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MATHEMATICAL MODELLING OF CABLE STAYED BRIDGES: EXISTENCE, UNIQUENESS, CONTINUOUS DEPENDENCE ON DATA, HOMOGENIZATION OF CABLE SYSTEMS*

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Abstract. A model of a cable stayed bridge is proposed. This model describes the behaviour of the center span, the part between pylons, hung on one row of cable stays. The existence, the uniqueness of a solution of a time independent problem and the continuous dependence on data are proved. The existence and the uniqueness of a solution of a linearized dynamic problem are proved. A homogenizing procedure making it possible to replace cables by a continuous system is proposed. A nonlinear dynamic problem connected with the homogenizing procedure is proposed and the existence and uniqueness of a solution are proved.

Keywords: cable stayed bridges, existence, uniqueness, continuous dependence on data, homogenization of cable systems

MSC 2000: 58D25, 35Q72

1. INTRODUCTION

The construction of cable stayed bridges is nowadays quite frequent in spite of being virtually unknown 40 years ago. The structure of cable stayed bridges resembles suspension bridges. Some models of suspension bridges have been studied in [1]–[4], [6], [8]–[10], [14].

In this paper we are going to study one model of a cable stayed bridge, depicted in Fig. 1, which describes the behaviour of the center span, the part of the deck between the pylons. More cable stayed bridge constructions are presented in the monograph [15].. The motion of the center span is described by two functions corresponding to the deflection and the torsion of any cross section of the center span.

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We are going to pay attention to the existence and uniqueness of a solution, and to its continuous dependence on the data. Let us notice that the cable stays are modelled as non-linear strings, which means the restoring force due to a cable is such that it strongly resists expansion, but does not restrict compression. Moreover, we are going to deal with some homogenizing techniques which make it possible to replace the cable stayed system with a continuous medium. The number of cable stays in Fig. 1 is relatively small, but these numbers are much larger in real constructions. Thus homogenization techniques can make numerical approximations of such problems easier.

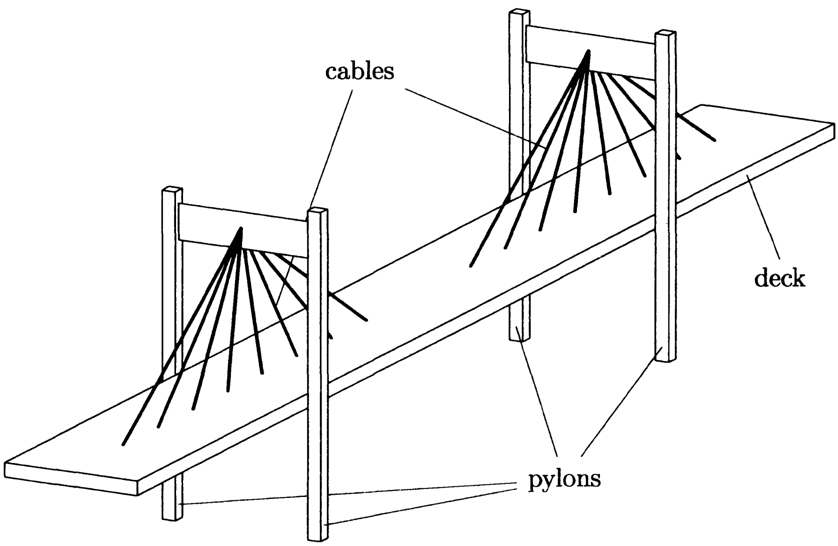

Figure 1.

2. Variational formulations

The main goal of this chapter is to formulate one problem which describes the behaviour of the center span suspended by one row of cable stays. This problem is depicted in Fig. 2.

Before writing down the variational equalities given in [12], let us mention that these equalities were derived from the Hamilton variational principle in the linear theory of elasticity. The derivation is based on the following hypotheses:

1. The central span is a homogeneous prism made of an orthotropic material whose symmetry axes are parallel to x, y, z, as depicted in Fig. 2.
2. Any cross section Q_x (see Fig. 2) perpendicular to the x-axis remains perpendicular to the deformed x-axis which is only allowed to move in the vertical

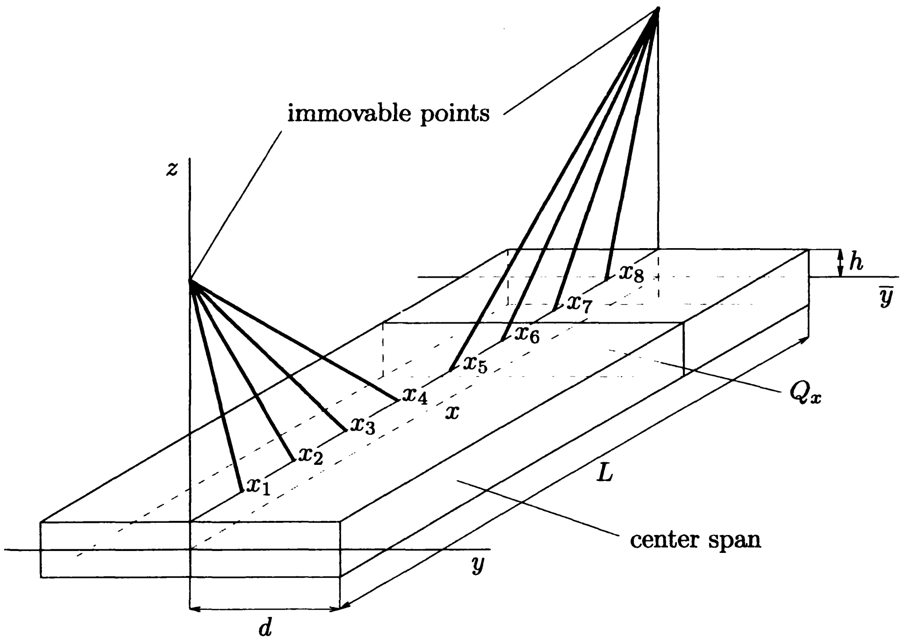

Figure 2.

direction. Moreover, the y, z-axes, perpendicular to each other before the deformation of any cross section, remain perpendicular after the deformation (see Fig. 3).

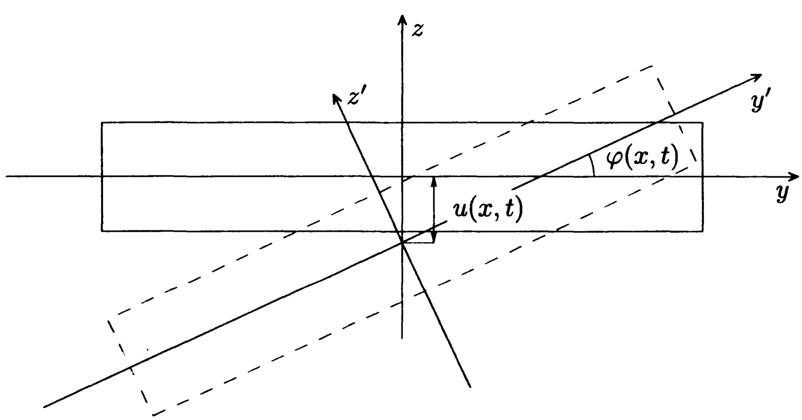

Figure 3.

Then the behaviour of the structure depicted in Fig. 2 can be described by $u(x,t)$, $\varphi(x,t)$ defined on $\langle 0, L\rangle \times \langle 0, T\rangle$. The function $u(x,t)$ corresponds to the deflection of the x-axis in the vertical direction and $\varphi(x,t)$ corresponds to the turning of Q_x round the x-axis.

Let us define bilinear forms

$$
\begin{aligned}
(u,v) &= \int_0^L uv\,\mathrm{d}x,\\
m_1(u,v) &= \int_0^L M_1 uv\,\mathrm{d}x,\\
m_2(\varphi,\psi) &= \int_0^L M_2\varphi\psi\,\mathrm{d}x,\\
k_1(u,v) &= \int_0^L K_1\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}\,\mathrm{d}x,\\
k_2(\varphi,\psi) &= \int_0^L K_2\frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x}\,\mathrm{d}x,
\end{aligned}
$$

where M_1, M_2, K_1, K_2 are constants which are given in [12] and generally depend on the material properties and the geometry of the prism representing the center span. The forms $m_1(\cdot,\cdot)$, $m_2(\cdot,\cdot)$ are connected with the kinetic energy of vertical and torsional vibrations, while the forms $k_1(\cdot,\cdot)$, $k_2(\cdot,\cdot)$ correspond to the deformation energy of these movements.

Let us define another bilinear form connected with the deformation energy of a row of cables stays, as depicted in Fig. 4,

$$
b(u,v) = \sum_{i=1}^{8} k_i u(x_i,t)v(x_i,t),
$$

where the coefficients k_i depend on the length and the stiffness of the cable stays attached to the center span in x_i, and the angles between those cables and the center span. The explicit shape of k_i is to be found in [12].

The cables in real constructions are stressed, which means that the length of these cables is shortened so that the cables loosen if the center span is bent upward by the value $d(x)$, as depicted in Fig 4. Moreover, the cables behave as non-linear strings, which means that the restoring force due to the cable attached at x_i is such that it resists expansion if the deflection of the center span at x_i in the upward direction is less than $d(x_i)$, but does not resist compression in the opposite case. Thus the deformation energy of the cable system in Fig. 2 is

$$
\frac{1}{2}b(g(u+d),g(u+d)),
$$

where g is a certain function. The function $g(x)$ applied in [12] is equal to $x^+ = \max\{0,x\}$, which corresponds to the loosening of cables. For mathematical reasons

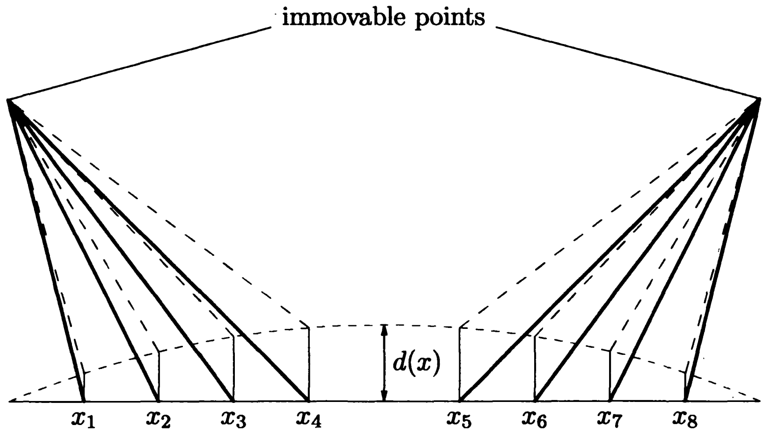

Figure 4.

we are going to deal with $g(x)$ defined as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0), \\ x - \varepsilon & \text{if } x \in (2\varepsilon, \infty), \end{cases} \tag{2.1}$$

and $g(x)$ is extended to the whole $\mathbb{R}$ so that it has continuous derivatives up to the order 2 and is convex.

This function is depicted by the solid line in Fig. 5 while the dashed line corresponds to x^+. On the interval $\langle 0, \varepsilon \rangle$ the function $g(x)$ describes the relaxation of cables at the moment when these cables start stretching. The center span is under

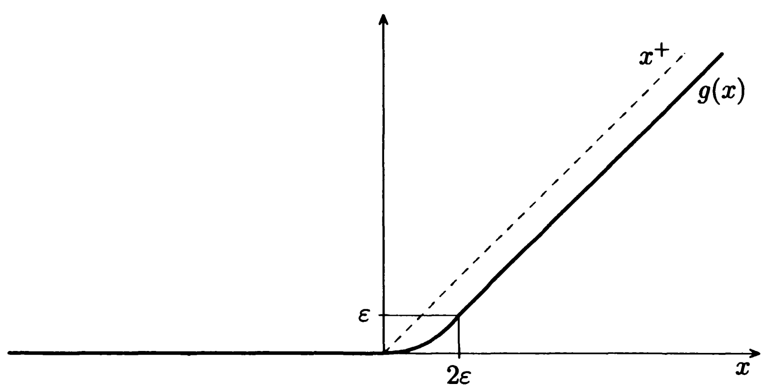

Figure 5.

the influence of the gravitational force represented by $F_1(x)$, $F_2(x)$ and the force of wind represented by $P_1(x,t)$, $P_2(x,t)$. Then the linear forms (F_1, u), (P_1, u), (F_2, φ),

(P_2, φ) correspond to the energy of external forces connected with the vertical and the torsional vibrations. The dynamic equilibrium of the system depicted in Fig. 2 is a stationary point of the functional

$$J(u, \varphi) = \int_0^T \mathcal{L}(u, \varphi)\,\mathrm{d}t,$$

where

$$\begin{aligned}\mathcal{L}(u, \varphi) = {} & \frac{1}{2}m_1\Big(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\Big) - \frac{1}{2}k_1(u, u) + \frac{1}{2}m_2\Big(\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial t}\Big) - \frac{1}{2}k_2(\varphi, \varphi) \\ & - \frac{1}{2}b(g(u + d), g(u + d)) + (F_1 + P_1, u) + (F_2 + P_2, \varphi).\end{aligned}$$

The functional above is defined on a set of sufficiently smooth functions on $\langle 0, L\rangle \times \langle 0, T\rangle$. These functions satisfy the conditions

$$\begin{gathered} u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) = 0, \quad t \in \langle 0, T\rangle, \\ u(x, 0) = \mu_0(x),\ \ u(x, T) = \mu_1(x),\ \ \varphi(x, 0) = \nu_0(x),\ \ \varphi(x, T) = \nu_1(x),\ \ x \in \langle 0, L\rangle, \end{gathered}$$

where μ_0, ν_1, ν_0, ν_1 are fixed functions on $\langle 0, L\rangle$.

Let us define bilinear forms

$$\delta_1(u, v) = \int_0^L \Theta_1 u v\,\mathrm{d}x,$$

$$\delta_2(\varphi, \psi) = \int_0^L \Theta_2 \varphi\psi\,\mathrm{d}x,$$

where Θ_1, Θ_2 are the damping coefficients for the vertical and torsional vibrations. Moreover, the forces induced by the wind can depend on φ, as depicted in Fig. 6. Thus these forces have to be described by $P_1(\varphi, x, t)$, $P_2(\varphi, x, t)$. Then the linear forms $(P_1(\varphi), v)$, $(P_2(\varphi), \psi)$ correspond to the energies of the external forces of wind.

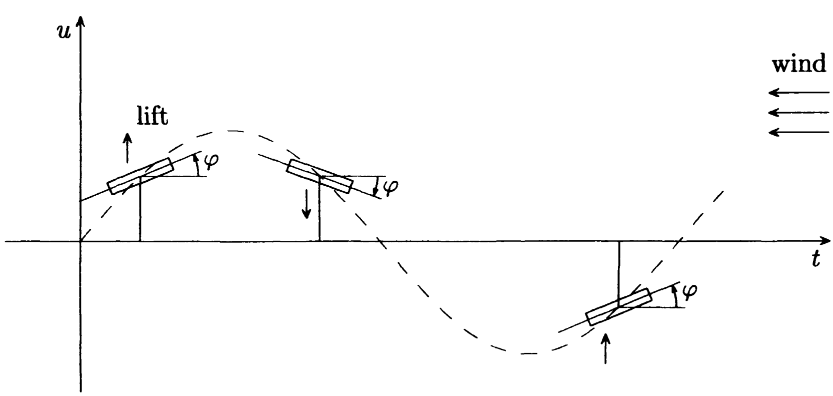

Figure 6.

The above variational principle yields variational equalities which can be generalized by adding the damping terms. These equalities read as follows:

$$
\begin{aligned}
(2.2)\qquad m_1\Big(\frac{\partial^2 u}{\partial t^2}, v\Big) + k_1(u,v) + \delta_1\Big(\frac{\partial u}{\partial t}, v\Big) + b(\tilde{g}(u+d), v) &= (F_1 + P_1(\varphi), v),\\
m_2\Big(\frac{\partial^2 \varphi}{\partial t^2}, \psi\Big) + k_2(\varphi,\psi) + \delta_2\Big(\frac{\partial \varphi}{\partial t}, \psi\Big) &= (F_2 + P_2(\varphi), \psi),
\end{aligned}
$$

where $2\tilde{g}(x) = (g^2(x))' = 2g(x)g'(x)$. The functions u, v, φ, ψ defined on $\langle 0, L\rangle \times \langle 0, T\rangle$ satisfy the conditions

$$
\begin{aligned}
(2.3)\qquad u(0,t) = u(L,t) = v(0,t) = v(L,t) = 0, &\quad t \in \langle 0, T\rangle,\\
\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, &\quad t \in \langle 0, T\rangle,\\
u(x,0) = u_0(x), \quad \frac{\partial}{\partial t}u(x,0) = u_1(x), &\quad x \in \langle 0, L\rangle,\\
\varphi(x,0) = \varphi_0(x), \quad \frac{\partial}{\partial t}\varphi(x,0) = \varphi_1(x), &\quad x \in \langle 0, L\rangle,
\end{aligned}
$$

where u_0, u_1, φ_0, φ_1 are fixed functions representing the initial conditions for (2.2). The functions u, φ are solutions to the above problem if (2.2) are fulfilled for any v, ψ satisfying (2.3).

It is sometimes useful to study the time independent problem connected with the above dynamic problem. The solution to that problem is a minimum of the functional

$$
\Phi(u,\varphi) = \frac{1}{2}k_1(u,u) + \frac{1}{2}k_2(\varphi,\varphi) + \frac{1}{2}b(g(u+d), g(u+d)) - (F_1, u) - (F_2, \varphi).
$$

This functional is defined on a set of sufficiently smooth functions on $\langle 0, L\rangle$. These functions satisfy the conditions

$$
u(0) = u(L) = \varphi(0) = \varphi(L) = 0.
$$

The above formulation is equivalent to the variational equalities

$$
\begin{aligned}
(2.4)\qquad k_1(u,v) + b(\tilde{g}(u+d), v) &= (F_1, v),\\
k_2(\varphi,\psi) &= (F_2, \psi).
\end{aligned}
$$

The functions u, v, φ, ψ defined on $\langle 0, L\rangle$ satisfy the conditions

$$
\begin{aligned}
(2.5)\qquad u(0) = u(L) = v(0) = v(L) &= 0,\\
\varphi(0) = \varphi(L) = \psi(0) = \psi(L) &= 0.
\end{aligned}
$$

The strictly mathematical formulation of these problems is given in the subsequent chapters.

3. Some preliminaries

The main goal of this chapter is to give a strictly mathematical formulation of the problems mentioned in the previous chapter. Let us recall some results from the theory of distributions with values in a Banach space V. If $\mathcal{D}((0,T))$ is the space of test functions with the usual topology, then distributions are linear continuous operators from $\mathcal{D}((0,T))$ to V with the weak topology, which means that if $u\colon \mathcal{D}((0,T)) \mapsto V$ is a distribution and $v \in V^*$, then $v(u)\colon \mathcal{D}((0,T)) \mapsto \mathbb{R}$ is a usual distribution from $D^*((0,T))$, the space of usual distributions. The symbol $\mathcal{D}^*((0,T),V)$ denotes the space of V-valued distributions on $(0,T)$. If $u\colon (0,T) \mapsto V$ belongs to $L^1((0,T),V)$, which is the space of Bochner integrable functions (see [5], [7]), we can define the expression

$$u(\varphi) = \int_0^T u\varphi \,\mathrm{d}s, \tag{3.1}$$

where $\varphi \in \mathcal{D}((0,T))$. This expression belongs to $\mathcal{D}^*((0,T),V)$ and the transformation from $L^1((0,T),V)$ to $\mathcal{D}^*((0,T),V)$ defined by (3.1) is an injective imbedding.

The derivative u' of u is defined as follows:

$$u'(\varphi) = -u(\varphi'), \quad \varphi \in \mathcal{D}((0,T))$$

and $u' \in \mathcal{D}^*((0,T),V)$ as well. A more thorough introduction to the theory of vector valued distributions and related topics as well as the proof of the following lemma can be found, for instance, in [5].

Lemma 3.1. *If $f \in L^1((0,T),V)$ and its distributional derivative f' belongs to $L^1((0,T),V)$, then $f \in C(\langle 0,T\rangle,V)$, the space of continuous functions from $\langle 0,T\rangle$ to V. Moreover, the equality $f(t_2)-f(t_1) = \int_{t_1}^{t_2} f'(t)\,\mathrm{d}s$ holds for any $t_1,t_2 \in \langle 0,T\rangle$.*

First of all let us generalize the definitions of $m_1(\cdot,\cdot)$, $m_2(\cdot,\cdot)$, $k_1(\cdot,\cdot)$, $k_2(\cdot,\cdot)$, $\delta_1(\cdot,\cdot)$, $\delta_2(\cdot,\cdot)$ by replacing the constants M_1, M_2, K_1, K_2, Θ_1, Θ_2 by bounded measurable non-negative functions defined on $(0,L)$. We will use the same symbols for these functions which satisfy the additional assumptions

$$M_1(x) \geqslant \varepsilon, \quad M_2(x) \geqslant \varepsilon, \quad K_1(x) \geqslant \varepsilon, \quad K_2(x) \geqslant \varepsilon, \quad x \in \langle 0,L\rangle, \tag{3.2}$$

where ε is a positive constant.

Let us denote

$$V_1 = H_0^1((0,L)) \cap H^2((0,L)), \quad V_2 = H_0^1((0,L)),$$
$$W = L^2((0,L)),$$

where $H_0^1((0,L))$, $H^2((0,L))$ are the Sobolev spaces formed by all functions in $L^2((0,T))$ whose first and second derivatives belong to $L^2((0,T))$, respectively. Moreover, the expression $u \in H_0^1((0,L))$ means that $u(0) = u(L) = 0$. Both V_1 and V_2 are Hilbert spaces equipped with the respective scalar products

$$\langle u, v\rangle_{V_1} = \int_0^L \{uv + u'v' + u''v''\}\, \mathrm{d}x, \quad \langle \varphi, \psi\rangle_{V_2} = \int_0^L \{\varphi\psi + \varphi'\psi'\}\, \mathrm{d}x.$$

By virtue of the Poincaré inequality (see [7]) there exists $C > 0$ such that the inequalities

$$C\|u\|_{V_1}^2 \leqslant k_1(u,u), \quad C\|\varphi\|_{V_2}^2 \leqslant k_2(\varphi,\varphi) \tag{3.3}$$

hold for any $u \in V_1$, $\varphi \in V_2$. The above definitions make the following natural imbeddings possible:

$$V_1 \subset W, \quad V_2 \subset W. \tag{3.4}$$

The bilinear form $b(\cdot,\cdot)$ connected with the cable systems is bounded on V_1 because this space can be imbedded in $C(\langle 0, L\rangle)$ and this imbedding is continuous (see [7]).

Let $L^2((0,T), X)$, where X is a Hilbert space, denote the space of all Bochner measurable functions $f\colon (0,T) \mapsto X$ satisfying

$$\left(\int_0^T \|f(s)^2\|_X\, \mathrm{d}s\right)^{\frac{1}{2}} = \|f\|_{L^2((0,T),X)} < \infty.$$

Let us recall that this is a Hilbert space equipped with the scalar product

$$\int_0^T \langle f(s), g(s)\rangle_X\, \mathrm{d}s.$$

These facts together with the imbeddings (3.4) make the following imbeddings possible:

$$\begin{aligned} &L^2((0,T), V_1) \subset L^2((0,T), W), \\ &L^2((0,T), V_2) \subset L^2((0,T), W). \end{aligned} \tag{3.5}$$

These spaces can be naturally imbedded into the spaces $\mathcal{D}^*((0,T), V_1)$, $\mathcal{D}^*((0,T), V_2)$, $\mathcal{D}^*((0,T), W)$.

Let us define the spaces

$$\begin{aligned} X_1 &= \{u \mid u \in L^2((0,T), V_1),\ u' \in L^2((0,T), V_1),\ u'' \in L^2((0,T), W)\}, \\ X_2 &= \{\varphi \mid \varphi \in L^2((0,T), V_2),\ \varphi' \in L^2((0,T), V_2),\ \varphi'' \in L^2((0,T), W)\}, \end{aligned}$$

where u', u'', φ', φ'' are distributional derivatives in $\mathcal{D}^*((0,T),V_1)$, $\mathcal{D}^*((0,T),W)$, $\mathcal{D}^*((0,T),V_2)$, $\mathcal{D}^*((0,T),W)$. We have applied the imbeddings (3.5) in the definitions of X_1, X_2.

4. Time independent problems: existence, uniqueness, continuous dependence on data

In this chapter we deal with a time independent problem. We will prove the existence, uniqueness and continuous dependence on the data.

Definition 4.1. Let $F_1, F_2 \in L^2((0,L))$, $d \in V_1$, then $u \in V_1$, $\varphi \in V_2$ are a solution to the problem $\mathcal{A}$ if (u,φ) is a minimum of the functional

$$\begin{aligned}(4.1)\qquad \Phi(u,\varphi) = \frac{1}{2}k_1(u,u) + \frac{1}{2}k_2(\varphi,\varphi) + \frac{1}{2}b(g(u+d),g(u+d))\\ - (F_1,u) - (F_2,\varphi)\end{aligned}$$

on $V_1 \times V_2$.

The above formulation is equivalent to the variational equalities

$$\begin{aligned}(4.2)\qquad k_1(u,v) + b(\tilde{g}(u+d),v) &= (F_1,v),\\ k_2(\varphi,\psi) &= (F_2,\psi)\end{aligned}$$

which hold for all $v \in V_1$, $\psi \in V_2$.

Theorem 4.1. *If the assumptions of Definition 4.1 are fulfilled, then there exist u,φ which are a solution to $\mathcal{A}$ and this solution is unique.*

Proof. The definition of Φ shows that this functional is convex and differentiable, so it is weakly lower semi-continuous (see [5]). If we apply the estimates (3.3), we have the inequality

$$(4.3)\qquad C_1(\|u\|_{V_1}^2 + \|\varphi\|_{V_2}^2) - C_2(\|u\|_{V_1} + \|\varphi\|_{V_2}) \leqslant \Phi(u,\varphi),$$

where C_1, C_2 are positive constants independent of u, φ. If $\|u\|_{V_1}, \|\varphi\|_{V_2} \to \infty$, then (4.3) yields that $\Phi(u,\varphi) \to \infty$, which means that Φ is coercive. The properties of Φ guarantee the existence of a solution to $\mathcal{A}$ (see [5]). The definition of $\Phi(u,\varphi)$ yields that this functional is strictly convex, which guarantees the uniqueness of $\mathcal{A}$. $\square$

Let us consider $k_j \in \langle 0,\infty), x_j \in \langle 0,L\rangle, j=1,\dots,8$,

$$K_1(x) \in L^\infty((0,L)), \quad K_1(x) \geqslant \varepsilon > 0,$$
$$K_2(x) \in L^\infty((0,L)), \quad K_2(x) \geqslant \varepsilon > 0,$$
$$F_1, F_2 \in L^2((0,L)),$$

where K_1, K_2 are the functions in the definitions of $k_1(\cdot,\cdot)$, $k_2(\cdot,\cdot)$ and x_j, k_j, $j=1,\dots,8$ are the terms in the definition of $b(\cdot,\cdot)$. Then by virtue of Theorem 4.1 we can define a transformation

$$P(k_1,\dots,k_8,x_1,\dots,x_8,K_1,K_2,F_1,F_2)$$

defined on $[\langle 0,\infty)]^8 \times [\langle 0,L\rangle]^8 \times [L^\infty((0,L))]^2 \times [L^2((0,L))]^2$ with the range $V_1 \times V_2$. This transformation assigns the solution (u,φ) of $\mathcal{A}$ to the above data.

Theorem 4.2. *The transformation P is continuous.*

Proof. Let

$$x_j^i \to x_j^0, \quad k_j^i \to k_j^0 \quad \text{in } \mathbb{R},\ j=1,\dots,8,$$
$$K_1^i \to K_1^0, \quad K_2^i \to K_2^0 \quad \text{in } L^\infty((0,L)),$$
$$F_1^i \to F_1^0, \quad F_2^i \to F_2^0 \quad \text{in } L^2((0,L)). \tag{4.4}$$

If $[u^i,\varphi^i] = P(k_1^i,\dots,k_8^i,x_1^i,\dots,x_8^i,K_1^i,K_2^i,F_1^i,F_2^i)$, then

$$\begin{aligned} k_1^i(u^i,u^i) + k_2^i(\varphi^i,\varphi^i) + b^i(\tilde{g}(u^i+d),u^i) - b^i(\tilde{g}(d),u^i) \\ = -b^i(\tilde{g}(d),u^i) + (F_1^i,u^i) + (F_2^i,\varphi^i), \end{aligned} \tag{4.5}$$

where $k_1^i(\cdot,\cdot)$, $k_2^i(\cdot,\cdot)$, $b^i(\cdot,\cdot)$ correspond to K_1^i, K_2^i, k_j^i, x_j^i, $j=1,\dots,8$. Since $b^i(\tilde{g}(u),v)$ is the derivative of the convex functional $\frac{1}{2}b^i(g(u),g(u))$ in the direction v, the inequality

$$b^i(\tilde{g}(u),u-v) - b^i(\tilde{g}(v),u-v) \geqslant 0 \tag{4.6}$$

holds for all $u,v \in V_1$ (see [5]). If we consider (3.3) and the last inequality, then (4.5) yields the inequality

$$\|u^i\|_{V_1}^2 + \|\varphi^i\|_{V_2}^2 \leqslant C_1 + C_2(\|u^i\|_{V_1} + \|\varphi^i\|_{V_2}), \tag{4.7}$$

where C_1, C_2 are positive constants common for all i. From (4.7) it follows that there exists a constant C such that

$$\|u^i\|_{V_1} \leqslant C, \quad \|\varphi^i\|_{V_2} \leqslant C, \tag{4.8}$$

which yields that there exist subsequences u^m, φ^m that weakly converge to u^0, φ^0 in V_1, V_2, respectively. If we consider that $V_1 \subset H^1((0,L)) \subset C^{\frac{1}{2}}(\langle 0,L\rangle)$, then by virtue of the Arzelà-Ascoli theorem the subsequence u^m strongly converges to u^0 in $C(\langle 0,L\rangle)$. If we consider that (u^m,φ^m) are solutions to the sequence of problems mentioned above, we have

$$\begin{aligned}
k_1^0(u^m,v) + b^0(\tilde{g}(u^m+d),v) &= k_1^0(u^m,v) - k_1^m(u^m,v) + (F_1^m - F_1^0,v) + b^0(\tilde{g}(u^m+d),v) \\
&\quad - b^m(\tilde{g}(u^m+d),v) + (F_1^0,v), \\
k_2^0(\varphi^m,\psi) &= k_2^0(\varphi^m,\psi) - k_2^m(\varphi^m,\psi) + (F_2^m - F_2^0,\psi) + (F_2^0,\psi),
\end{aligned} \tag{4.9}$$

where $v \in V_1$, $\psi \in V_2$. (4.4) and (4.8) yield

$$\begin{aligned}
k_1^0(u^m,v) - k_1^m(u^m,v) &\to 0 \\
k_2^0(\varphi^m,\psi) - k_2^m(\varphi^m,\psi) &\to 0.
\end{aligned} \tag{4.10}$$

The convergence of u^m in $C(\langle 0,\rangle)$ and (4.4) imply

$$b^0(\tilde{g}(u^m+d),v) - b^m(\tilde{g}(u^m+d),v) \to 0. \tag{4.11}$$

The weak convergence of u^m, φ^m to u^0, φ^0, (4.10) and (4.11) yield

$$\begin{aligned}
k^0(u^0,v) + b^0(\tilde{g}(u^0+d),v) &= (F_1^0,v), \\
k^0(\varphi^0,\psi) &= (F_2^0,\psi),
\end{aligned} \tag{4.12}$$

which means that (u^0,φ^0) is a solution to $\mathcal{A}$. Then we can write

$$\begin{aligned}
k_1^0(u^0-u^m,u^0-u^m) &= k_1^m(u^m,u^0-u^m) - k_1^0(u^m,u^0-u^m) \\
&\quad - b^0(\tilde{g}(u^0+d),u^0-u^m) \\
&\quad + b^m(\tilde{g}(u^m+d),u^0-u^m) \\
&\quad + (F_1^0 - F_1^m,u^0-u^m), \\
k_2^0(\varphi^0-\varphi^m,\varphi^0-\varphi^m) &= k_2^m(\varphi^m,\varphi^0-\varphi^m) - k_2^0(\varphi^m,\varphi^0-\varphi^m) \\
&\quad + (F_2^0 - F_2^m,\varphi^0-\varphi^m).
\end{aligned} \tag{4.13}$$

Due to the strong convergence of F_1^m, F_2^m to F_1^0, F_2^0 in $L^2((0,L))$, strong convergence of u^m, φ^m to u^0, φ^0 in $C(\langle 0,L\rangle)$, (4.4) and (4.8), the right-hand sides of (4.13) converge to zero. Then the estimates (3.3) yield the strong convergence of u^m, φ^m to u^0, φ^0 in V_1, V_2, respectively. If we consider that (u^0,φ^0) is the unique solution to $\mathcal{A}$, the whole sequence (u^i,φ^i) converges to (u^0,φ^0), which gives the desired result. □

Solving the last time independent problem, we admitted that the cables can loosen, which was described by the function g. It is evident that the cables are fully stressed in most cases so the inequalities

$$u(x_j) + d(x_j) \geqslant 2\varepsilon, \quad j = 1,\dots,8, \tag{4.14}$$

where ε is the term in (2.1), hold. Then the deformation energy of the cables is given by the term

$$\frac{1}{2}b(u+d-\varepsilon, u+d-\varepsilon), \tag{4.15}$$

which follows from the definition of g.

Let us settle a linear time independent problem which is connected with the problem $\mathcal{A}$.

Definition 4.2. Let $F_1, F_2 \in L^2((0,L))$, $d \in V_1$, then $(u,\varphi) \in V_1 \times V_2$ is a solution to the problem $\mathcal{A}_L$ if the variational equalities

$$\begin{aligned} k_1(u,v) + b(u+d-\varepsilon, v) &= (F_1, v),\\ k_2(\varphi,\psi) &= (F_2,\psi) \end{aligned} \tag{4.16}$$

hold for all $v \in V_1$, $\psi \in V_2$.

It is evident that the assertions similar to those we proved for $\mathcal{A}$ can be proved for the linear problem $\mathcal{A}_L$.

5. Linearized dynamic problems: formulation, existence, uniqueness

Let us assume that functions $P_1(y,x,t)$, $P_2(y,x,t)$ defined on $\mathbb{R} \times (0,L) \times (0,T)$ satisfy the modified Carathéodory conditions (MC):

1. $P_i(y,x,t)$, $\frac{\partial}{\partial y}P_i(y,x,t)$, $\frac{\partial}{\partial t}P_i(y,x,t)$, $i=1,2$ are continuous in y, t for almost every x and measurable for all y, t.
2. There exists $p(x) \in L^2((0,L))$ such that

$$|P_i(y,x,t)| \leqslant p(x), \quad \left|\frac{\partial}{\partial t}P_i(y,x,t)\right| \leqslant p(x), \quad i = 1,2.$$

3. There exists a constant C such that

$$\left|\frac{\partial}{\partial y} P_i(y,x,t)\right| \leqslant C, \quad i=1,2.$$

These assumptions guarantee that $P_i(\varphi(x,t),x,t)$, $\frac{\partial}{\partial t} P_i(\varphi(x,t),x,t)$, $\frac{\partial}{\partial y} P_i(\varphi(x,t),x,t)$, $i=1,2$ belong to $L^2((0,L)\times(0,T))$ for any measurable function $\varphi(x,t)$ defined on $(0,L)\times(0,T)$. If we consider the definitions of the bilinear forms introduced above and the fact that $u\in X_1$, $\varphi\in X_2$, $d\in V_1$, then the expressions $m_1(u'',v)$, $k_1(u,v)$, $b(\tilde{g}(u+d),v)$, $\delta_1(u',v)$, $m_2(\varphi'',\psi)$, $k_2(\varphi,\psi)$, $\delta_2(\varphi',\psi)$ belong to $L^2((0,T))$ for any $v\in V_1$, $\psi\in V_2$.

Definition 5.1. Let $d\in V_1$ and F_1, F_2 belong to $L^2((0,T))$ and let P_1, P_2 satisfy (MC). Then $(u,\varphi)\in X_1\times X_2$ is a solution to the problem $\mathcal{B}$ if the equalities

$$\begin{aligned} m_1(u'',v)+k_1(u,v)+\delta_1(u',v)+b(\tilde{g}(u+d),v) &= (F_1+P_1(\varphi),v),\\ m_2(\varphi'',\psi)+k_2(\varphi,\psi)+\delta_2(\varphi',\psi) &= (F_2+P_2(\varphi),\psi)\end{aligned} \tag{5.1}$$

hold in $L^2((0,T))$ for any $v\in V_1$, $\psi\in V_2$. Moreover, the initial conditions

$$\begin{aligned} u(0)=u_0,&\quad u'(0)=0,\\ \varphi(0)=\varphi_0,&\quad \varphi'(0)=0\end{aligned} \tag{5.2}$$

are fulfilled, where (u_0,φ_0) is a solution to the problem $\mathcal{A}$ with the right-hand sides F_1, F_2.

Let us notice that the expressions (5.2) in this definition are correct. The functions u, φ belong to $C(\langle 0,T\rangle,V_1)$, $C(\langle 0,T\rangle,V_2)$ and u', φ' to $C(\langle 0,T\rangle,W)$, which follows from Lemma 3.1.

The problem $\mathcal{B}$ corresponds to the situation when the central span quietly rests on the cables at the moment $t=0$. Then it starts moving under the influence of wind represented by P_1, P_2.

If the gravitational force fully stresses the cables, if the force of wind is small enough not to loosen the cables, and if we apply similar arguments as we did in the previous chapter, the following problem properly describes the behaviour of our model.

Definition 5.2. Let $d\in V_1$, let F_1, F_2 belong to $L^2((0,T))$ and let P_1, P_2 satisfy (MC). Then $(u,\varphi)\in X_1\times X_2$ is a solution to the problem $\mathcal{B}_L$ if the equalities

$$\begin{aligned} m_1(u'',v)+k_1(u,v)+\delta_1(u',v)+b(u+d-\varepsilon,v) &= (F_1+P_1(\varphi),v),\\ m_2(\varphi'',\psi)+k_2(\varphi,\psi)+\delta_2(\varphi',\psi) &= (F_2+P_2(\varphi),\psi)\end{aligned} \tag{5.3}$$

hold in $L^2((0,T))$ for any $v \in V_1$, $\psi \in V_2$. Moreover, the initial conditions

$$
\begin{aligned}
(5.4) \qquad\qquad & u(0) = u_0, \quad u'(0) = 0, \\
& \varphi(0) = \varphi_0, \quad \varphi'(0) = 0
\end{aligned}
$$

are fulfilled, where (u_0, φ_0) is a solution to the problem $\mathcal{A}_L$ with the right-hand sides F_1, F_2.

The proof of the existence will be based on the Galerkin method described, for instance, in [5], [11]. All functional spaces that we have dealt with are separable, which is essential for this method. Before we prove the basic result of this chapter, let us start with one auxiliary lemma.

Lemma 5.1. *Let $f_n \in L^2((0,T),X)$, where X is a separable Hilbert space and f_n weakly converge to f_0 in $L^2((0,T),X)$. Moreover, let there exist $C > 0$ such that $\|f_n(t)\|_X \leqslant C$ for any n and any $t \in (0,T)$. Then $\|f_0(t)\|_X \leqslant C$ for almost all $t \in (0,T)$.*

Proof. Let $x_n \in X$, $\|x_n\| > C$ be a dense subset in $\{x \in X,\ \|x\| > C\}$. Let us set

$$
\begin{aligned}
B_0 &= \{t \in (0,T) \mid \|f_0(t)\|_X > C\}, \\
B_j &= \Big\{t \in (0,T) \mid \|f_0(t) - x_j\|_X \leqslant \frac{\|x_j\|_X - C}{2}\Big\}.
\end{aligned}
$$

This yields $B_0 = \bigcup\limits_{n=1}^{\infty} B_n$. Due to the separability of X the sets B_n are measurable (see [16]), which means that there exists j such that $\mu(B_j) \neq 0$. If $\chi_j(s)$ is the characteristic function of B_j, then by virtue of the weak convergence of f_n the inequality

$$
\int_0^T \langle f_0(s), x_j\rangle_X\, \chi_j(s)\,\mathrm{d}s \leqslant C \cdot \mu(B_j)\|x_j\|_X
$$

holds. On the other hand, the definition of B_j yields

$$
\begin{aligned}
\int_0^T \langle f_0(s), x_j\rangle_X\, \chi_j(s)\,\mathrm{d}x &= \int_{B_j} (\langle f_0(s) - x_j, x_j\rangle_X + \|x_j\|_X^2)\,\mathrm{d}s \\
&\geqslant \int_{B_j} (\|x_j\|_X^2 - \|f_0(s) - x_j\|_X\, \|x_j\|_X)\,\mathrm{d}s \\
&\geqslant \int_{B_j} \|x_j\|_X \Big(\|x_j\|_X - \frac{\|x_j\|_X - C}{2}\Big)\,\mathrm{d}s \\
&= \mu(B_j)\|x_j\|_X\, \frac{C + \|x_j\|_X}{2}.
\end{aligned}
$$

If we compare these inequalities, we have $C \geqslant \frac{1}{2}(C + \|x_j\|_X)$, which contradicts $\|x_j\|_X > C$. $\square$

Theorem 5.1. *If the assumptions of Definition* 5.2 *are fulfilled, then there exists a solution to* $\mathcal{B}_L$ *and this solution is unique.*

Proof. The proof is divided into five steps:

1. We construct a sequence of approximate solutions based on the Galerkin method.
2. We establish a priori estimates which guarantee the existence of approximate solutions on $\langle 0, T\rangle$.
3. We establish a priori estimates which guarantee the existence of higher derivatives of the approximate solutions on $\langle 0, T\rangle$.
4. We select a subsequence which converges to a solution of $\mathcal{B}_L$.
5. We prove the initial conditions (5.4).

Step 1. Let v_i, ψ_i be sequences of linearly independent elements of V_1, V_2 and let the linear spans of these sequences be dense in V_1, V_2. Then the spans of these sequences are dense in W as well.

For any m let us consider the expressions

$$u_m(t) = \sum_{i=1}^{m} f_{im}(t) v_i,$$
$$\varphi_m(t) = \sum_{i=1}^{n} g_{im}(t) \psi_i$$

and moreover, let the equalities

$$v_1 = u_0 \quad \psi_1 = \varphi_0, \tag{5.5}$$

where (u_0, φ_0) is the solution to the problem $\mathcal{A}_L$, hold.

Let f_{im}, g_{im} be solutions to the system of ordinary differential equations

$$\begin{aligned} m_1(u_m''(t), v_i) + k_1(u_m(t), v_i) + \delta_1(u_m'(t), v_i) + b(u_m(t) + d - \varepsilon, v_i) \\ = (F_1 + P_1(\varphi_m(t)), v_i), \\ m_2(\varphi_m''(t), \psi_i) + k_2(\varphi_m(t), \psi_i) + \delta_2(\varphi_m'(t), \psi_i) \\ = (F_2 + P_2(\varphi_m(t)), \psi_i), \end{aligned} \tag{5.6}$$

$i = 1, \ldots, m$. Moreover, let f_{im}, g_{im} satisfy the initial conditions

$$\begin{aligned} &f_{1m}(0) = 1, \quad f_{im}(0) = 0, \\ &g_{1m}(0) = 1, \quad g_{im}(0) = 0, \quad i = 2, \ldots, m, \\ &f_{im}'(0) = 0, \quad g_{im}'(0) = 0, \quad i = 1, \ldots, m. \end{aligned} \tag{5.7}$$

If we consider the definition of $m_1(\cdot,\cdot)$, $m_2(\cdot,\cdot)$, $k_1(\cdot,\cdot)$, $k_2(\cdot,\cdot)$, $\delta_1(\cdot,\cdot)$, $\delta_2(\cdot,\cdot)$ and apply the Lebesgue dominanted convergence theorem, we can say that the functions $m_1\Big(\sum\limits_{i=1}^m y_i v_i, v_j\Big)$, $m_2\Big(\sum\limits_{i=1}^m y_i \psi_i, \psi_j\Big)$, $k_1\Big(\sum\limits_{i=1}^m y_i v_i, v_j\Big)$, $k_2\Big(\sum\limits_{i=1}^m y_i \psi_i, \psi_j\Big)$, $\delta_1\Big(\sum\limits_{i=1}^m y_i v_i, v_j\Big)$, $\delta_2\Big(\sum\limits_{i=1}^m y_i \psi_i, \psi_j\Big)$, $b\Big(\sum\limits_{i=1}^m y_i v_i + d - \varepsilon, v_j\Big)$ defined on $\mathbb{R}^m$ are continuous, where $j = 1,\dots,m$. If we consider (MC) and the Lebesgue dominanted convergence theorem, we can see that the functions

$$\left(F_1\Big(\sum_{i=1}^m y_i\psi_i, x, t\Big), v_j\right), \quad \left(F_2\Big(\sum_{i=1}^m y_i\psi_i, x, t\Big), \psi_j\right)$$

defined on $\mathbb{R}^m \times (0,T)$ are continuous on this set. This means that we can apply the theory of ordinary differential equations which guarantees the existence of a local solution to the system (5.6).

Step 2. If we multiply (5.6) by $f'_{jm}(t)$, $g'_{jm}(t)$ and sum these expressions, we have

$$\begin{aligned}(5.8)\quad & m_1(u''_m(t), u'_m(t)) + m_2(\varphi''_m(t), \varphi'_m(t)) + k_1(u_m(t), u'_m(t)) \\ &\quad + k_2(\varphi_m(t), \varphi'_m(t)) + \delta_1(u'_m(t), u'_m(t)) + \delta_2(\varphi'_m(t), \varphi'_m(t)) \\ &\quad + b(u_m(t) + d - \varepsilon, u'_m(t)) \\ &= (F_1 + P_1(\varphi_m(t)), u'_m(t)) + (F_2 + P_2(\varphi_m(t)), \varphi'_m(t)).\end{aligned}$$

The expression (5.8) yields

$$\begin{aligned}(5.9)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{m_1(u'_m(t), u'_m(t)) + m_2(\varphi'_m(t), \varphi'_m(t)) + k_1(u_m(t), u_m(t)) \\ &\quad + k_2(\varphi_m(t), \varphi_m(t)) + b(u_m(t) + d - \varepsilon, u_m(t) + d - \varepsilon)\} \\ &\quad + \delta_1(u'_m(t), u'_m(t)) + \delta_2(\varphi'_m(t), \varphi'(t)) \\ &= (F_1 + P_1(\varphi_m(t)), u'_m(t)) + (F_2 + P_2(\varphi_m(t)), \varphi'_m(t)),\end{aligned}$$

which results in the equality

$$\begin{aligned}(5.10)\quad & m_1(u'_m(t), u'_m(t)) + m_2(\varphi'_m(t), \varphi'_m(t)) + k_1(u_m(t), u_m(t)) \\ &\quad + k_2(\varphi_m(t), \varphi_m(t)) + b(u_m(t) + d - \varepsilon, u_m(t) + d - \varepsilon) \\ &= m_1(u'_m(0), u'_m(0)) + m_2(\varphi'_m(0), \varphi'_m(0)) + k_1(u_m(0), u_m(0)) \\ &\quad + k_2(\varphi_m(0), \varphi_m(0)) + b(u_m(0) + d - \varepsilon, u_m(0) + d - \varepsilon) \\ &\quad - 2\int_0^t \delta_1(u'_m(s), u'_m(s))\,\mathrm{d}s - 2\int_0^t \delta_2(\varphi'_m(s), \varphi'_m(s))\,\mathrm{d}s \\ &\quad + 2\int_0^t (F_1 + P_1(\varphi_m(s)), u'_m(s))\,\mathrm{d}s \\ &\quad + 2\int_0^t (F_2 + P_2(\varphi_m(s)), \varphi'_m(s))\,\mathrm{d}s.\end{aligned}$$

Relations (5.7), (5.10), (3.3) yield the inequality

$$\|u_m'(t)\|_W^2 + \|\varphi_m'(t)\|_W^2 + \|u_m(t)\|_{V_1}^2 + \|\varphi_m(t)\|_{V_2}^2 \leqslant C + C\int_0^t \{\|u_m'(s)\|_W^2 + \|\varphi_m'(s)\|_W^2\}\,\mathrm{d}s, \tag{5.11}$$

where C is a positive constant common for all m. Gronwall's inequality and (5.11) guarantee that the local solutions from *Step 1* exist on the whole interval $\langle 0, T\rangle$ and these solutions satisfy the following estimates:

$$\begin{gathered}\exists C > 0\ \forall m \in \mathbb{N}\ \forall t \in \langle 0, T\rangle,\\ \|u_m'(t)\|_W < C,\ \ \|\varphi_m'(t)\|_W < C,\ \ \|u_m(t)\|_{V_1} < C,\ \ \|\varphi_m(t)\|_{V_2} < C.\end{gathered} \tag{5.12}$$

Step 3. If we consider (5.5) and (5.7), then (5.6) implies the equalities

$$\begin{aligned} m_1\big(u_m''(0), v_i\big) &= (P_1(\varphi_m(0)), v_i),\\ m_2\big(\varphi_m''(0), \psi_i\big) &= (P_2(\varphi_m(0)), \psi_i),\end{aligned} \tag{5.13}$$

which yields the estimates

$$\|u_m''(0)\|_W < C, \quad \|\varphi_m''(0)\|_W < C, \tag{5.14}$$

where C is a constant common for all m.

If we differentiate the system (5.6) and multiply it by $f_{jm}''(t)$, $g_{jm}''(t)$, we have

$$\begin{aligned} & m_1(u_m'''(t), u_m''(t)) + m_2(\varphi_m'''(t), \varphi_m''(t))\\ &\quad + k_1(u_m'(t), u_m''(t)) + k_2(\varphi_m'(t), \varphi_m''(t)) + \delta_1(u_m''(t), u_m''(t))\\ &\quad + \delta_2(\varphi_m''(t), \varphi_m''(t)) + b(u_m'(t), u_m''(t))\\ &= \Big(\frac{\partial}{\partial y}P_1(\varphi_m(t))\varphi_m'(t), u_m''(t)\Big) + \Big(\frac{\partial}{\partial t}P_1(\varphi_m(t)), u_m''(t)\Big)\\ &\quad + \Big(\frac{\partial}{\partial y}P_2(\varphi_m(t))\varphi_m'(t), \varphi_m''(t)\Big) + \Big(\frac{\partial}{\partial t}P_2(\varphi_m(t)), \varphi_m''(t)\Big).\end{aligned} \tag{5.15}$$

The relation (5.15) yields

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{m_1(u_m''(t), u_m''(t)) + m_2(\varphi_m''(t), \varphi_m''(t))\\ &\quad + k_1(u_m'(t), u_m'(t)) + k_2(\varphi_m'(t), \varphi_m'(t)) + b(u_m'(t), u_m'(t))\}\\ &\quad + \delta_1(u_m''(t), u_m''(t)) + \delta_2(\varphi_m''(t), \varphi_m''(t))\\ &= \Big(\frac{\partial}{\partial y}P_1(\varphi_m(t))\varphi_m'(t), u_m''(t)\Big) + \Big(\frac{\partial}{\partial t}P_1(\varphi_m(t)), u_m''(t)\Big)\\ &\quad + \Big(\frac{\partial}{\partial y}P_2(\varphi_m(t))\varphi_m'(t), \varphi_m''(t)\Big) + \Big(\frac{\partial}{\partial t}P_2(\varphi_m(t)), \varphi_m''(t)\Big),\end{aligned} \tag{5.16}$$

which results in the equality

$$
\begin{aligned}
(5.17)\qquad & m_1(u''_m(t), u''_m(t)) + m_2(\varphi''_m(t), \varphi''_m(t)) \\
& \quad + k_1(u'_m(t), u'_m(t)) + k_2(\varphi'_m(t), \varphi'_m(t)) + b(u'_m(t), u'_m(t)) \\
& = m_1(u''_m(0), u''_m(0)) + m_2(\varphi''_m(0), \varphi''_m(0)) \\
& \quad + k_1(u'_m(0), u'_m(0)) + k_2(\varphi'_m(0), \varphi'_m(0)) + b(u'_m(0), u'_m(0)) \\
& \quad - 2\int_0^t \delta_1(u''_m(s), u''_m(s))\,\mathrm{d}s - 2\int_0^t \delta_2(\varphi''_m(s), \varphi''_m(s))\,\mathrm{d}s \\
& \quad + 2\int_0^t \Big(\frac{\partial}{\partial y} P_1(\varphi_m(s))\varphi'_m(s) + \frac{\partial}{\partial t} P_1(\varphi_m(s)), u''_m(s)\Big)\,\mathrm{d}s \\
& \quad + 2\int_0^t \Big(\frac{\partial}{\partial y} P_2(\varphi_m(s))\varphi'_m(s) + \frac{\partial}{\partial t} P_2(\varphi_m(s)), \varphi''_m(s)\Big)\,\mathrm{d}s.
\end{aligned}
$$

From (3.7), (5.13), (5.17), (3.3) we obtain the inequality

$$
\begin{aligned}
(5.18)\qquad & \|u''_m(t)\|^2_W + \|\varphi''_m(t)\|^2_W + \|u'_m(t)\|^2_{V_1} + \|\varphi'_m(t)\|^2_{V_2} \\
& \leqslant C + C\int_0^t \{\|u''_m(s)\|^2_W + \|\varphi''_m(s)\|^2_W\}\,\mathrm{d}s,
\end{aligned}
$$

where C is a positive constant common for all m. Gronwall's inequality and (5.18) guarantee that the approximate solutions satisfy the following estimates:

$$
\begin{aligned}
(5.19)\qquad & \exists C > 0\ \forall m \in \mathbb{N}\ \forall t \in \langle 0, T\rangle, \\
& \|u''_m(t)\|_W < C,\ \ \|\varphi''_m(t)\|_W < C,\ \ \|u'_m(t)\|_{V_1} < C,\ \ \|\varphi'_m(t)\|_{V_2} < C.
\end{aligned}
$$

Step 4. The estimates (5.12), (5.19) yield that the sequences $u_m(t)$, $u'_m(t)$ are bounded in $L^2((0,T), V_1)$, $\varphi_m(t)$, $\varphi'_m(t)$ are bounded in $L^2((0,T), V_2)$ and $u''_m(t)$, $\varphi''_m(t)$ are bounded in $L^2((0,T), W)$. These functional spaces are reflexive, which means that there exist subsequences $u_l(t)$, $u'_l(t)$, $u''_l(t)$ $\varphi_l(t)$, $\varphi'_l(t)$, $\varphi''_l(t)$ weakly converging to $u(t)$, $v(t)$, $w(t)$, $\varphi(t)$, $\psi(t)$, $\gamma(t)$ in the corresponding spaces. The weak convergence of these subsequences implies that the distributional derivatives of $u(t)$, $\varphi(t)$ in $L^2((0,T), V_1)$, $L^2((0,T), V_2)$, $L^2((0,T), W)$ exist and are equal to $v(t)$, $w(t)$, $\psi(t)$, $\gamma(t)$. In view of the above mentioned facts, we have

$$
\begin{aligned}
(5.20)\qquad & \lim_{l\to\infty}\int_0^T m_1(u''_l(t), v)\theta(t)\,\mathrm{d}t = \int_0^T m_1(u''(t), v)\theta(t)\,\mathrm{d}t, \\
& \lim_{l\to\infty}\int_0^T m_2(\varphi''_l(t), \psi)\theta(t)\,\mathrm{d}t = \int_0^T m_2(\varphi''(t), \psi)\theta(t)\,\mathrm{d}t,
\end{aligned}
$$

$$\lim_{l\to\infty}\int_0^T k_1(u_l(t),v)\theta(t)\,\mathrm{d}t = \int_0^T k_1(u(t),v)\theta(t)\,\mathrm{d}t,$$
$$\lim_{l\to\infty}\int_0^T k_2(\varphi_l(t),\psi)\theta(t)\,\mathrm{d}t = \int_0^T k_2(\varphi(t),\psi)\theta(t)\,\mathrm{d}t,$$
$$\lim_{l\to\infty}\int_0^T \delta_1(u_l'(t),v)\theta(t)\,\mathrm{d}t = \int_0^T \delta_1(u'(t),v)\theta(t)\,\mathrm{d}t,$$
$$\lim_{l\to\infty}\int_0^T \delta_2(\varphi_l'(t),\psi)\theta(t)\,\mathrm{d}t = \int_0^T \delta_2(\varphi'(t),\psi)\theta(t)\,\mathrm{d}t,$$
$$\lim_{l\to\infty}\int_0^T b(u_l(t)+d-\varepsilon,v)\theta(t)\,\mathrm{d}t = \int_0^T b(u(t)+d-\varepsilon,v)\theta(t)\,\mathrm{d}t,$$

where v, ψ, θ are arbitrary functions from V_1, V_2, $\mathcal{D}((0,T))$. Moreover, it follows from the definition of distributional derivatives that u_l, φ_l belong to $H^1((0,L)\times(0,T))$ and weakly converge to u, φ in this space. Taking into account that u_l, φ_l converge to u, φ in $L^2((0,T)\times(0,L))$, which follows from the Ehrling compactness theorem (see [7]), then by virtue of the Lebesgue dominanted convergence theorem we have

$$\lim_{l\to\infty}\int_0^T (P_1(\varphi_l(t)),v)\theta(t)\,\mathrm{d}t = \int_0^T (P_1(\varphi(t)),v)\theta(t)\,\mathrm{d}t,$$
$$\lim_{l\to\infty}\int_0^T (P_2(\varphi_l(t)),\psi)\theta(t)\,\mathrm{d}t = \int_0^T (P_2(\varphi(t)),\psi)\theta(t)\,\mathrm{d}t, \tag{5.21}$$

where v, ψ, θ are arbitrary.

If we consider (5.12), (5.19), (5.20), (5.21), Lemma 5.1 and the fact that the linear spans of v_i, ψ_i form dense subsets in V_1, V_2, then the equalities (5.3) hold.

Step 5. In view of the definition of X_1, X_2 we can write

$$\int_0^T \langle u'(t),v\rangle_{V_1}\theta(t)\,\mathrm{d}t = \int_0^T \langle u(t),v\rangle_{V_1}\theta'(t)\,\mathrm{d}t, \tag{5.22}$$
$$\int_0^T \langle \varphi'(t),\psi\rangle_{V_2}\theta(t)\,\mathrm{d}t = \int_0^T \langle \varphi(t),\psi\rangle_{V_2}\theta'(t)\,\mathrm{d}t,$$
$$\int_0^T \langle u''(t),v\rangle_{W}\theta(t)\,\mathrm{d}t = \int_0^T \langle u'(t),v\rangle_{W}\theta'(t)\,\mathrm{d}t,$$
$$\int_0^T \langle \varphi''(t),\psi\rangle_{W}\theta(t)\,\mathrm{d}t = \int_0^T \langle \varphi'(t),\psi\rangle_{W}\theta'(t)\,\mathrm{d}t,$$

where v, ψ, θ are arbitrary functions from V_1, V_2, $\mathcal{D}((0,T))$. From the definition of X_1, X_2 and (5.22) it follows that the expressions $\langle u(t),v\rangle_{V_1}$, $\langle\varphi(t),\psi\rangle_{V_2}$, $\langle u'(t),v\rangle_W$, $\langle\varphi'(t),\psi\rangle_W$, as functions defined on $(0,T)$, belong to $L^2((0,T))$ and have the generalized derivatives $\langle u'(t), v\rangle_{V_1}$, $\langle \varphi'(t), \psi\rangle_{V_2}$, $\langle u''(t), v\rangle_W$, $\langle \varphi''(t), \psi\rangle_W$ so that they belong to $H^1((0,T))$. Moreover, the sequences $\langle u_l(t), v\rangle_{V_1}$, $\langle \varphi_l(t), \psi\rangle_{V_2}$, $\langle u_l'(t), v\rangle_W$, $\langle \varphi_l'(t), \psi\rangle_W$ weakly converge to those functions in $H^1((0,T))$. Since the imbedding $H^1((0,T)) \subset C(\langle 0,T\rangle)$ is compact, those sequences strongly converge in $C(\langle 0,T\rangle)$. Then (5.7) yields the initial conditions (5.4).

Uniqueness. Let the problem $\mathcal{B}_L$ have two solutions (u_1,φ_1), (u_2,φ_2) and let $v = u_1 - u_2$, $\psi = \varphi_1 - \varphi_2$. By virtue of Lemma 3.1 we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\{m_1(v'(t),v'(t)) + m_2(\psi'(t),\psi'(t))\\ &\quad + k_1(v(t),v(t)) + k_2(\psi(t),\psi(t)) + b(v(t),v(t))\}\\ &\quad + \delta_1(v'(t),v'(t)) + \delta_2(\psi'(t),\psi'(t))\\ &= (P_1(\varphi_1(t)) - P_1(\varphi_2(t)), v'(t)) + (P_2(\varphi_1(t)) - P_2(\varphi_2(t)), \psi'(t)).\end{aligned}$$

The last expression yields the equality

$$\begin{aligned}(5.23)\qquad & m_1\big(v'(t),v'(t)\big) + m_2(\psi'(t),\psi'(t))\\ &\quad + k_1(v(t),v(t)) + k_2(\psi(t),\psi(t)) + b(v(t),v(t))\\ &= -2\int_0^t \delta_1(v'(s),v'(s))\,\mathrm{d}s - 2\int_0^t \delta_2(\psi'(s),\psi'(s))\,\mathrm{d}s\\ &\quad + 2\int_0^t (P_1(\varphi_1(s)) - P_1(\varphi_2(s)), v'(s))\,\mathrm{d}s\\ &\quad + 2\int_0^t (P_2(\varphi_1(s)) - P_2(\varphi_2(s)), \psi'(s))\,\mathrm{d}s.\end{aligned}$$

The conditions (MC) imply the inequalities

$$\begin{aligned}(5.24)\qquad & |(P_1(\varphi_1(t)) - P_1(\varphi_2(t)), v'(t))| \leqslant C(\|\psi(t)\|_{V_2}^2 + \|v'(t)\|_W^2),\\ & |(P_2(\varphi_1(t)) - P_2(\varphi_2(t)), \psi'(t))| \leqslant C(\|\psi(t)\|_{V_2}^2 + \|\psi'(t)\|_W^2),\end{aligned}$$

where C is a positive constant. If we consider (5.23), (5.24) and the inequalities (3.3), we can write

$$\begin{aligned}(5.25)\qquad & \|v'(t)\|_W^2 + \|\psi'(t)\|_W^2 + \|v(t)\|_{V_1}^2 + \|\psi(t)\|_{V_2}^2\\ &\leqslant C\int_0^t \{\|v'(s)\|_W^2 + \|\psi'(s)\|_W^2 + \|\psi(s)\|_{V_2}^2\}\,\mathrm{d}s,\end{aligned}$$

where C is a positive constant. By virtue of Gronwall's inequality, (5.25) yields the uniqueness of this problem. □

Proposition 5.1. *Let the assumptions of Theorem* 5.1 *be fulfilled, let* (u,φ) *be a solution to the problem* $\mathcal{B}_L$, *and let the functions* Θ_1, Θ_2 *describing the damping vanish. Then the equalities*

$$(5.26)\qquad \begin{aligned} E_1(t_2)-E_1(t_1) &= \int_{t_1}^{t_2} (F_1+P_1(\varphi(s)),u'(s))\,\mathrm{d}s,\\ E_2(t_2)-E_2(t_1) &= \int_{t_1}^{t_2} (F_2+P_2(\varphi(s)),\varphi'(s))\,\mathrm{d}s \end{aligned}$$

hold for any $t_1,t_2\in\langle 0,T\rangle$, *where*

$$\begin{aligned} E_1(t) &= \frac{1}{2}m_1(u'(t),u'(t))+\frac{1}{2}k_1(u(t),u(t))+\frac{1}{2}b(u(t)+d-\varepsilon,u(t)+d-\varepsilon),\\ E_2(t) &= \frac{1}{2}m_2(\varphi'(t),\varphi'(t))+\frac{1}{2}k_2(\varphi(t),\varphi(t)). \end{aligned}$$

Proof. Since (u,φ) are a solution to $\mathcal{B}_L$, we have

$$(5.27)\qquad \begin{aligned} m_1(u''(t),u'(t))+k_1(u(t),u'(t))+b(u(t)+d-\varepsilon,u'(t))&\\ =(F_1+P_1(\varphi(t)),u'(t))&,\\ m_2(\varphi''(t),\varphi'(t))+k_2(\varphi(t),\varphi'(t))=(F_2+P_2(\varphi(t)),\varphi'(t))&. \end{aligned}$$

The equalities (5.27) and Lemma 3.1 yield

$$(5.28)\qquad \begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{m_1(u'(t),u'(t))+k_1(u(t),u(t))+b(u(t)+d-\varepsilon,u(t)+d-\varepsilon)\}&\\ =(F_1+P_1(\varphi(t)),u'(t))&,\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{m_2(\varphi'(t),\varphi'(t))+k_2(\varphi(t),\varphi(t))\}=(F_2+P_2(\varphi(t)),\varphi'(t))&. \end{aligned}$$

After integrating the equation (5.28) from t_1 to t_2, we obtain the desired result. □

Let us notice that $E_1(t)$, $E_2(t)$ correspond to the total energy of the vertical and torsional vibrations of the system depicted in Fig. 2. The damping is neglected. The right-hand sides in (5.26) correspond to the change of the energy of the external forces. Thus the equations express the law of energy conservation.

Let $P(y,x,t)$ be defined on $\mathbb{R}\times(0,L)\times(0,T)$ and satisfy the assumptions (MC), then we can define

$$\begin{aligned} \|P\|_{\delta_1} &= \sup_{y\in\mathbb{R},\,t\in(0,T)} \|P(y,\cdot,t)\|_{L^2((0,L))},\\ \|P\|_{\delta_2} &= \sup_{y\in\mathbb{R},\,t\in(0,T)} \left\|\frac{\partial}{\partial y}P(y,\cdot,t)\right\|_{L^\infty((0,L))},\\ \|P\|_{\delta_3} &= \sup_{y\in\mathbb{R},\,t\in(0,T)} \left\|\frac{\partial}{\partial t}P(y,\cdot,t)\right\|_{L^2((0,L))},\\ \|P\|_{\delta} &= \|P\|_{\delta_1}+\|P\|_{\delta_2}+\|P\|_{\delta_3}. \end{aligned}$$

Proposition 5.2. *Let the assumptions of Theorem* 5.1 *be fulfilled and* (u_0, φ_0) *be a solution to the time independent problem* $\mathcal{A}_L$ *with the right-hand sides* F_1, F_2. *Then there exists a positive constant* C *independent of* F_1, F_2, P_1, P_2 *such that the inequalities*

$$\begin{aligned} \|u - u_0\|_{L^2((0,T),V_1)} &\leqslant C(\|P_1\|_\delta + \|P_2\|_\delta), \\ \|u'\|_{L^2((0,T),V_1)} &\leqslant C(\|P_1\|_\delta + \|P_2\|_\delta), \\ \|u''\|_{L^2((0,T),W)} &\leqslant C(\|P_1\|_\delta + \|P_2\|_\delta), \\ \|\varphi - \varphi_0\|_{L^2((0,T),V_2)} &\leqslant C(\|P_1\|_\delta + \|P_2\|_\delta), \\ \|\varphi'\|_{L^2((0,T),V_2)} &\leqslant C(\|P_1\|_\delta + \|P_2\|_\delta), \\ \|\varphi''\|_{L^2((0,T),W)} &\leqslant C(\|P_1\|_\delta + \|P_2\|_\delta) \end{aligned} \tag{5.29}$$

hold, where (u, φ) *is a solution to* $\mathcal{B}_L$ *and* u_0, φ_0 *in* (5.29) *are understood as constant functions defined on* $(0, T)$.

Proof. If the inequalities (5.29) hold for the approximate solution $u_m(t)$, $\varphi_m(t)$ in the proof of Theorem 5.1 with a constant C independent of m, then by virtue of the weak convergence of these approximate solutions the inequalities (5.29) remain fulfilled for the solutions u, φ to $\mathcal{B}_L$. Let us assume that the equations

$$\begin{aligned} k_1(u_0, v_j) + b(u_0 + d - \varepsilon, v_j) &= (F_1, v_j), \\ k_2(\varphi_0, \psi_j) &= (F_2, \psi_j) \end{aligned} \tag{5.30}$$

hold, where v_j, ψ_j are the same function as in the proof of Theorem 5.1. If we substitute (5.30) into (5.6), we have

$$\begin{aligned} m_1(u''_m(t), v_j) + k_1(u_m(t) - u^0, v_j) + \delta_1(u'_m(t), v_j) + b(u_m(t) - u_0, v_j) \\ = (P_1(\varphi_m(t)), v_j), \\ m_1(\varphi''_m(t), \psi_j) + k_2(\varphi_m(t) - \varphi_0, \psi_j) + \delta_2(\varphi'_m(t), \psi_j) \\ = (P_2(\varphi_m(t)), \psi_j). \end{aligned} \tag{5.31}$$

If we apply the same arguments as in Theorem 5.1, then (5.31) implies the equality

$$\begin{aligned} m_1(u'_m(t), u'_m(t)) + m_2(\varphi'_m(t), \varphi'_m(t)) + k_1(u_m(t) - u_0, u_m(t) - u_0) \\ + k_2(\varphi_m(t) - \varphi_0, \varphi_m(t) - \varphi_0) + b(u_m(t) - u_0, u_m(t) - u_0) \\ = -2\int_0^t \delta_1(u'_m(s), u'_m(s))\,\mathrm{d}s - 2\int_0^t \delta_2(\varphi'_m(s), \varphi'_m(s))\,\mathrm{d}s \\ + 2\int_0^t (P_1(\varphi_m(s)), u'_m(s))\,\mathrm{d}s + 2\int_0^t (P_2(\varphi_m(s)), \varphi'_m(s))\,\mathrm{d}s. \end{aligned} \tag{5.32}$$

The equality (5.32) yields the inequality

$$\begin{aligned}(5.33)\qquad &\|u_m'(t)\|_W^2 + \|\varphi_m'(t)\|_W^2 + \|u_m(t) - u_0\|_{V_1}^2 + \|\varphi_m(t) - \varphi_0\|_{V_2}^2 \\ &\leqslant C \int_0^t \{\|u_m'(s)\|_W^2 + \|\varphi_m'(s)\|_W^2 + \|P_1(\varphi_m(s))\|_W^2 \\ &\quad + \|P_2(\varphi_m(s))\|_W^2\}\, \mathrm{d}s.\end{aligned}$$

Now (5.33) and Gronwall's inequality imply the estimates 1 and 4 from (5.29).

Due to the equalities (5.13) and (5.17), we have the inequality

$$\begin{aligned}(5.34)\qquad &\|u_m''(t)\|_W^2 + \|\varphi_m''(t)\|_W^2 + \|u_m'(t)\|_{V_1}^2 + \|\varphi_m'(t)\|_{V_2}^2 \\ &\leqslant C(\|P_1\|_\delta + \|P_2\|_\delta) + C \int_0^t \{\|u_m''(s)\|_W^2 + \|\varphi_m''(s)\|_W^2\}\, \mathrm{d}s.\end{aligned}$$

The inequality (5.34) and Gronwall's inequality yield the estimates 2, 3, 5, 6 in (5.29).

□

Let $C^{\frac{1}{2}}(\langle 0, L\rangle \times \langle 0, T\rangle)$ be the space of Hölder continuous functions with the norm

$$(5.35)\quad \|u\|_{C^{\frac{1}{2}}(\langle 0,L\rangle\times\langle 0,T\rangle)} = \sup_{\substack{x\in\langle 0,L\rangle\\ y\in\langle 0,T\rangle}} |u(x,y)| + \sup_{\substack{x_1,x_2\in\langle 0,L\rangle\\ y_1,y_2\in\langle 0,T\rangle\\ x_1\neq x_2,\, y_1\neq y_2}} \frac{|u(x_1,y_1) - u(x_2,y_2)|}{|x_2 - x_1|^{\frac{1}{2}} + |y_2 - y_1|^{\frac{1}{2}}}.$$

Lemma 5.2. *If $u \in L^2((0,T),X)$, $u' \in L^2((0,T),X)$, where $X \subset H^1((0,L))$ is a continuous imbedding, then $u \in C^{\frac{1}{2}}(\langle 0,L\rangle \times \langle 0,T\rangle)$ and the inequality*

$$\|u\|_{C^{\frac{1}{2}}(\langle 0,L\rangle\times\langle 0,T\rangle)} \leqslant C(\|u\|_{L^2((0,T),X)} + \|u'\|_{L^2((0,T),X)})$$

holds, where C is a constant independent of u.

Proof. By virtue of Lemma 3.1, $u(t)$ is continuous as a function from $\langle 0,T\rangle$ to X and the inequality

$$(5.36)\qquad \|u(t_2) - u(t_1)\|_X = \left\| \int_{t_1}^{t_2} u'(s)\, \mathrm{d}s \right\|_X \leqslant \left(\int_0^T \|u'(s)\|_X^2\, \mathrm{d}s \right)^{\frac{1}{2}} |t_2 - t_1|^{\frac{1}{2}}$$

holds for any $t_1, t_2 \in \langle 0,T\rangle$. Moreover, the inequality

$$(5.37)\qquad \sup_{t\in\langle 0,T\rangle} \|u(t)\|_X \leqslant C(\|u\|_{L^2((0,T),X)} + \|u'\|_{L^2((0,T),X)})$$

holds, where C is independent of u. Let us admit that (5.37) does not hold, then there exists sequences $u_i, t_i \in \langle 0, T\rangle$ such that

$$
\begin{aligned}
(5.38) \qquad & \|u_i\|_{L^2((0,T),X)} + \|u_i'\|_{L^2((0,T),X)} = 1, \\
& \|u_i(t_i)\|_X \geqslant i,
\end{aligned}
$$

but the first relation in (5.38) contradicts (5.36). Let us estimate the terms on the right-hand side of the inequality

$$
(5.39) \qquad |u(x_1,t_1) - u(x_2,t_2)| \leqslant |u(x_1,t_1) - u(x_1,t_2)| + |u(x_1,t_2) - u(x_2,t_2)|.
$$

It is known that $H^1((0,L))$ is continuously imbedded in $C^{\frac{1}{2}}(\langle 0,L\rangle)$ with the norm

$$
\|v\|_{C^{\frac{1}{2}}(\langle 0,L\rangle)} = \sup_{x\in\langle 0,L\rangle} |v(x)| + \sup_{\substack{x_1,x_2\in\langle 0,L\rangle \\ x_1\neq x_2}} \frac{|v(x_2)-v(x_1)|}{|x_2-x_1|^{\frac{1}{2}}}.
$$

Hence we can write

$$
(5.40) \qquad |u(x_1,t_1) - u(x_1,t_2)| \leqslant C\|u(t_1) - u(t_2)\|_{H^1((0,L))}.
$$

From the last inequality and the estimate (5.36) we obtain

$$
(5.41) \qquad |u(x_1,t_1) - u(x_1,t_2)| \leqslant C\|u'\|_{L^2((0,T),X)}|t_2 - t_1|^{\frac{1}{2}}.
$$

Moreover, we have

$$
(5.42) \qquad |u(x_1,t_2) - u(x_2,t_2)| \leqslant C\|u(t_2)\|_{H^1((0,L))}|x_2 - x_1|^{\frac{1}{2}}.
$$

Then the desired result follows from (4.52), (5.37), (5.41) and (5.39). □

Due to the fact that V_1, V_2 are continuously imbedded in $H^1((0,L))$ and to the results of Theorem 5.1, we can see that both u and φ belong to $C^{\frac{1}{2}}(\langle 0,L\rangle \times \langle 0,T\rangle)$.

Proposition 5.3. *Let the assumptions of Theorem* 5.1 *be fulfilled, let* (u_0, φ_0) *be a solution to* $\mathcal{A}_L$ *with the right-hand sides* F_1, F_2, *let the inequalities*

$$
(u_0 + d)(x_j) > 2\varepsilon, \quad j = 1, \dots, 8
$$

hold and let the terms $\|P_1\|_\delta$, $\|P_2\|_\delta$ *be sufficiently small. Then the inequalities*

$$
(u + d)(x_j, t) > \varepsilon, \quad j = 1, \dots, 8
$$

hold for any $t \in \langle 0, T\rangle$.

Proof. The proof is an easy consequence of Proposition 5.2, Lemma 5.2 and the remark after that lemma. □

The last proposition describes the following fact: If the gravitation, represented by F_1, F_2, tightens the cables, then the strength of wind, represented by P_1, P_2, does not loosen the cables if this strength is sufficiently small. Similar problems were discussed in [2], [3], [10].

Let us note that Proposition 5.2 yields the existence of a solution to the nonlinear problem $\mathcal{B}$ if the terms $\|P_1\|_\delta$, $\|P_2\|_\delta$ are sufficiently small and the gravitation represented by F_1, F_2 tightens all cables.

6. Homogenization of cable systems: b-h convergence

The center span in our model depicted in Figure 2 is suspended by 8 cables, but it is obvious that the above theory works for any number of cables. Real constructions are suspended by much larger numbers of cables. The main goal of this chapter is to replace the cables with a continuous medium which asymptotically describes the behaviour of the cables. Let us define the bilinear form

$$h(u,v) = \int_0^L zuv \, \mathrm{d}x,$$

where $z \in L^\infty((0,L))$.

Definition 6.1. Let n_i be an increasing sequence of natural numbers, let $\{x_j^i\}_{j=1}^{n_i}$, $\{k_j^i\}_{j=1}^{n_i}$ satisfy $0 \leqslant x_1^i \leqslant x_2^i \leqslant \ldots \leqslant x_{n_i}^i \leqslant L$, $k_j^i \geqslant 0$ for any $i = 1, 2, \ldots$, and let $z \in L^\infty((0,L))$ satisfy $z(x) \geqslant 0$. Then $\{x_j^i\}_{j=1}^{n_i}$, $\{k_j^i\}_{j=1}^{n_i}$ b-h converge to z if the relation

$$\lim_{i\to\infty} \sum_{j=1}^{n_i} k_j^i f(x_j^i) = \int_0^L zf \, \mathrm{d}x$$

holds for all $f \in C(\langle 0,L \rangle)$. Moreover, let $b^i(u,v)$ denote the bilinear forms $\sum\limits_{j=1}^{n_i} k_j^i u(x_j^i) v(x_j^i)$.

This definition describe the process in which one cable system is being gradually replaced by another cable system with a larger number of thinner cables.

Definition 6.2. Let $F_1, F_2 \in L^2((0,L))$, $d \in V_1$. Then $(u,\varphi) \in V_1 \times V_2$ is a solution to the problem $\mathcal{E}$ if the equalities

$$k_1(u,v) + h(\tilde{g}(u+d),v) = (F_1,v),$$
$$k_2(\varphi,\psi) = (F_2,\psi)$$

hold for any $v \in V_1$, $\psi \in V_2$.

Definition 6.3. Let $F_1, F_2 \in L^2((0,L))$, $d \in V_1$. Then $(u,\varphi) \in V_1 \times V_2$ is a solution to the problem $\mathcal{E}_L$ if the equalities

$$\begin{aligned} k_1(u,v) + h(u+d-\varepsilon, v) &= (F_1, v), \\ k_2(\varphi,\psi) &= (F_2,\psi) \end{aligned}$$

hold for any $v \in V_1$, $\psi \in V_2$.

Let us notice that these problems are uniquely solvable. The proof of this assertion is parallel to the proof of Theorem 4.1.

Theorem 6.1. *Let $\{x_j^i\}_{j=1}^{n_i}$, $\{k_j^i\}_{j=1}^{n_i}$ b-h converge to $z \in L^\infty((0,L))$ and let (u^i,φ^i) be solutions to the sequence of the problems $\mathcal{A}$ with the bilinear forms $b^i(\cdot,\cdot)$ corresponding to $\{x_j^i\}_{j=1}^{n_i}$, $\{k_j^i\}_{j=1}^{n_i}$. Then*

$$\begin{aligned} u^i &\to u^0 \quad \textit{in } V_1, \\ \varphi^i &\to \varphi^0 \quad \textit{in } V_2, \end{aligned}$$

where (u^0,φ^0) is a solution to $\mathcal{E}$ with the bilinear form $h(\cdot,\cdot)$ corresponding to z.

P r o o f. The definition of the b-h convergence and the uniform boundedness theorem for functionals on $C(\langle 0,L\rangle)$ yield

$$|b^i(u,v)| \leqslant C\|u\|_{C(\langle 0,L\rangle)}\|v\|_{C(\langle 0,L\rangle)}, \tag{6.1}$$

where C is a positive constant independent of i, u, v.

From Definition 5.2 it follows that $\varphi^i = \varphi^0$ and we can only study the sequence u^i. This sequence satisfies the inequality

$$\begin{aligned} C\|u^i\|_{V_1}^2 &\leqslant k_1(u^i,u^i) + b^i(\tilde{g}(u^i+d),u^i) - b^i(\tilde{g}(d),u^i) \\ &= -b^i(\tilde{g}(d),u^i) + (F_1,u^i), \end{aligned} \tag{6.2}$$

where C is independent of i. The last inequality is a consequence of the estimate (4.6) which holds in this case for the same reasons.

Since V_1 is continuously imbedded to $C(\langle 0,L\rangle)$, the estimate (6.1) yields the inequality

$$-b^i(\tilde{g}(d),u^i) + (F_1,u^i) \leqslant C_1\|u^i\|_{V_1}, \tag{6.3}$$

where C_1 is independent of i. The inequalities (6.2), (6.3) yield that there exists a constant C such that

$$\|u^i\|_{V_1} \leqslant C, \tag{6.4}$$

where C is independent of i. From (6.4) it follows that there exists a subsequence u^j of the sequence u^i which weakly converges to u^0 in V_1. Taking into account that $V_1 \subset H^1((0,L))$ and $H^1((0,L))$ can be continuously imbedded in $C^{\frac{1}{2}}(\langle 0,L\rangle)$ (see [7]), by virtue of the Arzelà-Ascoli theorem we can assert that u^j converges to u^0 in $C(\langle 0,L\rangle)$. Moreover, u^j satisfy the equations

$$k_1(u^j,v) + h(\tilde{g}(u^j+d),v) = h(\tilde{g}(u^j+d),v) - b^j(\tilde{g}(u^j+d),v) + (F_1,v), \tag{6.5}$$

where v is an arbitrary element from V_1. Owing to the fact that $u^j \to u^0$ in $C(\langle 0,L\rangle)$, the estimate (6.1) and the b-h convergence, we have

$$\begin{aligned} h(\tilde{g}(u^j+d),v) - h(\tilde{g}(u^0+d),v) &\to 0,\\ h(\tilde{g}(u^0+d),v) - b^j(\tilde{g}(u^0+d),v) &\to 0,\\ b^j(\tilde{g}(u^0+d),v) - b^j(\tilde{g}(u^j+d),v) &\to 0 \end{aligned}$$

if $j \to \infty$, which yields

$$h(\tilde{g}(u^j+d),v) - b^j(\tilde{g}(u^j+d),v) \to 0 \tag{6.6}$$

if $j \to \infty$. From (6.5), (6.6) and the weak convergence of u^j to u^0 in V_1 it follows that (u^0,φ^0) is a solution to $\mathcal{E}$. This fact yields that there exists C such that the inequality

$$\begin{aligned} C\|u^j-u^0\|^2_{V_1} &\leqslant k_1(u^j-u^0,u^j-u^0)\\ &= h(\tilde{g}(u^0+d),u^j-u^0) - b^j(\tilde{g}(u^j+d),u^j-u^0) \end{aligned} \tag{6.7}$$

holds. If we consider (6.1), the weak convergence of u^j to u^0 in V_1, the strong convergence of u^j to u^0 in $C(\langle 0,L\rangle)$, then (3.3) and (6.6) imply the strong convergence of u^j to u^0 in V_1. Moreover, (u^0,φ^0) is the unique solution to $\mathcal{E}$, which means that the whole sequence u^i converges to u^0 in V_1. □

It is evident that we can prove the same theorem for the problems $\mathcal{A}_L$, $\mathcal{E}_L$. Let us formulate other two dynamic problems $\mathcal{F}$, $\mathcal{F}_L$ and study the connection between the problems $\mathcal{B}_L$, $\mathcal{F}_L$.

Definition 6.4. Let $d \in V_1$ and F_1, F_2 belong to $L^2((0,T))$ and let P_1, P_2 satisfy (MC). Then $u \in X_1$, $\varphi \in X_2$ are a solution to the problem $\mathcal{F}$ if the equalities

$$\begin{aligned} m_1(u'',v) + k_1(u,v) + \delta_1(u',v) + h(\tilde{g}(u+d),v) &= (F_1 + P_1(\varphi),v),\\ m_2(\varphi'',\psi) + k_2(\varphi,\psi) + \delta_2(\varphi',\psi) &= (F_2 + P_2(\varphi),\psi) \end{aligned}$$

hold in $L^2((0,T))$ for any $v \in V_1$, $\psi \in V_2$. Moreover, the initial conditions

$$\begin{aligned} u(0) &= u_0, \quad u'(0) = 0, \\ \varphi(0) &= \varphi_0, \quad \varphi'(0) = 0 \end{aligned}$$

are fulfilled, where u_0, φ_0 are a solution to the problem $\mathcal{E}$ with the right-hand sides F_1, F_2.

Definition 6.5. Let $d \in V_1$ and F_1, F_2 belong to $L^2((0,T))$ and let P_1, P_2 satisfy (MC). Then $(u, \varphi) \in X_1 \times X_2$ is a solution to the problem $\mathcal{F}_L$ if the equalities

$$\begin{aligned} m_1(u'', v) + k_1(u, v) + \delta_1(u', v) + h(u + d - \varepsilon, v) &= (F_1 + P_1(\varphi), v), \\ m_2(\varphi'', \psi) + k_2(\varphi, \psi) + \delta_2(\varphi', \psi) &= (F_2 + P_2(\varphi), \psi) \end{aligned} \tag{6.8}$$

hold in $L^2((0,T))$ for any $v \in V_1$, $\psi \in V_2$. Moreover, the initial conditions

$$\begin{aligned} u(0) &= u_0, \quad u'(0) = 0, \\ \varphi(0) &= \varphi_0, \quad \varphi'(0) = 0 \end{aligned} \tag{6.9}$$

are fulfilled, where (u_0, φ_0) is a solution to the problem $\mathcal{E}_L$ with the right-hand sides F_1, F_2.

It is evident that the problem $\mathcal{F}_L$ has a unique solution. The proof is parallel to that of Theorem 5.1.

Theorem 6.2. *Let the assumptions of Theorem* 5.1 *be fulfilled, let* $\{x_j^i\}_{j=1}^{n_i}$, $\{k_j^i\}_{j=1}^{n_i}$ *b-h converge to* $z \in L^\infty((0,L))$ *and let* u^i, φ^i *be solutions to the sequence of the problems* $\mathcal{B}_L$ *with the bilinear forms* $b^i(\cdot,\cdot)$ *corresponding to* $\{x_j^i\}_{j=1}^{n_i}$, $\{k_j^i\}_{j=1}^{n_i}$. *Then*

$$\begin{aligned} u^i &\to u^0, & \varphi^i &\to \varphi^0 & &\text{in } C(\langle 0,L\rangle \times \langle 0,T\rangle), \\ u^i &\rightharpoonup u^0, & \varphi^i &\rightharpoonup \varphi^0 & &\text{in } L^2((0,T),V_1),\ L^2((0,T),V_2), \\ u'^i &\rightharpoonup u'^0, & \varphi'^i &\rightharpoonup \varphi'^0 & &\text{in } L^2((0,T),V_1),\ L^2((0,T),V_2), \\ u''^i &\rightharpoonup u''^0, & \varphi''^i &\rightharpoonup \varphi''^0 & &\text{in } L^2((0,T),W), \end{aligned} \tag{6.10}$$

where (u^0, φ^0) *is a solution to* $\mathcal{F}_L$ *with the bilinear form* $h(\cdot,\cdot)$ *corresponding to* z.

P r o o f. If we change the form $b(\cdot,\cdot)$ for $b^i(\cdot,\cdot)$ in the equations (5.8), (5.9) and (5.10), we can see that all approximate solutions to the sequence of the problems $\mathcal{B}_L$ corresponding to $b^i(\cdot,\cdot)$ satisfy the inequality (5.11), where C is independent of i.

The equation (5.13) is fulfilled for the approximate solutions to the same sequence of problems and the constant C in (5.14) is independent of i. If we replace the form $b(\cdot,\cdot)$ by $b^i(\cdot,\cdot)$ in the equations (5.15), (5.16) and (5.17), we can see that the approximate solutions satisfy the inequality (5.18), where C is independent of i.

From the above considerations it follows that those approximate solutions satisfy the estimates (5.12) and (5.19) with C independent of i. This yields that the approximate solutions and their first derivatives are bounded in $L^2((0,T),V_1)$, $L^2((0,T),V_2)$ and their second derivatives in $L^2((0,T),W)$. So the sequences of solutions u^i, φ^i to the sequence of the problems $\mathcal{B}_L$ with the bilinear forms $b^i(\cdot,\cdot)$ satisfy the following estimates:

$$\begin{aligned}
(6.11)\quad & \exists C>0\ \forall i\in\mathbb{N}\\
& \|u^i\|_{L^2((0,T),V_1)}<C,\ \ \|u'^i\|_{L^2((0,T),V_1)}<C,\ \ \|u''^i\|_{L^2((0,T),W)}<C,\\
& \|\varphi^i\|_{L^2((0,T),V_2)}<C,\ \ \|\varphi'^i\|_{L^2((0,T),V_2)}<C,\ \ \|\varphi''^i\|_{L^2((0,T),W)}<C.
\end{aligned}$$

The estimates (6.11) yield that there exist $u^0\in L^2((0,T),V_1)$, $\varphi^0\in L^2((0,T),V_2)$, such that $u'^0\in L^2((0,T),V_1)$, $\varphi'^0\in L^2((0,T),V_2)$, $u''^0\in L^2((0,T),W)$, $\varphi''^0\in L^2((0,T),W)$, and subsequences u^j, φ^j of the sequences u^i, φ^i which satisfy

$$\begin{aligned}
(6.12)\qquad & u^j\rightharpoonup u^0,\quad \varphi^j\rightharpoonup\varphi^0 \quad \text{in } L^2((0,T),V_1),\ L^2((0,T),V_2),\\
& u'^j\rightharpoonup u'^0,\quad \varphi'^j\rightharpoonup\varphi'^0 \quad \text{in } L^2((0,T),V_1),\ L^2((0,T),V_2),\\
& u''^j\rightharpoonup u''^0,\ \varphi''^j\rightharpoonup\varphi''^0 \quad \text{in } L^2((0,T),W).
\end{aligned}$$

Since u^j, φ^j are the solutions to the problems $\mathcal{B}_L$ with the bilinear forms $b^j(\cdot,\cdot)$, then we have the equalities

$$\begin{aligned}
(6.13)\ & \int_0^T\{m_1(u''^j,v)+k_1(u^j,v)+\delta_1(u'^j,v)+h(u^j+d-\varepsilon,v)\}\,\mathrm{d}t\\
&\quad=\int_0^T\{(F_1+P_1(\varphi^j),v)+h(u^j+d-\varepsilon,v)-b^j(u^j+d-\varepsilon,v)\}\,\mathrm{d}t,\\
& \int_0^T\{m_2(\varphi''^j,\psi)+k_2(\varphi^j,\psi)+\delta_2(\varphi'^j,\psi)\}\,\mathrm{d}t\\
&\quad=\int_0^T(F_2+P_2(\varphi^j),\psi)\,\mathrm{d}t
\end{aligned}$$

that are fulfilled for any $v\in L^2((0,T),V_1)$, $\psi\in L^2((0,T),V_2)$, which is an easy consequence of Definition 5.2.

Lemma 5.2 yields that the sequences u^i, φ^i are bounded in $C^{\frac{1}{2}}(\langle 0,L\rangle\times\langle 0,T\rangle)$, so the subsequences u^j, φ^j strongly converge to u^0, φ^0 in $C(\langle 0,L\rangle\times\langle 0,T\rangle)$. Taking into account the definition of the b-h convergence, the inequality (6.1) and (MC), we have the relations

$$
\begin{aligned}
(6.14)\qquad & \lim_{j\to\infty}\int_0^T h(u^j+d-\varepsilon,v)-b^j(u^j+d-\varepsilon,v)\,\mathrm{d}t=0,\\
& \lim_{j\to\infty}\int_0^T (P_1(\varphi^j),v)\,\mathrm{d}t=\int_0^T (P_1(\varphi^0),v)\,\mathrm{d}t,\\
& \lim_{j\to\infty}\int_0^T (P_2(\varphi^j),\psi)\,\mathrm{d}t=\int_0^T (P_2(\varphi^0),\psi)\,\mathrm{d}t.
\end{aligned}
$$

From the equations (6.13), the relations (6.14), and (6.12) it follows that u^0, φ^0 satisfy the equations (6.8). If we follow the ideas of *Step 5* in Theorem 5.1 and consider Theorem 6.1, we can see that u^0, φ^0 satisfy the initial conditions (6.9), so (u^0,φ^0) is a solution to the problem $\mathcal{F}_L$. Since this solution is unique, we have (6.10). $\square$

Definition 6.6. Let $z_i \in L^\infty((0,L))$ be a sequence of functions satisfying $z_i(x)\geqslant 0$ and let $\{x_j\}_{j=1}^n$, $\{k_j\}_{j=1}^n$ be two finite sequences of numbers satisfying $x_j\in(0,L)$, $k_j>0$, $j=1,\dots,n$. Then the sequence z_i h-b converges to $\{x_j\}_{j=1}^n$, $\{k_j\}_{j=1}^n$ if for any $f\in C(\langle 0,L\rangle)$ the relation

$$\lim_{n\to\infty}\int_0^L z_i f\,\mathrm{d}x=\sum_{j=1}^n k_j f(x_j)$$

holds. Moreover, let $h^i(u,v)$ denote the bilinear form $\int_0^L z_i uv\,\mathrm{d}x$.

Theorem 6.3. *Let $z_i\in L^\infty((0,L))$ h-b converge to $\{x_j\}_{j=1}^n$, $\{k_j\}_{j=1}^n$ and let u^i, φ^j be solutions to the sequence of the problems $\mathcal{E}$ with the bilinear forms $h^i(\cdot,\cdot)$ corresponding to z_i. Then*

$$
\begin{aligned}
u^i &\to u^0 \quad \text{in } V_1,\\
\varphi^i &\to \varphi^0 \quad \text{in } V_2,
\end{aligned}
$$

where (u^0,φ^0) is a solution to the problem $\mathcal{A}$ with the bilinear form $b(\cdot,\cdot)$ corresponding to $\{x_j\}_{j=1}^n$, $\{k_j\}_{j=1}^n$.

Theorem 6.4. *Let the assumptions of Theorem 5.1 be fulfilled, let $z_i\in L^\infty((0,L))$ h-b converge to $\{x_j\}_{j=1}^n$, $\{k_j\}_{j=1}^n$, and u^i, φ^i be solutions to the sequence* *of the problems $\mathcal{F}_L$ with the bilinear forms $h^i(\cdot,\cdot)$ corresponding to z_i. Then*

$$\begin{aligned}
&u^i \to u^0, &&\varphi^i \to \varphi^0 &&\text{in } C(\langle 0,L\rangle \times \langle 0,T\rangle),\\
&u^i \rightharpoonup u^0, &&\varphi^i \rightharpoonup \varphi^0 &&\text{in } L^2((0,T),V_1),\ L^2((0,T),V_2),\\
&u'^i \rightharpoonup u'^0, &&\varphi'^i \rightharpoonup \varphi'^0 &&\text{in } L^2((0,T),V_1),\ L^2((0,T),V_2),\\
&u''^i \rightharpoonup u''^0, &&\varphi''^i \rightharpoonup \varphi''^0 &&\text{in } L^2((0,T),W),
\end{aligned}$$

where (u^0,φ^0) is a solution to $\mathcal{B}_L$ with the bilinear form $b(\cdot,\cdot)$ corresponding to $\{x_j^i\}_{j=1}^{n_i}$, $\{k_j^i\}_{j=1}^{n_i}$.

Proof. The proofs of Theorem 6.3 and Theorem 6.4 are parallel to the proofs of Theorem 6.1 and Theorem 6.2. We only have to replace $b^i(\cdot,\cdot)$, $h(\cdot,\cdot)$ by $h^i(\cdot,\cdot)$, $b(\cdot,\cdot)$. □

The above results show that the problems $\mathcal{E}$, $\mathcal{E}_L$, $\mathcal{F}_L$ approximate the behaviour of the construction depicted in Fig. 2 with both dense cable systems and distinct cables.

7. Two nonlinear dynamic problems: formulation, existence, uniqueness

In the preceding chapters above we studied some relations between the nonlinear problems $\mathcal{A}$ and $\mathcal{E}$ and between the linearized problems $\mathcal{B}_L$ and $\mathcal{F}_L$. These problems approximate each other. In this chapter we are going to study the two nonlinear problems $\mathcal{F}$, $\mathcal{G}$ connected with $\mathcal{B}$ and prove the existence and uniqueness of solutions to these problems.

Theorem 7.1. *If the assumptions of Definition 6.4 are fulfilled, then there exists a solution to $\mathcal{F}$ and this solution is unique.*

Proof. The proof is divided into the same five steps as in the proof of Theorem 5.1 and follows similar arguments so we briefly describe the differences.

Step 1. Let v_i, ψ_i be sequences of linearly independent elements of V_1, V_2 and let the linear spans of these sequences be dense in V_1, V_2 as well as in W.

For any m let

$$u_m(t) = \sum_{i=1}^{m} f_{im}(t)\, v_i,$$

$$\varphi_m(t) = \sum_{i=1}^{n} g_{im}(t)\psi_i$$

be approximate solutions which satisfy the system of ordinary differential equations

$$
\begin{aligned}
(7.1) \qquad & m_1(u_m''(t), v_i) + k_1\big(u_m(t), v_i\big) + \delta_1\big(u_m'(t), v_i\big) + h(\tilde{g}(u_m(t) + d), v_i) \\
&\qquad = (F_1 + P_1(\varphi_m(t)), v_i), \\
& m_2(\varphi_m''(t), \psi_i) + k_2(\varphi_m(t), \psi_i) + \delta_2(\varphi_m'(t), \psi_i) = (F_2 + P_2(\varphi_m(t)), \psi_i),
\end{aligned}
$$

$i = 1, \dots, m$. Moreover, f_{im}, g_{im} satisfy the initial conditions (5.5), (5.7). The existence of a local solution can be proved as in Theorem 5.1.

Step 2. If we multiply (7.1) by $f_{jm}'(t)$, $g_{jm}'(t)$ and sum these expressions, we have

$$
\begin{aligned}
(7.2) \qquad & m_1(u_m''(t), u_m'(t)) + m_2(\varphi_m''(t), \varphi_m'(t)) + k_1\big(u_m(t), u_m'(t)\big) \\
&\quad + k_2(\varphi_m(t), \varphi_m'(t)) + \delta_1(u_m'(t), u_m'(t)) + \delta_2(\varphi_m'(t), \varphi_m'(t)) \\
&\quad + h(\tilde{g}(u_m(t) + d), u_m'(t)) \\
&= (F_1 + P_1(\varphi_m(t)), u_m'(t)) + (F_2 + P_2(\varphi_m(t)), \varphi_m'(t)).
\end{aligned}
$$

The relation (7.2) yields

$$
\begin{aligned}
(7.3) \qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{m_1(u_m'(t), u_m'(t)) + m_2(\varphi_m'(t), \varphi_m'(t)) + k_1(u_m(t), u_m(t)) \\
&\quad + k_2(\varphi_m(t), \varphi_m(t)) + h(g(u_m(t) + d), g(u_m(t) + d))\} \\
&\quad + \delta_1(u_m'(t), u_m'(t)) + \delta_2(\varphi_m'(t), \varphi'(t)) \\
&= (F_1 + P_1(\varphi_m(t)), u_m'(t)) + (F_2 + P_2(\varphi_m(t)), \varphi_m'(t)).
\end{aligned}
$$

If we follow the arguments of *Step 2* in the proof of Theorem 5.1, we get the estimates (5.12).

Step 3. If we consider (5.5) and (5.7), then (7.1) implies the equalities (5.13) and the estimates (5.14).

If we differentiate the system (7.1) and multiply it by $f_{jm}''(t)$, $g_{jm}''(t)$, we obtain

$$
\begin{aligned}
(7.4) \qquad & m_1(u_m'''(t), u_m''(t)) + m_2(\varphi_m'''(t), \varphi_m''(t)) + k_1(u_m'(t), u_m''(t)) \\
&\quad + k_2(\varphi_m'(t), \varphi_m''(t)) + \delta_1(u_m''(t), u_m''(t)) + \delta_2(\varphi_m''(t), \varphi_m''(t)) \\
&\quad + h(\tilde{g}'(u_m(t) + d)u_m'(t), u_m''(t)) \\
&= \Big(\frac{\partial}{\partial y}P_1(\varphi_m(t))\varphi_m'(t), u_m''(t)\Big) + \Big(\frac{\partial}{\partial t}P_1(\varphi_m(t)), u_m''(t)\Big) \\
&\quad + \Big(\frac{\partial}{\partial y}P_2(\varphi_m(t))\varphi_m'(t), \varphi_m''(t)\Big) + \Big(\frac{\partial}{\partial t}P_2(\varphi_m(t)), \varphi_m''(t)\Big).
\end{aligned}
$$

The relation (7.4) yields

$$
\begin{aligned}
(7.5)\quad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{m_1(u''_m(t),u''_m(t)) + m_2(\varphi''_m(t),\varphi''_m(t)) + k_1(u'_m(t),u'_m(t)) \\
&\qquad + k_2(\varphi'_m(t),\varphi'_m(t)\Big\} + h(\tilde{g}'(u_m(t)+d)u'_m(t),u''_m(t)) \\
&\qquad + \delta_1(u''_m(t),u''_m(t)) + \delta_2(\varphi''_m(t),\varphi''_m(t)) \\
&\quad = \Big(\frac{\partial}{\partial y}P_1(\varphi_m(t))\varphi'_m(t),u''_m(t)\Big) + \Big(\frac{\partial}{\partial t}P_1(\varphi_m(t)),u''_m(t)\Big) \\
&\qquad + \Big(\frac{\partial}{\partial y}P_2(\varphi_m(t))\varphi'_m(t),\varphi''_m(t)\Big) + \Big(\frac{\partial}{\partial t}P_2(\varphi_m(t)),\varphi''_m(t)\Big),
\end{aligned}
$$

which results in the equality

$$
\begin{aligned}
(7.6)\quad & m_1(u''_m(t),u''_m(t)) + m_2(\varphi''_m(t),\varphi''_m(t)) + k_1(u'_m(t),u'_m(t)) \\
&\qquad + k_2(\varphi'_m(t),\varphi'_m(t)) \\
&\quad = m_1(u''_m(0),u''_m(0)) + m_2(\varphi''_m(0),\varphi''_m(0)) + k_1(u'_m(0),u'_m(0)) \\
&\qquad + k_2(\varphi'_m(0),\varphi'_m(0)) - 2\int_0^t h(\tilde{g}'(u_m(s)+d)u'_m(s),u''_m(s))\,\mathrm{d}s \\
&\qquad - 2\int_0^t \delta_1(u''_m(s),u''_m(s))\,\mathrm{d}s - 2\int_0^t \delta_2(\varphi''_m(s),\varphi''_m(s))\,\mathrm{d}s \\
&\qquad + 2\int_0^t \Big(\frac{\partial}{\partial y}P_1(\varphi_m(s))\varphi'_m(s) + \frac{\partial}{\partial t}P_1(\varphi_m(s)),u''_m(s)\Big)\,\mathrm{d}s \\
&\qquad + 2\int_0^t \Big(\frac{\partial}{\partial y}P_2(\varphi_m(s))\varphi'_m(s) + \frac{\partial}{\partial t}P_2(\varphi_m(s)),\varphi''_m(s)\Big)\,\mathrm{d}s.
\end{aligned}
$$

If we follow the arguments in the proof of Theorem 5.1 and consider the structure of the bilinear form $h(\cdot,\cdot)$ and the function g, we have the estimates (5.19).

Step 4. The proof of this step is almost parallel to the corresponding step in the proof of Theorem 5.1 with the exception represented by the equality

$$
(7.7)\qquad \lim_{l\to\infty}\int_0^T h(\tilde{g}(u_l(t)+d),v)\theta(t)\,\mathrm{d}t = \int_0^T h(\tilde{g}(u(t)+d),v)\theta(t)\,\mathrm{d}t,
$$

where v, ψ, θ are arbitrary functions from V_1, V_2, $\mathcal{D}((0,T))$. Moreover, from the definition of distributional derivatives it follows that u_l, φ_l belong to $H^1((0,L)\times(0,T))$ and weakly converge to u, φ in this space. If we note that u_l, φ_l converge to u, φ in $L^2((0,T)\times(0,L))$, which follows from the Ehrling compactness theorem (see [7]), then by virtue of the Lebesgue dominated convergence theorem the equality (7.7) is established.

The proof of the rest of this step and of *Step 5* is parallel to the corresponding parts in the proof of Theorem 5.1.

Uniqueness. Let the problem $\mathcal{F}$ have two solutions (u_1, φ_1), (u_2, φ_2) and let $v = u_1 - u_2$, $\psi = \varphi_1 - \varphi_2$. By virtue of Lemma 3.1 we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{m_1(v'(t), v'(t)) + m_2(\psi'(t), \psi'(t)) + k_1(v(t), v(t)) + k_2(\psi(t), \psi(t))\} \\
&\qquad\qquad + h(\tilde{g}(u_1(t) + d) - \tilde{g}(u_2(t) + d), v'(t)) \\
&\qquad\qquad + \delta_1(v'(t), v'(t)) + \delta_2(\psi'(t), \psi'(t)) \\
&\qquad\quad = (P_1(\varphi_1(t)) - P_1(\varphi_2(t)), v'(t)) + (P_2(\varphi_1(t)) - P_2(\varphi_2(t)), \psi'(t)).
\end{aligned}$$

This yields the equality

$$\begin{aligned}
m_1(v'(t), v'(t)) &+ m_2(\psi'(t), \psi'(t)) + k_1(v(t), v(t) + k_2(\psi(t), \psi(t)) \\
&= -2\int_0^t h(\tilde{g}(u_1(s) + d) - \tilde{g}(u_2(s) + d), v'(s))\,\mathrm{d}s \\
&\quad - 2\int_0^t \delta_1(v'(s), v'(s))\,\mathrm{d}s - 2\int_0^t \delta_2(\psi'(s), \psi'(s))\,\mathrm{d}s \\
&\quad + 2\int_0^t (P_1(\varphi_1(s)) - P_1(\varphi_2(s)), v'(s))\,\mathrm{d}s \\
&\quad + 2\int_0^t (P_2(\varphi_1(s)) - P_2(\varphi_2(s)), \psi'(s))\,\mathrm{d}s.
\end{aligned} \tag{7.8}$$

The conditions (MC), the structure of the bilinear form $h(\cdot,\cdot)$ and the properties of the function g imply the inequalities

$$\begin{aligned}
|(P_1(\varphi_1(t)) - P_1(\varphi_2(t)), v'(t))| &\leqslant C(\|\psi(t)\|_{V_2}^2 + \|v'(t)\|_W^2), \\
|(P_2(\varphi_1(t)) - P_2(\varphi_2(t)), \psi'(t))| &\leqslant C(\|\psi(t)\|_{V_2}^2 + \|\psi'(t)\|_W^2), \\
|h(\tilde{g}(u_1(t) + d) - \tilde{g}(u_2(t) + d), v'(t))| &\leqslant C(\|v(t)\|_{V_1}^2 + \|v'(t)\|_W^2),
\end{aligned} \tag{7.9}$$

where C is a positive constant. The rest of the proof is the same as for Theorem 5.1. □

Definition 7.1. Let $u_0 \in V_1$, $\varphi_0 \in V_2$, $u_1 \in W$, $\varphi_1 \in W$, $d \in V_1$ and F_1, F_2 belong to $L^2((0,T))$ and let P_1, P_2 satisfy (MC). Then $u \in X_1$, $\varphi \in X_2$ are a solution to the problem $\mathcal{G}$ if the equalities

$$\begin{aligned}
m_1(u'', v) + k_1(u, v) + \delta_1(u', v) + h(\tilde{g}(u + d), v) &= (F_1 + P_1(\varphi), v), \\
m_2(\varphi'', \psi) + k_2(\varphi, \psi) + \delta_2(\varphi', \psi) &= (F_2 + P_2(\varphi), \psi)
\end{aligned}$$

hold in $L^2((0,T))$ for any $v \in V_1$, $\psi \in V_2$. Moreover, the initial conditions

$$\begin{aligned}
u(0) = u_0, &\quad u'(0) = u_1, \\
\varphi(0) = \varphi_0, &\quad \varphi'(0) = \varphi_1
\end{aligned} \tag{7.10}$$

are fulfilled.

Theorem 7.2. *If the assumptions of Definition 7.2 are fulfilled and the assumptions*

$$K_1(x) \in C^2(\langle 0, L\rangle), \qquad K_2(x) \in C^1(\langle 0, L\rangle),$$
$$u_0(x) \in H^4((0,L)) \cap H_0^1((0,L)),$$
$$\frac{\mathrm{d}^2 u_0(0)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 u_0(L)}{\mathrm{d}x^2} = 0,$$
$$\varphi_0(x) \in H^2((0,L)) \cap H_0^1((0,L)) \tag{7.11}$$

hold, then there exists a solution to $\mathcal{G}$ and this solution is unique.

Proof. The proof is almost identical to the proof of Theorem 7.1 with only two changes.

First. Let v_i, ψ_i be sequences of linearly independent elements of V_1, V_2 and let the linear spans of these sequences be dense in V_1, V_2 as well as in W.

For any m let

$$u_m(t) = \sum_{i=1}^{m} f_{im}(t) v_i,$$
$$\varphi_m(t) = \sum_{i=1}^{n} g_{im}(t) \psi_i$$

be approximate solutions which satisfy the system of ordinary differential equations (7.1). The solution to this system satisfies the initial conditions

$$v_1 = u_0, \quad \psi_1 = \varphi_0,$$
$$f_{1m}(0) = 1, \quad f_{im}(0) = 0,$$
$$g_{1m}(0) = 1, \quad g_{im}(0) = 0,$$
$$i = 2, \ldots, m,$$
$$v_{m_1} = \sum_{i=1}^{m} f'_{im}(0) v_i, \quad v_{m_1} \to v_1 \text{ in } V_1,$$
$$\varphi_{m_1} = \sum_{i=1}^{m} g'_{im}(0) \psi_i, \quad \varphi_{m_1} \to \varphi_0 \text{ in } V_2. \tag{7.12}$$

Due to these conditions we can prove the estimates in *Step 2* of the proof of Theorem 7.1.

Second. We have to established the estimates (5.14). If we consider the assumptions (7.11), then we have

$$
\begin{aligned}
(7.13) \qquad k_1(u_0, v_i) &= \Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(K_1\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_0\Big), v_i\Big),\\
k_2(\varphi_0, \psi_i) &= -\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big(K_2\frac{\mathrm{d}}{\mathrm{d}x}\varphi_0\Big), \psi_i\Big).
\end{aligned}
$$

The equalities (7.13) together with the equations (7.1) and the initial condition (7.12) yield the relations

$$
\begin{aligned}
(7.14) \quad m_1(u_m''(0), v_i) = &-\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Big(K_1\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_0\Big), v_i\Big) - \delta_1(v_{m1}, v_i)\\
&- h(\tilde{g}(u_0 + d), v_i) + (F_1 + P_1(\varphi_0), v_i),\\
m_2(\varphi_m''(0), \psi_i) = &\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big(K_2\frac{\mathrm{d}}{\mathrm{d}x}\varphi_0\Big), \psi_i\Big) - \delta_2(\varphi_{m1}, \psi_i) + (F_2 + P_2(\varphi_0), \psi_i).
\end{aligned}
$$

Now (7.14) implies (5.14) and the estimates in *Step 3* of the proof of Theorem 7.1. The rest of the proof is parallel to the proof of Theorem 7.1. □

Proposition 7.1. *Let the assumptions of Theorem* 7.1 *be fulfilled, let* (u, φ) *be a solution to the problem* $\mathcal{F}$ *or* $\mathcal{G}$, *and let the functions* Θ_1, Θ_2 *describing the damping vanish. Then the equalities*

$$
\begin{aligned}
(7.15) \qquad E_1(t_2) - E_1(t_1) &= \int_{t_1}^{t_2} (F_1 + P_1(\varphi(s)), u'(s))\,\mathrm{d}s,\\
E_2(t_2) - E_2(t_1) &= \int_{t_1}^{t_2} (F_2 + P_2(\varphi(s)), \varphi'(s))\,\mathrm{d}s
\end{aligned}
$$

hold for any $t_1, t_2 \in \langle 0, T\rangle$, *where*

$$
\begin{aligned}
E_1(t) &= \frac{1}{2}m_1(u'(t), u'(t)) + \frac{1}{2}k_1(u(t), u(t)) + \frac{1}{2}h(g(u(t) + d), g(u(t) + d)),\\
E_2(t) &= \frac{1}{2}m_2(\varphi'(t), \varphi'(t)) + \frac{1}{2}k_2(\varphi(t), \varphi(t)).
\end{aligned}
$$

Proof. The proof is parallel to that of Proposition 5.1 and the equalities (7.15) can be interpreted in the same way. □

Let us notice one interesting fact. In [9] the authors describe the collapse of the Tacoma Narrows suspension bridge. The deck of this bridge was suspended by two rows of cables. The essential moment of this collapse was a rapid change of large vertical oscillations to torsional ones. The construction studied in this paper seems to be protected against such quick changes. The formulae (7.15) show that the internal energies of vertical and torsional oscillations remain separated in spite of the loosening of cables.

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