Ulf von Kalckreuth; Manfred Krtscha

Stable solutions to homogeneous difference-differential equations with constant coefficients: Analytical instruments and an application to monetary theory

Applications of Mathematics, Vol. 49 (2004), No. 4, 373–386

Persistent URL: `http://dml.cz/dmlcz/134574`

49 (2004) APPLICATIONS OF MATHEMATICS No. 4, 373–386

STABLE SOLUTIONS TO HOMOGENEOUS DIFFERENCE-DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS: ANALYTICAL INSTRUMENTS AND AN APPLICATION TO MONETARY THEORY

ULF VON KALCKREUTH, Frankfurt, MANFRED KRTSCHA, Karlsruhe

(Received August 5, 2002, in revised version July 2, 2003)

Abstract. In economic systems, reactions to external shocks often come with a delay. On the other hand, agents try to anticipate future developments. Both can lead to difference-differential equations with an advancing argument. These are more difficult to handle than either difference or differential equations, but they have the merit of added realism and increased credibility. This paper generalizes a model from monetary economics by von Kalckreuth and Schröder. Working out its stability properties, we present a general method for determining the stability of any solution to a homogeneous linear difference-differential equation with constant coefficients and advancing arguments.

Keywords: linear difference-differential equations, stability, monetary transmission

MSC 2000: 39A11, 39B99, 91BC2

1. Preliminaries

In economic systems, reactions to external shocks often come with a delay. On the other hand, agents try to anticipate future developments. Both can lead to difference-differential equations with an advancing argument. These are more difficult to handle than either difference or differential equations, but they have the merit of added realism and increased credibility. Working out a problem from monetary theory, we present a general method for determining the stability of any solution to a homogeneous linear difference-differential equation with constant coefficients and advancing arguments.

In the next section we start by presenting the generalisation of a monetary macroeconomic model by von Kalckreuth and Schröder [10]. In order for this model to have explanatory power, it is necessary to show that there is one and only one stable solution to the system. The following section introduces the reader into the theory of linear differential-difference equations with constant coefficients and presents a method of determining the stability of their solutions, making use of a hitherto almost ignored theorem by Hilb [9]. Our method is rather general and can be applied to a large class of dynamical problems. The last section applies our method to the economic problem at hand and draws conclusions.

2. A MODEL OF MONETARY TRANSMISSION

In order to investigate the interactions between the service life of capital, the term structure of interest rates and the impact of monetary policy on open economies, von Kalckreuth and Schröder [10] developed a dynamic macroeconomic model. This model considers an interest-rate structure within the framework of the Dornbusch [2] overshooting model. Whereas the central bank is able to influence the nominal short-term rate, aggregate demand depends on the real long-term rate. The interest-rate structure embodied in this model leads to an advancing argument in a system of functional equations. The authors solved this system by imposing additional restrictions.

Here, we want to investigate the *unrestricted* model dynamics. We show that there is a unique stable solution, which is identical to the solution described by von Kalckreuth and Schröder [10] in solving the restricted model.

The model contains the following equations:

$$\overline{M} - P(t) = \alpha_1 \overline{Y} - \alpha_2 i(t), \tag{2.1}$$

$$r(t) = i(t) - \dot{P}(t), \tag{2.2}$$

$$D(t) = \beta_0 + \beta_1 (E(t) - P(t)) + \beta_2 \overline{Y} - \beta_3 R_\Omega(t), \tag{2.3}$$

$$\dot{P}(t) = \Gamma(D(t) - \overline{Y}), \tag{2.4}$$

$$i(t) = i^* + \dot{E}(t), \tag{2.5}$$

$$\dot{R}_\Omega(t) = \frac{1}{\Omega}(r(t+\Omega) - r(t)). \tag{2.6}$$

A bar denotes a steady-state value. All coefficients in (2.1)–(2.6) are strictly positive. Eq. (2.1) is a money-market equation with M, P and Y being the logarithms of nominal money supply, price level and constant real income and i the nominal money-market interest rate. While P is restricted to be a globally continuous function of time, all the other endogenous variables are allowed to jump discontinuously in response to an unexpected shock at $t = 0$. Equation (2.2) defines the real money-market interest rate r as the difference between the short-term nominal interest rate i and inflation. In (2.3), the logarithm of the aggregate demand, D, depends on the logarithms of the real exchange rate $E - P$, the constant real income and the long-term real interest rate R_Ω, with maturity Ω. For simplicity, a uniform service life of capital is assumed. Equation (2.4) is a simple Phillips relationship, in which the rate of inflation $\dot{P} = \mathrm{d}P/\mathrm{d}t$ is determined by the ratio of the (variable) aggregate demand to the (constant) supply. Equation (2.5) represents the open interest-parity condition, with i^* being the given nominal short-term interest rate in the international money market. Equation (2.6), finally, relates the change in the real long-term interest rate to the difference between future short-term rates and present short-term rates.

The last equation brings in an advancing argument into the system. Because of its importance for our mathematical problem, the relationship between short and long-term rates will be derived here from first principles, as an arbitrage equation[1] for bonds of finite maturity.

Complete foresight in a perfect asset market implies the equality of the instantaneous real rates of return on bonds and investments in the money market. Consider a zero bond of arbitrary time to maturity Ω, issued at time t. Let N be the issue price and $R_\Omega(t)$ the long-term rate for bonds with time to maturity Ω. At time $t+\Omega$, then, the holder of the bond receives a payment of $N\exp(\Omega R_\Omega(t))$. Since there are no interest payments until maturity, according to the Hotelling rule the arbitrage condition is given by

$$\frac{\dot{K}(s)}{K(s)} = r(s), \tag{2.7}$$

with $K(s)$ as the real market value of a bond at any point in time s between t and $t + \Omega$. Taking into consideration the terminal condition that, at maturity, the real market value $K(s)$ is bound to be equal to the real value of the principal and the accrued interest, the general solution of (2.7) becomes

$$K(s) = N\exp\left(\Omega R_\Omega(t) - \int_s^{t+\Omega} r(\tau)\,\mathrm{d}\tau\right). \tag{2.8}$$

At the date of issue, however, the value of this expression must be equal to N, which immediately yields the arbitrage equation

$$R_\Omega(t) = \frac{1}{\Omega}\int_t^{t+\Omega} r(\tau)\,\mathrm{d}\tau. \tag{2.9}$$

[1] See, for example, [13]. Fisher and Turnovsky [5] also use this equation in the context of a dynamic macroeconomic model.

Equation (2.9) gives us the term structure of interest rates, according to the expectation theory, for the case of continuous interest compounding. The (continuous) long-term rate R_Ω is determined as the arithmetic mean of the short-term rates within the relevant time interval. The former thus anticipates the movement of the latter. Taking the derivative on both sides yields equation (2.6). The interest rate R_Ω, by definition, gives us the cost of capital of an investment project characterized by one single payment I at t and one single, certain return of V at the end of its lifetime Ω. Investigating the dynamic system for varying Ω thus permits us to describe the effects of a decreasing service life of capital, as a result of accelerated technical progress, on the dynamics of macroeconomic adjustment to various kinds of shocks.

We can reduce the system to one single dynamic equation. Long-run equilibrium is characterized by the conditions

$$\dot{P}(t) = 0 \quad \text{and} \quad \dot{E}(t) = 0. \tag{2.10}$$

Substituting these conditions into the system (2.1)–(2.6) readily yields a particular solution to the system, the steady-state solution. Given a shock to the system, e.g. by a monetary expansion, we can therefore concentrate on finding the solutions to the following set of homogeneous equations that describes the *deviations* from equilibrium:[2]

$$
\begin{aligned}
&(2.1') & P(t) &= \alpha_2 i(t),\\
&(2.2') & r(t) &= i(t) - \dot{P}(t),\\
&(2.3') & D(t) &= \beta_1 E(t) - \beta_1 P(t) - \beta_3 R_\Omega(t),\\
&(2.4') & \dot{P}(t) &= \Gamma D(t),\\
&(2.5') & i(t) &= \dot{E}(t),\\
&(2.6') & \dot{R}_\Omega(t) &= \frac{1}{\Omega}(r(t+\Omega) - r(t)).
\end{aligned}
$$

Repeated substitution yields the following homogeneous differential-difference equation with an advancing argument for the logarithm of the price level P:

$$\ddot{P}(t) = -AP(t+\Omega) + BP(t) + AC\dot{P}(t+\Omega) - BC\dot{P}(t), \tag{2.11}$$

[2] The levels of the variables in (2.1′)–(2.6′) are deviations from the steady state. In order to save notation, we will not introduce new symbols.

where

$$A = \frac{\Gamma\beta_3}{\Omega\alpha_2} > 0, \tag{2.12}$$

$$B = \frac{\Gamma}{\alpha_2}\left(\beta_1 + \frac{\beta_3}{\Omega}\right) > A > 0, \tag{2.13}$$

$$C = \alpha_2 > 0. \tag{2.14}$$

In order to solve the system analytically, von Kalckreuth and Schröder [10] used the following *additional* restriction on the time path of the deviation from equilibrium:

$$\dot{P}(t) = \lambda P(t), \quad \lambda \in \mathbb{R}, \;\; \lambda < 0. \tag{2.15}$$

This restriction requires the perfect foresight path to be of an especially simple, adaptive structure. Besides a family of exploding solutions, the system has one and only one stable solution. It may well be argued, however, that this restriction is essentially *ad hoc*, and it is not easily seen whether and how the results of the paper critically depend on constraining the solution to be exponential. Here, we do not intend to discuss the economic results of von Kalckreuth and Schröder [10], as they have been published elsewhere. Instead, *we want to investigate whether it is possible to drop the additional restriction without losing the results.* This is possible if there are no additional *stable* solutions to the system apart from the solution found by imposing (2.15).

In a model of perfect foresight, every market participant has the same expectation and these expectations coincide with the actual development. If there is more than one solution to a dynamical economic model of an asset market with perfect foresight, it has to be stated which of these solutions the agents will coordinate upon. Unstable solutions are generally excluded. Real variables, such as aggregate demand, real interest rates and real exchange rates, under normal economic circumstances remain bounded. A diverging real exchange rate, for example, would mean that in the course of time, the entire national product would sell for just a small amount of foreign money. In order to find the relevant solutions among various time paths that fulfil the system equations, it is therefore assumed that market participants do not coordinate on exploding solutions if there is a stable one.[3]

In dynamic systems describing physical phenomena, the initial conditions determine whether among several solutions a stable time path is attained or not. In economic systems with asset markets, it may be the other way round. With their

[3] This would naturally have to be different if the model was set up to explain stock market bubbles.

expectations, economic agents coordinate on the stable solution, and the initial condition, such as the immediate reaction of a stock price or the exchange rate to a shock, are determined by a jump discontinuity in order to set the system on the stable path. Asset prices jump because every market participant knows that they have to jump. If anybody were willing to pay more for that asset or sell it for less, he or she would be losing money. In other words, the initial conditions are not exogenous to the system, but they are endogenously determined. Working out the analytics of perfect foresight dynamics was a major advance in economic theory, due to, among others, Sargent and Wallace [14]. As concerns Dornbusch [2] overshooting model, see the papers by Wilson [15] and Gray and Turnovsky [6].[4]

If there is more than one stable solution, the model leaves the dynamics indeterminate. What is worse, the researcher has to explain by which mechanism the market participants manage to coordinate uniformly on a particular one of these solutions. We therefore want to know whether the stable solution found by von Kalckreuth and Schröder [10] under their restrictive assumption remains unique in the context of an unrestricted perfect foresight model. As we will see immediately below, the relevant initial conditions for the system in question are given not simply by an initial *value*, but by an initial *time path*. To the best of our knowledge, this is new in the economics literature. We will investigate the dynamic properties of our system using a method that can be applied to determine the stability of the solutions for any linear difference-differential equation with an advancing argument.

3. Linear difference-differential equations and a general method of investigating their stability

At this point it is convenient to review some of the basic properties of linear difference-differential equations. Consider the following simple example with positive constants c and δ, taken from Krtscha [12]:

$$f'(t+\delta) = -c\, f(t). \tag{3.1}$$

The positive constant δ plays an important role for the nature of the solution. For $\delta = 0$, the solution to (3.1) would be uniquely determined by the condition $f(t_0) = y_0$, whereas for $\delta > 0$ the solution needs more: it is uniquely determined by any given function $f\colon [t_0, t_0+\delta] \to \mathbb{R}$ that is continuous and differentiable in $(t_0, t_0+\delta)$,

[4] For a textbook treatment of perfect foresight dynamics that also covers the Dornbusch [2] overshooting model, see de la Fuente [7, Section 11.2].

implying the differentiability of f in (t_0, ∞). Then equation (3.1) leads to the formula

$$f(t+n\delta) = -c\int_0^t f(\tau + (n-1)\delta)\,\mathrm{d}\tau + f(n\delta),$$

by which we can regressively calculate $f(t)$ for all $n \in \mathbb{N}$. We want to give a specific example:

$$f'(t+1) = -f(t) \quad \text{with} \quad f(t) = 1-t, \quad t \in [0,1]. \tag{3.2}$$

Then, by means of the regression-formula for $t \in [0,1]$, we get

$$f(t+1) = -t + t^2/2,$$

implying $f(1) = 0$ and $f(2) = -\frac{1}{2}$, $f(t+2) = t^2/2 - t^3/3 - 1/2$, $f(t+3) = -t^3/3! + t^4/4! + t/2 - 1/3!$, and so on.

Now we may ask two questions:

- Is this special solution $f(t)$ to (3.1) stable in the following sense: "$f(t)$ converges if t tends to infinity"?
- Does every solution to the difference-differential equation (3.1) converge?

As to equation (3.2), it is almost obvious that every solution is stable, but concerning equation (3.1), we will see later that the answer depends on the constant c. But before we do so, we will show the consequences of a slight modification:

$$f'(t+1) = f(t) \quad \text{with} \quad f(t) = 1+t, \ \ t \in [0,1] \tag{3.3}$$

that has the obviously non-stable solution

$$f(t) = \sum_{j=0}^{n} \frac{(t-j+1)^j}{j!}, \quad n \leqslant t \leqslant n+1 \ \text{ for } n = 1,2,3,\ldots$$

We may ask whether it is possible to generate a stable solution to this difference-differential equation by choosing another suitable *rational* initial function $f\colon [0,1] \to \mathbb{R}$. As we will show, the answer is no. However, there are infinitely many *irrational* initial functions that imply a stable solution.

In order to prove our answers and to investigate the dynamic properties of the system described in Section 1, we revert to a theorem by Hilb [9] which has been almost forgotten because it appears too complicated for practical use. For the problem at hand, however, this theorem provides a crucial clue: *Any differentiable solution to our functional equation can be written as a uniformly convergent series.* This allows us to determine the stability properties of our differential-difference equation by inspecting the "characteristic roots", not unlike the treatment of linear differential equations. As we shall see, however, the characteristic equation of a homogeneous difference-differential equation ordinarily has infinitely many roots.

We present the basic theorem of Hilb [9] in a form adapted to our problem:

Theorem 1. *Consider the general homogeneous linear difference-differential equation with real constants k_{pq} and delays $0 = h_0 < h_1 < h_2 < \ldots < h_n$ in the form*

$$\sum_{p=0}^{m}\sum_{q=0}^{n} k_{pq} f^{(p)}(t + h_q) = 0,$$

where $f^{(0)}(t) = f(t)$. Consider further the characteristic equation that results from inserting $f(t) = \mathrm{e}^{zt}$ into the difference-differential equation:

$$\Pi(z) = \sum_{p=0}^{m}\sum_{q=0}^{n} k_{pq} z^p \mathrm{e}^{h_q z} = 0.$$

For the time being, let all the complex solutions z_ν to the characteristic equation be simple. Provided that the coefficients k_{m0} and k_{mn} are different from zero,[5] one can choose any m-times differentiable function $f\colon\ (t_0, t_0 + h_n) \to \mathbb{R}$ satisfying the difference-differential equation at $t = t_0 + h_n$. This choice uniquely determines the solution $f(t)$ to the difference-differential equation for all $t > t_0$, and $f(t)$ is given by the convergent series

$$f(t) = \sum_{\nu} \frac{-C_\nu(t_0)}{\Pi'(z_\nu)} \cdot \mathrm{e}^{z_\nu t}$$

with

$$\begin{aligned} C_\nu(t_0) = &\sum_{p=0}^{m}\sum_{q=0}^{n-1} k_{pq} z_\nu^p \mathrm{e}^{h_q z_\nu} \int_{t_0+h_q}^{t_0+h_n} \mathrm{e}^{-z_\nu \mu} f(\mu)\,\mathrm{d}\mu \\ &- \sum_{p=1}^{m}\sum_{q=0}^{n} k_{pq} z_\nu^p \sum_{s=0}^{p-1} \frac{f^{(s)}(t_0 + h_q)\mathrm{e}^{-z_\nu s_0}}{z_\nu^{s+1}}. \end{aligned}$$

The convergence is uniform, i.e. it does not depend on the $t \in [t_0, K]$ for any big K.

Hilb arrived at this result by the use of the residue theorem by Cauchy, and at the end of his 33-page paper he mentions that the formula for the coefficients C_ν holds even if the solutions to the characteristic equation are not simple. In this case,

[5] If this condition is not satisfied, f must in general be infinitely many times differentiable in $t_0 + h_n$.

however, the C_ν are of a polynomial form in t, the degree being $m-1$, if z_ν is an m-fold solution to the characteristic equation. The latter fact is also mentioned by Hadeler [8, p. 170] and Driver [3, p. 323].

The consequence of this convergence is, roughly speaking, that in order to study the behaviour of the solution $f(t)$ when t tends to infinity, we can stop adding up the above series at a sufficiently large sum index. We can prove the following theorem:

Theorem 2. *Let $M := \{z_\nu = u_\nu + \mathrm{i}v_\nu$ with real $u_\nu,\ v_\nu\}$ be the set of all solutions to the characteristic equation $\Pi(z) = 0$ in Theorem 1; let further the solution $f(t)$ to the corresponding difference-differential equation be generated by some $z_{\nu j} = u_{\nu j} + \mathrm{i}v_{\nu j} \in M$, i.e. $f(t)$ is a linear combination of the functions $\exp(z_{\nu j} \cdot t)$, which are independent, as was also proved by Hilb [9]. Then the solution $f(t)$ is stable if and only if all $u_{\nu j}$ are negative.*

P r o o f. Assuming there exists a positive $u_{\nu j}$, then it is obvious that $|f(t)|$ tends to infinity if t tends to infinity, because all $\exp(z_{\nu j} \cdot t)$ are independent. On the other hand, if all $u_{\nu j}$ are negative, then watching the solution $f(t)$ in $(t_0, K]$ by Theorem 1 we may take only a finite linear combination $f(t)$ of functions

$$\frac{C_{\nu j}(t_0)}{\Pi'(z_{\nu j})} \exp(z_{\nu j} \cdot t),$$

without making a big difference to the true solution $f(t)$. Then this linear combination $f(t)$ is stable because of the decreasing $|f(t)|$ to zero; for all $C_{\nu j}(t_0)$ are constant or utmost polynomial in t. □

Using Theorem 2, we are now in position to prove the assertions made at the start of this section. We insert a complex-valued function $f(t) = \mathrm{e}^{zt}$ with $z = a + \mathrm{i}b$ $(a, b \in \mathbb{R})$ and $f'(t) = z\mathrm{e}^{zt}$ in (3.1), and then proceed to solve the characteristic equation:

$$z \cdot \mathrm{e}^z = -c. \tag{3.4}$$

This equation has infinitely many complex solutions z_ν, and we now know that every differentiable solution to a linear functional differential equation with constant coefficients can be expressed by a convergent generalized Fourier series

$$\sum_\nu \Gamma_\nu \cdot \mathrm{e}^{z_\nu},$$

where the coefficients Γ_ν are constants if all z_ν are simple solutions to (3.4). However, it is nearly impossible to calculate all z_ν exactly. Hilb's formulas for the Γ_ν, moreover depending on the given function f in the starting interval $[t_0, t_0 + \delta]$, are too complicated for practical calculation. Nevertheless, it may be possible to decide under what conditions solutions $z_\nu = a_\nu + ib_\nu$ with negative a_ν exist, i.e. stable solutions to the differential equation with an advancing argument.

Returning to the general solution to (3.1), we obtain

$$(a + ib)e^{a+ib} = -c, \tag{3.5}$$

where b can be positive or negative. By splitting (3.5), we get the system

$$e^a(a\cos b - b\sin b) = -c, \tag{3.6}$$

$$b\cos b + a\sin b = 0. \tag{3.7}$$

For $\sin b \neq 0$, equation (3.7) implies $a = -b\cos b/\sin b$. Inserting this term into (3.6), we get

$$b/\sin b = c \cdot e^{b\cos b/\sin b}. \tag{3.8}$$

The graph of the left-hand side $b/\sin b$ of (3.8) is intersected by the graph of the right-hand side of (3.8), and we see infinitely many solutions b_k for $b > \pi$, all implying negative a_k, but only one solution a_1 for $b_1 \in [0, \pi]$. This solution implies a negative a_1 for $c < \pi/2$ and further a positive a_1 for $c > \pi/2$. The case $\sin b_1 = 0$ implies $b_1 = 0$ and, by (3.6), a negative a_1; this means a stable real solution.

Hence, by means of Theorem 2, we can state that every solution of (3.1) is stable if and only if $c \in (0, \pi/2)$. For $c = \pi/2$, we obtain $a_1 = 0$, $b_1 = \pi/2$, and the corresponding solution is non-stable, but periodic.

This example shows all possible types of solutions to a linear difference-differential equation with constant coefficients, and, because all solutions to the characteristic equation can be found graphically, it can be easily followed.

As to equation (3.3), there exist infinitely many $z_\nu = a_v + ib_\nu$ with negative a_ν, but also one real positive z_ν. If we want a stable solution, we have to take a linear combination of the $\exp(z_\nu \cdot t)$ with negative a_ν, but this is not a rational function.

4. Stability properties of the solutions to the dynamic monetary model

The examples in the preceding section served to introduce a solution method that is different from the use of the Laplace transform by which many examples are treated in Bellman and Cooke [1]. In Krtscha [11] and some other papers, the fixed-point principle of Banach is applied to prove the uniqueness of a given solution, see Driver [3, p. 255–267]. The disadvantage of the latter method is its limitation to local statements. However, in order to use our Theorem 2 we must find out where the characteristic roots are situated. For this purpose, we will use a "rotation method" which is applicable in cases where the rotation method described by Hale and Verduyn Lunel [4, p. 415, Theorem A.2] does not work since there exist roots to the right of the imaginary axis.

For the economic model developed in Section 2, it is again easy to find the real solutions z_ν to the characteristic equation graphically. However, as we are interested only in stable solutions, we have to show that all other solutions $z_\nu = a_v + ib_\nu$ have positive a_ν, implying not bounded solutions. If this can be achieved, there is a unique *stable* solution to the system that determines the model dynamics.

Theorem 3. *Given the conditions* (2.12), (2.13) and (2.14) *on the coefficients A, B and C, the characteristic equation of the difference-differential equation* (2.11) *admits no complex root with nonpositive real part, except a unique negative one.*

Proof. With

$$\Pi(z) := B - A\mathrm{e}^{\Omega z} - BCz + ACz\mathrm{e}^{\Omega z} - z^2 \tag{4.1}$$

we get the characteristic equation

$$\Pi(z) = 0 \tag{4.2}$$

It is not difficult to show graphically, as von Kalckreuth and Schröder [10] have done, that there is one and only one negative real solution to the characteristic equation. To complete the proof we now have to show that this stable solution is unique. □

Lemma. *There is no complex solution $z = -u + iv$ with $u \geqslant 0$ and $v \neq 0$ to the characteristic equation $B - A\mathrm{e}^{\Omega z} - BCz + ACz\mathrm{e}^{\Omega z} - z^2 = 0$ with $0 < A < B$ and $C > 0$.*

P r o o f. By inserting $z = -u + iv$ in the characteristic equation and splitting up into the real and the imaginary part, we obtain a system of two equations

$$\frac{B}{A} = \mathrm{e}^{-u\Omega}\cos v\Omega + \frac{1}{A}\cdot\frac{(u^2-v^2)(1+uC)+2uv^2C}{(1+uC)^2+v^2C^2}, \tag{4.3}$$

$$\frac{1}{A} = \mathrm{e}^{-u\Omega}\sin v\Omega\cdot\frac{(1+uC)^2+v^2C^2}{(v^2-u^2)vC+(1+uC)2uv}. \tag{4.4}$$

Since $\frac{B}{A} > 1$ and $\mathrm{e}^{-u\Omega}\cos v\Omega \leqslant 1$, equation (4.3) implies the inequality

$$(u^2-v^2)(1+uC)+2uv^2C > 0, \tag{4.5}$$

so that $u = 0$ is impossible. A purely imaginary solution to the characteristic equation, i.e. a periodic solution, can thus be excluded.

In order to consider solutions within the second and third quadrant of the complex plane, we watch the plane along a rotating ray, i.e. we define

$$v := mu, \quad \text{with } u > 0 \text{ and fixed real } m \neq 0,$$

and insert it in (4.4), to get

$$\frac{1}{A} = \mathrm{e}^{-u\Omega}\frac{\sin mu\Omega}{mu^2}\cdot\frac{(1+uC)^2+m^2u^2C^2}{(m^2+1)uC+2}.$$

Inserting this in (4.3), we obtain the equation

$$\frac{B}{A} = \mathrm{e}^{-u\Omega}\left[\cos mu\Omega + \frac{\sin mu\Omega}{m}\frac{(1-m^2)(1+uC)+2um^2C}{(m^2+1)uC+2}\right], \tag{4.6}$$

which implies the inequality

$$\frac{B}{A} \leqslant \mathrm{e}^{-u\Omega}\left[1+\left|\frac{\sin mu\Omega}{m}\right|\cdot\left|\frac{1-m^2+uC(1+m^2)}{2+uC(1+m^2)}\right|\right]. \tag{4.7}$$

Due to (4.5), the last ratio in (4.7) is positive, and as $1 - m^2 < 2$, this expression is bounded by 1. Hence (4.7) implies

$$\frac{B}{A} \leqslant \frac{1+\left|\frac{\sin mu\Omega}{m}\right|}{\mathrm{e}^{u\Omega}} \leqslant \frac{1+\frac{|m|u\Omega}{|m|}}{1+u\Omega+\frac{u^2\Omega^2}{2!}+\dots} \leqslant 1.$$

This last inequality is in contradiction to $A < B$. This proves our lemma, and also the fact that a non-exploding and non-trivial solution to the differential-difference equation can only be based upon a negative real solution of the characteristic equation to the differential difference equation. As there is one and only one such negative real characteristic root, the series of the non-exploding solution to our functional equation is reduced to

$$f(x) = -\frac{\exp(z_1 \cdot t)}{\Pi'(z_1)} C_1, \tag{4.8}$$

where the constant C_1, defined in Theorem 1, can be chosen in such a way that the initial condition for $t = 0$ be fulfilled. □

By imposing equation (2.15), von Kalckreuth and Schröder [10] confine the solution of system (2.1)–(2.6) to a simple exponential. Dropping this *ad-hoc* restriction and thus generalizing the system to an unrestricted perfect foresight model does not lead to additional stable solutions. The initial condition that corresponds to this perfect foresight equilibrium consists in the self fulfilling expectation on part of the market participants that the time path described by equation (4.8) will hold during the starting interval, $t \in [t_0, t_0 + \Omega]$. All the economic conclusions developed in von Kalckreuth and Schröder [10] with respect to monetary transmission and speed of adjustment under decreasing service life of capital remain intact and survive the generalization.

References

[1] *R. Bellman, K. L. Cooke*: Differential-Difference Equations. Academic Press, New York-London, 1963.

[2] *R. Dornbusch*: Expectation and exchange rate dynamics. Journal of Political Economics *84* (1976), 1161–1176.

[3] *R. D. Driver*: Ordinary and Delay Differential Equations. Springer-Verlag, New York-Heidelberg-Berlin, 1997.

[4] *J. K. Hale, S. M. Verduyn Lunel*: Introduction to Functional Differential Equations. Springer-Verlag, New York, 1993.

[5] *W. H. Fisher, S. J. Turnovsky*: Fiscal policy and the term structure of interest rates: an intertemporal analysis. Journal of Money, Credit and Banking *24* (1992), 1–26.

[6] *M. R. Gray, S. J. Turnovsky*: The stability of exchange rate dynamics under perfect myopic foresight. Int. Econ. Rev. *20* (1979), 643–660.

[7] *A. de la Fuente*: Mathematical Methods and Models for Economists. Cambridge University Press, Cambridge, 2000.

[8] *K. P. Hadeler*: Mathematik für Biologen. Springer-Verlag, Heidelberg, 1974.

[9] *E. Hilb*: Zur Theorie der linearen funktionalen Differentialgleichungen. Math. Ann. *78* (1918), 137–170.

[10] *U. von Kalckreuth, J. Schröder*: Monetary transmission in the new economy: service life of capital, transmission channels and the speed of adjustment. Jahrbuch für Wirtschaftswissenschaften (Review of Economics) *53* (2002), 125–141.

[11] *M. Krtscha*: Short-term and long-term interest rates in a monetary model of a closed economy. Operations Research 91. Physica Verlag Heidelberg, 1991.

[12] *M. Krtscha*: The dependence of the price level on the expansion of the money supply in closed economies. Mathematical Modelling in Economics. Springer-Verlag, Berlin, 1993, pp. 249–259.
[13] *J. H. McCulloch*: Measuring the term structure of interest rates. The Journal of Business *44* (1971), 19–31.
[14] *T. Sargent, N. Wallace*: The stability of models of money and growth with perfect foresight. Econometrica *41* (1973), 1043–1048.
[15] *C. A. Wilson*: Anticipated shocks and exchange rate dynamics. Journal of Political Economicy *87* (1979), 639–647.

Authors' addresses: *U. von Kalckreuth*, Economic Research Centre of the Deutsche Bundesbank, Deutsche Bundesbank, P.O. Box 10 06 02, D-60006 Frankfurt a.M., Germany, e-mail: `ulf.von-kalckreuth@bundesbank.de`; *M. Krtscha*, Institute for Mathematical Stochastics, University of Karlsruhe, Englerstr. 2, D-76128 Karlsruhe, Germany, e-mail: `manfred.krtscha@mathematik.uni-karlsruhe.de`.

STABLE SOLUTIONS TO HOMOGENEOUS DIFFERENCE-DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS: ANALYTICAL INSTRUMENTS AND AN APPLICATION TO MONETARY THEORY

Ulf von Kalckreuth, Frankfurt, Manfred Krtscha, Karlsruhe

(Received August 5, 2002, in revised version July 2, 2003)

Abstract. In economic systems, reactions to external shocks often come with a delay. On the other hand, agents try to anticipate future developments. Both can lead to difference-differential equations with an advancing argument. These are more difficult to handle than either difference or differential equations, but they have the merit of added realism and increased credibility. This paper generalizes a model from monetary economics by von Kalckreuth and Schröder. Working out its stability properties, we present a general method for determining the stability of any solution to a homogeneous linear difference-differential equation with constant coefficients and advancing arguments.

Keywords: linear difference-differential equations, stability, monetary transmission

MSC 2000: 39A11, 39B99, 91BC2

1. Preliminaries

In economic systems, reactions to external shocks often come with a delay. On the other hand, agents try to anticipate future developments. Both can lead to difference-differential equations with an advancing argument. These are more difficult to handle than either difference or differential equations, but they have the merit of added realism and increased credibility. Working out a problem from monetary theory, we present a general method for determining the stability of any solution to a homogeneous linear difference-differential equation with constant coefficients and advancing arguments.

In the next section we start by presenting the generalisation of a monetary macroeconomic model by von Kalckreuth and Schröder [10]. In order for this model to have explanatory power, it is necessary to show that there is one and only one stable solution to the system. The following section introduces the reader into the theory of linear differential-difference equations with constant coefficients and presents a method of determining the stability of their solutions, making use of a hitherto almost ignored theorem by Hilb [9]. Our method is rather general and can be applied to a large class of dynamical problems. The last section applies our method to the economic problem at hand and draws conclusions.

2. A model of monetary transmission

In order to investigate the interactions between the service life of capital, the term structure of interest rates and the impact of monetary policy on open economies, von Kalckreuth and Schröder [10] developed a dynamic macroeconomic model. This model considers an interest-rate structure within the framework of the Dornbusch [2] overshooting model. Whereas the central bank is able to influence the nominal short-term rate, aggregate demand depends on the real long-term rate. The interest-rate structure embodied in this model leads to an advancing argument in a system of functional equations. The authors solved this system by imposing additional restrictions.

Here, we want to investigate the *unrestricted* model dynamics. We show that there is a unique stable solution, which is identical to the solution described by von Kalckreuth and Schröder [10] in solving the restricted model.

The model contains the following equations:

$$\overline{M} - P(t) = \alpha_1 \overline{Y} - \alpha_2 i(t), \tag{2.1}$$

$$r(t) = i(t) - \dot{P}(t), \tag{2.2}$$

$$D(t) = \beta_0 + \beta_1 (E(t) - P(t)) + \beta_2 \overline{Y} - \beta_3 R_\Omega(t), \tag{2.3}$$

$$\dot{P}(t) = \Gamma(D(t) - \overline{Y}), \tag{2.4}$$

$$i(t) = i^* + \dot{E}(t), \tag{2.5}$$

$$\dot{R}_\Omega(t) = \frac{1}{\Omega}(r(t+\Omega) - r(t)). \tag{2.6}$$

A bar denotes a steady-state value. All coefficients in (2.1)–(2.6) are strictly positive. Eq. (2.1) is a money-market equation with M, P and Y being the logarithms of nominal money supply, price level and constant real income and i the nominal money-market interest rate. While P is restricted to be a globally continuous function of time, all the other endogenous variables are allowed to jump discontinuously in response to an unexpected shock at $t = 0$. Equation (2.2) defines the real money-market interest rate r as the difference between the short-term nominal interest rate i and inflation. In (2.3), the logarithm of the aggregate demand, D, depends on the logarithms of the real exchange rate $E - P$, the constant real income and the long-term real interest rate R_Ω, with maturity Ω. For simplicity, a uniform service life of capital is assumed. Equation (2.4) is a simple Phillips relationship, in which the rate of inflation $\dot{P} = \mathrm{d}P/\mathrm{d}t$ is determined by the ratio of the (variable) aggregate demand to the (constant) supply. Equation (2.5) represents the open interest-parity condition, with i^* being the given nominal short-term interest rate in the international money market. Equation (2.6), finally, relates the change in the real long-term interest rate to the difference between future short-term rates and present short-term rates.

The last equation brings in an advancing argument into the system. Because of its importance for our mathematical problem, the relationship between short and long-term rates will be derived here from first principles, as an arbitrage equation[1] for bonds of finite maturity.

Complete foresight in a perfect asset market implies the equality of the instantaneous real rates of return on bonds and investments in the money market. Consider a zero bond of arbitrary time to maturity Ω, issued at time t. Let N be the issue price and $R_\Omega(t)$ the long-term rate for bonds with time to maturity Ω. At time $t+\Omega$, then, the holder of the bond receives a payment of $N \exp(\Omega R_\Omega(t))$. Since there are no interest payments until maturity, according to the Hotelling rule the arbitrage condition is given by

$$\frac{\dot{K}(s)}{K(s)} = r(s), \tag{2.7}$$

with $K(s)$ as the real market value of a bond at any point in time s between t and $t + \Omega$. Taking into consideration the terminal condition that, at maturity, the real market value $K(s)$ is bound to be equal to the real value of the principal and the accrued interest, the general solution of (2.7) becomes

$$K(s) = N \exp\left(\Omega R_\Omega(t) - \int_s^{t+\Omega} r(\tau)\,\mathrm{d}\tau\right). \tag{2.8}$$

At the date of issue, however, the value of this expression must be equal to N, which immediately yields the arbitrage equation

$$R_\Omega(t) = \frac{1}{\Omega}\int_t^{t+\Omega} r(\tau)\,\mathrm{d}\tau. \tag{2.9}$$

[1] See, for example, [13]. Fisher and Turnovsky [5] also use this equation in the context of a dynamic macroeconomic model.

Equation (2.9) gives us the term structure of interest rates, according to the expectation theory, for the case of continuous interest compounding. The (continuous) long-term rate R_Ω is determined as the arithmetic mean of the short-term rates within the relevant time interval. The former thus anticipates the movement of the latter. Taking the derivative on both sides yields equation (2.6). The interest rate R_Ω, by definition, gives us the cost of capital of an investment project characterized by one single payment I at t and one single, certain return of V at the end of its lifetime Ω. Investigating the dynamic system for varying Ω thus permits us to describe the effects of a decreasing service life of capital, as a result of accelerated technical progress, on the dynamics of macroeconomic adjustment to various kinds of shocks.

We can reduce the system to one single dynamic equation. Long-run equilibrium is characterized by the conditions

$$\dot{P}(t) = 0 \quad \text{and} \quad \dot{E}(t) = 0. \tag{2.10}$$

Substituting these conditions into the system (2.1)–(2.6) readily yields a particular solution to the system, the steady-state solution. Given a shock to the system, e.g. by a monetary expansion, we can therefore concentrate on finding the solutions to the following set of homogeneous equations that describes the *deviations* from equilibrium:[2]

$$P(t) = \alpha_2 i(t), \tag{2.1'}$$
$$r(t) = i(t) - \dot{P}(t), \tag{2.2'}$$
$$D(t) = \beta_1 E(t) - \beta_1 P(t) - \beta_3 R_\Omega(t), \tag{2.3'}$$
$$\dot{P}(t) = \Gamma D(t), \tag{2.4'}$$
$$i(t) = \dot{E}(t), \tag{2.5'}$$
$$\dot{R}_\Omega(t) = \frac{1}{\Omega}(r(t+\Omega) - r(t)). \tag{2.6'}$$

Repeated substitution yields the following homogeneous differential-difference equation with an advancing argument for the logarithm of the price level P:

$$\ddot{P}(t) = -AP(t+\Omega) + BP(t) + AC\dot{P}(t+\Omega) - BC\dot{P}(t), \tag{2.11}$$

[2] The levels of the variables in (2.1′)–(2.6′) are deviations from the steady state. In order to save notation, we will not introduce new symbols.

where

$$A = \frac{\Gamma\beta_3}{\Omega\alpha_2} > 0, \tag{2.12}$$

$$B = \frac{\Gamma}{\alpha_2}\Big(\beta_1 + \frac{\beta_3}{\Omega}\Big) > A > 0, \tag{2.13}$$

$$C = \alpha_2 > 0. \tag{2.14}$$

In order to solve the system analytically, von Kalckreuth and Schröder [10] used the following *additional* restriction on the time path of the deviation from equilibrium:

$$\dot{P}(t) = \lambda P(t), \quad \lambda \in \mathbb{R}, \;\; \lambda < 0. \tag{2.15}$$

This restriction requires the perfect foresight path to be of an especially simple, adaptive structure. Besides a family of exploding solutions, the system has one and only one stable solution. It may well be argued, however, that this restriction is essentially *ad hoc*, and it is not easily seen whether and how the results of the paper critically depend on constraining the solution to be exponential. Here, we do not intend to discuss the economic results of von Kalckreuth and Schröder [10], as they have been published elsewhere. Instead, *we want to investigate whether it is possible to drop the additional restriction without losing the results.* This is possible if there are no additional *stable* solutions to the system apart from the solution found by imposing (2.15).

In a model of perfect foresight, every market participant has the same expectation and these expectations coincide with the actual development. If there is more than one solution to a dynamical economic model of an asset market with perfect foresight, it has to be stated which of these solutions the agents will coordinate upon. Unstable solutions are generally excluded. Real variables, such as aggregate demand, real interest rates and real exchange rates, under normal economic circumstances remain bounded. A diverging real exchange rate, for example, would mean that in the course of time, the entire national product would sell for just a small amount of foreign money. In order to find the relevant solutions among various time paths that fulfil the system equations, it is therefore assumed that market participants do not coordinate on exploding solutions if there is a stable one.[3]

In dynamic systems describing physical phenomena, the initial conditions determine whether among several solutions a stable time path is attained or not. In economic systems with asset markets, it may be the other way round. With their

[3] This would naturally have to be different if the model was set up to explain stock market bubbles.

expectations, economic agents coordinate on the stable solution, and the initial condition, such as the immediate reaction of a stock price or the exchange rate to a shock, are determined by a jump discontinuity in order to set the system on the stable path. Asset prices jump because every market participant knows that they have to jump. If anybody were willing to pay more for that asset or sell it for less, he or she would be losing money. In other words, the initial conditions are not exogenous to the system, but they are endogenously determined. Working out the analytics of perfect foresight dynamics was a major advance in economic theory, due to, among others, Sargent and Wallace [14]. As concerns Dornbusch [2] overshooting model, see the papers by Wilson [15] and Gray and Turnovsky [6].[4]

If there is more than one stable solution, the model leaves the dynamics indeterminate. What is worse, the researcher has to explain by which mechanism the market participants manage to coordinate uniformly on a particular one of these solutions. We therefore want to know whether the stable solution found by von Kalckreuth and Schröder [10] under their restrictive assumption remains unique in the context of an unrestricted perfect foresight model. As we will see immediately below, the relevant initial conditions for the system in question are given not simply by an initial *value*, but by an initial *time path*. To the best of our knowledge, this is new in the economics literature. We will investigate the dynamic properties of our system using a method that can be applied to determine the stability of the solutions for any linear difference-differential equation with an advancing argument.

3. Linear difference-differential equations and a general method of investigating their stability

At this point it is convenient to review some of the basic properties of linear difference-differential equations. Consider the following simple example with positive constants c and δ, taken from Krtscha [12]:

$$f'(t+\delta) = -c\,f(t). \tag{3.1}$$

The positive constant δ plays an important role for the nature of the solution. For $\delta = 0$, the solution to (3.1) would be uniquely determined by the condition $f(t_0) = y_0$, whereas for $\delta > 0$ the solution needs more: it is uniquely determined by any given function $f\colon\ [t_0, t_0+\delta] \to \mathbb{R}$ that is continuous and differentiable in $(t_0, t_0+\delta)$,

[4] For a textbook treatment of perfect foresight dynamics that also covers the Dornbusch [2] overshooting model, see de la Fuente [7, Section 11.2].

implying the differentiability of f in (t_0, ∞). Then equation (3.1) leads to the formula

$$f(t + n\delta) = -c \int_0^t f(\tau + (n-1)\delta)\, \mathrm{d}\tau + f(n\delta),$$

by which we can regressively calculate $f(t)$ for all $n \in \mathbb{N}$. We want to give a specific example:

$$f'(t+1) = -f(t) \quad \text{with} \quad f(t) = 1 - t, \quad t \in [0,1]. \tag{3.2}$$

Then, by means of the regression-formula for $t \in [0,1]$, we get

$$f(t+1) = -t + t^2/2,$$

implying $f(1) = 0$ and $f(2) = -\frac{1}{2}$, $f(t+2) = t^2/2 - t^3/3 - 1/2$, $f(t+3) = -t^3/3! + t^4/4! + t/2 - 1/3!$, and so on.

Now we may ask two questions:

- Is this special solution $f(t)$ to (3.1) stable in the following sense: "$f(t)$ converges if t tends to infinity"?
- Does every solution to the difference-differential equation (3.1) converge?

As to equation (3.2), it is almost obvious that every solution is stable, but concerning equation (3.1), we will see later that the answer depends on the constant c. But before we do so, we will show the consequences of a slight modification:

$$f'(t+1) = f(t) \quad \text{with} \quad f(t) = 1 + t, \;\; t \in [0,1] \tag{3.3}$$

that has the obviously non-stable solution

$$f(t) = \sum_{j=0}^{n} \frac{(t-j+1)^j}{j!}, \quad n \leqslant t \leqslant n+1 \;\text{ for }\; n = 1, 2, 3, \ldots$$

We may ask whether it is possible to generate a stable solution to this difference-differential equation by choosing another suitable *rational* initial function $f\colon [0,1] \to \mathbb{R}$. As we will show, the answer is no. However, there are infinitely many *irrational* initial functions that imply a stable solution.

In order to prove our answers and to investigate the dynamic properties of the system described in Section 1, we revert to a theorem by Hilb [9] which has been almost forgotten because it appears too complicated for practical use. For the problem at hand, however, this theorem provides a crucial clue: *Any differentiable solution to our functional equation can be written as a uniformly convergent series.* This allows us to determine the stability properties of our differential-difference equation by inspecting the "characteristic roots", not unlike the treatment of linear differential equations. As we shall see, however, the characteristic equation of a homogeneous difference-differential equation ordinarily has infinitely many roots.

We present the basic theorem of Hilb [9] in a form adapted to our problem:

Theorem 1. *Consider the general homogeneous linear difference-differential equation with real constants k_{pq} and delays $0 = h_0 < h_1 < h_2 < \ldots < h_n$ in the form*

$$\sum_{p=0}^{m} \sum_{q=0}^{n} k_{pq} f^{(p)}(t + h_q) = 0,$$

where $f^{(0)}(t) = f(t)$. Consider further the characteristic equation that results from inserting $f(t) = \mathrm{e}^{zt}$ into the difference-differential equation:

$$\Pi(z) = \sum_{p=0}^{m} \sum_{q=0}^{n} k_{pq} z^p \mathrm{e}^{h_q z} = 0.$$

For the time being, let all the complex solutions z_ν to the characteristic equation be simple. Provided that the coefficients k_{m0} and k_{mn} are different from zero,[5] *one can choose any m-times differentiable function $f\colon (t_0, t_0 + h_n) \to \mathbb{R}$ satisfying the difference-differential equation at $t = t_0 + h_n$. This choice uniquely determines the solution $f(t)$ to the difference-differential equation for all $t > t_0$, and $f(t)$ is given by the convergent series*

$$f(t) = \sum_{\nu} \frac{-C_\nu(t_0)}{\Pi'(z_\nu)} \cdot \mathrm{e}^{z_\nu t}$$

with

$$\begin{aligned} C_\nu(t_0) = {} & \sum_{p=0}^{m} \sum_{q=0}^{n-1} k_{pq} z_\nu^p \mathrm{e}^{h_q z_\nu} \int_{t_0+h_q}^{t_0+h_n} \mathrm{e}^{-z_\nu \mu} f(\mu) \,\mathrm{d}\mu \\ & - \sum_{p=1}^{m} \sum_{q=0}^{n} k_{pq} z_\nu^p \sum_{s=0}^{p-1} \frac{f^{(s)}(t_0 + h_q) \mathrm{e}^{-z_\nu s_0}}{z_\nu^{s+1}}. \end{aligned}$$

The convergence is uniform, i.e. it does not depend on the $t \in [t_0, K]$ for any big K.

Hilb arrived at this result by the use of the residue theorem by Cauchy, and at the end of his 33-page paper he mentions that the formula for the coefficients C_ν holds even if the solutions to the characteristic equation are not simple. In this case,

[5] If this condition is not satisfied, f must in general be infinitely many times differentiable in $t_0 + h_n$.

however, the C_ν are of a polynomial form in t, the degree being $m-1$, if z_ν is an m-fold solution to the characteristic equation. The latter fact is also mentioned by Hadeler [8, p. 170] and Driver [3, p. 323].

The consequence of this convergence is, roughly speaking, that in order to study the behaviour of the solution $f(t)$ when t tends to infinity, we can stop adding up the above series at a sufficiently large sum index. We can prove the following theorem:

Theorem 2. *Let $M := \{z_\nu = u_\nu + \mathrm{i}v_\nu$ with real $u_\nu,\ v_\nu\}$ be the set of all solutions to the characteristic equation $\Pi(z) = 0$ in Theorem 1; let further the solution $f(t)$ to the corresponding difference-differential equation be generated by some $z_{\nu j} = u_{\nu j} + \mathrm{i}v_{\nu j} \in M$, i.e. $f(t)$ is a linear combination of the functions $\exp(z_{\nu j}\cdot t)$, which are independent, as was also proved by Hilb* [9]. *Then the solution $f(t)$ is stable if and only if all $u_{\nu j}$ are negative.*

Proof. Assuming there exists a positive $u_{\nu j}$, then it is obvious that $|f(t)|$ tends to infinity if t tends to infinity, because all $\exp(z_{\nu j}\cdot t)$ are independent. On the other hand, if all $u_{\nu j}$ are negative, then watching the solution $f(t)$ in $(t_0, K]$ by Theorem 1 we may take only a finite linear combination $f(t)$ of functions

$$\frac{C_{\nu j}(t_0)}{\Pi'(z_{\nu j})}\exp(z_{\nu j}\cdot t),$$

without making a big difference to the true solution $f(t)$. Then this linear combination $f(t)$ is stable because of the decreasing $|f(t)|$ to zero; for all $C_{\nu j}(t_0)$ are constant or utmost polynomial in t. □

Using Theorem 2, we are now in position to prove the assertions made at the start of this section. We insert a complex-valued function $f(t) = \mathrm{e}^{zt}$ with $z = a + \mathrm{i}b$ $(a, b \in \mathbb{R})$ and $f'(t) = z\mathrm{e}^{zt}$ in (3.1), and then proceed to solve the characteristic equation:

$$z\cdot \mathrm{e}^{z} = -c. \tag{3.4}$$

This equation has infinitely many complex solutions z_ν, and we now know that every differentiable solution to a linear functional differential equation with constant coefficients can be expressed by a convergent generalized Fourier series

$$\sum_\nu \Gamma_\nu \cdot \mathrm{e}^{z_\nu},$$

where the coefficients Γ_ν are constants if all z_ν are simple solutions to (3.4). However, it is nearly impossible to calculate all z_ν exactly. Hilb's formulas for the Γ_ν, moreover depending on the given function f in the starting interval $[t_0, t_0+\delta]$, are too complicated for practical calculation. Nevertheless, it may be possible to decide under what conditions solutions $z_\nu = a_\nu + \mathrm{i}b_\nu$ with negative a_ν exist, i.e. stable solutions to the differential equation with an advancing argument.

Returning to the general solution to (3.1), we obtain

$$(a+\mathrm{i}b)\mathrm{e}^{a+\mathrm{i}b} = -c, \tag{3.5}$$

where b can be positive or negative. By splitting (3.5), we get the system

$$\mathrm{e}^{a}(a\cos b - b\sin b) = -c, \tag{3.6}$$

$$b\cos b + a\sin b = 0. \tag{3.7}$$

For $\sin b \neq 0$, equation (3.7) implies $a = -b\cos b/\sin b$. Inserting this term into (3.6), we get

$$b/\sin b = c\cdot \mathrm{e}^{b\cos b/\sin b}. \tag{3.8}$$

The graph of the left-hand side $b/\sin b$ of (3.8) is intersected by the graph of the right-hand side of (3.8), and we see infinitely many solutions b_k for $b > \pi$, all implying negative a_k, but only one solution a_1 for $b_1 \in [0, \pi]$. This solution implies a negative a_1 for $c < \pi/2$ and further a positive a_1 for $c > \pi/2$. The case $\sin b_1 = 0$ implies $b_1 = 0$ and, by (3.6), a negative a_1; this means a stable real solution.

Hence, by means of Theorem 2, we can state that every solution of (3.1) is stable if and only if $c \in (0, \pi/2)$. For $c = \pi/2$, we obtain $a_1 = 0$, $b_1 = \pi/2$, and the corresponding solution is non-stable, but periodic.

This example shows all possible types of solutions to a linear difference-differential equation with constant coefficients, and, because all solutions to the characteristic equation can be found graphically, it can be easily followed.

As to equation (3.3), there exist infinitely many $z_\nu = a_v + \mathrm{i}b_\nu$ with negative a_ν, but also one real positive z_ν. If we want a stable solution, we have to take a linear combination of the $\exp(z_\nu \cdot t)$ with negative a_ν, but this is not a rational function.

4. Stability properties of the solutions to the dynamic monetary model

The examples in the preceding section served to introduce a solution method that is different from the use of the Laplace transform by which many examples are treated in Bellman and Cooke [1]. In Krtscha [11] and some other papers, the fixed-point principle of Banach is applied to prove the uniqueness of a given solution, see Driver [3, p. 255–267]. The disadvantage of the latter method is its limitation to local statements. However, in order to use our Theorem 2 we must find out where the characteristic roots are situated. For this purpose, we will use a "rotation method" which is applicable in cases where the rotation method described by Hale and Verduyn Lunel [4, p. 415, Theorem A.2] does not work since there exist roots to the right of the imaginary axis.

For the economic model developed in Section 2, it is again easy to find the real solutions z_ν to the characteristic equation graphically. However, as we are interested only in stable solutions, we have to show that all other solutions $z_\nu = a_v + \mathrm{i}b_\nu$ have positive a_ν, implying not bounded solutions. If this can be achieved, there is a unique *stable* solution to the system that determines the model dynamics.

Theorem 3. *Given the conditions* (2.12), (2.13) and (2.14) *on the coefficients A, B and C, the characteristic equation of the difference-differential equation* (2.11) *admits no complex root with nonpositive real part, except a unique negative one.*

Proof. With

$$\Pi(z) := B - A\mathrm{e}^{\Omega z} - BCz + ACz\mathrm{e}^{\Omega z} - z^2 \tag{4.1}$$

we get the characteristic equation

$$\Pi(z) = 0 \tag{4.2}$$

It is not difficult to show graphically, as von Kalckreuth and Schröder [10] have done, that there is one and only one negative real solution to the characteristic equation. To complete the proof we now have to show that this stable solution is unique. □

Lemma. *There is no complex solution $z = -u + \mathrm{i}v$ with $u \geqslant 0$ and $v \neq 0$ to the characteristic equation $B - A\mathrm{e}^{\Omega z} - BCz + ACz\mathrm{e}^{\Omega z} - z^2 = 0$ with $0 < A < B$ and $C > 0$.*

P r o o f. By inserting $z = -u + \mathrm{i}v$ in the characteristic equation and splitting up into the real and the imaginary part, we obtain a system of two equations

$$\frac{B}{A} = \mathrm{e}^{-u\Omega}\cos v\Omega + \frac{1}{A}\cdot\frac{(u^2-v^2)(1+uC)+2uv^2C}{(1+uC)^2+v^2C^2}, \tag{4.3}$$

$$\frac{1}{A} = \mathrm{e}^{-u\Omega}\sin v\Omega\cdot\frac{(1+uC)^2+v^2C^2}{(v^2-u^2)vC+(1+uC)2uv}. \tag{4.4}$$

Since $\frac{B}{A} > 1$ and $\mathrm{e}^{-u\Omega}\cos v\Omega \leqslant 1$, equation (4.3) implies the inequality

$$(u^2-v^2)(1+uC)+2uv^2C > 0, \tag{4.5}$$

so that $u = 0$ is impossible. A purely imaginary solution to the characteristic equation, i.e. a periodic solution, can thus be excluded.

In order to consider solutions within the second and third quadrant of the complex plane, we watch the plane along a rotating ray, i.e. we define

$$v := mu, \quad \text{with } u > 0 \text{ and fixed real } m \neq 0,$$

and insert it in (4.4), to get

$$\frac{1}{A} = \mathrm{e}^{-u\Omega}\frac{\sin mu\Omega}{mu^2}\cdot\frac{(1+uC)^2+m^2u^2C^2}{(m^2+1)uC+2}.$$

Inserting this in (4.3), we obtain the equation

$$\frac{B}{A} = \mathrm{e}^{-u\Omega}\left[\cos mu\Omega + \frac{\sin mu\Omega}{m}\,\frac{(1-m^2)(1+uC)+2um^2C}{(m^2+1)uC+2}\right], \tag{4.6}$$

which implies the inequality

$$\frac{B}{A} \leqslant \mathrm{e}^{-u\Omega}\left[1 + \left|\frac{\sin mu\Omega}{m}\right|\cdot\left|\frac{1-m^2+uC(1+m^2)}{2+uC(1+m^2)}\right|\right]. \tag{4.7}$$

Due to (4.5), the last ratio in (4.7) is positive, and as $1 - m^2 < 2$, this expression is bounded by 1. Hence (4.7) implies

$$\frac{B}{A} \leqslant \frac{1+\left|\frac{\sin mu\Omega}{m}\right|}{\mathrm{e}^{u\Omega}} \leqslant \frac{1+\frac{|m|u\Omega}{|m|}}{1+u\Omega+\frac{u^2\Omega^2}{2!}+\ldots} \leqslant 1.$$

This last inequality is in contradiction to $A < B$. This proves our lemma, and also the fact that a non-exploding and non-trivial solution to the differential-difference equation can only be based upon a negative real solution of the characteristic equation to the differential difference equation. As there is one and only one such negative real characteristic root, the series of the non-exploding solution to our functional equation is reduced to

$$f(x) = -\frac{\exp(z_1 \cdot t)}{\Pi'(z_1)} C_1, \tag{4.8}$$

where the constant C_1, defined in Theorem 1, can be chosen in such a way that the initial condition for $t = 0$ be fulfilled. □

By imposing equation (2.15), von Kalckreuth and Schröder [10] confine the solution of system (2.1)–(2.6) to a simple exponential. Dropping this *ad-hoc* restriction and thus generalizing the system to an unrestricted perfect foresight model does not lead to additional stable solutions. The initial condition that corresponds to this perfect foresight equilibrium consists in the self fulfilling expectation on part of the market participants that the time path described by equation (4.8) will hold during the starting interval, $t \in [t_0, t_0 + \Omega]$. All the economic conclusions developed in von Kalckreuth and Schröder [10] with respect to monetary transmission and speed of adjustment under decreasing service life of capital remain intact and survive the generalization.

References

[1] *R. Bellman, K. L. Cooke*: Differential-Difference Equations. Academic Press, New York-London, 1963.

[2] *R. Dornbusch*: Expectation and exchange rate dynamics. Journal of Political Economics *84* (1976), 1161–1176.

[3] *R. D. Driver*: Ordinary and Delay Differential Equations. Springer-Verlag, New York-Heidelberg-Berlin, 1997.

[4] *J. K. Hale, S. M. Verduyn Lunel*: Introduction to Functional Differential Equations. Springer-Verlag, New York, 1993.

[5] *W. H. Fisher, S. J. Turnovsky*: Fiscal policy and the term structure of interest rates: an intertemporal analysis. Journal of Money, Credit and Banking *24* (1992), 1–26.

[6] *M. R. Gray, S. J. Turnovsky*: The stability of exchange rate dynamics under perfect myopic foresight. Int. Econ. Rev. *20* (1979), 643–660.

[7] *A. de la Fuente*: Mathematical Methods and Models for Economists. Cambridge University Press, Cambridge, 2000.

[8] *K. P. Hadeler*: Mathematik für Biologen. Springer-Verlag, Heidelberg, 1974.

[9] *E. Hilb*: Zur Theorie der linearen funktionalen Differentialgleichungen. Math. Ann. *78* (1918), 137–170.

[10] *U. von Kalckreuth, J. Schröder*: Monetary transmission in the new economy: service life of capital, transmission channels and the speed of adjustment. Jahrbuch für Wirtschaftswissenschaften (Review of Economics) *53* (2002), 125–141.

[11] *M. Krtscha*: Short-term and long-term interest rates in a monetary model of a closed economy. Operations Research 91. Physica Verlag Heidelberg, 1991.

[12] *M. Krtscha*: The dependence of the price level on the expansion of the money supply in closed economies. Mathematical Modelling in Economics. Springer-Verlag, Berlin, 1993, pp. 249–259.

[13] *J. H. McCulloch*: Measuring the term structure of interest rates. The Journal of Business *44* (1971), 19–31.

[14] *T. Sargent, N. Wallace*: The stability of models of money and growth with perfect foresight. Econometrica *41* (1973), 1043–1048.

[15] *C. A. Wilson*: Anticipated shocks and exchange rate dynamics. Journal of Political Economicy *87* (1979), 639–647.

Authors' addresses: *U. von Kalckreuth*, Economic Research Centre of the Deutsche Bundesbank, Deutsche Bundesbank, P.O. Box 10 06 02, D-60006 Frankfurt a.M., Germany, e-mail: `ulf.von-kalckreuth@bundesbank.de`; *M. Krtscha*, Institute for Mathematical Stochastics, University of Karlsruhe, Englerstr. 2, D-76128 Karlsruhe, Germany, e-mail: `manfred.krtscha@mathematik.uni-karlsruhe.de`.