability for large times at least for this finite dimensional problem we have to establish a first *a priori* estimate.

Since $\mathbf{v}^N \in L^2(I^*;X_N)$, we can test (2.33) with $\mathbf{v}^N$ and get

$$\frac{1}{2}d_t\|\mathbf{v}^N\|_2^2 + \left\langle \mathbf{S}(\mathbf{D}\mathbf{v}^N,\mathbf{E}), \mathbf{D}\mathbf{v}^N \right\rangle = \left\langle \mathbf{f}^N, \mathbf{v}^N \right\rangle - \chi^E \left\langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}\mathbf{v}^N \right\rangle. \tag{2.34}$$

Note that $\left\langle [\nabla\mathbf{v}^N]\mathbf{v}^N, \mathbf{v}^N \right\rangle = 0$ due to $\operatorname{div}\mathbf{v}^N = 0$. From the coercivity of $\mathbf{S}$ (cf. Lemma 2.19 (a)) and the pointwise inequalities

$$(1+y^2)^{\frac{q-2}{2}}y^2 \geqslant C(q)(y^q-1), \qquad (1+y^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}} \geqslant (1+y^2)^{\frac{p_\infty-2}{2}}$$

we deduce that the second term on the left-hand side of (2.34) is bounded from below by

$$C_2\int_\Omega (1+|\mathbf{E}|^2)(|\mathbf{D}\mathbf{v}^N|^{p(|\mathbf{E}|^2)} + |\mathbf{D}\mathbf{v}^N|^{p_\infty})\,\mathrm{d}x - C\int_\Omega 1+|\mathbf{E}|^2\,\mathrm{d}x.$$

The terms on the right-hand side of (2.34) are bounded from above by

$$\frac{C_2}{2}\int_\Omega (1+|\mathbf{E}|^2)|\mathbf{D}\mathbf{v}^N|^{p_\infty}\,\mathrm{d}x + C\|\mathbf{E}\|_2^2 + C\|\mathbf{f}\|_2^{p'_\infty}.$$

Integration over time and Gronwall's inequality thus imply

$$\max_{[0,T^*]}\|\mathbf{v}^N\|_2^2 + \int_0^{T^*}\int_\Omega |\mathbf{D}\mathbf{v}^N|^{p(|\mathbf{E}|^2)} + |\mathbf{D}\mathbf{v}^N|^{p_\infty}\,\mathrm{d}x\,\mathrm{d}t \leqslant C(T,\mathbf{f},\mathbf{v}_0,\mathbf{E}).$$

In particular we get

$$\|c_r^N\|_{L^\infty(I^*)} \leqslant C(T,\mathbf{f},\mathbf{v}_0,\mathbf{E}), \qquad 1 \leqslant r \leqslant N.$$

As a consequence we can iterate Carathéodory's theorem to push the solvability of the Galerkin system (2.33) up to any fixed time interval $I = [0,T]$. Hence, independently of N

$$\|\mathbf{v}^N\|_{L^\infty(I;L^2(\Omega))}^2 + \varrho_{p(|\mathbf{E}|^2),Q_T}(\mathbf{D}\mathbf{v}^N) + \|\nabla\mathbf{v}^N\|_{L^{p_\infty}(Q_T)}^{p_\infty} \leqslant C, \tag{2.35}$$

where we have also used Korn's inequality in $L^{p_\infty}(\Omega)$.

We got the first *a priori* estimate by using $\mathbf{v}^N$ as a test function. To derive our second *a priori* estimate we want to use $A\mathbf{v}^N$ as a test function. The special choice of base functions $\boldsymbol{\omega}^r$ ensures that we do not leave X_N, the space of admissible test functions. More explicitly we multiply the rth equation of the Galerkin system (2.33) by $\lambda_r c_r^N$ and use (2.32) to obtain

$$\begin{aligned}\big\langle \partial_t \mathbf{v}^N, A\mathbf{v}^N\big\rangle + \big\langle \mathbf{S}(\mathbf{D}\mathbf{v}^N,\mathbf{E}), \mathbf{D}(A\mathbf{v}^N)\big\rangle + \big\langle [\nabla\mathbf{v}^N]\mathbf{v}^N, A\mathbf{v}^N\big\rangle \\ = \big\langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{v}^N\big\rangle - \chi^E\big\langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}(A\mathbf{v}^N)\big\rangle.\end{aligned} \tag{2.36}$$

Due to the periodicity we have $A=-\Delta$, and thus

$$\int_\Omega [\nabla\mathbf{v}^N]\mathbf{v}^N\cdot A\mathbf{v}^N\,\mathrm{d}x = \int_\Omega \frac{\partial v_j^N}{\partial x_k}\frac{\partial v_i^N}{\partial x_j}\frac{\partial v_i^N}{\partial x_k}\,\mathrm{d}x \leqslant \|\nabla\mathbf{v}^N\|_3^3, \tag{2.37}$$

$$\begin{aligned}-\chi^{\mathbf{E}}\int_\Omega \mathbf{E}\otimes\mathbf{E}\cdot\mathbf{D}(A\mathbf{v}^N)\,\mathrm{d}x &= 2\chi^E\int_\Omega E_i\frac{\partial E_j}{\partial x_k}D_{ij}\Big(\frac{\partial\mathbf{v}^N}{\partial x_k}\Big)\,\mathrm{d}x \\ &\leqslant \frac{\gamma_1}{8}\int_\Omega(\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla\mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad + C(\gamma_1,\mathbf{E},\nabla\mathbf{E})\int_\Omega(\tilde{D}\mathbf{v}^N)^{2-p(|\mathbf{E}|^2)}\,\mathrm{d}x,\end{aligned} \tag{2.38}$$

$$\begin{aligned}\int_\Omega \mathbf{S}(\mathbf{D}\mathbf{v}^N,\mathbf{E})\cdot\mathbf{D}(A\mathbf{v}^N)\,\mathrm{d}x &= \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}^N,\mathbf{E})}{\partial D_{kl}}D_{kl}(\nabla\mathbf{v}^N)D_{ij}(\nabla\mathbf{v}^N)\,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}^N,\mathbf{E})}{\partial E_k}\nabla E_k D_{ij}(\nabla\mathbf{v}^N)\,\mathrm{d}x.\end{aligned} \tag{2.39}$$

The right-hand side of (2.39) is bounded from below by

$$\begin{aligned}&\frac{1}{2}\mathcal{I}(\mathbf{v}^N) + \frac{\gamma_1}{2}\int_\Omega(\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla\mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad - \frac{\gamma_1}{8}\int_\Omega(\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla\mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad - C(\gamma_1,\nabla\mathbf{E})\int_\Omega(\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)}(1+\ln(\tilde{D}\mathbf{v}^N)^2)^2\,\mathrm{d}x,\end{aligned}$$

where we used the definition of $\mathcal{I}$, (2.22) and Young's inequality. Thus we have

$$\begin{aligned}d_t\|\boldsymbol{\nabla}\mathbf{v}^N\|_2^2 + \mathcal{I}(\mathbf{v}^N) + \frac{\gamma_1}{2}\int_\Omega(\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla\mathbf{v}^N)|^2\,\mathrm{d}x \\ \leqslant C(1+\|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3 + |\langle\boldsymbol{\nabla}\mathbf{f}^N,\boldsymbol{\nabla}\mathbf{v}^N\rangle| + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N)),\end{aligned} \tag{2.40}$$

where we also used the estimate $\ln(1+y^2)\leqslant c(1+y^2)^{\frac14}$ and $p(|\mathbf{E}|^2)\leqslant p_0\leqslant 2$, $2-p(|\mathbf{E}|^2)\leqslant p(|\mathbf{E}|^2)$.

If $p > \frac{11}{5}$ one can show that $\|\nabla \mathbf{v}^N\|_3^3 \leqslant C_\varepsilon \|\nabla \mathbf{v}^N\|_p^p \|\nabla \mathbf{v}^N\|_2^2 + \varepsilon \mathcal{I}(\mathbf{v}^N)$ (see [30]), which enables us to apply Gronwall's inequality after absorbing $\varepsilon\mathcal{I}(\mathbf{v}^N)$ on the left-hand side. This would give us a global estimate. If $p > \frac{5}{3}$ we can show that $\|\nabla \mathbf{v}^N\|_3^3 \leqslant C_\varepsilon \|\nabla \mathbf{v}^N\|_p^p \|\nabla \mathbf{v}^N\|_2^R + \varepsilon \mathcal{I}(\mathbf{v}^N)$ for some constant $1 < R < \infty$ and thereafter absorb $\varepsilon\mathcal{I}(\mathbf{v}^N)$ on the left-hand side and apply a local version of Gronwall's inequality (cf. Lemma 2.52). This would give us an estimate for small times. Nevertheless we will not make use of these facts, since we are also interested in smaller values of p than $\frac{5}{3}$.

We will test immediately with "$\partial_t \mathbf{v}^N \partial_t$" to get in addition to (2.40) another estimate. Then we will use the resulting *two estimates at the same time* to derive quite strong *a priori* estimates for $\mathbf{v}^N$ for values up to $p > \frac{3}{2}$. Let us take the time derivative of the Galerkin system (2.33):

$$\begin{aligned} \left\langle \partial_t^2 \mathbf{v}^N, \boldsymbol{\omega}^r \right\rangle + \left\langle \partial_t \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}), \mathbf{D}\boldsymbol{\omega}^r \right\rangle + \left\langle \partial_t([\nabla \mathbf{v}^N]\mathbf{v}^N), \boldsymbol{\omega}^r \right\rangle \\ = \left\langle \partial_t \mathbf{f}^N, \boldsymbol{\omega}^r \right\rangle - \chi^E \left\langle \partial_t (\mathbf{E}\otimes\mathbf{E}), \mathbf{D}\boldsymbol{\omega}^r \right\rangle, \end{aligned} \tag{2.41}$$

for $1 \leqslant r \leqslant N$. Since $\mathbf{v}^N \in W^{2,2}(I; X_n)$, this makes sense and we can even test with $\partial_t \mathbf{v}^N \in W^{1,2}(I; X_n)$ resulting in

$$\begin{aligned} \frac{1}{2} d_t \|\partial_t \mathbf{v}^N\|_2^2 + \left\langle \partial_t \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}), \mathbf{D}(\partial_t \mathbf{v}^N) \right\rangle + \left\langle \partial_t([\nabla \mathbf{v}^N]\mathbf{v}^N), \partial_t \mathbf{v}^N \right\rangle \\ = \left\langle \partial_t \mathbf{f}^N, \partial_t \mathbf{v}^N \right\rangle - \chi^E \left\langle \partial_t(\mathbf{E}\otimes\mathbf{E}), \mathbf{D}(\partial_t \mathbf{v}^N) \right\rangle. \end{aligned}$$

Similarly as in (2.38) and (2.39) we get

$$\begin{aligned} -\chi^{\mathbf{E}} \int_\Omega \partial_t(\mathbf{E}\otimes\mathbf{E}) \cdot \mathbf{D}(\partial_t \mathbf{v}^N)\,\mathrm{d}x &\leqslant \frac{\gamma_1}{8} \int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2} |\mathbf{D}(\partial_t \mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad + C(\gamma_1, \mathbf{E}, \partial_t \mathbf{E}) \int_\Omega (\tilde{D}\mathbf{v}^N)^{2-p(|\mathbf{E}|^2)}\,\mathrm{d}x, \\ \int_\Omega \partial_t \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}) \cdot \mathbf{D}(\partial_t \mathbf{v}^N)\,\mathrm{d}x &= \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}^N, \mathbf{E})}{\partial D_{kl}} D_{kl}\,(\partial_t \mathbf{v}^N) D_{ij}(\partial_t \mathbf{v}^N)\,\mathrm{d}x \\ &\quad + \int_\Omega \frac{\partial S_{ij}(\mathbf{D}\mathbf{v}^N, \mathbf{E})}{\partial E_k}\, \partial_t E_k D_{ij}(\partial_t \mathbf{v}^N)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\mathcal{J}(\mathbf{v}^N) + \frac{\gamma_1}{2} \int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2} |\mathbf{D}(\partial_t \mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad - \frac{\gamma_1}{8} \int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2} |\mathbf{D}(\partial_t \mathbf{v}^N)|^2\,\mathrm{d}x \\ &\quad - C(\gamma_1, \partial_t \mathbf{E}) \int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)} (1 + \ln(\tilde{D}\mathbf{v}^N)^2)^2\,\mathrm{d}x, \end{aligned}$$

where we used the definition of $\mathcal{J}$, (2.23) and Young's inequality. This yields (cf. (2.40)), also using $\operatorname{div}\mathbf{v}^N = 0$ in the convective term,

$$
\begin{aligned}
(2.42)\qquad d_t\|\partial_t \mathbf{v}^N\|_2^2 + \mathcal{J}(\mathbf{v}^N) + \frac{\gamma_1}{2}\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\partial_t \mathbf{v}^N)|^2\,\mathrm{d}x \\
\leqslant C\big(1 + \big|\big\langle[\nabla\mathbf{v}^N]\partial_t\mathbf{v}^N, \partial_t\mathbf{v}^N\big\rangle\big| + \big|\big\langle\partial_t\mathbf{f}^N, \partial_t\mathbf{v}^N\big\rangle\big| + \|\nabla\mathbf{v}^N\|_3^3 \\
+ \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N)\big).
\end{aligned}
$$

Recall that

$$
\begin{aligned}
(2.43)\qquad d_t\|\boldsymbol{\nabla}\mathbf{v}^N\|_2^2 + \mathcal{I}(\mathbf{v}^N) + \frac{\gamma_1}{2}\int_\Omega (\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-2}|\mathbf{D}(\nabla\mathbf{v}^N)|^2\,\mathrm{d}x \\
\leqslant C\big(1 + \|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3 + \big|\big\langle\boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{v}^N\big\rangle\big| + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N)\big).
\end{aligned}
$$

At first sight, we have gained nothing. We have to control one more bad term, namely $|\langle[\nabla\mathbf{v}^N]\partial_t\mathbf{v}^N, \partial_t\mathbf{v}^N\rangle|$, but we only got more information about the time derivative of $\mathbf{v}^N$. But the critical term $\|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3$, which gave the lower bound for p, has no time derivatives. The next lemma shows that $\mathcal{J}(\mathbf{v}^N)$ reveals indeed more information.

Lemma 2.44. *Let $1 < q < \infty$, then for almost every $t \in I$*

$$
\begin{aligned}
(2.45)\qquad d_t(\|\tilde{D}\mathbf{v}(t)\|_q^q) &\leqslant C\mathcal{J}(t,\mathbf{v})^{\frac{1}{2}}(\varrho_{2q-p(|\mathbf{E}(t)|^2),\Omega}(\tilde{D}\mathbf{v}(t)))^{1/2} \\
&\leqslant \varepsilon\mathcal{J}(t,\mathbf{v}) + C_\varepsilon\varrho_{2q-p(|\mathbf{E}(t)|^2),\Omega}(\tilde{D}\mathbf{v}(t)),
\end{aligned}
$$

where $\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}) = \int_\Omega(\tilde{D}\mathbf{v})^{2q-p(|\mathbf{E}|^2)}\,\mathrm{d}x$ even if $2q - p(|\mathbf{E}|^2) < 1$.

Proof. Note that

$$
\partial_t((\tilde{D}\mathbf{v})^q) = q(\tilde{D}\mathbf{v})^{q-2}(D_{jk}\mathbf{v})(\partial_t D_{jk}\mathbf{v}).
$$

Hence

$$
\begin{aligned}
d_t(\|\tilde{D}\mathbf{v}\|_q^q) &\leqslant q\int_\Omega(\tilde{D}\mathbf{v})^{q-1}\|\partial_t\mathbf{D}\mathbf{v}\|\,\mathrm{d}x \\
&= q\int_\Omega(\tilde{D}\mathbf{v})^{\frac{p-2}{2}}|\mathbf{D}(\partial_t\mathbf{v})|(\tilde{D}\mathbf{v})^{q-\frac{1}{2}p}\,\mathrm{d}x \\
&\leqslant qC\mathcal{J}(\mathbf{v})^{\frac{1}{2}}(\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}))^{\frac{1}{2}},
\end{aligned}
$$

by Hölder's inequality, which proves the first assertion. The second follows from Young's inequality. $\square$

This lemma enables us to produce $d_t(\|\tilde{D}\mathbf{v}^N\|_q^q)$ on the left-hand side of (2.42) if we add $C\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v})$ to the right-hand side. Thus we have three terms to control:

$$\|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3, \quad |\langle[\nabla\mathbf{v}^N]\partial_t\mathbf{v}^N, \partial_t\mathbf{v}^N\rangle|, \quad \varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}^N). \tag{2.46}$$

The first and the second one will be easier to estimate for large q, but the third one for small q. The problem now is to find the *optimal* choice for q. We start by examining which values of q are needed for the first and the second term. In view of local Gronwall's inequality (cf. Lemma 2.52), we will be able to control arbitrary powers of $\|\tilde{D}\mathbf{v}^N\|_q^q$ and $\|\partial_t\mathbf{v}^N\|_2^2$.

Lemma 2.47. *Let $q > \frac{9-3p_\infty}{2}$, then there exists a constant $R_1 = R_1(p_\infty) > q$, such that*

$$\|\boldsymbol{\nabla}\mathbf{v}\|_3^3 \leqslant C_\varepsilon\|\tilde{D}\mathbf{v}\|_q^{R_1} + \varepsilon\mathcal{I}(\mathbf{v}) + \varepsilon.$$

P r o o f. If $q \geqslant 3$, then there is nothing to prove, so assume $q < 3$. We interpolate $L^3(\Omega) = [L^q(\Omega), L^{3p_\infty}(\Omega)]_\theta$ with $\theta = \frac{(3-q)p_\infty}{3p_\infty - q}$, $1-\theta = \frac{q(p_\infty-1)}{3p_\infty-q}$ and obtain

$$\|\boldsymbol{\nabla}\mathbf{v}\|_3^3 \leqslant \|\boldsymbol{\nabla}\mathbf{v}\|_q^{3(1-\theta)}\|\boldsymbol{\nabla}\mathbf{v}\|_{3p_\infty}^{3\theta}.$$

If $3\theta < p_\infty$, there exists an $\delta > 1$ such that

$$\begin{aligned}
\|\boldsymbol{\nabla}\mathbf{v}\|_3^3 &\leqslant C_\varepsilon\|\boldsymbol{\nabla}\mathbf{v}\|_q^{3(1-\theta)\delta'} + \varepsilon\|\boldsymbol{\nabla}\mathbf{v}\|_{3p_\infty}^{p_\infty} \\
&\leqslant C_\varepsilon\|\boldsymbol{\nabla}\mathbf{v}\|_q^{3(1-\theta)\delta'} + \varepsilon C\|\boldsymbol{\nabla}\mathbf{v}\|_{1,\frac{3p_\infty}{p_\infty+1}}^{p_\infty} \\
&\leqslant C_\varepsilon\|\boldsymbol{\nabla}\mathbf{v}\|_q^{3(1-\theta)\delta'} + \varepsilon C(\mathcal{I}(\mathbf{v}) + 1),
\end{aligned}$$

where we used Lemma 2.28. So by Korn's inequality

$$\|\boldsymbol{\nabla}\mathbf{v}\|_3^3 \leqslant C_{\varepsilon_2}\|\tilde{D}\mathbf{v}\|_q^{3(1-\theta)\delta'} + \varepsilon_2\mathcal{I}(\mathbf{v}) + \varepsilon_2.$$

We still have to verify $3\theta < p_\infty$, but this is equivalent to

$$\frac{3(3-q)p_\infty}{3p_\infty - q} < p_\infty \Longleftrightarrow \frac{9-3p_\infty}{2} < q,$$

which holds due to the assumption on q. $\square$

Lemma 2.48. *Let $q > \frac{9-3p_\infty}{2}$, then there exist constants $R_2 = R_2(p_\infty) > 2$ and $R_3 = R_3(p_\infty) > q$ such that*

$$|\langle[\nabla\mathbf{v}]\partial_t\mathbf{v}, \partial_t\mathbf{v}\rangle| \leqslant \varepsilon\mathcal{J}(\mathbf{v}) + C_\varepsilon(\|\partial_t\mathbf{v}\|_2^{R_2} + \|\tilde{D}\mathbf{v}\|_q^{R_3} + 1).$$

Proof. Note that Lemma 2.25 ($r \mapsto \frac{2q}{2-p_\infty+q}$) implies

$$\begin{aligned}\|\mathbf{D}(\partial_t\mathbf{v})\|_{\frac{2q}{2-p_\infty+q}} &\leqslant C\mathcal{J}(\mathbf{v})^{\frac{1}{2}}\|(\tilde{D}\mathbf{v})\|_{\frac{2q}{2-p_\infty}}^{\frac{2-p}{2}} \\ &\leqslant C\mathcal{J}(\mathbf{v})^{\frac{1}{2}}\|\tilde{D}\mathbf{v}\|_q^{\frac{2-p_\infty}{2}},\end{aligned} \tag{2.49}$$

where we used that $(1+y^2)^{(2-p)/4} \leqslant (1+y^2)^{(2-p_\infty)/4}$. Furthermore we have the embedding $W^{1,\frac{2q}{2-p_\infty+q}}(\Omega) \hookrightarrow L^{\frac{6q}{6-3p_\infty+q}}(\Omega)$. Since $\frac{9-3p_\infty}{2} < q$ is equivalent to $\frac{2q}{q-1} < \frac{6q}{6-3p_\infty+q}$, we can interpolate $L^{\frac{2q}{q-1}}(\Omega) = [L^2(\Omega), L^{\frac{6q}{6-3p_\infty+q}}(\Omega)]_\theta$. This and Korn's and Young's inequalities imply

$$\begin{aligned}|\langle[\nabla\mathbf{v}]\partial_t\mathbf{v}, \partial_t\mathbf{v}\rangle| &\leqslant \|\partial_t\mathbf{v}\|_{\frac{2q}{q-1}}^2\|\boldsymbol{\nabla}\mathbf{v}\|_q \\ &\leqslant C\|\partial_t\mathbf{v}\|_2^{2(1-\theta)}\|\partial_t\mathbf{v}\|_{\frac{6q}{6-3p_\infty+q}}^{2\theta}\|\boldsymbol{\nabla}\mathbf{v}\|_q \\ &\leqslant C\|\partial_t\mathbf{v}\|_2^{2(1-\theta)}\|\partial_t\boldsymbol{\nabla}\mathbf{v}\|_{\frac{2q}{2-p_\infty+q}}^{2\theta}\|\boldsymbol{\nabla}\mathbf{v}\|_q \\ &\overset{(2.49)}{\leqslant} C\|\partial_t\mathbf{v}\|_2^{2(1-\theta)}(\mathcal{J}(\mathbf{v})^{\frac{1}{2}}\|\tilde{D}\mathbf{v}\|_q^{\frac{2-p_\infty}{2}})^{2\theta}\|\boldsymbol{\nabla}\mathbf{v}\|_q \\ &\leqslant \varepsilon\mathcal{J}(\mathbf{v}) + C_\varepsilon(\|\partial_t\mathbf{v}\|_2^{R_2} + \|\tilde{D}\mathbf{v}\|_q^{R_3} + 1).\end{aligned}$$

□

It is indeed interesting that both terms $|\langle[\nabla\mathbf{v}]\partial_t\mathbf{v}, \partial_t\mathbf{v}\rangle|$ and $\|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3$ require the same bound for q, which is $q > \frac{1}{2}(9-3p_\infty)$. Now we have to find the upper bound for q, in order to control $\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}^N)$. For that we require $q \leqslant \frac{1}{2}(3+p_\infty)$ and obtain

$$\int_\Omega |\tilde{D}\mathbf{v}^N|^{2q-p(|\mathbf{E}|^2)}\,\mathrm{d}x \leqslant \int_\Omega |\tilde{D}\mathbf{v}^N|^{2q-p_\infty}\,\mathrm{d}x = \|\tilde{D}\mathbf{v}^N\|_{2q-p_\infty}^{2q-p_\infty} \leqslant C(\|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3 + 1),$$

since $2q-p_\infty \leqslant 3$. That means that $\varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}^N)$ can be controlled if $\|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3$ can be controlled. But for $p_\infty > \frac{3}{2}$ we can always find q such that

$$\frac{9-3p_\infty}{2} < q \leqslant \frac{3+p_\infty}{2}.$$

Thus all terms in (2.46) can be controlled under this condition. It remains to control the terms involving $\mathbf{f}^N$ in (2.42) and (2.43), which is easily established by

$$
\begin{aligned}
|\langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{v}^N\rangle| &\leqslant \|P^N\mathbf{f}\|_{1,2}\,\|\boldsymbol{\nabla}\mathbf{v}^N\|_2 \leqslant C\|\mathbf{f}\|_{1,2}\,\|\boldsymbol{\nabla}\mathbf{v}^N\|_2 \\
&\leqslant C\|\mathbf{f}\|_{1,2}^2 + C\|\tilde{D}\mathbf{v}^N\|_q^2, \\
|\langle \partial_t\mathbf{f}^N, \partial_t\mathbf{v}^N\rangle| &\leqslant \|P^N(\partial_t\mathbf{f})\|_2\,\|\partial_t\mathbf{v}^N\|_2 \leqslant C\|\partial_t\mathbf{f}\|_2\,\|\partial_t\mathbf{v}^N\|_2 \\
&\leqslant C\|\partial_t\mathbf{f}\|_2^2 + C\|\partial_t\mathbf{v}^N\|_2^2.
\end{aligned}
$$

Finally we have, since $p(|\mathbf{E}|^2) \leqslant p_0 \leqslant 2 \leqslant q$,

$$
\varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N) \leqslant \|\tilde{D}\mathbf{v}^N\|_{p_0}^{p_0} \leqslant C\|\tilde{D}\mathbf{v}^N\|_q^2. \tag{2.50}
$$

Hence by Lemma 2.47, Lemma 2.48, Korn's inequality, and the above calculations we get, for $\max\bigl(2, \frac{9-3p_\infty}{2}\bigr) < q \leqslant \frac{3+p_\infty}{2}$,

$$
\begin{aligned}
d_t\|\partial_t\mathbf{v}^N\|_2^2 &+ d_t(\|\tilde{D}\mathbf{v}(t)\|_q^q) + d_t\|\boldsymbol{\nabla}\mathbf{v}^N\|_2^2 + \mathcal{I}(\mathbf{v}^N) + \frac{\gamma_1}{2}\mathcal{J}(\mathbf{v}^N) \\
&\leqslant C(1 + |\langle[\nabla\mathbf{v}]\partial_t\mathbf{v}^N, \partial_t\mathbf{v}^N\rangle| + |\langle\partial_t\mathbf{f}^N, \partial_t\mathbf{v}^N\rangle| + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N) \\
&\quad + |\langle\boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{v}^N\rangle| + \|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3 + \varrho_{2q-p(|\mathbf{E}|^2),\Omega}(\tilde{D}\mathbf{v}^N)) \\
&\leqslant C(1 + \|\tilde{D}\mathbf{v}^N\|_q^{\max(R_1,R_3,2)} + \|\partial_t\mathbf{v}^N\|_2^{\max(R_2,2)} + \|\mathbf{f}\|_{1,2}^2 + \|\partial_t\mathbf{f}\|_2^2).
\end{aligned}
$$

The following lemma ensures that for small times T' we get boundedness (uniformly with respect to N) of the following expressions, for $\max\bigl(2, \frac{9-3p_\infty}{2}\bigr) < q \leqslant \frac{3+p_\infty}{2}$:

$$
\begin{aligned}
&\|\partial_t\mathbf{v}^N\|_{L^\infty(I';L^2(\Omega))}^2, &&\|\boldsymbol{\nabla}\mathbf{v}^N\|_{L^\infty(I';L^q(\Omega))}^q, \\
&\|\mathcal{I}(\mathbf{v}^N)\|_{L^1(I')}, &&\|\mathcal{J}(\mathbf{v}^N)\|_{L^1(I')},
\end{aligned} \tag{2.51}
$$

where $I' = [0, T']$. These *a priori* estimates in turn are sufficient to pass to the limit $N \to \infty$ to get a solution $\mathbf{v}$ of our original problem (2.2).

Lemma 2.52 (local version of Gronwall's lemma). *Let $T, \alpha, c_0 > 0$ be given constants and let $0 < h \in C([0,T])$, $0 \leqslant f \in C^1([0,T])$ satisfy*

$$
f'(t) \leqslant h(t) + c_0 f(t)^{1+\alpha}. \tag{2.53}
$$

Then

$$
f(t) \leqslant H(t) + H(t_0)\bigl((1 - \alpha c_0 H(t_0)^\alpha\, t)^{-\frac{1}{\alpha}} - 1\bigr)
$$

for all $t \in [0, t_0)$, where

$$
H(t) := f(0) + \int_0^t h(s)\,\mathrm{d}s,
$$

and where t_0 is defined by the condition $\alpha c_0 H(t_0)^\alpha\, t_0 = 1$.

Proof. Define $a\colon [0,t_0) \to \mathbb{R}^{\geqslant 0}$ by

$$a(t) := H(t_0)\big((1-\alpha c_0 H(t_0)^\alpha\, t)^{-\frac{1}{\alpha}} - 1\big).$$

Then a solves

$$\begin{aligned} a'(t) &= c_0(H(t_0)+a(t))^{1+\alpha}, \\ a(0) &= 0. \end{aligned}$$

Setting $z(t) := H(t) + a(t)$ we see that for all $t \in [0,t_0)$ holds

$$\begin{aligned} z'(t) = h(t) + a'(t) &= h(t) + c_0(H(t_0)+a(t))^{1+\alpha} \\ &> h(t) + c_0(H(t)+a(t))^{1+\alpha} = h(t) + c_0 z(t)^{1+\alpha}. \end{aligned}$$

Since $z(0) \geqslant f(0)$ we get from this and (2.53) that $f'(0) < z'(0)$. Consequently, there exists $t' > 0$ such that for all $t \in [0,t']$ holds $f(t) \leqslant z(t)$. Iterating this argument we obtain the assertion of the lemma. $\square$

In order to derive the last estimate from Theorem 2.13 we go once more into (2.36) and move the term with the time derivative to the right-hand side. This gives

$$\mathcal{I}(\mathbf{v}^N) \leqslant C\big(1 + \|\boldsymbol{\nabla}\mathbf{v}^N\|_3^3 + |\langle \boldsymbol{\nabla}\mathbf{f}^N, \boldsymbol{\nabla}\mathbf{v}^N\rangle| + \varrho_{p(|\mathbf{E}|^2),\Omega}(\mathbf{D}\mathbf{v}^N) + |\langle \partial_t \mathbf{v}^N, -\Delta\mathbf{v}^N\rangle|\big).$$

Using

$$\|\mathbf{f}^N\|_{L^\infty(I;W^{1,2}(\Omega))} = \|P^N\mathbf{f}\|_{L^\infty(I;W^{1,2}(\Omega))} \leqslant C\|\mathbf{f}\|_{L^\infty(I;W^{1,2}(\Omega))} \leqslant C,$$

together with (2.50), (2.51) and Lemma 2.47, for $q > \max\big(2, \frac{1}{2}(9-3p_\infty)\big)$, we get

$$\mathcal{I}(\mathbf{v}^N) \leqslant C\big(1 + |\langle \partial_t\mathbf{v}^N, -\Delta\mathbf{v}^N\rangle|\big). \tag{2.54}$$

The following lemma gives control of the remaining term $|\langle \partial_t\mathbf{v}^N, -\Delta\mathbf{v}^N\rangle|$.

Lemma 2.55. *For $1 < p_\infty \leqslant 2$ there holds*

$$|\langle \partial_t \mathbf{v}, \Delta\mathbf{v}\rangle| \leqslant C\|\partial_t\mathbf{v}\|_2^{\frac{4(p_\infty-1)}{3p_\infty-2}}\, \mathcal{J}(\mathbf{v})^{\frac{2-p_\infty}{2(3p_\infty-2)}}(\mathcal{I}(\mathbf{v})+1)^{\frac{p_\infty+2}{2(3p_\infty-2)}}.$$

Proof. With the help of Lemma 2.28 we conclude

$$\begin{aligned} |\langle \partial_t\mathbf{v}, \Delta\mathbf{v}\rangle| &\leqslant \|\partial_t\mathbf{v}\|_{\frac{3p_\infty}{2p_\infty-1}}\|\mathbf{v}\|_{2,\frac{3p_\infty}{p_\infty+1}} \\ &\leqslant C\|\partial_t\mathbf{v}\|_{\frac{3p_\infty}{2p_\infty-1}}(\mathcal{I}(\mathbf{v})+1)^{1/p_\infty} \\ &\leqslant C\|\partial_t\mathbf{v}\|_2^{1-\theta}\|\partial_t\mathbf{v}\|_{1,\frac{3p_\infty}{1+p_\infty}}^{\theta}(\mathcal{I}(\mathbf{v})+1)^{1/p_\infty} \\ &\leqslant C\|\partial_t\mathbf{v}\|_2^{1-\theta}\big(\mathcal{J}(\mathbf{v})^{\frac{1}{2}}(\mathcal{I}(\mathbf{v})+1)^{\frac{2-p_\infty}{2p_\infty}}\big)^{\theta}(\mathcal{I}(\mathbf{v})+1)^{1/p_\infty}, \end{aligned}$$

where we used the interpolation $L^{\frac{3p_\infty}{2p_\infty-1}}(\Omega) = [L^2(\Omega), L^{3p_\infty}(\Omega)]_\theta$ with $\theta = \frac{2-p_\infty}{3p_\infty-2}$, $1-\theta = \frac{4p_\infty-4}{3p_\infty-2}$. Consequently $\frac{2-p_\infty}{2p_\infty}\theta + \frac{1}{p_\infty} = \frac{p_\infty+2}{2(3p_\infty-2)}$. This proves the lemma. $\square$

This lemma, (2.54) and (2.51) imply

$$1+\mathcal{I}(\mathbf{v}^N) \leqslant C\big(1+\mathcal{J}(\mathbf{v}^N)^{\frac{2-p_\infty}{2(3p_\infty-2)}}(\mathcal{I}(\mathbf{v}^N)+1)^{\frac{p_\infty+2}{2(3p_\infty-2)}}\big).$$

Thus by Young's inequality, which is applicable for $p_\infty > \frac{6}{5}$, we get

$$\mathcal{I}(\mathbf{v}^N) \leqslant C\big(1+\mathcal{J}(\mathbf{v}^N)^{\frac{2-p_\infty}{5p_\infty-6}}\big), \tag{2.56}$$

which raised to the power $\frac{5p_\infty-6}{2-p_\infty}$ gives, in view of (2.51),

$$\mathcal{I}(\mathbf{v}^N)^{\frac{5p_\infty-6}{2-p_\infty}} \leqslant C\big(1+\mathcal{J}(\mathbf{v}^N)\big) \leqslant C.$$

This and (2.51) implies that the following expressions are bounded independently on N, for $\max\big(2, \frac{9-3p_\infty}{2}\big) < q \leqslant \frac{3+p_\infty}{2}$,

$$\begin{aligned} &\|\partial_t \mathbf{v}^N\|^2_{L^\infty(I';L^2(\Omega))}, \quad \|\nabla \mathbf{v}^N\|^q_{L^\infty(I';L^q(\Omega))}, \\ &\|\mathcal{I}(\mathbf{v}^N)\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I')}, \quad \|\mathcal{J}(\mathbf{v}^N)\|_{L^1(I')}. \end{aligned} \tag{2.57}$$

2.2. Passage to the limit

From (2.57) and Lemma 2.28 it follows that

$$\|\mathbf{v}^N\|_{L^{p_\infty \frac{5p_\infty-6}{2-p_\infty}}(I';W^{2,p_\infty}(\Omega))} \leqslant C, \tag{2.58}$$

$$\|\partial_t \mathbf{v}^N\|_{L^\infty(I';L^2(\Omega))} + \|\partial_t \mathbf{v}^N\|_{L^{p_\infty}(I';W^{1,\frac{3p_\infty}{p_\infty+1}}(\Omega))} \leqslant C, \tag{2.59}$$

since $\langle \mathbf{v}^N, 1\rangle = 0$. Thus we can pick a subsequence (still denoted by $\mathbf{v}^N$) with

$$\mathbf{v}^N \rightharpoonup \mathbf{v} \quad \text{in } L^{p_\infty \frac{5p_\infty-6}{2-p_\infty}}(I';W^{2,p_\infty}(\Omega)), \tag{2.60}$$

$$\partial_t \mathbf{v}^N \rightharpoonup \partial_t \mathbf{v} \quad \text{in } L^\infty(I';L^2(\Omega)) \cap L^{p_\infty}(I';W^{1,\frac{3p_\infty}{p_\infty+1}}(\Omega)), \tag{2.61}$$

where we have used the fact that the weak limit of distributions on $I\times\Omega$ is unique. Since $W^{2,p}(\Omega) \hookrightarrow\hookrightarrow W^{1,2}(\Omega)$ for $p > \frac{6}{5}$, the lemma of Aubin-Lions implies the existence of a subsequence such that

$$\nabla \mathbf{v}^N \to \nabla \mathbf{v} \quad \text{in } L^2(I'\times\Omega). \tag{2.62}$$

As a consequence we get convergence of the convective term

$$[\nabla \mathbf{v}^N]\mathbf{v}^N \to [\nabla \mathbf{v}]\mathbf{v} \quad \text{in } L^1(I'\times\Omega). \tag{2.63}$$

Observe that we have due to Lemma 2.19 (c) (with $\mathbf{B} = \mathbf{0}$) and $p(|\mathbf{E}|^2) \leqslant p_0 \leqslant 2$

$$
\begin{aligned}
(2.64) \qquad \|\mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E})\|_{L^2(I'\times\Omega)} &\leqslant C(\mathbf{E})\|(\tilde{D}\mathbf{v}^N)^{p(|\mathbf{E}|^2)-1}\|_{L^2(I'\times\Omega)} \\
&\leqslant C(1 + \|\boldsymbol{\nabla}\mathbf{v}^N\|_{L^2(I'\times\Omega)}) \leqslant C.
\end{aligned}
$$

On the other hand by (2.62) $\mathbf{D}\mathbf{v}^N \to \mathbf{D}\mathbf{v}$ a.e. in $I' \times \Omega$, so

$$
(2.65) \qquad \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}) \to \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}) \quad \text{a.e. in } I' \times \Omega
$$

due to the continuity properties of $\mathbf{S}$. Now Vitali's convergence theorem, (2.64) and (2.65) imply

$$
(2.66) \qquad \mathbf{S}(\mathbf{D}\mathbf{v}^N, \mathbf{E}) \to \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}) \quad \text{in } L^1(I' \times \Omega).
$$

Now we can easily pass to the limit in the Galerkin system (2.33). Indeed, choose $\boldsymbol{\omega}^r$ and $\varphi \in C_0^\infty(I')$, then we can conclude from (2.33), (2.61), (2.63), and (2.66) that

$$
\begin{aligned}
&\int_{I'} \varphi(\langle \partial_t \mathbf{v}, \boldsymbol{\omega}^r\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}), \mathbf{D}\boldsymbol{\omega}^r\rangle + \langle [\nabla\mathbf{v}]\mathbf{v}, \boldsymbol{\omega}^r\rangle)\,\mathrm{d}t \\
&\qquad = \int_{I'} \varphi(\langle \mathbf{f}, \boldsymbol{\omega}^r\rangle - \chi^E\langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}\boldsymbol{\omega}^r\rangle)\,\mathrm{d}t.
\end{aligned}
$$

Furthermore $\mathbf{v}$ fulfills

$$
\|\partial_t\mathbf{v}\|_{L^2(I'\times\Omega)} + \|\mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E})\|_{L^1(I'\times\Omega)} + \|[\nabla\mathbf{v}]\mathbf{v}\|_{L^{\frac{4}{3}}(I'\times\Omega)} \leqslant C.
$$

Since $\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \ldots\}$ is dense in $W^{s,2}(\Omega) \cap V_{p_\infty}$ and $W^{s,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for $s > \frac{5}{2}$, we deduce that

$$
\begin{aligned}
&\int_{I'} \varphi(\langle \partial_t \mathbf{v}, \boldsymbol{\omega}\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}), \mathbf{D}\boldsymbol{\omega}\rangle + \langle [\nabla\mathbf{v}], \mathbf{v}, \boldsymbol{\omega}\rangle)\,\mathrm{d}t \\
&\qquad = \int_{I'} \varphi(\langle \mathbf{f}, \boldsymbol{\omega}\rangle - \chi^E\langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}\boldsymbol{\omega}\rangle)\,\mathrm{d}t
\end{aligned}
$$

is fulfilled for all $\boldsymbol{\omega} \in W^{s,2}(\Omega) \cap V_{p_\infty}$, especially for all $\boldsymbol{\omega} \in \mathcal{V}$. Note that

$$
\langle \partial_t\mathbf{v}, \boldsymbol{\omega}\rangle,\ \langle \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}), \mathbf{D}\boldsymbol{\omega}\rangle,\ \langle [\nabla\mathbf{v}]\mathbf{v}, \boldsymbol{\omega}\rangle,\ \langle \mathbf{f}, \boldsymbol{\omega}\rangle,\ \langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}\boldsymbol{\omega}\rangle \in L^1(I')
$$

and thus we obtain for all $\boldsymbol{\omega} \in \mathcal{V}$ and a.e. $t \in I'$

$$
(2.67) \qquad \langle \partial_t\mathbf{v}, \boldsymbol{\omega}\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}, \mathbf{E}), \mathbf{D}\boldsymbol{\omega}\rangle + \langle [\nabla\mathbf{v}]\mathbf{v}, \boldsymbol{\omega}\rangle = \langle \mathbf{f}, \boldsymbol{\omega}\rangle - \chi^E\langle \mathbf{E}\otimes\mathbf{E}, \mathbf{D}\boldsymbol{\omega}\rangle.
$$

It remains to show that $\mathbf{v}(0)=\mathbf{v}_0$. The embedding $W^{1,2}(I')\hookrightarrow C^{\frac12}(\overline{I'})$ and the interpolation $L^\infty(I')=[L^2(I'),W^{1,2}(I')]_{\frac12}$ imply

$$\begin{aligned}(2.68)\quad \|P^N\mathbf{v}_0-\mathbf{v}(0)\|_2&=\|\mathbf{v}^N(0)-\mathbf{v}(0)\|_2\\ &\leqslant C\underbrace{\|\mathbf{v}^N-\mathbf{v}\|^{\frac12}_{L^2(I';L^2(\Omega))}}_{\to 0}\underbrace{\|\partial_t\mathbf{v}^N-\partial_t\mathbf{v}\|^{\frac12}_{L^2(I';L^2(\Omega))}}_{\leqslant C}\to 0.\end{aligned}$$

Since $P^N\mathbf{v}_0\to\mathbf{v}_0$ in $L^2(\Omega)$ we get $\mathbf{v}(0)=\mathbf{v}_0$. Overall we have shown by (2.67) and (2.68) that $\mathbf{v}$ satisfies (2.2) in the weak sense. It remains to prove the estimates for $\mathbf{v}$, $\mathcal{I}(\mathbf{v})$ and $\mathcal{J}(\mathbf{v})$. First of all, from (2.60) and (2.61) it follows that

$$(2.69)\qquad \|\partial_t\mathbf{v}\|_{L^\infty(I';L^2(\Omega))}+\|\mathbf{v}\|_{L^{p_\infty\frac{5p_\infty-6}{2-p_\infty}}(I';W^{2,p_\infty}(\Omega))}\leqslant C.$$

The passage to the limit in the expressions $\|\mathcal{I}(\mathbf{v}^N)\|_{L^{\frac{5p_\infty-6}{2-p_\infty}}(I')}$ and $\|\mathcal{J}(\mathbf{v}^N)\|_{L^1(I')}$ is possible, since due to (2.62), (2.58), (2.59) and the convexity of $\mathcal{I}$ and $\mathcal{J}$ in $\mathbf{D}(\nabla\mathbf{v})$ and $\mathbf{D}(\partial_t\mathbf{v})$, respectively, we can use De Giorgi's semicontinuity theorem (cf. [23], p. 132) and a version of it (cf. [12]) to obtain

$$(2.70)\qquad \int_0^{T'}\mathcal{I}(t,\mathbf{v})^{\frac{5p_\infty-6}{2-p_\infty}}+\mathcal{J}(t,\mathbf{v})\,\mathrm{d}t\leqslant C.$$

Moreover from this, (2.30) and Young's inequality we get

$$(2.71)\qquad \int_0^{T'}\|\partial_t\mathbf{v}\|_{1,\frac{3p_\infty}{p_\infty+1}}^{\frac{p_\infty(5p_\infty-6)}{(3p_\infty-2)(p_\infty-1)}}\,\mathrm{d}t\leqslant C\int_0^{T'}\mathcal{I}(t,\mathbf{v})^{\frac{5p_\infty-6}{2-p_\infty}}+\mathcal{J}(t,\mathbf{v})\,\mathrm{d}t\leqslant C.$$

In order to obtain the estimate for $\partial_t^2\mathbf{v}$ we differentiate (2.67) with respect to time in the sense of distributions, which yields for all $\boldsymbol{\omega}\in\mathcal{V}$ and all $\varphi\in C_0^\infty(I')$

$$\begin{aligned}(2.72)\quad \int_0^{T'}\langle\partial_t^2\mathbf{v},\boldsymbol{\omega}\rangle\varphi\,\mathrm{d}t=\int_0^{T'}&-\langle\partial_t\mathbf{S}(\mathbf{D}\mathbf{v},\mathbf{E}),\mathbf{D}\boldsymbol{\omega}\rangle\varphi+\langle 2\mathbf{v}\otimes\partial_t\mathbf{v},\mathbf{D}\boldsymbol{\omega}\rangle\varphi\\ &+\langle\partial_t\mathbf{f},\boldsymbol{\omega}\rangle\varphi-2\chi^E\langle\mathbf{E}\otimes\partial_t\mathbf{E},\mathbf{D}\boldsymbol{\omega}\rangle\varphi\,\mathrm{d}t.\end{aligned}$$

From (2.6), (2.7) and $p(|\mathbf{E}|^2)\leqslant p_0\leqslant 2$ we get

$$\|\partial_t\mathbf{S}(\mathbf{D}\mathbf{v},\mathbf{E})\|_2^2\leqslant C\big(1+\mathcal{J}(\mathbf{v})+\|\nabla\mathbf{v}|_2^2+\|\nabla\mathbf{v}\|_3^3\big),$$

which due to (2.69) and (2.70) belongs to $L^1(I')$. From (2.69) and the assumptions on the data we easily see that also the other three terms on the right-hand side of (2.72) belong to $L^1(I')$ if $\boldsymbol{\omega}\in L^2(I';V_2)$. This implies $\partial_t^2\mathbf{v}\in L^2(I';(V_2)^*)$. From (2.69), (2.71) and the parabolic embedding (cf. [14]) we finally get $\mathbf{v}\in C(I';V_r)$, $1\leqslant r<6(p_\infty-1)$. This finishes the proof of Theorem 2.13.

3. Time discretization

Now we discuss a time discretization of the system (2.2) under the additional assumption that

$$p = \text{const.}$$

and consequently we have to modify our basic assumptions on $\mathbf{S}$. We assume that the following *monotonicity condition*

$$\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}} B_{ij}B_{kl} \geqslant \gamma_1(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}|\mathbf{B}|^2, \tag{3.1}$$

is satisfied for all $\mathbf{B},\mathbf{D} \in X := \{\mathbf{D} \in \mathbb{R}^{3\times 3}_{\text{sym}},\ \operatorname{tr}\mathbf{D} = 0\}$, and that the following *growth conditions* are satisfied for $i,j,k,l,n = 1,2,3$,

$$\Big|\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial D_{kl}}\Big| \leqslant \gamma_2(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}, \tag{3.2}$$

$$\Big|\frac{\partial S_{ij}(\mathbf{D},\mathbf{E})}{\partial E_n}\Big| \leqslant \gamma_3|\mathbf{E}|(1+|\mathbf{E}|^2)(1+|\mathbf{D}|^2)^{\frac{p-1}{2}}. \tag{3.3}$$

For the numerical analysis we need some additional notation. Let $I_k = \{t_m\}_{m=0}^M$ be a given net in an interval $I = [0,t_M]$ with a constant *time-step* size $k := t_m - t_{m-1}$. We denote by $d_t\mathbf{v}^m := k^{-1}(\mathbf{v}^m - \mathbf{v}^{m-1})$ the divided difference in time. By $l^q(I_k;X)$ we denote the space of functions $\{\varphi^m\}_{m=0}^M$ with finite norm $\Big(k\sum\limits_{m=0}^{M}\|\varphi^m\|_X^q\Big)^{1/p}$. For $q=\infty$, functions $\{\varphi^m\}_{m=0}^M$ need to satisfy the bound $\max\limits_{0\leqslant m\leqslant M}\|\varphi^m\|_X < \infty$.

The problem (2.2) is approximated by a time discretization by means of the *implicit Euler scheme*:

Algorithm 3.4. *Let there be given a time-step size $k>0$ and the corresponding net $I_k = \{t_m\}_{m=0}^M$. For $m \geqslant 1$ and $\mathbf{v}^{m-1}$ given from the previous step, compute an iterate $\mathbf{v}^m$ that solves*

$$\begin{aligned} d_t\mathbf{v}^m - \operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{v}^m,\mathbf{E}(t_m)) + [\nabla\mathbf{v}^m]\mathbf{v}^m + \nabla\pi^m &\\ = \mathbf{f}(t_m) + \chi^E[\nabla\mathbf{E}(t_m)]\mathbf{E}(t_m)&, \\ \operatorname{div}\mathbf{v}^m = 0&, \\ \mathbf{v}^0 = \mathbf{v}_0&, \end{aligned} \tag{3.5}$$

endowed with space-periodic boundary conditions (2.6).

The main result of this section is:

Theorem 3.6. *Assume that the extra stress tensor* $\mathbf{S}$ *satisfies* (3.1)–(3.3) *and* $\mathbf{S}(\mathbf{0},\mathbf{E})=\mathbf{0}$. *Let* $\mathbf{v}_0 \in W^{2,2}(\Omega)\cap V_p$ *be a given initial velocity,* $\mathbf{f}\in C(I;W^{1,2}(\Omega))$, $\partial_t\mathbf{f}\in C(I;L^2(\Omega))$ *be a given force,* $\mathbf{E}\in C^1(\bar{I};C^1(\Omega))$ *be a given electric field. Let* $\mathbf{v}$ *be a strong solution of the problem* (2.2) *on the interval* $I'=[0,T']$ *for* $p\in\left[\frac{5}{3},2\right]$ *satisfying* (2.14) *and* (2.15). *Suppose that* $\mathbf{v}^m$ *is a weak solution of the problem* (3.5) *satisfying* (3.19) *and* $t_M\leqslant T'$. *Then for all*

$$\alpha<\alpha_0(p):=\frac{5p-6}{4(p-1)} \tag{3.7}$$

there exists a constant C *that only depends on* $\mathbf{v}_0$, $\mathbf{f}$, Ω, T' *and* α *but not on the time-step size* k, *such that the following error estimate is valid, provided that the time-step size is chosen sufficiently small, i.e.* $k\leqslant k_0(p,T')$,

$$\max_{1\leqslant m\leqslant M}\|\mathbf{v}(t_m)-\mathbf{v}^m\|_2^2+k\sum_{m=1}^{M}\|\mathbf{D}(\mathbf{v}(t_m)-\mathbf{v}^m)\|_p^2\leqslant Ck^{2\alpha}. \tag{3.8}$$

Remark 3.9. With a more refined technique (cf. [13]) one can show that the assertion of the theorem holds for $p\in\left(\frac{11+\sqrt{21}}{10},2\right]\approx(1.5583,2]$.

Before we start with the proof of Theorem 3.6 we need some additional properties of quantities related to $\mathbf{S}$. Due to (3.1)–(3.3) we get that $\mathcal{I}(t,\mathbf{v})$ and $\mathcal{J}(t,\mathbf{v})$ defined in (2.9) and (2.10) satisfy the analogue of (2.22) and (2.23), i.e.

$$\begin{aligned}\mathcal{I}(t,\mathbf{v})&\geqslant\gamma_1\int_\Omega(\tilde{D}\mathbf{v}(t))^{p-2}|\mathbf{D}(\nabla\mathbf{v})(t)\|^2\,\mathrm{d}x,\\ \mathcal{J}(t,\mathbf{v})&\geqslant\gamma_1\int_\Omega(\tilde{D}\mathbf{v}(t))^{p-2}|\mathbf{D}(\partial_t\mathbf{v})(t)|^2\,\mathrm{d}x.\end{aligned}$$

The discrete analogue for $\mathcal{J}(\mathbf{v})$ for a function defined on a net I_k reads as follows

$$\mathcal{K}(\mathbf{v}^m):=\int_\Omega\int_0^1\frac{\partial S_{ij}(\mathbf{D}(s\mathbf{v}^m+(1-s)\mathbf{v}^{m-1}),\mathbf{E}(t_m))}{\partial D_{kl}}\,\mathrm{d}s\,D_{ij}(d_t\mathbf{v}^m)D_{kl}(d_t\mathbf{v}^m)\,\mathrm{d}x,$$

which due to (3.1) and Lemma 2.17 satisfies

$$\mathcal{K}(\mathbf{v}^m)\geqslant C_3\int_\Omega(1+|\mathbf{D}\mathbf{v}^m|^2+|\mathbf{D}\mathbf{v}^{m-1}|^2)^{\frac{p-2}{2}}|\mathbf{D}(d_t\mathbf{v}^m)|^2\,\mathrm{d}x. \tag{3.10}$$

Lemma 3.11. *Let* $\mathbf{S}$ *satisfy* (3.1) *and* (3.2). *Then for all (sufficiently smooth)* $\mathbf{v}$ *with* $\int_\Omega \mathbf{v}\,\mathrm{d}x = 0$, *for all* $1 \leqslant q < \infty$, *and almost every* $t \in I$ *there holds:*

$$\|\nabla \mathbf{v}(t)\|^2_{\frac{6q}{6-3p+q}} + \|\mathbf{D}(\nabla \mathbf{v})(t)\|^2_{\frac{2q}{2-p+q}} \leqslant C\mathcal{I}(t,\mathbf{v})\|\tilde{D}\mathbf{v}(t)\|_q^{2-p}, \tag{3.12}$$

$$\|\partial_t \mathbf{v}(t)\|^2_{\frac{6q}{6-3p+q}} + \|\mathbf{D}(\partial_t \mathbf{v})(t)\|^2_{\frac{2q}{2-p+q}} \leqslant C\mathcal{J}(t,\mathbf{v})\|\tilde{D}\mathbf{v}(t)\|_q^{2-p}. \tag{3.13}$$

Proof. Lemma 2.25 $\left(r \mapsto \frac{2q}{2-p+q}\right)$ and $p = \text{const.}$ imply

$$\begin{aligned}\|\mathbf{D}(\nabla \mathbf{v})\|_{\frac{2q}{2-p+q}} &\leqslant C\mathcal{I}(\mathbf{v})^{\frac12}\|(\tilde{D}\mathbf{v})^{\frac{2-p}{2}}\|_{\frac{2q}{2-p}}\\ &\leqslant C\mathcal{I}(\mathbf{v})^{\frac12}\|\tilde{D}\mathbf{v}\|_q^{\frac{2-p}{2}},\end{aligned}$$

which together with the embedding $W^{2,\frac{2q}{2-p+q}}(\Omega) \hookrightarrow W^{1,\frac{6q}{6-3p+q}}(\Omega)$ proves the first assertion. The second assertion follows analogously. $\square$

Since $\mathcal{K}(\mathbf{v}^m)$ is the discrete version of $\mathcal{J}(\mathbf{v})$ we immediately obtain in the same way as in Lemma 3.11 and Lemma 2.28:

Lemma 3.14. *Let* $\mathbf{S}$ *satisfy* (3.1) *and* (3.2). *For all (sufficiently smooth)* $\mathbf{v}^m$ *with* $\int_\Omega \mathbf{v}^m\,\mathrm{d}x = 0$ *there holds for all* $q \in [1,\infty)$:

$$\begin{aligned}&\|d_t\mathbf{v}^m\|^2_{\frac{6q}{6-3p+q}} + \|\mathbf{D}(d_t\mathbf{v}^m)\|^2_{\frac{2q}{2-p+q}} \\ &\qquad\leqslant C\mathcal{K}(\mathbf{v}^m)(\|\tilde{D}\mathbf{v}^m\|_q + \|\tilde{D}\mathbf{v}^{m-1}\|_q)^{2-p},\end{aligned} \tag{3.15}$$

$$\|d_t\mathbf{v}^m\|_{3p}^p + \|d_t\nabla\mathbf{v}^m\|^p_{\frac{3p}{p+1}} \leqslant C(1+\mathcal{I}(\mathbf{v}^m)+\mathcal{I}(\mathbf{v}^{m-1}))^{\frac{2-p}{2}}\mathcal{K}(\mathbf{v}^m)^{p/2}, \tag{3.16}$$

$$\leqslant C(1+\mathcal{I}(\mathbf{v}^m)+\mathcal{I}(\mathbf{v}^{m-1})+\mathcal{K}(\mathbf{v}^m)). \tag{3.17}$$

The following lemma ensures the solvability of the problem (3.5).

Lemma 3.18. *Let* $\mathbf{S}$, $\mathbf{v}_0$, $\mathbf{f}$ *and* $\mathbf{E}$ *satisfy the assumptions of Theorem* 3.6. *Then there exists a weak solution* $\mathbf{v}^m$ *of* (3.5) *satisfying*

$$\max_{1\leqslant m\leqslant M}\|\mathbf{v}^m\|_2^2 + k\sum_{m=1}^{M}\|\mathbf{D}\mathbf{v}^m\|_p^p \leqslant C(\mathbf{f},\mathbf{v}_0,\mathbf{E}), \tag{3.19}$$

whenever $p > \frac32$.

P r o o f. First of all note that the strategy employed in the proof of Theorem 2.13 to ensure the existence of strong solutions is not applicable in the *discrete case*, since there is no discrete version of the local Gronwall's inequality. For $p > \frac{9}{5}$ the estimate (3.19) is sufficient to ensure the existence of weak solutions using the theory of monotone operators (cf. [29]). For this we must view (3.5), with k and m fixed, as a steady system with the discrete time derivative as the right-hand side. In order to prove the lemma for $p > \frac{3}{2}$ we proceed as follows (cf. [22], [41]). We approximate (3.5) by the mollified system

$$\begin{aligned} d_t \mathbf{v}_n^m - \operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{v}_n^m, \mathbf{E}(t_m)) + [\nabla \mathbf{v}^m](\mathbf{v}_n^m)_{1/n} + \nabla \pi_n^m \\ = \mathbf{f}_n(t_m) + \chi^E [\nabla \mathbf{E}_n(t_m)] \mathbf{E}_n(t_m), \\ \operatorname{div} \mathbf{v}_n^m = 0, \end{aligned} \tag{3.20}$$

where $(\mathbf{v}_n^m)_{1/n} = w_{1/n} * \mathbf{v}_n^m$ is the usual mollification. Now we fix m and k and move the discrete time derivative to the right-hand side and view (3.20) as a steady system. Using the Galerkin method and the theory of monotone operators[9] it is easy to show that there exists a weak solution to (3.20) satisfying the estimate (3.19). The *key* observation is that

$$[\nabla \mathbf{v}^m](\mathbf{v}_n^m)_{1/n} \text{ is bounded in } L^{\frac{3p}{6-p}}(\Omega)$$

uniformly with respect to n. To take advantage of this property we must use L^∞-test functions which ensure the almost everywhere convergence of $\mathbf{D}\mathbf{v}_n^m$. This argument is elaborated in detail in [41] and one can follow exactly the argumentation there. As a result one obtains that $\mathbf{D}\mathbf{v}_n^m$ converges a.e. in Ω to $\mathbf{D}\mathbf{v}^m$, which together with Vitali's convergence theorem enables the limiting process in the weak formulation of (3.20). $\square$

In order to verify Theorem 3.6 we have to deal with two problems. Namely that the discrete solution $\mathbf{v}^m$ of the problem (3.5) is only weak and secondly that the information about $\partial_t^2 \mathbf{v}$ is also weak. Thus we introduce an auxiliary problem to split these problems subsequently. We follow the procedure introduced in [37] and consider the following auxiliary problem:

Algorithm 3.21. *Suppose that* $\mathbf{v}$ *is a strong solution to the problem* 2.2 *with the properties stated in Theorem* 2.13. *Then determine* $\mathbf{V}^m$, $m = 1, \ldots, M$, *that*

[9] Note that the mollified convective term maps the space V_p into $W^{-1,p}(\Omega)$ for $p > \frac{3}{2}$.

solves

$$
\begin{aligned}
(3.22)\qquad d_t\mathbf{V}^m - \operatorname{div}\mathbf{S}(\mathbf{D}\mathbf{V}^m,\mathbf{E}(t_m)) + [\nabla\mathbf{V}^m]\mathbf{v}(t_m) + \nabla\Pi^m&\\
= \mathbf{f}(t_m) + \chi^E[\nabla\mathbf{E}(t_m)]\mathbf{E}(t_m)&,\\
\operatorname{div}\mathbf{V}^m = 0&,\\
\mathbf{V}^0 = \mathbf{v}_0&,
\end{aligned}
$$

endowed with space-periodic boundary conditions (2.12).

We have linearized the convective term with respect to the continuous solution $\mathbf{v}(t_m)$, for which we have good regularity properties. The hope is that $\mathbf{V}^m$ *inherits the regularity* from $\mathbf{v}$. In fact this is the case at the expense of restricting ourselves to a smaller range of p's.

Proposition 3.23. *Let* $\mathbf{S}$, $\mathbf{v}_0$, $\mathbf{f}$ *and* $\mathbf{E}$ *satisfy the assumptions of Theorem* 3.6. *Let* $\mathbf{v}$ *defined on* $I=[0,T']$ *be the strong solution ensured by this theorem and let* $t_M < T'$. *Then there exists a strong solution* $\mathbf{V}^m$ *of the problem* (3.22) *whenever* $p\in\left[\frac{5}{3},2\right]$. *This solution satisfies*

$$
(3.24)\qquad \max_{1\leqslant m\leqslant M}\|d_t\mathbf{V}^m\|_2^2 + k\sum_{m=1}^{M}\left(\mathcal{I}(\mathbf{V}^m)^{\frac{5p-6}{2-p}} + \mathcal{K}(\mathbf{V}^m)\right) \leqslant C(\mathbf{f},\mathbf{v}_0,\mathbf{E}).
$$

In particular we have that for all $1<r<6(p-1)$ *it holds*

$$
\begin{aligned}
(3.25)\qquad \mathbf{V}^m &\in l^{p\frac{5p-6}{2-p}}\left(I_k;W^{2,\frac{3p}{p+1}}(\Omega)\right)\cap l^\infty(I_k;V_r),\\
d_t\mathbf{V}^m &\in l^{\frac{p(5p-6)}{(3p-2)(p-1)}}\left(I_k;W^{1,\frac{3p}{p+1}}(\Omega)\right)\cap l^\infty(I_k;L^2(\Omega)).
\end{aligned}
$$

P r o o f. The existence of a strong solution $\mathbf{V}^m$ of (3.22) follows from the regularity in (3.25) using the Galerkin approach with eigenfunctions of the Stokes operator as a basis. The regularity (3.25) follows in the same way as in the proof of Theorem 2.13 from (3.24) using also Lemma 3.14. Thus we shall only derive these estimates. For all missing details in the following computations we refer to [30, Section 5.3].

First of all we test the weak formulation of (3.22), which reads for all $\boldsymbol{\varphi}\in V_p$

$$
\begin{aligned}
(3.26)\qquad \langle d_t\mathbf{V}^m,\boldsymbol{\varphi}\rangle + \langle\mathbf{S}(\mathbf{D}\mathbf{V}^m,\mathbf{E}(t_m)),\mathbf{D}\boldsymbol{\varphi}\rangle + \langle[\nabla\mathbf{V}^m]\mathbf{v}(t_m),\boldsymbol{\varphi}\rangle&\\
= \langle\mathbf{f}(t_m),\boldsymbol{\varphi}\rangle - \chi^E\langle\mathbf{E}(t_m)\otimes\mathbf{E}(t_m),\mathbf{D}\boldsymbol{\varphi}\rangle&,
\end{aligned}
$$

with $\mathbf{V}^m$ and sum up over all iteration steps to obtain the first *a priori* estimate

$$
(3.27)\qquad \max_{1\leqslant m\leqslant M}\|\mathbf{V}^m\|_2^2 + k\sum_{m=1}^{M}\|\mathbf{D}\mathbf{V}^m\|_p^p \leqslant C,
$$

where we used that $\langle[\nabla\mathbf{V}^m]\mathbf{v}(t_m),\mathbf{V}^m\rangle = 0$.

The next step is to use in (3.26) $-\Delta \mathbf{V}^m$ as a test function. Again we use that $\operatorname{div} \mathbf{v}(t_m) = 0$ in the linearized convective term, the properties of $\mathbf{S}$ (cf. (3.1)–(3.3)), the definition of $\mathcal{I}(\mathbf{V}^m)$ and obtain, after summation up to level $N \in \{1, \dots, M\}$,

$$\begin{aligned}
(3.28) \qquad & \|\nabla \mathbf{V}^N\|_2^2 + k \sum_{m=1}^{N} \mathcal{I}(\mathbf{V}^m) \\
& \leqslant C\biggl(1 + k \sum_{m=1}^{N} \int_\Omega |\nabla \mathbf{v}(t_m)|\, |\nabla \mathbf{V}^m|^2 \, \mathrm{d}x \\
& \quad + k \sum_{m=1}^{N} \int_\Omega \Bigl| \frac{\partial S_{ij}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m))}{\partial E_n} \nabla E_n \cdot D_{ij}(\nabla \mathbf{V}^m) \Bigr| \, \mathrm{d}x \biggr).
\end{aligned}$$

The last term on the right-hand side can be bounded by (cf. (3.3))

$$(3.29) \qquad \varepsilon k \sum_{m=1}^{N} \mathcal{I}(\mathbf{V}^m) + Ck \sum_{m=1}^{N} \|\tilde{D}\mathbf{V}^m\|_p^p,$$

where the first term is absorbed in the left-hand side of (3.28). The second term on the right-hand side in (3.28) can, for $1 < r < 6(p-1)$, $\alpha \in (0,1)$, be estimated by

$$(3.30) \qquad \|\nabla \mathbf{v}(t_m)\|_r \, \|\nabla \mathbf{V}^m\|_{2r'}^2 \leqslant C \|\nabla \mathbf{V}^m\|_{2r'}^2 = C \|\nabla \mathbf{V}^m\|_{2r'}^{2(\alpha + 1 - \alpha)},$$

where r' is the dual exponent to r and where we used $\mathbf{v} \in C(I; V_r)$. Now, for $p > \frac{4}{3}$ and $\frac{3p}{3p-2} < r < 6(p-1)$ we interpolate $L^{2r'}(\Omega)$ both between $L^2(\Omega)$ and $L^{3p}(\Omega)$ and between $L^p(\Omega)$ and $L^{3p}(\Omega)$, which gives

$$\begin{aligned}
(3.31) \qquad & \|\nabla \mathbf{V}^m\|_{2r'} \leqslant \|\nabla \mathbf{V}^m\|_2^{\frac{r(3p-2)-3p}{r(3p-2)}} \, \|\nabla \mathbf{V}^m\|_{3p}^{\frac{3p}{r(3p-2)}}, \\
& \|\nabla \mathbf{V}^m\|_{2r'} \leqslant \|\nabla \mathbf{V}^m\|_p^{\frac{1}{4}\frac{r(3p-2)-3p}{r}} \, \|\nabla \mathbf{V}^m\|_{3p}^{\frac{3}{4}\frac{r(2-p)+p}{r}}.
\end{aligned}$$

Using also (2.29) the right-hand side of (3.30) can be estimated by

$$(3.32) \qquad C(1 + \|\nabla \mathbf{V}^m\|_2^2)^{Q_1} \, \|\nabla \mathbf{V}^m\|_p^{p\,Q_2} (1 + \mathcal{I}(\mathbf{V}^m))^{Q_3},$$

where

$$\begin{aligned}
Q_1 &= (1-\alpha)\frac{r(3p-2)-3p}{r(3p-2)}, \qquad Q_2 = \alpha \frac{1}{2p}\frac{r(3p-2)-3p}{r}, \\
Q_3 &= (1-\alpha)\frac{2}{p}\frac{3p}{r(3p-2)} + \alpha \frac{3}{2p}\frac{r(2-p)+p}{r}.
\end{aligned}$$

Young's inequality together with the requirements

$$Q_2 \cdot \delta = \frac{1}{1+\varepsilon}, \quad Q_3 \cdot \delta' = 1, \quad \frac{1}{\delta} + \frac{1}{\delta'} = 1$$

for any prescribed $\varepsilon > 0$ yields

$$1 + \|\nabla \mathbf{V}^N\|_2^2 + k \sum_{m=1}^{N} \mathcal{I}(\mathbf{V}^m) \leqslant C\biggl(1 + k \sum_{m=1}^{N} \|\nabla \mathbf{V}^m\|_p^{\frac{p}{1+\varepsilon}} (1 + \|\nabla \mathbf{V}^m\|_2^2)^{\lambda_\varepsilon(r)}\biggr),$$

where

$$\lambda_\varepsilon(r) \searrow \lambda = \frac{2(p-1)(2-p)}{3p^2 - 5p + 1} \quad \text{for } \varepsilon \searrow 0, \ \ r \nearrow 6(p-1).$$

In view of (3.27) we have to check whether $\lambda < 1$, which holds for $p \in \bigl(\frac{11+\sqrt{21}}{10}, 2\bigr]$. Therefore we can employ discrete Gronwall's lemma and obtain our second *a priori* estimate

$$\max_{1\leqslant m \leqslant M} \|\nabla \mathbf{V}^m\|_2^2 + k \sum_{m=1}^{M} \mathcal{I}(\mathbf{V}^m) \leqslant C. \tag{3.33}$$

Now we want to use "$d_t^2 \mathbf{V}^m$" as a test function in (3.26). This in fact will give us the lower bound $p \geqslant \frac{5}{3}$. Firstly, we have to introduce $\mathbf{V}^{-1}$. For that we set for all $\boldsymbol{\varphi} \in V_p$

$$\begin{aligned}\frac{1}{k}\langle \mathbf{V}^0 - \mathbf{V}^{-1}, \boldsymbol{\varphi}\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{V}^0, \mathbf{E}(0)), \mathbf{D}\boldsymbol{\varphi}\rangle + \langle [\nabla \mathbf{V}^0]\mathbf{V}^0, \boldsymbol{\varphi}\rangle \\ = \langle \mathbf{f}(0), \boldsymbol{\varphi}\rangle - \chi^E \langle \mathbf{E}(0) \otimes \mathbf{E}(0), \mathbf{D}\boldsymbol{\varphi}\rangle.\end{aligned}$$

Using $\mathbf{V}^0 = \mathbf{v}_0$, $p \leqslant 2$ and the assumption on $\mathbf{v}_0$ and $\mathbf{E}$ we obtain

$$\begin{aligned}\|d_t \mathbf{V}^0\|_2^2 \leqslant C(\|\mathbf{f}(0)\|_2^2 + \|[\nabla \mathbf{v}_0]\mathbf{v}_0\|_2^2 + \|\operatorname{div} \mathbf{S}(\mathbf{D}\mathbf{v}_0, \mathbf{E}(0))\|_2^2 \\ + \|\mathbf{E}(0) \otimes \mathbf{D}\mathbf{E}(0)\|_2^2) \leqslant C.\end{aligned} \tag{3.34}$$

Now we can take the discrete time derivative of the weak formulation (3.26), use $d_t \mathbf{V}^m$ as a test function, and sum up to level $N \in \{1, \ldots, M\}$, to obtain

$$\begin{aligned}&\|d_t \mathbf{V}^N\|_2^2 \\ &\quad + \frac{1}{k} \sum_{m=1}^{N} \int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_{m-1})))\mathbf{D}(\mathbf{V}^m - \mathbf{V}^{m-1}) \, \mathrm{d}x \\ &\quad \leqslant C\biggl(1 + k \sum_{m=1}^{N} \biggl| \int_\Omega [\nabla \mathbf{V}^m] d_t \mathbf{v}(t_{m-1}) \cdot d_t \mathbf{V}^m \, \mathrm{d}x \biggr|\biggr),\end{aligned} \tag{3.35}$$

where we used (3.34). From the formula $d_t\mathbf{v}(t_m) = k^{-1}\int_{t_{m-1}}^{t_m} \partial_t \mathbf{v}(s)\,\mathrm{d}s$ and $(2.15)_2$ we deduce

$$\|d_t\mathbf{v}(t_m)\|_2 \leqslant \operatorname*{ess\,sup}_I \|\partial_t \mathbf{v}\|_2 \leqslant C, \tag{3.36}$$

and thus we can bound the last term in (3.35) by

$$\begin{aligned} \|d_t\mathbf{v}(t_{m-1})\|_2 \, \| \, |\nabla \mathbf{v}^m| \, |d_t\mathbf{V}^m| \, \|_2 &\leqslant C\|\nabla \mathbf{V}^m\|_4 \, \|d_t\mathbf{V}^m\|_4 \\ &\leqslant \varepsilon\mathcal{K}(\mathbf{V}^m) + C\mathcal{I}(\mathbf{V}^m), \end{aligned} \tag{3.37}$$

where we used (3.15), (3.12) with $q = 2$, (3.33) and Young's inequality. However we have to check whether

$$\frac{12}{8-3p} \geqslant 4 \Longleftrightarrow p \geqslant \frac{5}{3},$$

which is the lower bound from the proposition. Furthermore we have for the second term on the left-hand side of (3.35)

$$\begin{aligned} &k^{-1}\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_{m-1}))) \cdot \mathbf{D}(\mathbf{V}^m - \mathbf{V}^{m-1})\,\mathrm{d}x \\ &= k^{-1}\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_m))) \cdot \mathbf{D}(\mathbf{V}^m - \mathbf{V}^{m-1})\,\mathrm{d}x \\ &\quad + k^{-1}\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^{m-1}, \mathbf{E}(t_{m-1}))) \cdot \mathbf{D}(\mathbf{V}^m - \mathbf{V}^{m-1})\,\mathrm{d}x \\ &= k\mathcal{K}(\mathbf{V}^m) \\ &\quad + k\int_\Omega\int_0^1 \frac{S_{ij}(\mathbf{D}\mathbf{V}^{m-1}, (1-\tau)\mathbf{E}(t_{m-1}) + \tau\mathbf{E}(t_m))}{\partial E_n}\,\mathrm{d}\tau d_t E_n(t_m) D_{ij}(d_t\mathbf{V}^m)\,\mathrm{d}x. \end{aligned}$$

The last term is moved to the right-hand side and there estimated by

$$\varepsilon k\mathcal{K}(\mathbf{V}^m) + Ck(\|\tilde{D}\mathbf{V}^m\|_p^p + \|\tilde{D}\mathbf{V}^{m-1}\|_p^p), \tag{3.38}$$

where we used (3.3) and (3.10). Note that the last term is finite after summation over m, due to (3.27). Alltogether, we have therefore derived our third *a priori* estimate

$$\max_{1\leqslant m\leqslant M} \|d_t\mathbf{V}^m\|_2^2 + k\sum_{m=1}^M \mathcal{K}(\mathbf{V}^m) \leqslant C. \tag{3.39}$$

Using $-\Delta \mathbf{V}^m$ as a test function in (3.26), where also the term with the discrete time derivative is estimated, yields for $p > \frac{3}{2}$ and $\frac{6}{3p-2} < r < 6(p-1)$ (cf. (3.28)–(3.30))

$$
\begin{aligned}
(3.40)\qquad 1 + \mathcal{I}(\mathbf{V}^m) &\leqslant C\big(1 + \varepsilon\mathcal{I}(\mathbf{V}^m) + \|\tilde{D}\mathbf{V}^m\|_p^p + \|\nabla\mathbf{V}^m\|_{2r'}^2 \\
&\qquad + \|d_t\mathbf{V}^m\|_{\frac{3p}{2p-1}} \|\nabla^2\mathbf{V}^m\|_{\frac{3p}{p+1}}\big) \\
&\leqslant C\big(1 + C_\varepsilon\|\nabla\mathbf{V}^m\|_2^2 + \varepsilon\mathcal{I}(\mathbf{V}^m)(1 + \|\tilde{D}\mathbf{V}^m\|_2^{2-p}) \\
&\qquad + \|d_t\mathbf{V}^m\|_{\frac{3p}{2p-1}} \|\nabla^2\mathbf{V}^m\|_{\frac{3p}{p+1}}\big) \\
&\leqslant C\big(C_\varepsilon + \varepsilon\mathcal{I}(\mathbf{V}^m) + \|d_t\mathbf{V}^m\|_{\frac{3p}{2p-1}} (1 + \mathcal{I}(\mathbf{V}^m))^{1/p}\big),
\end{aligned}
$$

where we used $\mathbf{V}^m \in l^\infty(I_k; W^{1,2}\Omega)$ and $p \leqslant 2$; the interpolation of $L^{2r'}(\Omega)$ between $L^2(\Omega)$ and $L^{\frac{12}{8-3p}}(\Omega)$, which is possible for $p > \frac{3}{2}$, and (3.12) with $q = 2$; again $\mathbf{V}^m \in l^\infty(I_k; W^{1,2}(\Omega))$ and finally (2.29). For ε sufficiently small we can absorb the term $c\varepsilon\mathcal{I}(\mathbf{V}^m)$ into the left-hand side of (3.40). Thus we get

$$
(3.41)\qquad (1 + \mathcal{I}(\mathbf{V}^m))^{\frac{p-1}{p}} \leqslant C\big(1 + \|d_t\mathbf{V}^m\|_{\frac{3p}{2p-1}}\big).
$$

Now we interpolate $L^{\frac{3p}{2p-1}}(\Omega)$ between $L^2(\Omega)$ and $L^{3p}(\Omega)$, and use that $d_t\mathbf{V}^m \in l^\infty(I_k; L^2(\Omega))$ and (3.16), to arrive at

$$
(3.42)\qquad (1 + \mathcal{I}(\mathbf{V}^m))^{\frac{p-1}{p}} \leqslant C\big(1 + \mathcal{K}(\mathbf{V}^m)^{\lambda/2}(1 + \mathcal{I}(\mathbf{V}^m) + \mathcal{I}(\mathbf{V}^{m-1}))^{\lambda\,\frac{2-p}{2p}}\big),
$$

with $\lambda = \frac{2-p}{3p-2}$. We raise this inequality to the power γ and apply Young's inequality to get

$$
\begin{aligned}
(3.43)\qquad & (1 + \mathcal{I}(\mathbf{V}^m))^{\gamma\frac{p-1}{p}} \\
&\leqslant C\big(1 + \mathcal{K}(\mathbf{V}^m)^{\gamma\frac{\lambda}{2}}(1 + \mathcal{I}(\mathbf{V}^m) + \mathcal{I}(\mathbf{V}^{m-1}))^{\gamma\lambda\frac{2-p}{2p}}\big) \\
&\leqslant C\big(1 + C_\varepsilon\mathcal{K}(\mathbf{V}^m) + \varepsilon(1 + \mathcal{I}(\mathbf{V}^m) + \mathcal{I}(\mathbf{V}^{m-1}))^{\frac{2\gamma}{2-\gamma\lambda}\,\lambda\,\frac{2-p}{2p}}\big).
\end{aligned}
$$

We now require $\gamma\frac{p-1}{p} = \frac{2\gamma}{2-\gamma\lambda}\,\lambda\,\frac{2-p}{2p}$, which gives $\gamma = \frac{p}{p-1}\frac{5p-6}{2-p}$. With this γ and ε sufficiently small we can absorb the last term in (3.43) into the left-hand side after summation over all time steps. Thus we have derived

$$
(3.44)\qquad k\sum_{m=0}^{M} \mathcal{I}(\mathbf{V}^m)^{\frac{5p-6}{2-p}} \leqslant C\Big(1 + k\sum_{m=0}^{M}\mathcal{K}(\mathbf{V}^m)\Big) \leqslant C.
$$

The proof is complete. $\square$

Proposition 3.23 shows that the solution $\mathbf{V}^m$ of (3.22) has the same regularity properties as the solution $\mathbf{v}$ of the problem (2.2). Thus we can *split* the error into two parts, namely

$$\mathbf{v}(t_m) - \mathbf{v}^m = (\mathbf{v}(t_m) - \mathbf{V}^m) + (\mathbf{V}^m - \mathbf{v}^m) =: \boldsymbol{\eta}^m + \mathbf{e}^m. \tag{3.45}$$

Before we discuss these errors we need one more property of $\mathbf{S}$.

Lemma 3.46. *Let* $\mathbf{S}$ *satisfy* (3.1) *and* (3.2). *Then for all (sufficiently smooth)* $\mathbf{v}$, $\mathbf{w}$, *for all* $1 \leqslant r < \infty$, *and almost every* $t \in I'$ *there holds*

$$\begin{aligned}\|\mathbf{D}(\mathbf{v}(t)-\mathbf{w}(t))\|^2_{\frac{2r}{2-p+r}} \leqslant C\langle \mathbf{S}(\mathbf{D}\mathbf{v}(t),\mathbf{E}(t)) - \mathbf{S}(\mathbf{D}\mathbf{w}(t),\mathbf{E}(t)), \mathbf{D}(\mathbf{v}(t)-\mathbf{w}(t))\rangle \\ \times (1 + \|\mathbf{D}\mathbf{v}(t)\|_r + \|\mathbf{D}(\mathbf{v}(t)-\mathbf{w}(t))\|_r)^{2-p}.\end{aligned}$$

Proof. We have using Lemma 2.19

$$\begin{aligned}\|\mathbf{D}(\mathbf{v}-\mathbf{w})\|^{\frac{2r}{2-p+r}}_{\frac{2r}{2-p+r}} &= \int_\Omega ((1+|\mathbf{D}\mathbf{v}|+|\mathbf{D}(\mathbf{v}-\mathbf{w})|)^{p-2}|\mathbf{D}(\mathbf{v}-\mathbf{w})|^2)^{\frac{r}{2-p+r}} \\ &\quad \times (1+|\mathbf{D}\mathbf{v}|+|\mathbf{D}(\mathbf{v}-\mathbf{w})|)^{\frac{(2-p)r}{2-p+r}}\,\mathrm{d}x \\ &\leqslant \left(\int_\Omega (\mathbf{S}(\mathbf{D}\mathbf{v},\mathbf{E}) - \mathbf{S}(\mathbf{D}\mathbf{w},\mathbf{E}))\cdot \mathbf{D}(\mathbf{v}-\mathbf{w})\,\mathrm{d}x\right)^{\frac{r}{2-p+r}} \\ &\quad \times \left(\int_\Omega (1+|\mathbf{D}\mathbf{v}|+|\mathbf{D}(\mathbf{v}-\mathbf{w})|)^r\,\mathrm{d}x\right)^{\frac{2-p}{2-p+r}},\end{aligned}$$

which immediately gives the assertion. □

Let us first discuss the error $\boldsymbol{\eta}^m$, where we can take advantage of the regularity properties for $\mathbf{v}$ and $\mathbf{V}^m$. The error $\boldsymbol{\eta}^m$ is governed by the following system, which holds for all $\boldsymbol{\varphi} \in V_p$,

$$\begin{aligned}&\langle d_t\boldsymbol{\eta}^m, \boldsymbol{\varphi}\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{v}(t_m),\mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{V}^m,\mathbf{E}(t_m)), \mathbf{D}\boldsymbol{\varphi}\rangle \\ &\qquad + \langle [\nabla\boldsymbol{\eta}^m]\mathbf{v}(t_m), \boldsymbol{\varphi}\rangle = \langle \mathbf{R}^m, \boldsymbol{\varphi}\rangle,\end{aligned} \tag{3.47}$$

supplemented with

$$\mathbf{R}^m := d_t\mathbf{v}(t_m) - \partial_t\mathbf{v}(t_m) = -\frac{1}{k}\int_{t_{m-1}}^{t_m} (s-t_{m-1})\partial_t^2\mathbf{v}(s)\,\mathrm{d}s. \tag{3.48}$$

From (3.48) and (2.15) we compute that

$$\|\mathbf{R}^m\|_2^2 \leqslant C \sup_{s\in[t_{m-1},t_m]} \|\partial_t\mathbf{v}(s)\|_2^2, \tag{3.49}$$

$$\|\mathbf{R}^m\|^2_{(\mathbf{V}_2)^*} \leqslant Ck\int_{t_{m-1}}^{t_m} \|\partial_t^2\mathbf{v}(s)\|^2_{(\mathbf{V}_2)^*}\,\mathrm{d}s. \tag{3.50}$$

If we use $\boldsymbol{\eta}^m$ as a test function in (3.47) and sum over all iteration steps, we obtain, for $1 < r < 6(p-1)$,

$$\max_{1\leqslant m\leqslant M} \|\boldsymbol{\eta}^m\|_2^2 + k\sum_{m=1}^{M}\big(\|\mathbf{D}\boldsymbol{\eta}^m\|^2_{\frac{2r}{2-p+r}} + \|\mathbf{D}\boldsymbol{\eta}^m\|_p^2\big) \leqslant C(r)k\sum_{m=1}^{M}\langle \mathbf{R}^m, \boldsymbol{\eta}^m\rangle, \tag{3.51}$$

where we have used Lemma 3.46 and $\mathbf{v}(t_m), \mathbf{V}^m \in l^\infty(I_k; V_r)$. We can bound the term on the right-hand side with the help of the embedding $W^{1,\frac{2r}{2-p+r}}(\Omega) \hookrightarrow W^{\frac{2r-6+3p}{2r},2}(\Omega)$ and the interpolation of $W^{\frac{2r-6+3p}{2r},2}(\Omega)$ between $W^{1,2}(\Omega)$ and $L^2(\Omega)$ as follows

$$\begin{aligned}\langle \mathbf{R}^m, \boldsymbol{\eta}^m\rangle &\leqslant \|\mathbf{R}^m\|_2^{1-\frac{2r-6+3p}{2r}}\|\mathbf{R}^m\|_{(V_2)^*}^{\frac{2r-6+3p}{2r}}\|\boldsymbol{\eta}^m\|_{V_{\frac{2r}{2-p+r}}}\\ &\leqslant C(\mathbf{f},\mathbf{v}_0)\|\mathbf{R}^m\|_{(V_2)^*}^{\frac{2r-6+3p}{r}} + \frac{1}{2}\|\mathbf{D}\boldsymbol{\eta}^m\|^2_{\frac{2r}{2-p+r}},\end{aligned} \tag{3.52}$$

where we also used Korn's and Young's inequalities and (3.49). Now, we move the last term in (3.52) to the left-hand side of (3.51) and it remains to bound the first term in (3.52). Note that

$$\frac{2r-6+3p}{2r} =: \tilde{\alpha}(p,r) \nearrow \alpha_0(p) := \frac{5p-6}{4(p-1)}, \quad \text{for} \quad r \nearrow 6(p-1). \tag{3.53}$$

From (3.50) and $(2.15)_3$ we derive

$$\begin{aligned}k\sum_{m=1}^{M}\|\mathbf{R}^m\|_{(V_2)^*}^{2\tilde{\alpha}(p,r)} &\leqslant Ck^{2\tilde{\alpha}(p,r)}\bigg(\sum_{m=1}^{M}\int_{t_{m-1}}^{t_m}\|\partial_t^2\mathbf{v}(s)\|^2_{(V_2)^*}\,\mathrm{d}s\bigg)^{\tilde{\alpha}(p,r)}\\ &\leqslant Ck^{2\tilde{\alpha}(p,r)},\end{aligned}$$

which together with (3.51), (3.52) yields

$$\max_{1\leqslant m\leqslant M}\|\boldsymbol{\eta}^m\|_2^2 + k\sum_{m=1}^{M}\|\mathbf{D}\boldsymbol{\eta}^m\|_p^2 \leqslant C(r)k^{2\tilde{\alpha}(p,r)}, \tag{3.54}$$

with $\tilde{\alpha}(p,r)$ defined in (3.53).

We still have to deal with the error $\mathbf{e}^m$, which is governed by the system

$$\langle d_t\mathbf{e}^m, \boldsymbol{\varphi}\rangle + \langle \mathbf{S}(\mathbf{D}\mathbf{V}^m, \mathbf{E}(t_m)) - \mathbf{S}(\mathbf{D}\mathbf{v}^m, \mathbf{E}(t_m)), \mathbf{D}\boldsymbol{\varphi}\rangle = \langle \mathbf{r}^m, \boldsymbol{\varphi}\rangle, \tag{3.55}$$

which holds for all $\boldsymbol{\varphi} \in V_p$, and where

$$\text{(3.56)} \qquad \begin{aligned} -\mathbf{r}^m &= [\nabla \mathbf{V}^m]\mathbf{v}(t_m) - [\nabla \mathbf{v}^m]\mathbf{v}^m \\ &= [\nabla \mathbf{V}^m]\boldsymbol{\eta}^m + [\nabla \mathbf{V}^m]\mathbf{e}^m + [\nabla \mathbf{e}^m]\mathbf{v}^m. \end{aligned}$$

If we use in (3.55) the test function $\mathbf{e}^m$ and sum over all iteration steps, we get

$$\text{(3.57)} \quad \begin{aligned} \max_{1\leqslant m\leqslant M} \|\mathbf{e}^m\|_2^2 &+ k\sum_{m=1}^M \frac{\|\mathbf{D}\mathbf{e}^m\|_p^2}{C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p}} \\ &\leqslant Ck\sum_{m=1}^M \int_\Omega |\boldsymbol{\eta}^m|\,|\mathbf{e}^m|\,|\nabla \mathbf{V}^m|\,\mathrm{d}x + Ck\sum_{m=1}^M \int_\Omega |\mathbf{e}^m|^2\,|\nabla \mathbf{V}^m|\,\mathrm{d}x \\ &=: Ck\sum_{m=1}^M (I_1^m + I_2^m). \end{aligned}$$

For the lower bound of the elliptic term we used Lemma 3.46 with $r = p$ and the uniform bound for $\nabla \mathbf{V}^m \in l^\infty(I_k; L^p(\Omega))$. With the help of Hölder's inequality, the interpolation inequality

$$\|\mathbf{v}\|_{2r'} \leqslant \|\mathbf{v}\|_2^{1-\lambda}\|\nabla \mathbf{v}\|_p^\lambda$$

with $\lambda = \frac{3p}{r(5p-6)}$ and $\nabla \mathbf{V}^m \in l^\infty(I_k; L^r(\Omega))$, $\frac{3p}{5p-6} < r < 6(p-1)$, we find that the term I_1^m is bounded by

$$\text{(3.58)} \quad \begin{aligned} &\|\nabla \mathbf{V}^m\|_r\,\|\mathbf{e}^m\|_{2r'}\|\boldsymbol{\eta}^m\|_{2r'} \\ &\leqslant C\|\boldsymbol{\eta}^m\|_2^{1-\lambda}\|\nabla\boldsymbol{\eta}^m\|_p^\lambda\,\|\mathbf{e}^m\|_2^{1-\lambda}\frac{\|\mathbf{D}\mathbf{e}^m\|_p^\lambda}{(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^{\lambda/2}}(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^{\lambda/2} \\ &\leqslant C\|\mathbf{e}^m\|_2\,\|\boldsymbol{\eta}^m\|_2(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^{\frac{\lambda}{2(1-\lambda)}} + \frac{\frac{1}{2}\|\mathbf{D}\mathbf{e}^m\|_p}{(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^{1/2}}\|\mathbf{D}\boldsymbol{\eta}^m\|_p \\ &\leqslant C\|\boldsymbol{\eta}^m\|_2^2 + C(1 + \|\mathbf{D}\mathbf{e}^m\|_p^p)^{\frac{(2-p)\lambda}{p(1-\lambda)}}\|\mathbf{e}^m\|_2^2 + C\|\mathbf{D}\boldsymbol{\eta}^m\|_p^2 + \frac{\frac{1}{2}\|\mathbf{D}\mathbf{e}^m\|_p^2}{C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p}}. \end{aligned}$$

The last term on the right-hand side is absorbed into the left-hand side of (3.57). For the first term and the third term in the last line of (3.58) we use estimate (3.54). The term I_2^m is easier. We get

$$\text{(3.59)} \quad \begin{aligned} \|\nabla \mathbf{V}^m\|_r\|\mathbf{e}^m\|_{2r'}^2 &\leqslant C\|\mathbf{e}^m\|_2^{2(1-\lambda)}\frac{\|\mathbf{D}\mathbf{e}^m\|_p^{2\lambda}}{(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^\lambda}(C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p})^\lambda \\ &\leqslant C(1 + \|\mathbf{D}\mathbf{e}^m\|_p^p)^{\frac{(2-p)\lambda}{p(1-\lambda)}}\|\mathbf{e}^m\|_2^2 + \frac{1}{2}\frac{\|\mathbf{D}\mathbf{e}^m\|_p^2}{C + \|\mathbf{D}\mathbf{e}^m\|_p^{2-p}}. \end{aligned}$$

Thus we arrive at

$$
\max_{1\leqslant m\leqslant M}\|\mathbf{e}^m\|_2^2 + k\sum_{m=1}^{M}\frac{\|\mathbf{D}\mathbf{e}^m\|_p^2}{C+\|\mathbf{D}\mathbf{e}^m\|_p^{2-p}} \tag{3.60}
$$

$$
\leqslant Ck^{2\tilde{\alpha}(p,r)} + k\sum_{m=1}^{M}(C+\|\mathbf{D}\mathbf{e}^m\|_p^p)^{\frac{2-p}{p}\frac{\lambda}{1-\lambda}}\|\mathbf{e}^m\|_2^2
$$

and we can use the discrete Gronwall's lemma whenever $\frac{2-p}{p}\frac{\lambda}{1-\lambda} < 1$, where $\lambda = \frac{3p}{r(5p-6)}$, $1 < r < 6(p-1)$. One easily computes that this requirement is equivalent to $p > \frac{11+\sqrt{21}}{10}$. After the application of the discrete Gronwall's lemma we obtain that the left-hand side of (3.60) is bounded by $Ck^{2\tilde{\alpha}(p,r)}$, with $\tilde{\alpha}(p,r)$ given by (3.53). We can always choose r such that $2\tilde{\alpha}(p,r) > 1$ and we readily obtain that

$$
\max_{1\leqslant m\leqslant M}\|\mathbf{D}\mathbf{e}^m\|_p^2 \leqslant C
$$

and in turn we derive

$$
\max_{1\leqslant m\leqslant M}\|\mathbf{e}^m\|_2^2 + k\sum_{m=1}^{M}\|\mathbf{D}\mathbf{e}^m\|_p^2 \leqslant C(r)k^{\tilde{\alpha}(p,r)}. \tag{3.61}
$$

Since the same estimates hold for $\boldsymbol{\eta}^m$ we have furnished the proof of Theorem 3.6.

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References

[1] *B. Abu-Jdayil, P. O. Brunn*: Effects of nonuniform electric field on slit flow of an electrorheological fluid. J. Rheol. *39* (1995), 1327–1341.

[2] *B. Abu-Jdayil, P. O. Brunn*: Effects of electrode morphology on the slit flow of an electrorheological fluid. J. Non-Newtonian Fluid Mech. *63* (1966), 45–61.

[3] *B. Abu-Jdayil, P. O. Brunn*: Study of the flow behaviour of electrorheological fluids at shear- and flow- mode. Chem. Eng. and Proc. *36* (1997), 281–289.

[4] *W. Bao, J. W. Barrett*: A priori and a posteriori error bounds for a nonconforming linear finite element approximation of a non-Newtonian flow. RAIRO Modél. Math. Anal. Numér. *32* (1998), 843–858.

[5] *J. Baranger, K. Najib, and D. Sandri*: Numerical analysis of a three-fields model for a quasi-Newtonian flow. Comput. Methods Appl. Mech. Engrg. *109* (1993), 281–292.

[6] *H. Bellout, F. Bloom, and J. Nečas*: Young measure-valued solutions for non-Newtonian incompressible fluids. Comm. Partial Differential Equations *19* (1994), 1763–1803.

[7] *R. Bloodworth*: Electrorgeological fluids based on polyurethane dispersions. In: Electrorheological Fluids (R. Tao, G. D. Roy, eds.). World Scientific, 1994, pp. 67–83.
[8] *R. Bloodworth, E. Wendt*: Materials for ER-fluids. Int. J. Mod. Phys. B *23/24* (1996), 2951–2964.
[9] *B. D. Coleman, W. Noll*: The thermodynamics of elastic materials with heat conduction and viscosity. Arch. Rational Mech. Anal. *13* (1963), 167–178.
[10] *L. Diening*: Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. Math. Inequ. Appl. *7* (2004), 245–253. Preprint 2002-02, University Freiburg.
[11] *L. Diening*: Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. Math. Nachr. *268* (2004), 31–43.
[12] *L. Diening*: Theoretical and numerical results for electrorheological fluids. PhD. thesis. University Freiburg, 2002.
[13] *L. Diening, A. Prohl, and M. Růžička*: On time discretizations for generalized Newtonian fluids. In: Nonlinear Problems in Mathematical Physics and Related Topics II. In honour of Professor O. A. Ladyzhenskaya (M. Sh. Birman, S. Hildebrandt, V. Solonnikov, and N. N. Uraltseva, eds.). Kluwer/Plenum, New York, 2002, pp. 89–118.
[14] *L. Diening, M. Růžička*: Strong solutions for generalized Newtonian fluids. J. Math. Fluid. Mech. Accepted. Preprint 2003-8, University Freiburg.
[15] *L. Diening, M. Růžička*: Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. J. Reine Angew. Math. *563* (2003), 197–220.
[16] *L. Diening, M. Růžička*: Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, Part I. J. Math. Anal. Appl. (2004), 559–571.
[17] *L. Diening, M. Růžička*: Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, Part II. J. Math. Anal. Appl. (2004), 572–588.
[18] *W. Eckart*: Theoretische Untersuchungen von elektrorheologischen Flüssigkeiten bei homogenen und inhomogenen elektrischen Feldern. Shaker Verlag, Aachen, 2000.
[19] *W. Eckart, M. Růžička*: Modeling micropolar electrorheological fluids. Accepted. Preprint 2003-11, University Freiburg.
[20] *A. C. Eringen, G. Maugin*: Electrodynamics of Continua, Vol. I and II. Springer-Verlag, New York, 1989.
[21] *J. Frehse, J. Málek*: Problems due to the no-slip boundary in incompressible fluid dynamics. In: Geometric Analysis and Nonlinear Partial Differential Equations. Springer-Verlag, Berlin, 2003, pp. 559–571.
[22] *J. Frehse, J. Málek, and M. Steinhauer*: An existence result for fluids with shear dependent viscosity—steady flows. Nonlinear Anal. *30* (1997), 3041–3049.
[23] *M. Giaquinta, G. Modica, and J. Souček*: Cartesian currents in the calculus of variations. II. Variational integrals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Vol. 38. Springer-Verlag, Berlin, 1998.
[24] *E. Giusti*: Direct Methods in the Calculus of Variations. Unione Matematica Italiana, Bologna, 1994. (In Italian.)
[25] *R. A. Grot*: Relativistic continuum physics: Electromagnetic interactions. In: Continuum Physics (A. C. Eringen, ed.). Academic Press, 1976, pp. 130–221.
[26] *T. C. Halsey, J. E. Martin, and D. Adolf*: Rheology of Electrorheological Fluids. Phys. Rev. Letters *68* (1992), 1519–1522.
[27] *K. Hutter, A. A. F. van de Ven*: Field Matter Interactions in Thermoelastic Solids. Lecture Notes in Physics, Vol. 88. Springer-Verlag, Berlin, 1978.
[28] *O. Kováčik, J. Rákosník*: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslovak Math. J. *41* (1991), 592–618.

[29] *J. L. Lions*: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Dunod, Paris, 1969. (In French.)
[30] *J. Málek, J. Nečas, M. Rokyta, and M. Růžička*: Weak and Measure-Valued Solutions to Evolutionary PDEs. Applied Mathematics and Mathematical Computations, Vol. 13. Chapman & Hall, London, 1996.
[31] *J. Málek, J. Nečas, and M. Růžička*: On the non-Newtonian incompressible fluids. Math. Models Methods Appl. Sci. *3* (1993), 35–63.
[32] *J. Málek, J. Nečas, and M. Růžička*: On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains. The case $p \geqslant 2$. Adv. Differential Equations *6* (2001), 257–302.
[33] *J. Málek, K. R. Rajagopal, and M. Růžička*: Existence and regularity of solutions and the stability of the rest state for fluids with shear dependent viscosity. Math. Models Methods Appl. Sci. *5* (1995), 789–812.
[34] *A. Milani, R. Picard*: Decomposition theorems and their application to non-linear electro- and magneto-static boundary value problems. Lecture Notes in Math., Vol. 1357. Springer-Verlag, 1988, pp. 317–340.
[35] *Y. H. Pao*: Electromagnetic forces in deformable continua. Mechanics Today, Vol. 4 (S. Nemat-Nasser, ed.). Pergamon Press, 1978, pp. 209–306.
[36] *M. Parthasarathy, D. J. Klingenberg*: Mechanism and models. Materials, Sciences and Engineering *R17* (1966), 57–103.
[37] *A. Prohl, M. Růžička*: On fully implicit space-time discretization for motions of incompressible fluids with shear dependent viscosities: The case $p \leqslant 2$. SIAM J. Numer. Anal. *39* (2001), 241–249.
[38] *K. R. Rajagopal, M. Růžička*: On the modelling of electrorheological materials. Mech. Research Comm. *23* (1996), 401–407.
[39] *K. R. Rajagopal, M. Růžička*: Mathematical modelling of electrorheological materials. Cont. Mech. and Thermodynamics *13* (2001), 59–78.
[40] Helsinki research group on variable exponent Lebesgue and Sobolev spaces. http://www.math.helsinki.fi/analysis/varsobgroup/.
[41] *M. Růžička*: A note on steady flow of fluids with shear dependent viscosity. Proceedings of the Second World Congress of Nonlinear Analysts (Athens, 1996). Nonlinear Anal. *30* (1997), 3029–3039.
[42] *M. Růžička*: Flow of shear dependent electrorheological fluids: Unsteady space periodic case. In: Applied Nonlinear Analysis (A. Sequeira, ed.). Kluwer/Plenum, New York, 1999, pp. 485–504.
[43] *M. Růžička*: Electrorheological fluids: Modeling and mathematical theory. RIMS Kokyuroku *1146* (2000), 16–38.
[44] *M. Růžička*: Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics, Vol. 1748. Springer-Verlag, Berlin, 2000.
[45] *C. Truesdell, W. Noll*: The Non-Linear Field Theories of Mechanics. Handbuch der Physik, Vol. III/3. Springer-Verlag, New York, 1965.
[46] *T. Wunderlich*: Der Einfluß der Elektrodenoberfläche und der Strömungsform auf den elektrorheologischen Effekt. PhD. thesis. University Erlangen-Nürnberg, 2000.
[47] *T. Wunderlich, P. O. Brunn*: Pressure drop measurements inside a flat channel—with flush mounted and protruding electrodes of varable length—using an electrorheological fluid. Experiments in Fluids *28* (2000), 455–461.

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