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SOME NEW REGULARITY CRITERIA FOR THE NAVIER-STOKES EQUATIONS CONTAINING GRADIENT OF THE VELOCITY*

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Abstract. We study the nonstationary Navier-Stokes equations in the entire three-dimensional space and give some criteria on certain components of gradient of the velocity which ensure its global-in-time smoothness.

Keywords: Navier-Stokes equations, regularity of systems of PDE's

MSC 2000: 35Q35, 76D05

1. INTRODUCTION

Consider the three-dimensional Cauchy problem for the Navier-Stokes equations, i.e. the system of PDE's (as the numerical values of the constant viscosity and the constant density do not play any role here, they are assumed to be equal to 1)

$$
\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla p = \mathbf{0} \\ \operatorname{div}\mathbf{u} = 0 \end{aligned}\right\} \text{ in } (0,T)\times\mathbb{R}^3 \tag{1.1}
$$
$$
\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3,
$$

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where $\mathbf{u}$: $(0,T)\times\mathbb{R}^3\to\mathbb{R}^3$ is the velocity field, p: $(0,T)\times\mathbb{R}^3\to\mathbb{R}$ is the pressure, $0<T\leqslant\infty$, $\mathbf{u}_0$: $\mathbb{R}^3\to\mathbb{R}^3$ with $\operatorname{div}\mathbf{u}_0=0$ is the initial velocity. For simplicity, the external force is taken to be zero.

It is well known that for $\mathbf{u}_0\in L^2(\mathbb{R}^3)$ with $\operatorname{div}\mathbf{u}_0=0$ there exists at least one weak solution (see [7] or also [5] for other types of domains). Nevertheless, the fundamental question of the uniqueness and regularity of such solutions is still open. On the other hand, there are many criteria which ensure that the weak solution is a strong one and thus unique in the class of all weak solutions satisfying the energy inequality. Let us summarize here some of them

- $\mathbf{u}\in L^t(I;L^s)$, $2/t+3/s\leqslant 1$, $2\leqslant t\leqslant\infty$, $3\leqslant s\leqslant\infty$ (see [16], for the case $s=3$ see [14], [4])
- $u_3\in L^t(I;L^s)$, $2/t+3/s\leqslant\frac{1}{2}$, $4\leqslant t\leqslant\infty$, $6<s\leqslant\infty$ (see [9])
- $u_3\in L^{t_1}(I;L^{s_1})$, $u_1,u_2\in L^{t_2}(I;L^{s_2})$,
  $2\leqslant s_2,t_2\leqslant\infty$
  $2\leqslant t_1\leqslant\infty$, $3<s_1\leqslant\infty$, $2/t_1+3/s_1\leqslant 1$
  $(2/t_2+3/s_2)+(2/t_1+3/s_1)\leqslant 2$
  $2/t_1+2/t_2\leqslant 1$, $2/s_1+2/s_2<1$

(see [10]; the proofs in [9] and [10] are done for the suitable weak solutions as local regularity criteria; nevertheless one can easily transform the proofs for the Cauchy problem to get global regularity criteria)

- $\omega_1,\omega_2\in L^t(I;L^s)$, $2/t+3/s\leqslant 2$, $1<t\leqslant\infty$, $\frac{3}{2}<s<\infty$ (see [2])
  (We denote by ω_i the ith component of the vorticity.)
- $\nabla v_1,\nabla v_2\in L^t(I;L^s)$, $2/t+3/s\leqslant 1$, $2\leqslant t\leqslant\infty$, $3\leqslant s\leqslant\infty$ (see [2])
- $p\in L^t(I;L^s)$, $2/t+3/s\leqslant 2$, $1\leqslant t\leqslant\infty$, $\frac{3}{2}<s\leqslant\infty$ (see [3])
- $\nabla u_3\in L^t(I;L^s)$, $2/t+3/s\leqslant\frac{3}{2}$, $\frac{4}{3}\leqslant t\leqslant\infty$, $2\leqslant s\leqslant\infty$ (see [12], independently also [18])
- p_- bounded from below, see [15]
  (By p_- we understand the negative part of the pressure.)
- $p_-\in L^{t_1}(I;L^{s_1}(U))$, $2/t_1+3/s_1\leqslant 2$, $1<t_1\leqslant\infty$, $\frac{3}{2}<s_1\leqslant\infty$ and
  $\mathbf{u}\in L^{t_2}(I;L^{s_2}(V))$, $2/t_2+3/s_2\leqslant 1$, $3\leqslant t_2\leqslant\infty$, $3<s_2\leqslant\infty$ with
  $U=\{(\mathbf{x},t)\in Q_T;\ t_0-r^2/\varrho^2<t<t_0,\ \varrho\sqrt{t_0-t}<|\mathbf{x}-\mathbf{x}_0|<r\}$,
  $V=\{(\mathbf{x},t)\in Q_T;\ t_0-r^2/\varrho^2<t<t_0,\ |\mathbf{x}-\mathbf{x}_0|<\varrho\sqrt{t_0-t}\}$ (see [8])[1].

V. Scheffer investigated in [13] for the first time partial regularity of weak solutions and studied the Hausdorff dimension of the set of their possible singularities. His approach, later on adapted by [1], forms the basic idea of the regularity criteria in [8], [9] and [10].

[1] This implies that the point $(\mathbf{x}_0,t_0)$ is a regular point; it is not obvious how to transform this local regularity criterion into a global one.

In what follows, we denote by $L^p(\mathbb{R}^3)$ the Lebesgue spaces, $1 \leqslant p \leqslant \infty$, by $W^{k,p}(\mathbb{R}^3)$ the Sobolev spaces for $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, both endowed with the standard norms $\|\cdot\|_{p,\mathbb{R}^3}$ and $\|\cdot\|_{k,p,\mathbb{R}^3}$, respectively. The anisotropic Lebesgue spaces $L^t(0,T;L^s(\mathbb{R}^3))$ will be denoted, for brevity, by $L^{t,s}(Q_T)$, $1 \leqslant t,s \leqslant \infty$, $Q_T = (0,T) \times \mathbb{R}^3$. If no misunderstanding can occur we will omit writing Q_T and $\mathbb{R}^3$, respectively.

All generic constants will be denoted by C. Their values can vary, even on the same line or in the same formula.

We will also use the summation convention; unless otherwise stated, the summation over repeated indices will be used, from 1 to 3.

2. Main theorems

The main goal is to prove the following four theorems.

Theorem 1. *Let* $\mathbf{u}$ *be a weak solution to the Navier-Stokes equations* (1.1) *corresponding to the initial condition* $\mathbf{u}_0 \in W^{1,2}$ *with* $\operatorname{div} \mathbf{u}_0 = 0$ *such that* $\mathbf{u}$ *satisfies the energy inequality. Moreover let* $u_3 \in L^{t_1,s_1}$, $2/t_1+3/s_1 \leqslant 1$, $2 \leqslant t_1 \leqslant \infty$, $3 < s_1 \leqslant \infty$ *and one of the following conditions holds true*

(a) $\partial u_1/\partial x_3$, $\partial u_2/\partial x_3$ *belong to* L^{t_2,s_2} *with* $2/t_2 + 3/s_2 \leqslant 2$, $1 \leqslant t_2 \leqslant \infty$, $\frac{3}{2} < s_2 \leqslant \infty$,

(b) $\partial u_1/\partial x_2$, $\partial u_2/\partial x_1$ *belong to* L^{t_3,s_3} *with* $2/t_3+3/s_3 \leqslant 2$, $2 \leqslant t_3 \leqslant \infty$, $2 \leqslant s_3 \leqslant 3$,

(c) $\partial u_2/\partial x_3 \in L^{t_4,s_4}$, $\partial u_1/\partial x_2 \in L^{t_5,s_5}$, $2/t_i + 3/s_i \leqslant 2$, $i = 4,5$, $1 \leqslant t_4 \leqslant \infty$, $\frac{3}{2} < s_4 \leqslant \infty$, $2 \leqslant t_5 \leqslant \infty$, $2 \leqslant s_5 \leqslant 3$.

Then $(\mathbf{u}, p)$ *with* p *the corresponding pressure is the strong solution to the Navier-Stokes equations which is unique in the class of all weak solutions satisfying the energy inequality.*

Remark 1. Note that in (b) it might be interesting to replace the conditions on $\partial u_1/\partial x_2$ and $\partial u_2/\partial x_1$ by the same condition on ω_3. Unfortunately, this does not seem to be possible, at least by the present technique.

Remark 2. In part (a) we can replace the assumptions on $\partial u_1/\partial x_3$, $\partial u_2/\partial x_3$ by analogous assumptions on $\partial u_2/\partial x_3$, $\partial u_2/\partial x_2$, or $\partial u_2/\partial x_3$, $\partial u_1/\partial x_1$, or $\partial u_1/\partial x_3$, $\partial u_2/\partial x_2$, or $\partial u_1/\partial x_3$, $\partial u_1/\partial x_1$. Similarly, instead of (c), we can assume $\partial u_1/\partial x_3 \in L^{t_4,s_4}$, $\partial u_2/\partial x_1 \in L^{t_5,s_5}$.

Remark 3. It will be clear from the proof why s_3 and s_5 satisfy more restrictive conditions than s_2 and s_4. For s_3 and $s_5 > 3$ or from $(\frac{18}{11}, 2)$ we can still obtain some conditions implying the regularity; however these conditions are more restrictive, i.e. they do not lie on the same scale as those in Theorem 1; see the note at the end of Step 3 (ii) in the proof of Theorem 1 below.

Remark 4. The limit cases, i.e. in (a) $u_3 \in L^{\infty,3}$, in (b) $s_2 = \frac{3}{2}$, $t_2 = \infty$ and in (c) $s_4 = \frac{3}{2}$, $t_4 = \infty$ do not imply the regularity. We have to add the assumption that the above mentioned norms are sufficiently small. The same holds also for the limit case in Theorem 3 below.

In the following Theorems 2–4 we assume similarly as in Theorem 1 that $\mathbf{u}$ is a weak solution to the Navier-Stokes equations (1.1) corresponding to the initial condition $\mathbf{u}_0 \in W^{1,2}$ with $\operatorname{div} \mathbf{u}_0 = 0$ such that $\mathbf{u}$ satisfies the energy inequality.

Theorem 2. *Let $\partial u_3/\partial x_3 \in L^{\infty,\infty}$. Then $(\mathbf{u}, p)$ with p the corresponding pressure is the strong solution to the Navier-Stokes equations which is unique in the class of all weak solutions satisfying the energy inequality.*

Theorem 3. *Let $\partial u_3/\partial x_3, \partial u_2/\partial x_2 \in L^{t_1,s_1}$, $2/t_1 + 3/s_1 \leqslant 2$, $1 \leqslant t_1 \leqslant \infty$, $\frac{3}{2} < s_1 \leqslant \infty$. Then $(\mathbf{u}, p)$ with p the corresponding pressure is the strong solution to the Navier-Stokes equations which is unique in the class of all weak solutions satisfying the energy inequality.*

Theorem 4. *Let one of the following conditions be satisfied*

(i) $\partial \mathbf{u}/\partial x_3 \in L^{t_1,s_1}$, $2/t_1 + 3/s_1 \leqslant \frac{3}{2}$, $\frac{4}{3} \leqslant t_1 \leqslant \infty$, $2 \leqslant s_1 \leqslant \infty$, *or*

(ii) $\partial u_3/\partial x_3 \in L^{t_2,s_2}$, $2/t_2 + 3/s_2 \leqslant 1$, $2 \leqslant t_2 \leqslant \infty$, $3 \leqslant s_2 \leqslant \infty$ *and* $\partial u_i/\partial x_3 \in L^{t_3,s_3}$, $2/t_3 + 3/s_3 \leqslant 2$, $1 \leqslant t_3 \leqslant \infty$, $\frac{3}{2} < s_3 \leqslant \infty$, $i = 1, 2$.

Then $(\mathbf{u}, p)$ with p the corresponding pressure is the strong solution to the Navier-Stokes equations which is unique in the class of all weak solutions satisfying the energy inequality.

Remark 5. Note that the regularity assumption in Theorem 2 can be written as $\partial u_3/\partial x_3 \in L^{t,s}$ with $2/t + 3/s = 0$.

Remark 6. Comparing results from [2] with any of the results from Theorem 3–4, we see that we require here less in the sense that we need only three (or two) components of the gradient to satisfy less restrictive conditions than in the above cited paper.

Remark 7. Let us also note that, even though we consider here the right-hand side of the Navier-Stokes equations to be zero, similar results as presented in Theorems 1–4 hold also if some $\mathbf{f} \neq \mathbf{0}$ appears in the right-hand side; only the smoothness of the solution depends on the smoothness of $\mathbf{f}$.

For a moment, let $(\mathbf{u}, p)$ be a smooth solution to the Navier-Stokes equations such that $\mathbf{u} \in L^2(0,T;W^{k,2})$, $\mathbf{u}_t \in L^2(0,T;W^{k-2,2})$, $k \geqslant 3$. Then we have the following equation for the pressure

$$-\Delta p = \operatorname{div}\operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \quad \text{in } (0,T) \times \mathbb{R}^3 \tag{3.1}$$

and thus

Lemma 1. *The following estimates for the pressure hold true*

$$\|p\|_q(t) \leqslant C\|\mathbf{u}\|_{2q}^2(t),$$
$$\left\|\frac{\partial p}{\partial x_i}\right\|_q(t) \leqslant C \sum_{j,k=1}^{3} \left\|\frac{\partial(u_j u_k)}{\partial x_i}\right\|_q(t)$$

for $1 < q < \infty$.

Proof. This is an easy consequence of equation (3.1), standard L^q estimates for the Laplace equation in the entire space (i.e. the Marcinkiewicz multiplier theorem, see e.g. [17]) and the fact that $\nabla p(t) \in L^2$. □

Next, let us consider our weak solution to the Navier-Stokes equations as given in Theorems 1–4. As $\mathbf{u}_0 \in W^{1,2}$, we know (see [6]) that there is $t_0 > 0$ such that there exists a smooth solution to the Navier-Stokes equations on $(0, t_0)$ corresponding to the initial condition $\mathbf{u}_0$. Moreover, since this solution is unique in the class of all weak solutions satisfying the energy inequality, it coincides with "our" weak solution on this time interval. Denote by t^* the supremum of all $\bar{t} > 0$ such that on $(0, \bar{t})$ there is a smooth solution to the Navier-Stokes equations. Note that $t^* > 0$. Assume now $t^* < \infty$. Evidently on any compact subinterval of $(0, t^*)$ "our" weak solution coincides with this smooth solution (and it is, due to the absence of the right-hand side, $C^\infty([\delta, t^* - \delta] \times \mathbb{R}^3)$, $0 < \delta < t^*$).

If we show that some norm of $\mathbf{u}$ (or $\nabla \mathbf{u}$), sufficient to ensure the smoothness of the Navier-Stokes equations, remains bounded independently of t as $t \to t^*$, we can extend our solution (due to the result from [6]) after the time instant t^* which would contradict the definition of t^* and thus $t^* = \infty$. In the following sections we will show such estimates. We will always work on some subintervals of $(0, t^*)$ and thus all equations will be satisfied pointwise. Before starting with these estimates let us recall some useful inequalities. We have (for the proof see [11])

Lemma 2. *Let h be a function such that $h \in L^q$ and $\nabla h \in L^s$, $s \in [1,\infty]$, $r \geqslant q$ and $r \leqslant \infty$ if $s > 3$, $r < \infty$ if $s = 3$ and $r \leqslant 3s(3-s)^{-1}$ if $s < 3$. Then there exists a constant C such that*

$$\|h\|_r \leqslant C\|\nabla h\|_s^a \|h\|_q^{1-a}, \quad a \in [0,1],$$

where $1/r = a(1/s - \frac{1}{3}) + (1-a)1/q$.

Recall also that if $\operatorname{div} \mathbf{u} = 0$ then

$$C_1 \|\operatorname{curl} \mathbf{u}\|_q \leqslant \|\nabla \mathbf{u}\|_q \leqslant C_2(q)\|\operatorname{curl} \mathbf{u}\|_q, \tag{3.2}$$

$1 < q < \infty$ (and C_1 remains bounded if $q \to 1$ or $q \to \infty$ while $C_2(q) \to \infty$ in this case).

4. Proof of Theorem 1

We will proceed in several steps:

Step 1: Estimates of the vorticity

Let us recall that $\omega = \operatorname{curl} \mathbf{u}$ satisfies the following system

$$\frac{\partial \omega}{\partial t} - \Delta\omega + \mathbf{u}\cdot\nabla\omega - \omega\cdot\nabla\mathbf{u} = \mathbf{0} \quad \text{in } (0,T)\times\mathbb{R}^3$$
$$\omega(0,\mathbf{x}) = \operatorname{curl}\mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3.$$

Multiply the equation by ω and integrate over $\mathbb{R}^3$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 = \int_{\mathbb{R}^3} \omega_i \frac{\partial u_j}{\partial x_i}\omega_j.$$

If $j = 3$ then

$$\int_{\mathbb{R}^3} \omega_i \frac{\partial u_3}{\partial x_i}\omega_3 = -\int_{\mathbb{R}^3} u_3\omega_i \frac{\partial \omega_3}{\partial x_i}$$

and recalling that $\omega_i = \varepsilon_{ijk}\partial u_k/\partial x_j$ (ε_{ijk} is the Levi-Cività skew-symmetric tensor) we get

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3}\omega_i\frac{\partial u_j}{\partial x_i}\omega_j &= \int_{\mathbb{R}^3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_1} - \int_{\mathbb{R}^3}\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3}\\ &+ \int_{\mathbb{R}^3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_2} - \int_{\mathbb{R}^3}\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3}\\ &+ \int_{\mathbb{R}^3} c_{ijklm}u_3\frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l\partial x_m}\end{aligned}$$

with c_{ijklm} a constant matrix. Thus

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 = \int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_1} - \int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3}$$
$$+ \int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_2} - \int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3}$$
$$+ \int_{\mathbb{R}^3} c_{ijklm} u_3 \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_m}.$$

Step 2: Estimates of u_3

Now

$$\left|\int_{\mathbb{R}^3} u_3 \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_m}\right| \leqslant \|\nabla^2 \mathbf{u}\|_2 \|u_3\|_s \|\nabla \mathbf{u}\|_{2s(s-2)^{-1}} \quad \text{(by means of (3.2))}$$
$$\leqslant C\|\nabla\omega\|_2^{(s+3)s^{-1}} \|\omega\|_2^{(s-3)s^{-1}} \|u_3\|_s$$
$$\leqslant \frac{1}{C}\|\nabla\omega\|_2^2 + C\|\omega\|_2^2 \|u_3\|_s^{2s(s-3)^{-1}},$$

i.e. if $u_3 \in L^{t,s}$, $2/t + 3/s \leqslant 1$, $s > 3$, we can estimate this term by putting the first term to the left-hand side and applying the Gronwall inequality to the other one; if $s = 3$ we need that the $L^{\infty,3}$ norm of u_3 is sufficiently small.

Step 3: Estimates of ∇u_i, $i = 1, 2$

(i) $\partial u_1/\partial x_3$, $\partial u_2/\partial x_3$

Evidently, using Lemma 2 the last remaining terms can be estimated as follows $(i, j, k, l = 1, 2)$

$$\left|\int_{\mathbb{R}^3} \frac{\partial u_i}{\partial x_3}\frac{\partial u_j}{\partial x_3}\frac{\partial u_k}{\partial x_l}\right| \leqslant \left\|\frac{\partial u_i}{\partial x_3}\right\|_s \|\nabla \mathbf{u}\|^2_{2s(s-1)^{-1}} \leqslant C\|\nabla\omega\|_2^{3/s}\|\omega\|_2^{(2s-3)s^{-1}}\left\|\frac{\partial u_i}{\partial x_3}\right\|_s$$
$$\leqslant \frac{1}{C}\|\nabla\omega\|_2^2 + C\|\omega\|_2^2 \left\|\frac{\partial u_i}{\partial x_3}\right\|_s^{2s(2s-3)^{-1}}$$

and if $\partial u_i/\partial x_3 \in L^{t,s}$, $2/t + 3/s \leqslant 2$ we put the first term to the left-hand side and estimate the other term by means of the Gronwall inequality. Thus part (a) with $\partial u_1/\partial x_3$, $\partial u_2/\partial x_3$ of Theorem 1 is shown. Similarly, using also the continuity equation, we can show the first part of Remark 2.

(ii) $\partial u_1/\partial x_2$, $\partial u_2/\partial x_1$

Here we have to integrate by parts in two terms. We get

$$\int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_1} + \int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_2}$$
$$= -2\int_{\mathbb{R}^3} \frac{\partial^2 u_2}{\partial x_1 \partial x_3}\frac{\partial u_2}{\partial x_3} u_1 - 2\int_{\mathbb{R}^3} \frac{\partial^2 u_1}{\partial x_2 \partial x_3}\frac{\partial u_1}{\partial x_3} u_2$$

$$= 2\int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3} + 2\int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_3}$$
$$+ 2\int_{\mathbb{R}^3} \frac{\partial^2 u_2}{\partial x_3^2}\frac{\partial u_2}{\partial x_1} u_1 + 2\int_{\mathbb{R}^3} \frac{\partial^2 u_1}{\partial x_3^2}\frac{\partial u_1}{\partial x_2} u_2.$$

The first two terms can be estimated as above. For the other two we get $(i, j = 1, 2,\ i \neq j)$

$$I = \left|\int_{\mathbb{R}^3} \frac{\partial^2 u_i}{\partial x_3^2}\frac{\partial u_i}{\partial x_j} u_j\right| \leqslant \left\|\frac{\partial u_i}{\partial x_j}\right\|_s \left\|\frac{\partial^2 u_i}{\partial x_3^2}\right\|_2 \|u_j\|_{2s(s-2)^{-1}}.$$

Now for $2 \leqslant s \leqslant 3$ (i.e. $6 \leqslant 2s(s-2)^{-1} \leqslant \infty$) we can apply Lemma 2 to get

$$I \leqslant C\left\|\frac{\partial u_i}{\partial x_j}\right\|_s \|\nabla\omega\|_2^{3/s}\|\omega\|_2^{(2s-3)s^{-1}} \leqslant \frac{1}{C}\|\nabla\omega\|_2^2 + C\left\|\partial u_i/\partial x_j\right\|_s^{2s(2s-3)^{-1}} \|\omega\|_2^2$$

and we estimate this term as above. For $s > 3$ we proceed as in [12], but the result is more restrictive ($\partial u_i/\partial x_j \in L^{6s(5s-6)^{-1},s}$, $s > 3$) or for $s < 2$ we can estimate the term by $\|\partial u_i/\partial x_j\|_2\|\nabla^2\mathbf{u}\|_2\|\mathbf{u}\|_\infty$ and interpolate the L^2-norm between L^s and L^6; we get again a more restrictive condition ($\partial u_i/\partial x_j \in L^{8s(11s-18)^{-1},s}$, $\frac{18}{11} \leqslant s \leqslant 2$).

(iii) Proof of (c)

We can combine parts (i) and (ii) to show (c) as well as the second part of Remark 2. Theorem 1 is proved. □

5. Proofs of Theorems 2–4

Proof of Theorem 2.

It is enough to show (see [9] or [10]) that $u_3 \in L^{t,s}$ for $2/t + 3/s \leqslant \frac{1}{2}$, $s > 6$. To this aim let us multiply the equation for u_3 by $|u_3|^4 u_3$ and integrate over $\mathbb{R}^3$. Then

$$\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\|u_3\|_6^6 + \frac{5}{9}\|\nabla|u_3|^3\|_2^2 = -\int \frac{\partial p}{\partial x_3}|u_3|^4 u_3 \equiv I_1.$$

Now, integrating by parts in the term on the right-hand side we obtain

$$|I_1| \leqslant C\int_{\mathbb{R}^3} |p|\left|\frac{\partial u_3}{\partial x_3}\right||u_3|^4 \leqslant \|u_3\|_6^4\left\|\frac{\partial u_3}{\partial x_3}\right\|_\infty \|\mathbf{u}\|_6^2.$$

If $\frac{\partial u_3}{\partial x_3}$ is bounded in $L^{\infty,\infty}$, we get that

$$\|u_3\|_{L^{\infty,6}} + \|\nabla|u_3|^3\|_{L^{2,2}} \leqslant C.$$

But $\|u_3\|_{L^{6,18}} \leqslant C\|\nabla|u_3|^3\|_{L^{2,2}}$ and thus Theorem 2 is shown. □

Proof of Theorem 3.

The idea is more or less the same as previously. It is enough to show that $\mathbf{u}$ is bounded in $L^{t,s}$ for $2/t + 3/s \leqslant 1$, $s \geqslant 3$. To this aim, let us multiply the ith component of the Navier-Stokes equations by $|u_i|u_i$ and integrate over $\mathbb{R}^3$. We get

$$\sum_{i=1}^{3}\Big(\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i\|_3^3 + \frac{8}{9}\|\nabla|u_i|^{\frac{3}{2}}\|_2^2\Big) = -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\frac{\partial p}{\partial x_i}|u_i|u_i \equiv I_2.$$

We integrate by parts on the right-hand side and use the continuity equation. Then

$$\begin{aligned}
|I_2| &\leqslant C\sum_{i=1}^{3}\int_{\mathbb{R}^3}|p|\Big|\frac{\partial u_i}{\partial x_i}\Big|\,|u_i| \\
&\leqslant C\sum_{i=1}^{3}\Big(\Big\|\frac{\partial u_2}{\partial x_2}\Big\|_s + \Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s\Big)\|u_i\|_{3s(s-1)^{-1}}\|\mathbf{u}\|_{3s(s-1)^{-1}}^2 \\
&\leqslant C\sum_{i=1}^{3}\Big(\Big\|\frac{\partial u_2}{\partial x_2}\Big\|_s + \Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s\Big)\|u_i\|_{3s(s-1)^{-1}}^3 \\
&\leqslant \sum_{i=1}^{3}\Big(\frac{4}{9}\|\nabla|u_i|^{\frac{3}{2}}\|_2^2 + C\Big(\Big\|\frac{\partial u_2}{\partial x_2}\Big\|_s^{2s(2s-3)^{-1}} + \Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s^{2s(2s-3)^{-1}}\Big)\|u_i\|_3^3\Big).
\end{aligned}$$

After employing the Gronwall inequality, under the assumption that $\partial u_2/\partial x_2$ and $\partial u_3/\partial x_3$ are bounded in $L^{t,s}$, $2/t + 3/s \leqslant 2$, $s > \frac{3}{2}$, we get

$$\|\mathbf{u}\|_{L^{\infty,3}} + \sum_{i=1}^{3}\|\nabla|u_i|^{\frac{3}{2}}\|_{L^{2,2}} \leqslant C$$

and thus $\mathbf{u}$ is bounded in $L^{\infty,3}$ which gives the global-in-time regularity of the solution. For $s = \frac{3}{2}$ we have to assume that the corresponding norms are sufficiently small. □

Proof of Theorem 4.

We will now use Theorem 1 part (a). Since we know that in both cases $\partial u_i/\partial x_3$, $i = 1, 2$, satisfy the assumptions of Theorem 1, it is enough to verify that $u_3 \in L^{t,s}$ for $2/t + 3/s \leqslant 1$, $s > 3$. To this aim we multiply the equation for u_3 by $|u_3|u_3$ and integrate over $\mathbb{R}^3$. Then

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|u_3\|_3^3 + \frac{8}{9}\|\nabla|u_3|^{\frac{3}{2}}\|_2^2 = -\int_{\mathbb{R}^3}\frac{\partial p}{\partial x_3}u_3|u_3| \equiv I_3.$$

Now

$$|I_3| \leqslant C\int_{\mathbb{R}^3} |p|\left|\frac{\partial u_3}{\partial x_3}\right||u_3| \leqslant C\left\|\frac{\partial u_3}{\partial x_3}\right\|_s \|u_3\|_3\|\mathbf{u}\|^2_{6s(2s-3)^{-1}}$$
$$\leqslant C\Big(\left\|\frac{\partial u_3}{\partial x_3}\right\|_s^{2s(s-3)^{-1}} + \|\mathbf{u}\|_6^2\Big)\|\mathbf{u}\|_2^{(s-3)s^{-1}}\|u_3\|_3$$

and using the Gronwall inequality we finish the proof of the case (ii) as u_3 is bounded in $L^{3,9}$.

To prove (i) we will use Lemma 1. We proceed as above but we do not integrate by parts on the right-hand side and get

(a) $s \geqslant 6$

$$|I_3| \leqslant \left\|\frac{\partial p}{\partial x_3}\right\|_3 \|u_3\|_3^2 \leqslant C\sum_{i=1}^3\left\|\frac{\partial u_i}{\partial x_3}\right\|_s \|\mathbf{u}\|_{3s(s-3)^{-1}}\|u_3\|_3^2$$
$$\leqslant C\sum_{i=1}^3 \|u_3\|_3^2\left\|\frac{\partial u_i}{\partial x_3}\right\|_s \|\mathbf{u}\|_6^{(s+6)/(2s)}\|\mathbf{u}\|_2^{(s-6)/(2s)}$$
$$\leqslant C\sum_{i=1}^3 \|u_3\|_3^2\|\mathbf{u}\|_2^{(s-6)/(2s)}\Big(\left\|\frac{\partial u_i}{\partial x_3}\right\|_s^{4s(3s-6)^{-1}} + \|\mathbf{u}\|_6^2\Big)$$

and if $\partial u_i/\partial x_3 \in L^{t,s}$, $2/t+3/s \leqslant \frac{3}{2}$, $s \geqslant 6$, we can estimate this term by means of the Gronwall inequality.

(b) $2 \leqslant s < 6$

If $2 < s < 6$ then

$$|I_3| \leqslant \left\|\frac{\partial p}{\partial x_3}\right\|_{\frac{3}{2}} \|u_3\|_6^2 \leqslant C\sum_{i=1}^3\left\|\frac{\partial u_i}{\partial x_3}\right\|_s \|\mathbf{u}\|_{3s(2s-3)^{-1}}\|u_3\|_3^{\frac{1}{2}}\|u_3\|_9^{\frac{3}{2}}$$
$$\leqslant \frac{4}{9}\|\nabla|u_3|^{\frac{3}{2}}\|_2^2 + C\sum_{i=1}^3 \|u_3\|_3\|\mathbf{u}\|_2^{(3s-6)s^{-1}}\Big(\left\|\frac{\partial u_i}{\partial x_3}\right\|_s^{4s(3s-6)^{-1}} + \|\mathbf{u}\|_6^2\Big),$$

i.e. again after employing the Gronwall inequality we get that u_3 is bounded in $L^{3,9}$ and thus the solution is smooth. Similarly we proceed for $s = 2$. Theorem 4 is proved. □

Remark 8. Note that in part (ii) we could replace the assumption on $\partial u_1/\partial x_3$ and $\partial u_2/\partial x_3$ by any assumption from Theorem 1 (a), (b), (c) or from Remark 2. But these results seem to be less interesting. Namely, we interpret the results of Theorem 4 as follows. If we control the flow in the "additional" third dimension, we get the regularity; this is in accordance with the expectation since in two space dimensions any weak solution is a strong one provided the data are smooth enough.

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SOME NEW REGULARITY CRITERIA FOR THE NAVIER-STOKES EQUATIONS CONTAINING GRADIENT OF THE VELOCITY*

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Abstract. We study the nonstationary Navier-Stokes equations in the entire three-dimensional space and give some criteria on certain components of gradient of the velocity which ensure its global-in-time smoothness.

Keywords: Navier-Stokes equations, regularity of systems of PDE's

MSC 2000: 35Q35, 76D05

1. Introduction

Consider the three-dimensional Cauchy problem for the Navier-Stokes equations, i.e. the system of PDE's (as the numerical values of the constant viscosity and the constant density do not play any role here, they are assumed to be equal to 1)

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla p = \mathbf{0} \\ \operatorname{div}\mathbf{u} = 0 \end{aligned}\right\} \text{ in } (0,T)\times\mathbb{R}^3 \tag{1.1}$$

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3,$$

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where $\mathbf{u}\colon (0,T)\times\mathbb{R}^3\to\mathbb{R}^3$ is the velocity field, $p\colon (0,T)\times\mathbb{R}^3\to\mathbb{R}$ is the pressure, $0<T\leqslant\infty$, $\mathbf{u}_0\colon \mathbb{R}^3\to\mathbb{R}^3$ with $\operatorname{div}\mathbf{u}_0=0$ is the initial velocity. For simplicity, the external force is taken to be zero.

It is well known that for $\mathbf{u}_0\in L^2(\mathbb{R}^3)$ with $\operatorname{div}\mathbf{u}_0=0$ there exists at least one weak solution (see [7] or also [5] for other types of domains). Nevertheless, the fundamental question of the uniqueness and regularity of such solutions is still open. On the other hand, there are many criteria which ensure that the weak solution is a strong one and thus unique in the class of all weak solutions satisfying the energy inequality. Let us summarize here some of them

- $\mathbf{u}\in L^t(I;L^s)$, $2/t+3/s\leqslant 1$, $2\leqslant t\leqslant\infty$, $3\leqslant s\leqslant\infty$ (see [16], for the case $s=3$ see [14], [4])
- $u_3\in L^t(I;L^s)$, $2/t+3/s\leqslant\frac12$, $4\leqslant t\leqslant\infty$, $6<s\leqslant\infty$ (see [9])
- $u_3\in L^{t_1}(I;L^{s_1})$, $u_1,u_2\in L^{t_2}(I;L^{s_2})$,
  $2\leqslant s_2,t_2\leqslant\infty$
  $2\leqslant t_1\leqslant\infty$, $3<s_1\leqslant\infty$, $2/t_1+3/s_1\leqslant 1$
  $(2/t_2+3/s_2)+(2/t_1+3/s_1)\leqslant 2$
  $2/t_1+2/t_2\leqslant 1$, $2/s_1+2/s_2<1$

(see [10]; the proofs in [9] and [10] are done for the suitable weak solutions as local regularity criteria; nevertheless one can easily transform the proofs for the Cauchy problem to get global regularity criteria)

- $\omega_1,\omega_2\in L^t(I;L^s)$, $2/t+3/s\leqslant 2$, $1<t\leqslant\infty$, $\frac32<s<\infty$ (see [2])
  (We denote by ω_i the ith component of the vorticity.)
- $\nabla v_1,\nabla v_2\in L^t(I;L^s)$, $2/t+3/s\leqslant 1$, $2\leqslant t\leqslant\infty$, $3\leqslant s\leqslant\infty$ (see [2])
- $p\in L^t(I;L^s)$, $2/t+3/s\leqslant 2$, $1\leqslant t\leqslant\infty$, $\frac32<s\leqslant\infty$ (see [3])
- $\nabla u_3\in L^t(I;L^s)$, $2/t+3/s\leqslant\frac32$, $\frac43\leqslant t\leqslant\infty$, $2\leqslant s\leqslant\infty$ (see [12], independently also [18])
- p_- bounded from below, see [15]
  (By p_- we understand the negative part of the pressure.)
- $p_-\in L^{t_1}(I;L^{s_1}(U))$, $2/t_1+3/s_1\leqslant 2$, $1<t_1\leqslant\infty$, $\frac32<s_1\leqslant\infty$ and
  $\mathbf{u}\in L^{t_2}(I;L^{s_2}(V))$, $2/t_2+3/s_2\leqslant 1$, $3\leqslant t_2\leqslant\infty$, $3<s_2\leqslant\infty$ with
  $U=\{(\mathbf{x},t)\in Q_T;\ t_0-r^2/\varrho^2<t<t_0,\ \varrho\sqrt{t_0-t}<|\mathbf{x}-\mathbf{x}_0|<r\}$,
  $V=\{(\mathbf{x},t)\in Q_T;\ t_0-r^2/\varrho^2<t<t_0,\ |\mathbf{x}-\mathbf{x}_0|<\varrho\sqrt{t_0-t}\}$ (see [8])[1].

V. Scheffer investigated in [13] for the first time partial regularity of weak solutions and studied the Hausdorff dimension of the set of their possible singularities. His approach, later on adapted by [1], forms the basic idea of the regularity criteria in [8], [9] and [10].

[1] This implies that the point $(\mathbf{x}_0,t_0)$ is a regular point; it is not obvious how to transform this local regularity criterion into a global one.

In what follows, we denote by $L^p(\mathbb{R}^3)$ the Lebesgue spaces, $1 \leqslant p \leqslant \infty$, by $W^{k,p}(\mathbb{R}^3)$ the Sobolev spaces for $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, both endowed with the standard norms $\|\cdot\|_{p,\mathbb{R}^3}$ and $\|\cdot\|_{k,p,\mathbb{R}^3}$, respectively. The anisotropic Lebesgue spaces $L^t(0,T;L^s(\mathbb{R}^3))$ will be denoted, for brevity, by $L^{t,s}(Q_T)$, $1 \leqslant t,s \leqslant \infty$, $Q_T = (0,T) \times \mathbb{R}^3$. If no misunderstanding can occur we will omit writing Q_T and $\mathbb{R}^3$, respectively.

All generic constants will be denoted by C. Their values can vary, even on the same line or in the same formula.

We will also use the summation convention; unless otherwise stated, the summation over repeated indices will be used, from 1 to 3.

2. Main theorems

The main goal is to prove the following four theorems.

Theorem 1. *Let* $\mathbf{u}$ *be a weak solution to the Navier-Stokes equations* (1.1) *corresponding to the initial condition* $\mathbf{u}_0 \in W^{1,2}$ *with* $\operatorname{div} \mathbf{u}_0 = 0$ *such that* $\mathbf{u}$ *satisfies the energy inequality. Moreover let* $u_3 \in L^{t_1,s_1}$, $2/t_1 + 3/s_1 \leqslant 1$, $2 \leqslant t_1 \leqslant \infty$, $3 < s_1 \leqslant \infty$ *and one of the following conditions holds true*

(a) $\partial u_1/\partial x_3$, $\partial u_2/\partial x_3$ *belong to* L^{t_2,s_2} *with* $2/t_2 + 3/s_2 \leqslant 2$, $1 \leqslant t_2 \leqslant \infty$, $\frac{3}{2} < s_2 \leqslant \infty$,

(b) $\partial u_1/\partial x_2$, $\partial u_2/\partial x_1$ *belong to* L^{t_3,s_3} *with* $2/t_3 + 3/s_3 \leqslant 2$, $2 \leqslant t_3 \leqslant \infty$, $2 \leqslant s_3 \leqslant 3$,

(c) $\partial u_2/\partial x_3 \in L^{t_4,s_4}$, $\partial u_1/\partial x_2 \in L^{t_5,s_5}$, $2/t_i + 3/s_i \leqslant 2$, $i = 4,5$, $1 \leqslant t_4 \leqslant \infty$, $\frac{3}{2} < s_4 \leqslant \infty$, $2 \leqslant t_5 \leqslant \infty$, $2 \leqslant s_5 \leqslant 3$.

Then $(\mathbf{u}, p)$ *with* p *the corresponding pressure is the strong solution to the Navier-Stokes equations which is unique in the class of all weak solutions satisfying the energy inequality.*

R e m a r k 1. Note that in (b) it might be interesting to replace the conditions on $\partial u_1/\partial x_2$ and $\partial u_2/\partial x_1$ by the same condition on ω_3. Unfortunately, this does not seem to be possible, at least by the present technique.

R e m a r k 2. In part (a) we can replace the assumptions on $\partial u_1/\partial x_3$, $\partial u_2/\partial x_3$ by analogous assumptions on $\partial u_2/\partial x_3$, $\partial u_2/\partial x_2$, or $\partial u_2/\partial x_3$, $\partial u_1/\partial x_1$, or $\partial u_1/\partial x_3$, $\partial u_2/\partial x_2$, or $\partial u_1/\partial x_3$, $\partial u_1/\partial x_1$. Similarly, instead of (c), we can assume $\partial u_1/\partial x_3 \in L^{t_4,s_4}$, $\partial u_2/\partial x_1 \in L^{t_5,s_5}$.

R e m a r k 3. It will be clear from the proof why s_3 and s_5 satisfy more restrictive conditions than s_2 and s_4. For s_3 and $s_5 > 3$ or from $(\frac{18}{11}, 2)$ we can still obtain some conditions implying the regularity; however these conditions are more restrictive, i.e. they do not lie on the same scale as those in Theorem 1; see the note at the end of Step 3 (ii) in the proof of Theorem 1 below.

R e m a r k 4. The limit cases, i.e. in (a) $u_3 \in L^{\infty,3}$, in (b) $s_2 = \frac{3}{2}$, $t_2 = \infty$ and in (c) $s_4 = \frac{3}{2}$, $t_4 = \infty$ do not imply the regularity. We have to add the assumption that the above mentioned norms are sufficiently small. The same holds also for the limit case in Theorem 3 below.

In the following Theorems 2–4 we assume similarly as in Theorem 1 that $\mathbf{u}$ is a weak solution to the Navier-Stokes equations (1.1) corresponding to the initial condition $\mathbf{u}_0 \in W^{1,2}$ with $\operatorname{div} \mathbf{u}_0 = 0$ such that $\mathbf{u}$ satisfies the energy inequality.

Theorem 2. *Let $\partial u_3/\partial x_3 \in L^{\infty,\infty}$. Then $(\mathbf{u}, p)$ with p the corresponding pressure is the strong solution to the Navier-Stokes equations which is unique in the class of all weak solutions satisfying the energy inequality.*

Theorem 3. *Let $\partial u_3/\partial x_3, \partial u_2/\partial x_2 \in L^{t_1,s_1}$, $2/t_1 + 3/s_1 \leqslant 2$, $1 \leqslant t_1 \leqslant \infty$, $\frac{3}{2} < s_1 \leqslant \infty$. Then $(\mathbf{u}, p)$ with p the corresponding pressure is the strong solution to the Navier-Stokes equations which is unique in the class of all weak solutions satisfying the energy inequality.*

Theorem 4. *Let one of the following conditions be satisfied*
(i) $\partial \mathbf{u}/\partial x_3 \in L^{t_1,s_1}$, $2/t_1 + 3/s_1 \leqslant \frac{3}{2}$, $\frac{4}{3} \leqslant t_1 \leqslant \infty$, $2 \leqslant s_1 \leqslant \infty$, *or*
(ii) $\partial u_3/\partial x_3 \in L^{t_2,s_2}$, $2/t_2 + 3/s_2 \leqslant 1$, $2 \leqslant t_2 \leqslant \infty$, $3 \leqslant s_2 \leqslant \infty$ *and* $\partial u_i/\partial x_3 \in L^{t_3,s_3}$, $2/t_3 + 3/s_3 \leqslant 2$, $1 \leqslant t_3 \leqslant \infty$, $\frac{3}{2} < s_3 \leqslant \infty$, $i = 1, 2$.

Then $(\mathbf{u}, p)$ with p the corresponding pressure is the strong solution to the Navier-Stokes equations which is unique in the class of all weak solutions satisfying the energy inequality.

R e m a r k 5. Note that the regularity assumption in Theorem 2 can be written as $\partial u_3/\partial x_3 \in L^{t,s}$ with $2/t + 3/s = 0$.

R e m a r k 6. Comparing results from [2] with any of the results from Theorem 3–4, we see that we require here less in the sense that we need only three (or two) components of the gradient to satisfy less restrictive conditions than in the above cited paper.

R e m a r k 7. Let us also note that, even though we consider here the right-hand side of the Navier-Stokes equations to be zero, similar results as presented in Theorems 1–4 hold also if some $\mathbf{f} \neq \mathbf{0}$ appears in the right-hand side; only the smoothness of the solution depends on the smoothness of $\mathbf{f}$.

3. Auxiliary results

For a moment, let $(\mathbf{u}, p)$ be a smooth solution to the Navier-Stokes equations such that $\mathbf{u} \in L^2(0,T;W^{k,2})$, $\mathbf{u}_t \in L^2(0,T;W^{k-2,2})$, $k \geqslant 3$. Then we have the following equation for the pressure

$$-\Delta p = \operatorname{div}\operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \quad \text{in } (0,T) \times \mathbb{R}^3 \tag{3.1}$$

and thus

Lemma 1. *The following estimates for the pressure hold true*

$$\|p\|_q(t) \leqslant C\|\mathbf{u}\|_{2q}^2(t),$$
$$\Big\|\frac{\partial p}{\partial x_i}\Big\|_q(t) \leqslant C \sum_{j,k=1}^{3} \Big\|\frac{\partial(u_j u_k)}{\partial x_i}\Big\|_q(t)$$

for $1 < q < \infty$.

P r o o f. This is an easy consequence of equation (3.1), standard L^q estimates for the Laplace equation in the entire space (i.e. the Marcinkiewicz multiplier theorem, see e.g. [17]) and the fact that $\nabla p(t) \in L^2$. □

Next, let us consider our weak solution to the Navier-Stokes equations as given in Theorems 1–4. As $\mathbf{u}_0 \in W^{1,2}$, we know (see [6]) that there is $t_0 > 0$ such that there exists a smooth solution to the Navier-Stokes equations on $(0, t_0)$ corresponding to the initial condition $\mathbf{u}_0$. Moreover, since this solution is unique in the class of all weak solutions satisfying the energy inequality, it coincides with "our" weak solution on this time interval. Denote by t^* the supremum of all $\bar{t} > 0$ such that on $(0, \bar{t})$ there is a smooth solution to the Navier-Stokes equations. Note that $t^* > 0$. Assume now $t^* < \infty$. Evidently on any compact subinterval of $(0, t^*)$ "our" weak solution coincides with this smooth solution (and it is, due to the absence of the right-hand side, $C^\infty([\delta, t^* - \delta] \times \mathbb{R}^3)$, $0 < \delta < t^*$).

If we show that some norm of $\mathbf{u}$ (or $\nabla\mathbf{u}$), sufficient to ensure the smoothness of the Navier-Stokes equations, remains bounded independently of t as $t \to t^*$, we can extend our solution (due to the result from [6]) after the time instant t^* which would contradict the definition of t^* and thus $t^* = \infty$. In the following sections we will show such estimates. We will always work on some subintervals of $(0, t^*)$ and thus all equations will be satisfied pointwise. Before starting with these estimates let us recall some useful inequalities. We have (for the proof see [11])

Lemma 2. *Let h be a function such that $h \in L^q$ and $\nabla h \in L^s$, $s \in [1,\infty]$, $r \geqslant q$ and $r \leqslant \infty$ if $s > 3$, $r < \infty$ if $s = 3$ and $r \leqslant 3s(3-s)^{-1}$ if $s < 3$. Then there exists a constant C such that*

$$\|h\|_r \leqslant C\|\nabla h\|_s^a \|h\|_q^{1-a}, \quad a \in [0,1],$$

where $1/r = a(1/s - \frac{1}{3}) + (1-a)1/q$.

Recall also that if $\operatorname{div} \mathbf{u} = 0$ then

$$C_1\|\operatorname{curl} \mathbf{u}\|_q \leqslant \|\nabla \mathbf{u}\|_q \leqslant C_2(q)\|\operatorname{curl} \mathbf{u}\|_q, \tag{3.2}$$

$1 < q < \infty$ (and C_1 remains bounded if $q \to 1$ or $q \to \infty$ while $C_2(q) \to \infty$ in this case).

4. Proof of Theorem 1

We will proceed in several steps:

Step 1: Estimates of the vorticity

Let us recall that $\omega = \operatorname{curl} \mathbf{u}$ satisfies the following system

$$\frac{\partial \omega}{\partial t} - \Delta\omega + \mathbf{u} \cdot \nabla\omega - \omega \cdot \nabla \mathbf{u} = \mathbf{0} \quad \text{in } (0,T) \times \mathbb{R}^3$$
$$\omega(0,\mathbf{x}) = \operatorname{curl} \mathbf{u}_0(\mathbf{x}) \quad \text{in } \mathbb{R}^3.$$

Multiply the equation by ω and integrate over $\mathbb{R}^3$. Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_2^2 + \|\nabla\omega\|_2^2 = \int_{\mathbb{R}^3} \omega_i \frac{\partial u_j}{\partial x_i}\omega_j.$$

If $j = 3$ then

$$\int_{\mathbb{R}^3} \omega_i \frac{\partial u_3}{\partial x_i}\omega_3 = -\int_{\mathbb{R}^3} u_3\omega_i \frac{\partial \omega_3}{\partial x_i}$$

and recalling that $\omega_i = \varepsilon_{ijk}\partial u_k/\partial x_j$ (ε_{ijk} is the Levi-Cività skew-symmetric tensor) we get

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3} \omega_i \frac{\partial u_j}{\partial x_i}\omega_j = &\int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_1} - \int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3}\\ &+ \int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_2} - \int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3}\\ &+ \int_{\mathbb{R}^3} c_{ijklm} u_3 \frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l \partial x_m}\end{aligned}$$

with c_{ijklm} a constant matrix. Thus

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_2^2+\|\nabla\omega\|_2^2 = &\int_{\mathbb{R}^3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_1}-\int_{\mathbb{R}^3}\frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3}\\ &+\int_{\mathbb{R}^3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_2}-\int_{\mathbb{R}^3}\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3}\\ &+\int_{\mathbb{R}^3}c_{ijklm}u_3\frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l\partial x_m}.\end{aligned}$$

Step 2: Estimates of u_3

Now

$$\begin{aligned}\left|\int_{\mathbb{R}^3}u_3\frac{\partial u_i}{\partial x_j}\frac{\partial^2 u_k}{\partial x_l\partial x_m}\right| &\leqslant \|\nabla^2\mathbf{u}\|_2\|u_3\|_s\|\nabla\mathbf{u}\|_{2s(s-2)^{-1}} \quad \text{(by means of (3.2))}\\ &\leqslant C\|\nabla\omega\|_2^{(s+3)s^{-1}}\|\omega\|_2^{(s-3)s^{-1}}\|u_3\|_s\\ &\leqslant \frac{1}{C}\|\nabla\omega\|_2^2+C\|\omega\|_2^2\|u_3\|_s^{2s(s-3)^{-1}},\end{aligned}$$

i.e. if $u_3\in L^{t,s}$, $2/t+3/s\leqslant 1$, $s>3$, we can estimate this term by putting the first term to the left-hand side and applying the Gronwall inequality to the other one; if $s=3$ we need that the $L^{\infty,3}$ norm of u_3 is sufficiently small.

Step 3: Estimates of ∇u_i, $i=1,2$

(i) $\partial u_1/\partial x_3$, $\partial u_2/\partial x_3$

Evidently, using Lemma 2 the last remaining terms can be estimated as follows $(i,j,k,l=1,2)$

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\frac{\partial u_i}{\partial x_3}\frac{\partial u_j}{\partial x_3}\frac{\partial u_k}{\partial x_l}\right| &\leqslant \left\|\frac{\partial u_i}{\partial x_3}\right\|_s\|\nabla\mathbf{u}\|_{2s(s-1)^{-1}}^2\leqslant C\|\nabla\omega\|_2^{3/s}\|\omega\|_2^{(2s-3)s^{-1}}\left\|\frac{\partial u_i}{\partial x_3}\right\|_s\\ &\leqslant \frac{1}{C}\|\nabla\omega\|_2^2+C\|\omega\|_2^2\left\|\frac{\partial u_i}{\partial x_3}\right\|_s^{2s(2s-3)^{-1}}\end{aligned}$$

and if $\partial u_i/\partial x_3\in L^{t,s}$, $2/t+3/s\leqslant 2$ we put the first term to the left-hand side and estimate the other term by means of the Gronwall inequality. Thus part (a) with $\partial u_1/\partial x_3$, $\partial u_2/\partial x_3$ of Theorem 1 is shown. Similarly, using also the continuity equation, we can show the first part of Remark 2.

(ii) $\partial u_1/\partial x_2$, $\partial u_2/\partial x_1$

Here we have to integrate by parts in two terms. We get

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_1}+\int_{\mathbb{R}^3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_2}\\ &\qquad= -2\int_{\mathbb{R}^3}\frac{\partial^2 u_2}{\partial x_1\partial x_3}\frac{\partial u_2}{\partial x_3}u_1-2\int_{\mathbb{R}^3}\frac{\partial^2 u_1}{\partial x_2\partial x_3}\frac{\partial u_1}{\partial x_3}u_2\end{aligned}$$

$$= 2\int_{\mathbb{R}^3} \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3}\frac{\partial u_1}{\partial x_3} + 2\int_{\mathbb{R}^3} \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3}\frac{\partial u_2}{\partial x_3}$$
$$+ 2\int_{\mathbb{R}^3} \frac{\partial^2 u_2}{\partial x_3^2}\frac{\partial u_2}{\partial x_1}u_1 + 2\int_{\mathbb{R}^3} \frac{\partial^2 u_1}{\partial x_3^2}\frac{\partial u_1}{\partial x_2}u_2.$$

The first two terms can be estimated as above. For the other two we get $(i,j = 1,2,\ i \neq j)$

$$I = \left|\int_{\mathbb{R}^3} \frac{\partial^2 u_i}{\partial x_3^2}\frac{\partial u_i}{\partial x_j}u_j\right| \leqslant \Big\|\frac{\partial u_i}{\partial x_j}\Big\|_s \Big\|\frac{\partial^2 u_i}{\partial x_3^2}\Big\|_2 \|u_j\|_{2s(s-2)^{-1}}.$$

Now for $2 \leqslant s \leqslant 3$ (i.e. $6 \leqslant 2s(s-2)^{-1} \leqslant \infty$) we can apply Lemma 2 to get

$$I \leqslant C\Big\|\frac{\partial u_i}{\partial x_j}\Big\|_s \|\nabla\omega\|_2^{3/s}\|\omega\|_2^{(2s-3)s^{-1}} \leqslant \frac{1}{C}\|\nabla\omega\|_2^2 + C\Big\|\partial u_i/\partial x_j\Big\|_s^{2s(2s-3)^{-1}}\|\omega\|_2^2$$

and we estimate this term as above. For $s > 3$ we proceed as in [12], but the result is more restrictive ($\partial u_i/\partial x_j \in L^{6s(5s-6)^{-1},s}$, $s > 3$) or for $s < 2$ we can estimate the term by $\|\partial u_i/\partial x_j\|_2\|\nabla^2\mathbf{u}\|_2\|\mathbf{u}\|_\infty$ and interpolate the L^2-norm between L^s and L^6; we get again a more restrictive condition ($\partial u_i/\partial x_j \in L^{8s(11s-18)^{-1},s}$, $\frac{18}{11} \leqslant s \leqslant 2$).

(iii) Proof of (c)
We can combine parts (i) and (ii) to show (c) as well as the second part of Remark 2. Theorem 1 is proved. □

5. Proofs of Theorems 2–4

P r o o f of Theorem 2.

It is enough to show (see [9] or [10]) that $u_3 \in L^{t,s}$ for $2/t + 3/s \leqslant \frac{1}{2}$, $s > 6$. To this aim let us multiply the equation for u_3 by $|u_3|^4u_3$ and integrate over $\mathbb{R}^3$. Then

$$\frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\|u_3\|_6^6 + \frac{5}{9}\|\nabla|u_3|^3\|_2^2 = -\int \frac{\partial p}{\partial x_3}|u_3|^4u_3 \equiv I_1.$$

Now, integrating by parts in the term on the right-hand side we obtain

$$|I_1| \leqslant C\int_{\mathbb{R}^3} |p|\Big|\frac{\partial u_3}{\partial x_3}\Big||u_3|^4 \leqslant \|u_3\|_6^4\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_\infty \|\mathbf{u}\|_6^2.$$

If $\frac{\partial u_3}{\partial x_3}$ is bounded in $L^{\infty,\infty}$, we get that

$$\|u_3\|_{L^{\infty,6}} + \|\nabla|u_3|^3\|_{L^{2,2}} \leqslant C.$$

But $\|u_3\|_{L^{6,18}} \leqslant C\|\nabla|u_3|^3\|_{L^{2,2}}$ and thus Theorem 2 is shown. □

P r o o f of Theorem 3.

The idea is more or less the same as previously. It is enough to show that $\mathbf{u}$ is bounded in $L^{t,s}$ for $2/t+3/s\leqslant 1$, $s\geqslant 3$. To this aim, let us multiply the ith component of the Navier-Stokes equations by $|u_i|u_i$ and integrate over $\mathbb{R}^3$. We get

$$\sum_{i=1}^{3}\Big(\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|u_i\|_3^3+\frac{8}{9}\|\nabla|u_i|^{\frac{3}{2}}\|_2^2\Big)=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\frac{\partial p}{\partial x_i}|u_i|u_i\equiv I_2.$$

We integrate by parts on the right-hand side and use the continuity equation. Then

$$\begin{aligned}|I_2|&\leqslant C\sum_{i=1}^{3}\int_{\mathbb{R}^3}|p|\Big|\frac{\partial u_i}{\partial x_i}\Big|\,|u_i|\\&\leqslant C\sum_{i=1}^{3}\Big(\Big\|\frac{\partial u_2}{\partial x_2}\Big\|_s+\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s\Big)\|u_i\|_{3s(s-1)^{-1}}\|\mathbf{u}\|^2_{3s(s-1)^{-1}}\\&\leqslant C\sum_{i=1}^{3}\Big(\Big\|\frac{\partial u_2}{\partial x_2}\Big\|_s+\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s\Big)\|u_i\|^3_{3s(s-1)^{-1}}\\&\leqslant\sum_{i=1}^{3}\Big(\frac{4}{9}\|\nabla|u_i|^{\frac{3}{2}}\|_2^2+C\Big(\Big\|\frac{\partial u_2}{\partial x_2}\Big\|_s^{2s(2s-3)^{-1}}+\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s^{2s(2s-3)^{-1}}\Big)\|u_i\|_3^3\Big).\end{aligned}$$

After employing the Gronwall inequality, under the assumption that $\partial u_2/\partial x_2$ and $\partial u_3/\partial x_3$ are bounded in $L^{t,s}$, $2/t+3/s\leqslant 2$, $s>\frac{3}{2}$, we get

$$\|\mathbf{u}\|_{L^{\infty,3}}+\sum_{i=1}^{3}\|\nabla|u_i|^{\frac{3}{2}}\|_{L^{2,2}}\leqslant C$$

and thus $\mathbf{u}$ is bounded in $L^{\infty,3}$ which gives the global-in-time regularity of the solution. For $s=\frac{3}{2}$ we have to assume that the corresponding norms are sufficiently small. □

P r o o f of Theorem 4.

We will now use Theorem 1 part (a). Since we know that in both cases $\partial u_i/\partial x_3$, $i=1,2$, satisfy the assumptions of Theorem 1, it is enough to verify that $u_3\in L^{t,s}$ for $2/t+3/s\leqslant 1$, $s>3$. To this aim we multiply the equation for u_3 by $|u_3|u_3$ and integrate over $\mathbb{R}^3$. Then

$$\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\|u_3\|_3^3+\frac{8}{9}\|\nabla|u_3|^{\frac{3}{2}}\|_2^2=-\int_{\mathbb{R}^3}\frac{\partial p}{\partial x_3}u_3|u_3|\equiv I_3.$$

Now

$$|I_3| \leqslant C\int_{\mathbb{R}^3} |p|\Big|\frac{\partial u_3}{\partial x_3}\Big||u_3| \leqslant C\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s \|u_3\|_3 \|\mathbf{u}\|^2_{6s(2s-3)^{-1}}$$
$$\leqslant C\Big(\Big\|\frac{\partial u_3}{\partial x_3}\Big\|_s^{2s(s-3)^{-1}} + \|\mathbf{u}\|_6^2\Big)\|\mathbf{u}\|_2^{(s-3)s^{-1}}\|u_3\|_3$$

and using the Gronwall inequality we finish the proof of the case (ii) as u_3 is bounded in $L^{3,9}$.

To prove (i) we will use Lemma 1. We proceed as above but we do not integrate by parts on the right-hand side and get

(a) $s \geqslant 6$

$$|I_3| \leqslant \Big\|\frac{\partial p}{\partial x_3}\Big\|_3 \|u_3\|_3^2 \leqslant C\sum_{i=1}^3 \Big\|\frac{\partial u_i}{\partial x_3}\Big\|_s \|\mathbf{u}\|_{3s(s-3)^{-1}}\|u_3\|_3^2$$
$$\leqslant C\sum_{i=1}^3 \|u_3\|_3^2 \Big\|\frac{\partial u_i}{\partial x_3}\Big\|_s \|\mathbf{u}\|_6^{(s+6)/(2s)}\|\mathbf{u}\|_2^{(s-6)/(2s)}$$
$$\leqslant C\sum_{i=1}^3 \|u_3\|_3^2\|\mathbf{u}\|_2^{(s-6)/(2s)}\Big(\Big\|\frac{\partial u_i}{\partial x_3}\Big\|_s^{4s(3s-6)^{-1}} + \|\mathbf{u}\|_6^2\Big)$$

and if $\partial u_i/\partial x_3 \in L^{t,s}$, $2/t+3/s \leqslant \frac{3}{2}$, $s \geqslant 6$, we can estimate this term by means of the Gronwall inequality.

(b) $2 \leqslant s < 6$

If $2 < s < 6$ then

$$|I_3| \leqslant \Big\|\frac{\partial p}{\partial x_3}\Big\|_{\frac{3}{2}} \|u_3\|_6^2 \leqslant C\sum_{i=1}^3 \Big\|\frac{\partial u_i}{\partial x_3}\Big\|_s \|\mathbf{u}\|_{3s(2s-3)^{-1}}\|u_3\|_3^{\frac{1}{2}}\|u_3\|_9^{\frac{3}{2}}$$
$$\leqslant \frac{4}{9}\|\nabla|u_3|^{\frac{3}{2}}\|_2^2 + C\sum_{i=1}^3 \|u_3\|_3\|\mathbf{u}\|_2^{(3s-6)s^{-1}}\Big(\Big\|\frac{\partial u_i}{\partial x_3}\Big\|_s^{4s(3s-6)^{-1}} + \|\mathbf{u}\|_6^2\Big),$$

i.e. again after employing the Gronwall inequality we get that u_3 is bounded in $L^{3,9}$ and thus the solution is smooth. Similarly we proceed for $s = 2$. Theorem 4 is proved. □

Remark 8. Note that in part (ii) we could replace the assumption on $\partial u_1/\partial x_3$ and $\partial u_2/\partial x_3$ by any assumption from Theorem 1 (a), (b), (c) or from Remark 2. But these results seem to be less interesting. Namely, we interpret the results of Theorem 4 as follows. If we control the flow in the "additional" third dimension, we get the regularity; this is in accordance with the expectation since in two space dimensions any weak solution is a strong one provided the data are smooth enough.

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