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ON SOME ITERATED MEANS ARISING IN HOMOGENIZATION THEORY

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Abstract. We consider iteration of arithmetic and power means and discuss methods for determining their limit. These means appear naturally in connection with some problems in homogenization theory.

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1. INTRODUCTION

Means have interested mathematicians since antiquity. The most common means are power means, in particular the arithmetic, geometric and harmonic mean (with or without weights). For positive numbers a and b these means (with equal weights) are defined by the formulæ

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b},$$

respectively. There are many examples of iterated means. For example, if we start with any two given numbers a and b and repeatedly form the arithmetic and the harmonic mean, in that order, i.e. we reiterate the map which to the pair (a, b) assigns a new pair (a', b') given by

$$a' = \frac{a+b}{2}, \quad b' = \frac{2ab}{a+b}, \tag{1}$$

then we obtain sequences a', a'', a''', ... and b', b'', b''', ... These sequences converge to the common limit $G = \sqrt{ab}$. The algorithm defined by (1) is sometimes called the Newton algorithm (see [1]). Another algorithm, the algorithm studied by Gauss for the arithmetic-geometric mean, is obtained by taking instead successively the arithmetic and the geometric mean (in that order). Thus the map will in this case be

$$a' = \frac{a+b}{2}, \quad b' = \sqrt{ab}. \tag{2}$$

A more general algorithm which contains both the Newton algorithm and the Gauss algorithm can be found in [17].

In this paper we will study a related type of iterated means: Let $x > 0$, let $m, n \in \{1, 2, \ldots\}$, $p > 1$, $0 \leqslant v \leqslant 1$ and consider the weighted means $A_m, P_m \colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\begin{aligned} A_m(x) &= \alpha x + \beta, \\ P_m(x) &= \left(\gamma x^{\frac{1}{1-p}} + \delta\right)^{1-p}, \end{aligned}$$

where $\gamma = v^{\frac{1}{mn}}$, $\alpha = v^{\frac{n-1}{mn}}$ and $\gamma + \delta = \alpha + \beta = 1$. The iterated means $(A_m P_m)^m(x)$ and $(P_m A_m)^m$, and their limits, were introduced and studied in [7], [14] and [15] (here "multiplication" between A_m and P_m and "exponentiation" in m denote composition in the obvious way).

They appear naturally in connection with estimation of certain effective properties for multiscaled materials involving the so-called reiterated homogenization (see e.g. [4], [12], [13] and the references given there). Moreover, these iterated means converge to the same limit as m goes to infinity. Generally this limit is difficult to find, but in the case $p = 2$, it turns out to be the mean

$$1 + \frac{nv(x-1)}{n + (1-v)(x-1)},$$

which is the well known Hashin-Shtrikman lower bound if $x > 1$ and Hashin-Shtrikman upper bound if $x < 1$ (see [11]). This fact was proved in [14] by obtaining explicit formulæ for these means.

In this paper we derive the same formulæ by another method. Moreover, we discuss some alternative presentations of these means and their common limit in general.

The paper's organization is as follows. In Section 2 we review the corresponding result from [14] indicating a matrix proof for it and consider also the case $n = 1$ (Example 3). Section 3 is about the general case. In particular, in Remark 4 it is indicated that the expressions $(A_m P_m)^m$ and $(P_m A_m)^m$ are conjugate (as operators) to certain products of power means. This is important, because (see again Remark 5) it opens up a connection with the theory of *interpolation functions* (see [5], in particular Section 5.4, and the references to papers by J. Peetre given there). Finally, we consider some limiting cases in Section 4.

2. THE CASES $p = 2$ AND $n = 1$

We first note that if

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = A \begin{bmatrix} a \\ b \end{bmatrix},$$

where

$$A = \begin{bmatrix} k & l \\ s & r \end{bmatrix},$$

then

$$\frac{a'}{b'} = \frac{ka/b + l}{sa/b + r}.$$

Thus, if

$$F(x) = \frac{kx + l}{sx + r},$$

then the iterate F^m is given by

$$F^m(x) = \frac{a^{(m)}}{b^{(m)}}, \tag{3}$$

where

$$\begin{bmatrix} a^{(m)} \\ b^{(m)} \end{bmatrix} = A^m \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

Now, assume in addition that $l = 1 - k$ and $s = 1 - r$, i.e. the matrix A is row-stochastic. (We call a matrix row-stochastic if all elements are non-negative numbers and if the sum of the elements on any row is unity. Replacing the word row by column we, similarly, obtain the notion of column-stochastic.) By finding eigenvectors and eigenvalues we obtain that

$$A = PDP^{-1},$$

where

$$P = \begin{bmatrix} 1 & 1-k \\ 1 & r-1 \end{bmatrix}, \quad P^{-1} = \frac{1}{k+r-2} \begin{bmatrix} r-1 & k-1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & k+r-1 \end{bmatrix}.$$

Thus

$$A^m = PD^mP^{-1},$$

where

$$D^m = \begin{bmatrix} 1 & 0 \\ 0 & (k+r-1)^m \end{bmatrix}.$$

Accordingly we have a closed form expression for the iterate F^m. We observe that

$$A^m - I = P(D^m - I)P^{-1} = \frac{(k+r-1)^m - 1}{k+r-2} \begin{bmatrix} k-1 & -(k-1) \\ -(r-1) & r-1 \end{bmatrix},$$

i.e.

$$A^m = \frac{(k+r-1)^m - 1}{k+r-2} \begin{bmatrix} k-1 & -(k-1) \\ -(r-1) & r-1 \end{bmatrix} + I.$$

Multiplying by $[x, 1]^T$ we obtain the equations

$$\begin{aligned} a^m &= s((k-1)(x-1)) + x, \\ b^m &= s(-(r-1)(x-1)) + 1, \end{aligned}$$

where

$$s = \frac{(k+r-1)^m - 1}{k+r-2}.$$

This gives

$$\begin{aligned} \frac{a^m}{b^m} - 1 &= \frac{s((k-1)(x-1)) + x - (s(-(r-1)(x-1)) + 1)}{s(-(r-1)(x-1)) + 1} \\ &= \frac{(x-1)(s(k+r-2)+1)}{s(1-r)(x-1)+1}, \end{aligned}$$

and we get that

$$\left(\frac{a^m}{b^m} - 1\right)^{-1} = \frac{1}{(x-1)(s(k+r-2)+1)} + \frac{s(1-r)}{s(k+r-2)+1}.$$

Thus we have obtained the following result:

Theorem 1. *We have*

$$\left(\frac{a^m}{b^m} - 1\right)^{-1} = \frac{1}{(k+r-1)^m}\frac{1}{x-1} + \frac{((k+r-1)^m - 1)(1-r)}{(k+r-2)(k+r-1)^m}. \tag{4}$$

In the case $p = 2$ where

$$\begin{aligned} P_m A_m(x) &= \frac{\alpha x + \beta}{\alpha\delta x + (\gamma + \beta\delta)}, \\ A_m P_m(x) &= \frac{\beta\delta x + (\alpha + \beta\gamma)}{\delta x + \gamma}, \end{aligned}$$

the corresponding matrices are

$$A = \begin{bmatrix} \alpha & \beta \\ \alpha\delta & \gamma + \beta\delta \end{bmatrix}$$

and

$$A = \begin{bmatrix} \alpha + \beta\delta & \beta\gamma \\ \delta & \gamma \end{bmatrix},$$

respectively (which both are row-stochastic). Thus by (4) we obtain the following result:

Corollary 2. *For $p = 2$ we have*

$$\frac{1}{(P_m A_m)^m(x) - 1} = \left(\frac{1}{\gamma\alpha}\right)^m \frac{1}{x-1} + \frac{1 - \left(\frac{1}{\gamma\alpha}\right)^m}{1 - \frac{1}{\gamma\alpha}} \frac{\delta}{\gamma},$$
$$\frac{1}{(A_m P_m)^m(x) - 1} = \left(\frac{1}{\gamma\alpha}\right)^m \frac{1}{x-1} + \frac{1 - \left(\frac{1}{\gamma\alpha}\right)^m}{1 - \frac{1}{\gamma\alpha}} \frac{\delta}{\alpha\gamma}.$$

Remark 1. A different proof of this theorem can be found in [14]. In our opinion the above proof is more straightforward and offers a better understanding of the final result.

Remark 2. From a mathematical point of view the theory of Markov chains is about iteration of column-stochastic square matrices. Thus the iterated means $(P_m A_m)^m$ and $(A_m P_m)^m$ could be interpreted as Markov chains for $p = 2$. In the case of general p we can now say that $(P_m A_m)^m$ and $(A_m P_m)^m$ represent some kind of generalized Markov chains.

Remark 3. By using the equalities $\gamma = v^{\frac{1}{mn}}$, $\alpha = v^{\frac{n-1}{mn}}$ and $\gamma + \delta = \alpha + \beta = 1$, we obtain

$$\lim_{m\to\infty} (P_m A_m)^m = \lim_{m\to\infty} (A_m P_m)^m = 1 + \frac{nv(x-1)}{n + (1-v)(x-1)} \tag{5}$$

(cf. Section 1).

Let

$$U_m(x) = A_m P_m(x),$$
$$V_m(x) = P_m A_m(x).$$

The general case $p \neq 2$ is much more difficult to handle. However, in the case $n = 1$ we can give explicit formulæ for any m.

Example 3. If $n = 1$ then $\alpha = 1$ and $\beta = 0$ and so one gets

$$U_m(x) = V_m(x) = \left(\gamma x^{\frac{1}{1-p}} + \delta\right)^{1-p}.$$

Let us compute the square of U_m:

$$U_m^2(x) = \left(\gamma\left(\gamma x^{\frac{1}{1-p}} + \delta\right) + \delta\right)^{1-p} = \left(\gamma^2 x^{\frac{1}{1-p}} + \gamma\delta + \delta\right)^{1-p},$$

and similarly we obtain

$$\begin{aligned} U_m^j(x) &= \left(\gamma\left(\gamma x^{\frac{1}{1-p}} + \delta\right) + \delta\right)^{1-p} = \left(\gamma^j x^{\frac{1}{1-p}} + (\gamma^{j-1} + \gamma^{j-2} + \ldots + 1)\delta\right)^{1-p} \\ &= \left(\gamma^j x^{\frac{1}{1-p}} + \frac{\gamma^j - 1}{\gamma - 1}\delta\right)^{1-p} = \left(\gamma^j x^{\frac{1}{1-p}} + 1 - \gamma^j\right)^{1-p}. \end{aligned}$$

In particular, for $j = m$ we obtain that

$$U_m^m(x) = V_m^m(x) = \left(\gamma^m x^{\frac{1}{1-p}} + 1 - \gamma^m\right)^{1-p} = \left(v x^{\frac{1}{1-p}} + 1 - v\right)^{1-p}.$$

3. On the general case

By putting $x = t^{1-p}$ and $y = s^{1-p}$ in the formulæ $y = U_m(x)$ and $y = V_m(x)$ we obtain the relations

$$s^{1-p} = \alpha(\gamma t + \delta)^{1-p} + \beta \tag{6}$$

and

$$s^{1-p} = \left(\gamma(\alpha t^{1-p} + \beta)^{\frac{1}{1-p}} + \delta\right)^{1-p}, \tag{7}$$

respectively. Equivalently we can write this as

$$s = (\alpha(\gamma t + \delta)^{1-p} + \beta)^{\frac{1}{1-p}} \tag{8}$$

and

$$s = \gamma(\alpha t^{1-p} + \beta)^{\frac{1}{1-p}} + \delta, \tag{9}$$

respectively.

In deriving (6) and (7) we have transformed the two equations involved separately. Let us apply the same transformation to both variables in equations (8) and (9), namely let us set $t = t_1^{1-q}$ and $s = s_1^{1-p} = s_1^{\frac{1}{1-q}}$, where q is the conjugate exponent of p, i.e. $q = p/(p-1)$. Then (8) gives

$$s_1^{1-p} = \big(\alpha\big(\gamma t_1^{1-q} + \delta\big)^{1-p} + \beta\big)^{\frac{1}{1-p}}$$

or

$$s_1 = \alpha\big(\gamma t_1^{1-q} + \delta\big)^{\frac{1}{1-q}} + \beta.$$

Similarly from (9) we derive

$$s_1^{1-q} = \gamma\big(\alpha t_1^{(1-q)(1-p)} + \beta\big)^{\frac{1}{1-p}} + \delta$$

or

$$s_1 = (\gamma(\alpha t_1 + \beta)^{1-q} + \delta)^{\frac{1}{1-q}}.$$

Let us now rewrite the two equations derived side by side, beginning with the last one:

$$\begin{aligned} s_1 &= (\gamma(\alpha t_1 + \beta)^{1-q} + \delta)^{\frac{1}{1-q}}, \\ s_1 &= \alpha(\gamma t_1^{1-q} + \delta)^{\frac{1}{1-q}} + \beta, \end{aligned}$$

and compare them with (8) and (9):

$$\begin{aligned} s &= (\alpha(\gamma t + \delta)^{1-p} + \beta)^{\frac{1}{1-p}}, \\ s &= \gamma(\alpha t^{1-p} + \beta)^{\frac{1}{1-p}} + \delta. \end{aligned}$$

Then a remarkable fact evolves. The two sets of equations have exactly the same shape! Only q has been replaced by p and, moreover, α and γ are interchanged and, likewise, β and δ. Returning to the self-dual case $p = q = 2$, where the equations are written in terms of homographic transformations, we see that they are a reflection of this duality.

Put $T_p(t) = t^p$ and use the notation $B_p(t) = (\alpha t^p + \beta)^{\frac{1}{p}}$ and $B_{-q}(t) = (\gamma t^{-q} + \delta)^{-\frac{1}{q}}$. We observe that

$$B_p = T_p^{-1} A_m T_p \quad \text{and} \quad B_{-q} = T_p^{-1} P_m T_p.$$

It is now manifested that our iteration of the original reiteration of the means A_m and P_m becomes reiteration of the power means B. More precisely, we have

$$B_p B_{-q} = T_p^{-1}(A_m P_m) T_p \quad \text{and} \quad B_{-q} B_p = T_p^{-1}(P_m A_m) T_p.$$

Iterating we find

$$(B_pB_{-q})^m = T_p^{-1}(A_mP_m)^mT_p \quad \text{and} \quad (B_{-q}B_p)^m = T_p^{-1}(P_mA_m)^mT_p.$$

Remark 4. This shows that $(A_mP_m)^m$ and $(P_mA_m)^m$, as well as their limits, are conjugate operations to the products $(B_pB_{-q})^m$ and $(B_{-q}B_p)^m$, respectively.

Remark 5 (On the connection with interpolation functions). Remark 4 opens up interesting new vistas. We refer the reader to the excellent book [5], in particular Section 5.4, and the references to papers by J. Peetre given there; the latter can be easily identified by looking up in MatSciNet, for example. Roughly speaking, it is seen that the expressions mentioned in Remark 4, as well as their limits when $m \to \infty$, arise by repeatedly forming certain K_p- and J_q-functionals and passing to the limit. We intend to return to this connection in subsequent publications.

In the case $p \neq 2$ and $n \neq 1$ we have not found any explicit formulæ as above, but we have done some numerical experiments also for such cases.

Example 4 (cf. [7]). Let $p = 5$, $v = 0.5$, $n = 2$, $x = 0.01$. In the case $m = 100$ we obtain that

$$(A_mP_m)^m = 0.3704 \quad \text{and} \quad (P_mA_m)^m = 0.3714.$$

These values have been found by using Excel.

4. Limiting cases

Our main result in this section reads:

Theorem 5. *We have*

$$\lim_{m\to\infty}(P_mA_m)^m = \lim_{m\to\infty}(A_mP_m)^m = z,$$

where z is the solution of the first order differential equation

$$\begin{cases} -nv\,\dfrac{\mathrm{d}z}{\mathrm{d}v} = (p-n)z + (1-p)z^{\frac{p}{p-1}} + n - 1 \\ z(1) = x \end{cases} \tag{10}$$

Moreover,

$$(A_mP_m)^m \leqslant z \leqslant (P_mA_m)^m \tag{11}$$

for all m.

Proof. Using that $\gamma = v^{\frac{1}{mn}}$ and $\alpha = v^{\frac{n-1}{mn}}$ we obtain the relations $\alpha = (1-\delta)^{n-1}$. Differentiating $V_m(x) = P_m(A_m(x))$ and $U_m(x) = A_m(P_m(x))$ with respect to δ we obtain the estimate

$$\begin{aligned} V_m(x) &= x + \big((1-p)x^{\frac{p}{p-1}} + x(p-n) + n - 1\big)\delta + O(\delta^2),\\ U_m(x) &= x + \big((1-p)x^{\frac{p}{p-1}} + (p-n)x + n - 1\big)\delta + O(\delta^2), \end{aligned}$$

where K_i depends only on p, n and x. Observe that the first two terms in the power series representation for $V_m(x)$ and $U_m(x)$ are equal. We now continue by using similar ideas to those used for the so called *differential effective medium theory*, a theory which was initiated by Bruggeman in 1935 [8], and further developed in the 80's (see [2]). Letting y_m $(= (P_m A_m)^m)$ be obtained iteratively by

$$y_m = y_{m-1} + \big((1-p)y_{m-1}^{\frac{p}{p-1}} + y_{m-1}(p-n) + n - 1\big)\delta + O(\delta^2), \tag{12}$$

and putting $t = \lim_{m\to\infty} m\delta$, we observe that (12) is a discretization of the ordinary differential equation

$$y'(t) = \big(y(t)^{\frac{p}{p-1}}(1-p) + y(t)(p-n) + n - 1\big). \tag{13}$$

The convergence $y_m \to y(t)$ follows by the standard Euler or tangent line method for discretization of differential equations (see e.g. [6, pp. 399–410]). In our case $\delta = 1 - v^{\frac{1}{mn}}$, so

$$m\delta \to \lim_{m\to\infty}\big((1-v^{\frac{1}{mn}})m\big) = -\frac{1}{n}\ln v \; (= t).$$

Thus, using that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -nv$$

and setting $z(v(t)) = y(t)$ we obtain (10). The initial condition $z(1) = x$ follows by inserting $\delta = 1 - 1^{\frac{1}{mn}} = 0$ into any iteration of $A_m P_m$ and $P_m A_m$. For the proof of the inequality (11), we refer to [14] or [15]. □

We see that (10) is a separable differential equation. Thus, letting $J(x)$ be a primitive of

$$K(x) = \frac{1}{x(p-n) + x^{\frac{p}{p-1}}(1-p) + n - 1},$$

i.e.

$$J(x) = \int K(x)\,\mathrm{d}x,$$

we can find the limit z from the relation

$$J(z) = J(x) - \frac{1}{n}\ln v. \tag{14}$$

R e m a r k 6. Actually, since $x = 1$ is a singular point, we have to consider one primitive of $K(x)$ for $x > 1$ and another for $x < 1$. However, this is not an obstacle, since z always lies on the same side of 1 as x (see Section 4.2 below).

R e m a r k 7. In the case $p = 2$ we have

$$J(z) = \int \frac{1}{-(z-1)(z-1+n)}\,\mathrm{d}z = \frac{1}{n}\ln\frac{z-1+n}{z-1}.$$

Thus (14) implies

$$\ln\frac{z-1+n}{z-1} = \ln\frac{x-1+n}{v(x-1)},$$

which (certainly) yields the same expression for z as in (5).

R e m a r k 8. The inequalities (11) in Theorem 5 are generalizations of the inequality

$$A_1P_1 \leqslant P_1A_1,$$

which has been known for a long time, see e.g. [3].

4.1. The case when p is an integer

In the case $p \neq 2$ and $n \neq 1$ it seems difficult to find an explicit formula for the limit z. When p is an integer > 1 we can make the substitution $x = u^{p-1}$, $\mathrm{d}x = (p-1)u^{p-2}\,\mathrm{d}u$ and obtain

$$J(x) = \int \frac{G(u)}{F(u)}\,\mathrm{d}u,$$

where

$$G(u) = (p-1)u^{p-2}$$

and

$$F(u) = (p-n)u^{p-1} + u^p(1-p) + n - 1.$$

We note that

$$\begin{aligned} F(u) &= (p-n)(u^{p-1}-1) + (1-p)(u^p-1) \\ &= (u-1)\left((p-n)\sum_{i=0}^{p-2} u^i + (1-p)\sum_{i=0}^{p-1} u^i\right) = -(u-1)H(u), \end{aligned}$$

where

$$H(u) = (n-1)\sum_{i=0}^{p-2} u^i + (p-1)u^{p-1}.$$

Moreover,

$$\left|(n-1)\sum_{i=0}^{p-2} u^i\right| < |(p-1)u^{p-1}|$$

on each circle $|u| = r > n-1$. [This is clear for $n = 1$ or $p = 2$. For $n > 1$ and $p > 2$ we have that $|u| > 1$, so

$$\left|\sum_{i=0}^{p-2} u^i\right| \leqslant 1 + \sum_{i=1}^{p-2} |u|^i < |u| + \sum_{i=1}^{p-2} |u|^{p-2} \leqslant (p-1)|u|^{p-2}.$$

Thus

$$\left|(n-1)\sum_{i=0}^{p-2} u^i\right| < |u|(p-1)|u|^{p-2} = |(p-1)u^{p-1}|.]$$

Thus, by Rouché's Theorem all zeros of $H(u)$ lie on the closed disc $|u| \leqslant n-1$.

Moreover, we observe that $F(u)$ has p disjoint zeros when $n > 1$. This is seen by the fact that the only zeros of $F'(u) = (p-1)u^{p-2}(p-n-pu)$ are 0 and $(p-n)/p$, none of which are zeros of $F(u)$. Thus

$$\frac{G(u)}{F(u)} = -\frac{1}{n(u-1)} + \sum_{i=1}^{p-1} \frac{M_i}{u-\gamma_i}, \tag{15}$$

where $\gamma_1, \ldots, \gamma_{p-1}$ are the zeros of $H(u)$ and the residues M_i are found by the formula

$$M_i = \frac{G(\gamma_i)}{F'(\gamma_i)} = \frac{1}{p-n-p\gamma_i}.$$

We see directly that none of these zeros can be positive real numbers. Now, assume that $p \geqslant n > 1$. We note that $H(0) = n-1$ and

$$H(-(n-1)) = \frac{1}{n}((1-n)^{p-1}(n(p-2)+1) - (1-n)),$$

the latter being positive for odd p and negative for even p. Since $F'(u)$ has the same sign for negative values of u, this shows that none of the zeros are real for odd p and precisely one of them is real (actually negative) when p is even. We also note that if $\gamma = a + bi$ is a zero of $H(u)$ then so is its conjugate $\overline{\gamma} = a - bi$. The sum of each pair of the corresponding conjugated terms in (15) is real. Indeed,

$$\frac{\frac{1}{p-n-p(a+bi)}}{u-(a+bi)} + \frac{\frac{1}{p-n-p(a-bi)}}{u-(a-bi)} = \frac{2(p-n-pa)(u-a) - 2pb^2}{((p-n-pa)^2 + p^2b^2)((u-a)^2+b^2)}.$$

Thus (15) can be rewritten in the form

$$\frac{G(u)}{F(u)} = -\frac{1}{n(u-1)} + \sum_i \frac{A_i 2(u-a_i)+B_i}{(u-a_i)^2+b_i^2}.$$

By integrating (and using that $x = u^{p-1}$) we obtain that $J(x)$ is of the form

$$(16)\quad J(x) = -\frac{1}{n}\ln\left(x^{\frac{1}{p-1}}-1\right)+\sum_i\left(A_i\ln\left(\left(x^{\frac{1}{p-1}}-a_i\right)^2+b_i^2\right)+\frac{B_i}{b_i}\arctan\frac{x^{\frac{1}{p-1}}-a_i}{b_i}\right).$$

Example 6. As in Example 4 we set $p = 5$, $v = 0.5$, $n = 2$, $x = 0.01$. By numerical computation we find all zeros of $H(u)$ and next find z by solving (14) numerically (by using Mathematica):

$$z = 0.370957.$$

Note that this value agrees with (11) and Example 4 (as it should).

4.2. The cases $p \to 1$ and $p \to \infty$

Since we are dealing with power means, it is easy to see that $(P_m A_m)^m$ lies between 1 and x, increase with p and is monotone in v, so this must also be true for the limit z (see e.g. [9] or [10]). Let us keep this in mind in the treatment below.

We observe that the term $z^{\frac{p}{p-1}}(1-p) \to 0$ as $p \to 1$ if $0 < x < 1$. Solving (10) we obtain that

$$z \to \left(1 - v^{\frac{n-1}{n}}\right) + v^{\frac{n-1}{n}} x$$

(i.e. the arithmetic mean of x and 1 with weights $v^{\frac{n-1}{n}}$ and $1 - v^{\frac{n-1}{n}}$).

Moreover, the monotonicity of z shows that the limit of z as $p \to 1$ exists even for $x > 1$, but in this case (10) yields that

$$z \to 1 \quad \text{for all } 0 \leqslant v < 1.$$

Thus, in general we have

$$z \to \min\{1, A_1\}$$

as $p \to 1$.

Remark 9. This result implies that we can interchange the order of limits:

$$\lim_{p\to 1}\lim_{m\to\infty}(P_m A_m)^m = \lim_{m\to\infty}\lim_{p\to 1}(P_m A_m)^m.$$

To see this we use the standard properties of power-means [9]:

$$P_m \to \min\{1, x\} \quad \text{as } p \to 1$$

and the fact that

$$\lim_{m\to\infty} (\lim_{p\to 1} P_m A_m)^m = \lim_{m\to\infty} (\min\{1, A_m\})^m = \lim_{m\to\infty} (\min\{1, A_m^m\}) = \min\{1, A_1\}.$$

As $p \to \infty$ the right-hand side of (10) goes to $-\ln z + (n-1)(1-z)$ and (10) reduces to

$$\begin{cases} -2v\dfrac{\mathrm{d}z}{\mathrm{d}v} = -\ln z + (n-1)(1-z), \\ z(1) = x. \end{cases} \tag{17}$$

Observe that P_m converges to the geometric mean

$$G_m = x^{v^{1/(nm)}}.$$

Moreover, by using the same procedure as in the proof of Theorem 5 we obtain that the limit

$$\lim_{m\to\infty} (G_m A_m)^m = \lim_{m\to\infty} (A_m G_m)^m = z \tag{18}$$

is the solution of the differential equation (17).

Remark 10. One of the interesting phenomena with (18) is that in the limit we obtain a Geometric-Arithmetic type mean.

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ON SOME ITERATED MEANS ARISING IN HOMOGENIZATION THEORY

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Abstract. We consider iteration of arithmetic and power means and discuss methods for determining their limit. These means appear naturally in connection with some problems in homogenization theory.

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1. Introduction

Means have interested mathematicians since antiquity. The most common means are power means, in particular the arithmetic, geometric and harmonic mean (with or without weights). For positive numbers a and b these means (with equal weights) are defined by the formulæ

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b},$$

respectively. There are many examples of iterated means. For example, if we start with any two given numbers a and b and repeatedly form the arithmetic and the harmonic mean, in that order, i.e. we reiterate the map which to the pair (a,b) assigns a new pair (a',b') given by

$$a' = \frac{a+b}{2}, \quad b' = \frac{2ab}{a+b}, \tag{1}$$

then we obtain sequences a', a'', a''', … and b', b'', b''', … These sequences converge to the common limit $G = \sqrt{ab}$. The algorithm defined by (1) is sometimes called the Newton algorithm (see [1]). Another algorithm, the algorithm studied by Gauss for the arithmetic-geometric mean, is obtained by taking instead successively the arithmetic and the geometric mean (in that order). Thus the map will in this case be

$$a' = \frac{a+b}{2}, \quad b' = \sqrt{ab}. \tag{2}$$

A more general algorithm which contains both the Newton algorithm and the Gauss algorithm can be found in [17].

In this paper we will study a related type of iterated means: Let $x > 0$, let $m, n \in \{1, 2, \ldots\}$, $p > 1$, $0 \leqslant v \leqslant 1$ and consider the weighted means $A_m, P_m\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\begin{aligned} A_m(x) &= \alpha x + \beta, \\ P_m(x) &= \left(\gamma x^{\frac{1}{1-p}} + \delta\right)^{1-p}, \end{aligned}$$

where $\gamma = v^{\frac{1}{mn}}$, $\alpha = v^{\frac{n-1}{mn}}$ and $\gamma + \delta = \alpha + \beta = 1$. The iterated means $(A_m P_m)^m(x)$ and $(P_m A_m)^m$, and their limits, were introduced and studied in [7], [14] and [15] (here "multiplication" between A_m and P_m and "exponentiation" in m denote composition in the obvious way).

They appear naturally in connection with estimation of certain effective properties for multiscaled materials involving the so-called reiterated homogenization (see e.g. [4], [12], [13] and the references given there). Moreover, these iterated means converge to the same limit as m goes to infinity. Generally this limit is difficult to find, but in the case $p = 2$, it turns out to be the mean

$$1 + \frac{nv(x-1)}{n + (1-v)(x-1)},$$

which is the well known Hashin-Shtrikman lower bound if $x > 1$ and Hashin-Shtrikman upper bound if $x < 1$ (see [11]). This fact was proved in [14] by obtaining explicit formulæ for these means.

In this paper we derive the same formulæ by another method. Moreover, we discuss some alternative presentations of these means and their common limit in general.

The paper's organization is as follows. In Section 2 we review the corresponding result from [14] indicating a matrix proof for it and consider also the case $n = 1$ (Example 3). Section 3 is about the general case. In particular, in Remark 4 it is indicated that the expressions $(A_m P_m)^m$ and $(P_m A_m)^m$ are conjugate (as operators) to certain products of power means. This is important, because (see again Remark 5) it opens up a connection with the theory of *interpolation functions* (see [5], in particular Section 5.4, and the references to papers by J. Peetre given there). Finally, we consider some limiting cases in Section 4.

2. The cases $p = 2$ and $n = 1$

We first note that if

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = A \begin{bmatrix} a \\ b \end{bmatrix},$$

where

$$A = \begin{bmatrix} k & l \\ s & r \end{bmatrix},$$

then

$$\frac{a'}{b'} = \frac{ka/b + l}{sa/b + r}.$$

Thus, if

$$F(x) = \frac{kx + l}{sx + r},$$

then the iterate F^m is given by

$$F^m(x) = \frac{a^{(m)}}{b^{(m)}}, \tag{3}$$

where

$$\begin{bmatrix} a^{(m)} \\ b^{(m)} \end{bmatrix} = A^m \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

Now, assume in addition that $l = 1 - k$ and $s = 1 - r$, i.e. the matrix A is row-stochastic. (We call a matrix row-stochastic if all elements are non-negative numbers and if the sum of the elements on any row is unity. Replacing the word row by column we, similarly, obtain the notion of column-stochastic.) By finding eigenvectors and eigenvalues we obtain that

$$A = PDP^{-1},$$

where

$$P = \begin{bmatrix} 1 & 1-k \\ 1 & r-1 \end{bmatrix}, \quad P^{-1} = \frac{1}{k+r-2} \begin{bmatrix} r-1 & k-1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & k+r-1 \end{bmatrix}.$$

Thus

$$A^m = PD^mP^{-1},$$

where

$$D^m = \begin{bmatrix} 1 & 0 \\ 0 & (k+r-1)^m \end{bmatrix}.$$

Accordingly we have a closed form expression for the iterate F^m. We observe that

$$A^m - I = P(D^m - I)P^{-1} = \frac{(k+r-1)^m - 1}{k+r-2} \begin{bmatrix} k-1 & -(k-1) \\ -(r-1) & r-1 \end{bmatrix},$$

i.e.

$$A^m = \frac{(k+r-1)^m - 1}{k+r-2} \begin{bmatrix} k-1 & -(k-1) \\ -(r-1) & r-1 \end{bmatrix} + I.$$

Multiplying by $[x, 1]^T$ we obtain the equations

$$\begin{aligned} a^m &= s((k-1)(x-1)) + x, \\ b^m &= s(-(r-1)(x-1)) + 1, \end{aligned}$$

where

$$s = \frac{(k+r-1)^m - 1}{k+r-2}.$$

This gives

$$\begin{aligned} \frac{a^m}{b^m} - 1 &= \frac{s((k-1)(x-1)) + x - (s(-(r-1)(x-1)) + 1)}{s(-(r-1)(x-1)) + 1} \\ &= \frac{(x-1)(s(k+r-2)+1)}{s(1-r)(x-1)+1}, \end{aligned}$$

and we get that

$$\left(\frac{a^m}{b^m} - 1\right)^{-1} = \frac{1}{(x-1)(s(k+r-2)+1)} + \frac{s(1-r)}{s(k+r-2)+1}.$$

Thus we have obtained the following result:

Theorem 1. *We have*

$$\left(\frac{a^m}{b^m} - 1\right)^{-1} = \frac{1}{(k+r-1)^m} \frac{1}{x-1} + \frac{((k+r-1)^m - 1)(1-r)}{(k+r-2)(k+r-1)^m}. \tag{4}$$

In the case $p = 2$ where

$$\begin{aligned} P_m A_m(x) &= \frac{\alpha x + \beta}{\alpha\delta x + (\gamma + \beta\delta)}, \\ A_m P_m(x) &= \frac{\beta\delta x + (\alpha + \beta\gamma)}{\delta x + \gamma}, \end{aligned}$$

the corresponding matrices are

$$A = \begin{bmatrix} \alpha & \beta \\ \alpha\delta & \gamma + \beta\delta \end{bmatrix}$$

and

$$A = \begin{bmatrix} \alpha + \beta\delta & \beta\gamma \\ \delta & \gamma \end{bmatrix},$$

respectively (which both are row-stochastic). Thus by (4) we obtain the following result:

Corollary 2. *For $p = 2$ we have*

$$\frac{1}{(P_m A_m)^m(x) - 1} = \left(\frac{1}{\gamma\alpha}\right)^m \frac{1}{x-1} + \frac{1 - \left(\frac{1}{\gamma\alpha}\right)^m}{1 - \frac{1}{\gamma\alpha}} \frac{\delta}{\gamma},$$

$$\frac{1}{(A_m P_m)^m(x) - 1} = \left(\frac{1}{\gamma\alpha}\right)^m \frac{1}{x-1} + \frac{1 - \left(\frac{1}{\gamma\alpha}\right)^m}{1 - \frac{1}{\gamma\alpha}} \frac{\delta}{\alpha\gamma}.$$

Remark 1. A different proof of this theorem can be found in [14]. In our opinion the above proof is more straightforward and offers a better understanding of the final result.

Remark 2. From a mathematical point of view the theory of Markov chains is about iteration of column-stochastic square matrices. Thus the iterated means $(P_m A_m)^m$ and $(A_m P_m)^m$ could be interpreted as Markov chains for $p = 2$. In the case of general p we can now say that $(P_m A_m)^m$ and $(A_m P_m)^m$ represent some kind of generalized Markov chains.

Remark 3. By using the equalities $\gamma = v^{\frac{1}{mn}}$, $\alpha = v^{\frac{n-1}{mn}}$ and $\gamma + \delta = \alpha + \beta = 1$, we obtain

$$\lim_{m\to\infty} (P_m A_m)^m = \lim_{m\to\infty} (A_m P_m)^m = 1 + \frac{nv(x-1)}{n + (1-v)(x-1)} \tag{5}$$

(cf. Section 1).

Let

$$U_m(x) = A_m P_m(x),$$
$$V_m(x) = P_m A_m(x).$$

The general case $p \neq 2$ is much more difficult to handle. However, in the case $n = 1$ we can give explicit formulæ for any m.

E x a m p l e 3. If $n=1$ then $\alpha=1$ and $\beta=0$ and so one gets

$$U_m(x)=V_m(x)=\left(\gamma x^{\frac{1}{1-p}}+\delta\right)^{1-p}.$$

Let us compute the square of U_m:

$$U_m^2(x)=\left(\gamma\left(\gamma x^{\frac{1}{1-p}}+\delta\right)+\delta\right)^{1-p}=\left(\gamma^2 x^{\frac{1}{1-p}}+\gamma\delta+\delta\right)^{1-p},$$

and similarly we obtain

$$\begin{aligned} U_m^j(x)&=\left(\gamma\left(\gamma x^{\frac{1}{1-p}}+\delta\right)+\delta\right)^{1-p}=\left(\gamma^j x^{\frac{1}{1-p}}+(\gamma^{j-1}+\gamma^{j-2}+\ldots+1)\delta\right)^{1-p}\\ &=\left(\gamma^j x^{\frac{1}{1-p}}+\frac{\gamma^j-1}{\gamma-1}\delta\right)^{1-p}=\left(\gamma^j x^{\frac{1}{1-p}}+1-\gamma^j\right)^{1-p}. \end{aligned}$$

In particular, for $j=m$ we obtain that

$$U_m^m(x)=V_m^m(x)=\left(\gamma^m x^{\frac{1}{1-p}}+1-\gamma^m\right)^{1-p}=\left(vx^{\frac{1}{1-p}}+1-v\right)^{1-p}.$$

3. On the general case

By putting $x=t^{1-p}$ and $y=s^{1-p}$ in the formulæ $y=U_m(x)$ and $y=V_m(x)$ we obtain the relations

$$s^{1-p}=\alpha(\gamma t+\delta)^{1-p}+\beta \tag{6}$$

and

$$s^{1-p}=\left(\gamma(\alpha t^{1-p}+\beta)^{\frac{1}{1-p}}+\delta\right)^{1-p}, \tag{7}$$

respectively. Equivalently we can write this as

$$s=(\alpha(\gamma t+\delta)^{1-p}+\beta)^{\frac{1}{1-p}} \tag{8}$$

and

$$s=\gamma(\alpha t^{1-p}+\beta)^{\frac{1}{1-p}}+\delta, \tag{9}$$

respectively.

In deriving (6) and (7) we have transformed the two equations involved separately. Let us apply the same transformation to both variables in equations (8) and (9), namely let us set $t = t_1^{1-q}$ and $s = s_1^{1-p} = s_1^{\frac{1}{1-q}}$, where q is the conjugate exponent of p, i.e. $q = p/(p-1)$. Then (8) gives

$$s_1^{1-p} = \left(\alpha\left(\gamma t_1^{1-q} + \delta\right)^{1-p} + \beta\right)^{\frac{1}{1-p}}$$

or

$$s_1 = \alpha\left(\gamma t_1^{1-q} + \delta\right)^{\frac{1}{1-q}} + \beta.$$

Similarly from (9) we derive

$$s_1^{1-q} = \gamma\left(\alpha t_1^{(1-q)(1-p)} + \beta\right)^{\frac{1}{1-p}} + \delta$$

or

$$s_1 = (\gamma(\alpha t_1 + \beta)^{1-q} + \delta)^{\frac{1}{1-q}}.$$

Let us now rewrite the two equations derived side by side, beginning with the last one:

$$\begin{aligned} s_1 &= (\gamma(\alpha t_1 + \beta)^{1-q} + \delta)^{\frac{1}{1-q}}, \\ s_1 &= \alpha(\gamma t_1^{1-q} + \delta)^{\frac{1}{1-q}} + \beta, \end{aligned}$$

and compare them with (8) and (9):

$$\begin{aligned} s &= (\alpha(\gamma t + \delta)^{1-p} + \beta)^{\frac{1}{1-p}}, \\ s &= \gamma(\alpha t^{1-p} + \beta)^{\frac{1}{1-p}} + \delta. \end{aligned}$$

Then a remarkable fact evolves. The two sets of equations have exactly the same shape! Only q has been replaced by p and, moreover, α and γ are interchanged and, likewise, β and δ. Returning to the self-dual case $p = q = 2$, where the equations are written in terms of homographic transformations, we see that they are a reflection of this duality.

Put $T_p(t) = t^p$ and use the notation $B_p(t) = (\alpha t^p + \beta)^{\frac{1}{p}}$ and $B_{-q}(t) = (\gamma t^{-q} + \delta)^{-\frac{1}{q}}$. We observe that

$$B_p = T_p^{-1} A_m T_p \quad \text{and} \quad B_{-q} = T_p^{-1} P_m T_p.$$

It is now manifested that our iteration of the original reiteration of the means A_m and P_m becomes reiteration of the power means B. More precisely, we have

$$B_p B_{-q} = T_p^{-1}(A_m P_m) T_p \quad \text{and} \quad B_{-q} B_p = T_p^{-1}(P_m A_m) T_p.$$

Iterating we find

$$(B_pB_{-q})^m = T_p^{-1}(A_mP_m)^mT_p \quad \text{and} \quad (B_{-q}B_p)^m = T_p^{-1}(P_mA_m)^mT_p.$$

R e m a r k 4. This shows that $(A_mP_m)^m$ and $(P_mA_m)^m$, as well as their limits, are conjugate operations to the products $(B_pB_{-q})^m$ and $(B_{-q}B_p)^m$, respectively.

R e m a r k 5 (On the connection with interpolation functions). Remark 4 opens up interesting new vistas. We refer the reader to the excellent book [5], in particular Section 5.4, and the references to papers by J. Peetre given there; the latter can be easily identified by looking up in MatSciNet, for example. Roughly speaking, it is seen that the expressions mentioned in Remark 4, as well as their limits when $m \to \infty$, arise by repeatedly forming certain K_p- and J_q-functionals and passing to the limit. We intend to return to this connection in subsequent publications.

In the case $p \neq 2$ and $n \neq 1$ we have not found any explicit formulæ as above, but we have done some numerical experiments also for such cases.

E x a m p l e 4 (cf. [7]). Let $p = 5$, $v = 0.5$, $n = 2$, $x = 0.01$. In the case $m = 100$ we obtain that

$$(A_mP_m)^m = 0.3704 \quad \text{and} \quad (P_mA_m)^m = 0.3714.$$

These values have been found by using Excel.

4. Limiting cases

Our main result in this section reads:

Theorem 5. *We have*

$$\lim_{m\to\infty}(P_mA_m)^m = \lim_{m\to\infty}(A_mP_m)^m = z,$$

where z is the solution of the first order differential equation

$$\begin{cases} -nv\,\dfrac{\mathrm{d}z}{\mathrm{d}v} = (p-n)z + (1-p)z^{\frac{p}{p-1}} + n - 1 \\ z(1) = x \end{cases} \tag{10}$$

Moreover,

$$(A_mP_m)^m \leqslant z \leqslant (P_mA_m)^m \tag{11}$$

for all m.

P r o o f. Using that $\gamma = v^{\frac{1}{mn}}$ and $\alpha = v^{\frac{n-1}{mn}}$ we obtain the relations $\alpha = (1-\delta)^{n-1}$. Differentiating $V_m(x) = P_m(A_m(x))$ and $U_m(x) = A_m(P_m(x))$ with respect to δ we obtain the estimate

$$V_m(x) = x + \big((1-p)x^{\frac{p}{p-1}} + x(p-n) + n - 1\big)\delta + O(\delta^2),$$
$$U_m(x) = x + \big((1-p)x^{\frac{p}{p-1}} + (p-n)x + n - 1\big)\delta + O(\delta^2),$$

where K_i depends only on p, n and x. Observe that the first two terms in the power series representation for $V_m(x)$ and $U_m(x)$ are equal. We now continue by using similar ideas to those used for the so called *differential effective medium theory*, a theory which was initiated by Bruggeman in 1935 [8], and further developed in the 80's (see [2]). Letting y_m $(= (P_m A_m)^m)$ be obtained iteratively by

$$y_m = y_{m-1} + \big((1-p)y_{m-1}^{\frac{p}{p-1}} + y_{m-1}(p-n) + n - 1\big)\delta + O(\delta^2), \tag{12}$$

and putting $t = \lim_{m\to\infty} m\delta$, we observe that (12) is a discretization of the ordinary differential equation

$$y'(t) = \big(y(t)^{\frac{p}{p-1}}(1-p) + y(t)(p-n) + n - 1\big). \tag{13}$$

The convergence $y_m \to y(t)$ follows by the standard Euler or tangent line method for discretization of differential equations (see e.g. [6, pp. 399–410]). In our case $\delta = 1 - v^{\frac{1}{mn}}$, so

$$m\delta \to \lim_{m\to\infty}\big(\big(1 - v^{\frac{1}{mn}}\big)m\big) = -\frac{1}{n}\ln v \ (= t).$$

Thus, using that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -nv$$

and setting $z(v(t)) = y(t)$ we obtain (10). The initial condition $z(1) = x$ follows by inserting $\delta = 1 - 1^{\frac{1}{mn}} = 0$ into any iteration of $A_m P_m$ and $P_m A_m$. For the proof of the inequality (11), we refer to [14] or [15]. $\square$

We see that (10) is a separable differential equation. Thus, letting $J(x)$ be a primitive of

$$K(x) = \frac{1}{x(p-n) + x^{\frac{p}{p-1}}(1-p) + n - 1},$$

i.e.

$$J(x) = \int K(x)\,\mathrm{d}x,$$

we can find the limit z from the relation

$$J(z) = J(x) - \frac{1}{n}\ln v. \tag{14}$$

R e m a r k 6. Actually, since $x = 1$ is a singular point, we have to consider one primitive of $K(x)$ for $x > 1$ and another for $x < 1$. However, this is not an obstacle, since z always lies on the same side of 1 as x (see Section 4.2 below).

R e m a r k 7. In the case $p = 2$ we have

$$J(z) = \int \frac{1}{-(z-1)(z-1+n)}\,\mathrm{d}z = \frac{1}{n}\ln\frac{z-1+n}{z-1}.$$

Thus (14) implies

$$\ln\frac{z-1+n}{z-1} = \ln\frac{x-1+n}{v(x-1)},$$

which (certainly) yields the same expression for z as in (5).

R e m a r k 8. The inequalities (11) in Theorem 5 are generalizations of the inequality

$$A_1P_1 \leqslant P_1A_1,$$

which has been known for a long time, see e.g. [3].

4.1. The case when p is an integer

In the case $p \neq 2$ and $n \neq 1$ it seems difficult to find an explicit formula for the limit z. When p is an integer > 1 we can make the substitution $x = u^{p-1}$, $\mathrm{d}x = (p-1)u^{p-2}\,\mathrm{d}u$ and obtain

$$J(x) = \int \frac{G(u)}{F(u)}\,\mathrm{d}u,$$

where

$$G(u) = (p-1)u^{p-2}$$

and

$$F(u) = (p-n)u^{p-1} + u^p(1-p) + n - 1.$$

We note that

$$\begin{aligned} F(u) &= (p-n)(u^{p-1}-1) + (1-p)(u^p-1) \\ &= (u-1)\Bigl((p-n)\sum_{i=0}^{p-2}u^i + (1-p)\sum_{i=0}^{p-1}u^i\Bigr) = -(u-1)H(u), \end{aligned}$$

where

$$H(u) = (n-1)\sum_{i=0}^{p-2} u^i + (p-1)u^{p-1}.$$

Moreover,

$$\left|(n-1)\sum_{i=0}^{p-2} u^i\right| < |(p-1)u^{p-1}|$$

on each circle $|u| = r > n-1$. [This is clear for $n = 1$ or $p = 2$. For $n > 1$ and $p > 2$ we have that $|u| > 1$, so

$$\left|\sum_{i=0}^{p-2} u^i\right| \leqslant 1 + \sum_{i=1}^{p-2} |u|^i < |u| + \sum_{i=1}^{p-2} |u|^{p-2} \leqslant (p-1)|u|^{p-2}.$$

Thus

$$\left|(n-1)\sum_{i=0}^{p-2} u^i\right| < |u|(p-1)|u|^{p-2} = |(p-1)u^{p-1}|.]$$

Thus, by Rouché's Theorem all zeros of $H(u)$ lie on the closed disc $|u| \leqslant n-1$.

Moreover, we observe that $F(u)$ has p disjoint zeros when $n > 1$. This is seen by the fact that the only zeros of $F'(u) = (p-1)u^{p-2}(p-n-pu)$ are 0 and $(p-n)/p$, none of which are zeros of $F(u)$. Thus

$$\frac{G(u)}{F(u)} = -\frac{1}{n(u-1)} + \sum_{i=1}^{p-1} \frac{M_i}{u-\gamma_i}, \tag{15}$$

where $\gamma_1, \ldots, \gamma_{p-1}$ are the zeros of $H(u)$ and the residues M_i are found by the formula

$$M_i = \frac{G(\gamma_i)}{F'(\gamma_i)} = \frac{1}{p-n-p\gamma_i}.$$

We see directly that none of these zeros can be positive real numbers. Now, assume that $p \geqslant n > 1$. We note that $H(0) = n-1$ and

$$H(-(n-1)) = \frac{1}{n}((1-n)^{p-1}(n(p-2)+1) - (1-n)),$$

the latter being positive for odd p and negative for even p. Since $F'(u)$ has the same sign for negative values of u, this shows that none of the zeros are real for odd p and precisely one of them is real (actually negative) when p is even. We also note that if $\gamma = a + b\mathrm{i}$ is a zero of $H(u)$ then so is its conjugate $\overline{\gamma} = a - b\mathrm{i}$. The sum of each pair of the corresponding conjugated terms in (15) is real. Indeed,

$$\frac{\dfrac{1}{p-n-p(a+b\mathrm{i})}}{u-(a+b\mathrm{i})} + \frac{\dfrac{1}{p-n-p(a-b\mathrm{i})}}{u-(a-b\mathrm{i})} = \frac{2(p-n-pa)(u-a) - 2pb^2}{((p-n-pa)^2 + p^2b^2)((u-a)^2+b^2)}.$$

Thus (15) can be rewritten in the form

$$\frac{G(u)}{F(u)} = -\frac{1}{n(u-1)} + \sum_i \frac{A_i 2(u-a_i) + B_i}{(u-a_i)^2 + b_i^2}.$$

By integrating (and using that $x = u^{p-1}$) we obtain that $J(x)$ is of the form

$$(16)\quad J(x) = -\frac{1}{n}\ln\bigl(x^{\frac{1}{p-1}}-1\bigr)+\sum_i\Bigl(A_i\ln\bigl(\bigl(x^{\frac{1}{p-1}}-a_i\bigr)^2+b_i^2\bigr)+\frac{B_i}{b_i}\arctan\frac{x^{\frac{1}{p-1}}-a_i}{b_i}\Bigr).$$

Example 6. As in Example 4 we set $p = 5$, $v = 0.5$, $n = 2$, $x = 0.01$. By numerical computation we find all zeros of $H(u)$ and next find z by solving (14) numerically (by using Mathematica):

$$z = 0.370957.$$

Note that this value agrees with (11) and Example 4 (as it should).

4.2. The cases $p \to 1$ and $p \to \infty$

Since we are dealing with power means, it is easy to see that $(P_m A_m)^m$ lies between 1 and x, increase with p and is monotone in v, so this must also be true for the limit z (see e.g. [9] or [10]). Let us keep this in mind in the treatment below.

We observe that the term $z^{\frac{p}{p-1}}(1-p) \to 0$ as $p \to 1$ if $0 < x < 1$. Solving (10) we obtain that

$$z \to \bigl(1 - v^{\frac{n-1}{n}}\bigr) + v^{\frac{n-1}{n}} x$$

(i.e. the arithmetic mean of x and 1 with weights $v^{\frac{n-1}{n}}$ and $1 - v^{\frac{n-1}{n}}$).

Moreover, the monotonicity of z shows that the limit of z as $p \to 1$ exists even for $x > 1$, but in this case (10) yields that

$$z \to 1 \quad \text{for all } 0 \leqslant v < 1.$$

Thus, in general we have

$$z \to \min\{1, A_1\}$$

as $p \to 1$.

Remark 9. This result implies that we can interchange the order of limits:

$$\lim_{p\to 1}\lim_{m\to\infty}(P_m A_m)^m = \lim_{m\to\infty}\lim_{p\to 1}(P_m A_m)^m.$$

To see this we use the standard properties of power-means [9]:

$$P_m \to \min\{1, x\} \quad \text{as } p \to 1$$

and the fact that

$$\lim_{m\to\infty} (\lim_{p\to 1} P_m A_m)^m = \lim_{m\to\infty} (\min\{1, A_m\})^m = \lim_{m\to\infty} (\min\{1, A_m^m\}) = \min\{1, A_1\}.$$

As $p \to \infty$ the right-hand side of (10) goes to $-\ln z + (n-1)(1-z)$ and (10) reduces to

$$\begin{cases} -2v\dfrac{\mathrm{d}z}{\mathrm{d}v} = -\ln z + (n-1)(1-z), \\ z(1) = x. \end{cases} \tag{17}$$

Observe that P_m converges to the geometric mean

$$G_m = x^{v^{1/(nm)}}.$$

Moreover, by using the same procedure as in the proof of Theorem 5 we obtain that the limit

$$\lim_{m\to\infty} (G_m A_m)^m = \lim_{m\to\infty} (A_m G_m)^m = z \tag{18}$$

is the solution of the differential equation (17).

Remark 10. One of the interesting phenomena with (18) is that in the limit we obtain a Geometric-Arithmetic type mean.

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