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BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Algorithms for finding an approximate solution of boundary value problems for systems of functional ordinary differential equations are studied. Sufficient conditions for consistency and convergence of these methods are given. In the last section, a construction of methods of arbitrary order is presented.

Keywords: boundary value problems, functional differential equations, difference method, consistency, convergence, methods of arbitrary order

MSC 2000: 34K10

1. Introduction

Let $\mathbb{R}^q$ denote the real q-dimensional space with a norm $\|\cdot\|$. For real numbers a_1, b_1, $a_1 < b_1$ and an integer $i \geqslant 0$, let $C^i([a_1, b_1], \mathbb{R}^q)$ denote the space of functions with continuous derivatives up to the order i on $[a_1, b_1]$ into $\mathbb{R}^q$ and $C(\cdot, \mathbb{R}^q) := C^0(\cdot, \mathbb{R}^q)$. Let $J = [a, b]$, $\overline{J} = [a - a_0, a] \cup [b, b + b_0]$, $\tilde{J} = J \cup \overline{J}$, $a < b$, $a_0, b_0 \geqslant 0$ and let $\psi \in C^1(\overline{J}, \mathbb{R}^q)$ be given. By $\mathcal{C}^i(\tilde{J}, \mathbb{R}^q)$ we denote the class of functions $x \in C(\tilde{J}, \mathbb{R}^q)$ which are identical with $\psi^{(i)}$ on $\overline{J}$, $i = 0, 1$; $\mathcal{C}(\tilde{J}, \mathbb{R}^q) := \mathcal{C}^0(\tilde{J}, \mathbb{R}^q)$.

For given $\psi \in C^1(\overline{J}, \mathbb{R}^q)$ and $f\colon \mathcal{C}(\tilde{J}, \mathbb{R}^q) \times \mathcal{C}^1(\tilde{J}, \mathbb{R}^q) \to L^\infty(J, \mathbb{R}^q)$, where $L^\infty(J, \mathbb{R}^q)$ denotes the space of bounded measurable functions on J with values in $\mathbb{R}^q$, we consider the system of functional ordinary differential equations of the form

$$y''(t) = f(y, y')(t), \quad t \in J \tag{1a}$$

subject to the boundary condition

$$y(t) = \psi(t), \quad t \in \overline{J}. \tag{1b}$$

By a solution of (1) we mean a function $y\colon \tilde{J} \to \mathbb{R}^q$ which has an absolutely continuous first derivative on $\tilde{J}$, satisfies (1b) and y'' equals f on J except on a set of Lebesgue measure zero. Indeed, if f is continuous, then the solution of (1) reduces to the solution in the classical sense.

Notice that equation (1a) is a very general type of equation. It includes, as a special case, the system of ordinary differential equations of the form

$$y''(t) = f_0(t, y(t), y'(t)), \quad t \in J, \tag{2}$$

with $f_0\colon J \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, so we have $a_0 = b_0 = 0$ and $\bar{J} = \{a, b\}$. The system of differential equations of the form

$$y''(t) = f_1(t, y(\alpha_1(t)), \ldots, y(\alpha_p(t)), y'(\beta_1(t)), \ldots, y'(\beta_q(t))), \quad t \in J, \tag{3}$$

is also a special case of (1a) with $f_1\colon J \times (\mathbb{R}^m)^{p+q} \to \mathbb{R}^m$, $\alpha_i, \beta_j \in C(J, \tilde{J})$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. In this case,

$$a - a_0 = \min(\inf_{t\in J} \alpha_i(t), \inf_{t\in J} \beta_j(t)), \qquad b + b_0 = \max(\sup_{t\in J} \alpha_i(t), \sup_{t\in J} \beta_j(t)),$$
$$i = 1, 2, \ldots, p, \;\; j = 1, 2, \ldots, q.$$

If $\alpha_i(t) \leqslant t$, $\beta_j(t) \leqslant t$, $t \in J$, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, then $b_0 = 0$ and (3) is a problem of delay type. Integro-differential equations of the Volterra or Fredholm type are also special cases of (1a), for example,

$$y''(t) = f_2\left(t, y(\alpha(t)), y'(\beta(t)), \int_a^{\gamma(t)} k(t, \tau, y(\tau), y'(\tau))\,\mathrm{d}\tau\right), \quad t \in J, \tag{4}$$

with $f_2\colon J \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $k\colon J \times J \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ for $\alpha, \beta \in C(J, \tilde{J})$ and $\gamma \in C(J, J)$.

The existence and uniqueness of solutions for problems of type (1) has been investigated in [3], [4], [8], [13], [14], [20], [21], [22], [23]. We quote only a few papers in which numerical methods are used for problems which are special cases of (1). We can indicate that a shooting method (see for example [12], [19], [22]) and a finite difference method (see [5], [9], [12], [16], [17]) are frequently used for finding a numerical solution of problems of type (1). A collocation method [7] or iterative sequences [11] can also be used (see also [6]).

Condition (1b) can be placed into the operator f to have the boundary problem with boundary conditions at the points a and b. For $i = 0, 1$ and $y \in C(J, \mathbb{R}^q)$ such that $y(a) = \psi^{(i)}(a)$, $y(b) = \psi^{(i)}(b)$ we define operators T_i by

$$(T_i y)(t) := \begin{cases} \psi^{(i)}(t), & t \in \bar{J}, \\ y(t), & t \in J. \end{cases}$$

Notice that for $\psi \in C^1(\bar{J}, \mathbb{R}^q)$ the operator T_i maps $C(J, \mathbb{R}^q)$ subject to the above boundary conditions into $\mathcal{C}^i(\tilde{J}, \mathbb{R}^q)$, $i = 0, 1$. Under the above notation, problem (1) takes the form

$$y''(t) = f(T_0 y, T_1 y')(t), \quad t \in J, \tag{5a}$$

$$y(a) = \psi(a), \quad y(b) = \psi(b). \tag{5b}$$

Indeed, if y is a solution of (5), then $T_0 y$ is the corresponding solution of problem (1).

Assume that (5) has a solution φ. The purpose of this paper is a numerical approximation of φ which will be denoted by y^h. By z^h we denote an approximation of φ'. Here h is a constant stepsize, $Nh = b - a$, $t_n = a + nh$, $n = 0, 1, \ldots, N$. It must be emphasized that the approximate solution y^h of (5) must be computed at all points $t \in J$ and not only at the points t_n. Denote by $\psi_h^{(i)}$ a continuous approximation of $\psi^{(i)}$, $i = 0, 1$. For $y \in \mathbb{R}^q$ we define two operators T_0^h and T_1^h by the relations

$$(T_0^h y)(t) := \begin{cases} \psi_h(t) & t \in \bar{J}, \\ y(t) & t \in J, \end{cases}$$

$$(T_1^h y)(t) := \begin{cases} \psi_h'(t) & t \in \bar{J} \setminus \{a, b\}, \\ y(t) & t \in J. \end{cases}$$

Under the above notation a numerical solution of (5) may be constructed by the difference method of the form

$$\begin{cases} y^h(t) = \psi_h(t) \ \text{ for } \ t = a \ \text{ or } \ t = b, \\ y^h(t_n + rh) - 2y^h(t_n) + y^h(t_n - rh) = h^2 F_{(h,r)}(T_0^h y^h, T_1^h z^h)(t_n), \\ \qquad r \in [0, 1], \ \ n = 1, 2, \ldots, N-1, \end{cases} \tag{6}$$

$$\begin{aligned} z^h(t_n + rh) &= \frac{1}{2h}[y^h(t_{n+1} + rh) - y^h(t_{n-1} + rh)], \\ & r \in [0, 1], \ \ n = 1, 2, \ldots, N-2. \end{aligned} \tag{7}$$

Here the operators $F_{(h,r)} \colon \mathcal{C}(\tilde{J}, \mathbb{R}^q) \times \tilde{\mathcal{C}}(\tilde{J}, \mathbb{R}^q) \to L^\infty(J, \mathbb{R}^q)$ are defined for $(h, r) \in H \times [0, 1]$ with $H = [0, h_0]$ for some $h_0 > 0$, where $\tilde{\mathcal{C}}(\tilde{J}, \mathbb{R}^q)$ denotes the class of piecewise continuous functions from $\tilde{J}$ into $\mathbb{R}^q$ In addition, we assume that $F_{(h,0)}(T_0^h y^h, T_1^h z^h)(t) = \theta$, where θ is the zero element in $\mathbb{R}^q$.

Formulas (6)–(7) can be obtained in the following way. Observe that equation (5a) is identical with the system of first order differential equations

$$\begin{cases} y'(t) = z(t), \\ z'(t) = f(T_0 y, T_1 z)(t), \quad t \in J. \end{cases}$$

Now, if we apply to this system the approximations

$$\begin{cases} rhz^h(t) = y^h(t+rh) - y^h(t), \\ rh(z')^h(t) = z^h(t) - z^h(t-rh), \end{cases}$$

then we obtain (6). If we apply the approximations

$$\begin{cases} hz^h(t) = y^h(t) - y^h(t-h), \\ rh(z')^h(t) = z^h(t+rh) - z^h(t), \end{cases}$$

then, instead of (6), the following expression will arise:

$$(8)\quad \begin{cases} y^h(t) = \psi_h(t) \ \text{ for } \ \mathrm{t} = a \ \text{ or } \ t = b, \\ y^h(t_n+rh) - y^h(t_n) - y^h(t_{n-1}+rh) + y^h(t_{n-1}) = h^2 F_{(h,r)}(T_0^h y^h, T_1^h z^h)(t_n), \\ \qquad r \in [0,1], \ \ n = 1, 2, \ldots, N-1. \end{cases}$$

Similarly, using the approximations

$$\begin{cases} rhz^h(t) = y^h(t+rh) - y^h(t), \\ rh(z')^h(t) = z^h(t+rh) - z^h(t), \end{cases}$$

instead of (6) the expression

$$(9)\qquad \begin{cases} y^h(t) = \psi_h(t) \ \text{ for } \ t = a \ \text{ or } \ t = b, \\ y^h(t_n + 2rh) - 2y^h(t_n + rh) + y^h(t_n) = h^2 F_{(h,r)}(T_0^h y^h, T_1^h z^h)(t_n), \\ \qquad r \in \left[0, \frac{1}{2}\right], \ \ n = 0, 1, \ldots, N-1 \end{cases}$$

will be obtained. Basing on the above we see that a numerical solution of (5) may be described by different formulas because different approximations for z^h can be used. In this paper, only the method (6)–(7) will be analysed.

It is simple to see that if

$$F_{(h,r)}(x_1, x_2)(t) = A(r)\overline{F}_h(x_1, x_2)(t), \quad A(0) = 0 \quad \text{and} \quad A(1) \neq 0,$$

then, for $n = 1, 2, \ldots, N-1$ and $r \in [0,1]$, the second formula in (6) can be rewritten in the equivalent form

$$(6')\quad \begin{cases} y^h(t_{n+1}) - 2y^h(t_n) + y^h(t_{n-1}) = A(1)h^2 \overline{F}_h(T_0^h y^h, T_1^h z^h)(t_n), \\ y^h(t_n + rh) - 2y^h(t_n) + y^h(t_n - rh) = \overline{A}(r)[y^h(t_{n+1}) - 2y^h(t_n) + y^h(t_{n-1})] \end{cases}$$

with $\overline{A}(r) = A(r)/A(1)$. Moreover, in this case, (7) and (6′) yield

$$(7') \quad \begin{cases} z^h(t_n) = \dfrac{1}{2h}[y^h(t_{n+1}) - y^h(t_{n-1})], \\ z^h(t_n + rh) - 2z^h(t_n) + z^h(t_n - rh) = \overline{A}(r)[z^h(t_{n+1}) - 2z^h(t_n) + z^h(t_{n-1})]. \end{cases}$$

Notice that for the linear approximation

$$y^h(t_n + rh) = (1 - r)y^h(t_n) + ry^h(t_{n+1}), \quad r \in [0, 1],$$

we need to take $\overline{A}(r) = r$, while using the quadratic interpolation

$$\begin{aligned} y^h(t_n + rh) = {} & \frac{r(r-1)}{2} y^h(t_{n-1}) - (r^2 - 1)y^h(t_n) \\ & + \frac{r(r+1)}{2} y^h(t_{n+1}), \quad r \in [-1, 1], \end{aligned}$$

we find $\overline{A}(r) = r^2$.

Taking $\overline{F}_h = f^h$, we have the simplest numerical method for solving (5) (here f^h denotes an approximation of f). If the mapping F has the form

$$(10) \qquad F_{(h,r)}(T_0^h y^h, T_1^h z^h)(t) = \sum_{i=0}^{2} B_i(t, h, r) f^h(T_0^h y^h, T_1^h z^h)(t - h + ih),$$

where $B_i \colon J \times H \times [0, 1] \to \mathbb{R}$ are bounded and $B_i(t, h, 0) = 0$, then (6) yields nonstationary linear methods with variable coefficients (see Example II).

The paper is organized as follows. The problem of consistency of the method (6)–(7) is briefly considered in Section 3. In Section 4, sufficient conditions under which the procedure (6)–(7) is convergent are given. Error estimates are also discussed. The last section deals with a construction of methods of a fixed order and some new algorithms of the corresponding order are produced. Finally, we note that in literature, numerical methods have been considered for special cases of (1), usually when the operator f has the form $f(y, y')(t) = f(t, y(t))$ or $f(y, y')(t) = f(t, y(t), y(\alpha(t)))$ or $f(y, y')(t) = f(t, y(t), y(\beta(t, y(t))))$.

2. Lemmas

In this part we investigate a difference problem of type (6). A description of its solution will be given by the corresponding formulas. To formulate a convergence theorem also some properties of this solution will be stated.

For given $g_n\colon [0,1] \to \mathbb{R}^q$, $g_n(0) = \theta$, $n = 0,1,\dots,N$, we define two sequences $\{g^{(k)}\}$ and $\{s_i\}$ by the relations

$$\begin{cases} g^{(1)}(r) := g_1(r), \\ g^{(k+1)}(r) := 2g^{(k)}(1) - g^{(k)}(1-r) + g_{k+1}(r), \quad k = 1,2,\dots,N-1 \end{cases}$$

for $r \in [0,1]$, and

$$s_i := \begin{cases} 1 & \text{if } i \text{ is odd}, \\ 0 & \text{if } i \text{ is even}. \end{cases}$$

Taking into account the definition, we see that

$$g^{(k+1)}(0) = g^{(k)}(1), \quad k = 1,2,\dots,N-1.$$

Lemma 1. *Assume that $g_n(0) = \theta$, $n = 0,1,\dots,N$. Then the solution y^h of the difference problem*

$$\begin{cases} y^h(t_n + rh) - 2y^h(t_n) + y^h(t_n - rh) = g_n(r), \\ \qquad r \in [0,1], \ n = 1,2,\dots,N-1, \\ y^h(a) = \psi_h(a), \quad y^h(b) = \psi_h(b) \end{cases} \tag{11}$$

can be expressed by

$$y^h(t_n + rh) = \begin{cases} \dfrac{n+1}{N}\,\bar{\psi}_N - (n-1)\psi_h(a) - y^h(t_1 - rh) + g^{(n)}(r), \\ \qquad n = 1,3,\dots,N-1-s_N, \quad r \in [0,1], \\ \dfrac{n}{N}\,\bar{\psi}_N - n\psi_h(a) + y^h(a+rh) + g^{(n)}(r), \\ \qquad n = 2,4,\dots,N-2+s_N, \quad r \in [0,1], \end{cases} \tag{12}$$

where

$$\bar{\psi}_N = \psi_h(b) + (N-1)\psi_h(a) - g^{(N-1)}(1).$$

P r o o f. Using induction with respect to n, it is simple to prove that

$$y^h(t_n + rh) = \begin{cases} (n+1)y^h(t_1) - (n-1)\psi_h(a) - y^h(t_1 - rh) + g^{(n)}(r), \\ \qquad n = 1,3,\dots,N-1-s_N, \quad r \in [0,1], \\ ny^h(t_1) - n\psi_h(a) + y^h(a+rh) + g^{(n)}(r), \\ \qquad n = 2,4,\dots,N-2+s_N, \quad r \in [0,1]. \end{cases} \tag{13}$$

Indeed, (13) is true for $n = 1$. Assume that it is true for $n = k$ and let k be odd. Thus (11) and (13) yield

$$\begin{aligned}y^h(t_{k+1} + rh) &= 2y^h(t_{k+1}) - y^h(t_k + (1-r)h) + g_{k+1}(r)\\ &= 2[(k+1)y^h(t_1) - (k-1)\psi_h(a) - y^h(a) + g^{(k)}(1)]\\ &\quad - [(k+1)y^h(t_1) - (k-1)\psi_h(a) - y^h(a + rh)\\ &\quad + g^{(k)}(1-r)] + g_{k+1}(r),\end{aligned}$$

which proves that formula (13) is true (for example, if $k = 1$, then (13) holds for $n = 2$). For k even, (13) is also true.

Our next objective is to show that (12) holds. Assume that N is even, hence $s_N = 0$. Take $n = N - 1$ and $r = 1$. By applying (13) we obtain the equality

$$y^h(t_N) = \psi_h(b) = Ny^h(t_1) - (N-2)\psi_h(a) - \psi_h(a) + g^{(N-1)}(1)$$

and hence we have immediately

$$y^h(t_1) = \frac{1}{N}\bar{\psi}_N.$$

This and (13) yield (12). For N odd, the proof is similar. □

R e m a r k 1. Relation (12) implies that

$$y^h(t_n + h) = y^h(t_{n+1} + 0h) = \lim_{r\to 0^+} y^h(t_{n+1} + rh), \quad n = 1, 2, \ldots, N-1,$$

if $g^{(n)}$ is continuous.

R e m a r k 2. Let $q = 1$, $g_n(r) = a_n r$, $r \in [0, 1]$, $a_n \in \mathbb{R}$, $n = 1, 2, \ldots, N$. According to the definition of $\{g^{(k)}\}$, we have immediately

$$g^{(k)}(r) = \sum_{i=1}^{k} (k + r - i)a_i, \quad k = 1, 2, \ldots, N-1.$$

Now, after some calculations, (12) takes the form

$$y^h(t_n + rh) = \begin{cases} \dfrac{n+1}{N}\,\psi_h(b) + \left(2 - \dfrac{n+1}{N}\right)\psi_h(a) - y^h(t_1 - rh) - \displaystyle\sum_{i=1}^{N-1} \bar{d}_{ni}(r)a_i,\\ \qquad n = 1, 3, \ldots, N-1-s_N,\\ \dfrac{n}{N}\,\psi_h(b) - \dfrac{n}{N}\,\psi_h(a) + y^h(a + rh) - \displaystyle\sum_{i=1}^{N-1} \bar{d}_{ni}(r)a_i,\\ \qquad n = 2, 4, \ldots, N-2+s_N,\end{cases}$$

where

$$\bar{d}_{ni}(r) := \begin{cases} i + s_n - r - \dfrac{i}{N}(n + s_n) & \text{if } i \leqslant n, \\[2ex] n + s_n - \dfrac{i}{N}(n + s_n) & \text{if } i > n. \end{cases}$$

Lemma 2. *For $n = 1, 2 \ldots, N$ and $r \in [0, 1]$ we have the identity*

$$\begin{aligned} g^{(n)}(r) = {} & 2 \sum_{i=1}^{n-1} \left[\frac{i+1}{2} \right] g_{n-i}(1) \\ & + \sum_{i=1}^{n} (-1)^{2n+1-i} g_{n+1-i}\bigl(s_i r + (1 - r)(1 - s_i)\bigr), \end{aligned} \tag{14}$$

where $[\,\cdot\,]$ denotes the integer part of the argument and $\sum\limits_{1}^{0} \ldots = 0$.

P r o o f. This formula can be proved by induction. $\square$

R e m a r k 3. It is simple to see that

$$g^{(n)}(1) = \sum_{i=1}^{n} i g_{n+1-i}(1), \quad n = 1, 2, \ldots, N.$$

Lemma 3. *Let*

$$d_{ni} := \begin{cases} i + s_i - \dfrac{i(n+1)}{N} & \text{if } i \leqslant n - 1, \\[2ex] n + 1 - \dfrac{i(n+1)}{N} & \text{if } n \leqslant i, \end{cases} \qquad \text{for } n = 1, 3, \ldots, N - 1 - s_N,$$

$$d_{ni} := \begin{cases} i - s_i - \dfrac{in}{N} & \text{if } i \leqslant n - 1, \\[2ex] n - \dfrac{in}{N} & \text{if } n \leqslant i, \end{cases} \qquad \text{for } n = 2, 4, \ldots, N - 2 + s_N.$$

Then we have

$$\begin{aligned} -\frac{n + s_n}{N} \sum_{i=1}^{N-1} i g_{N-i}(1) + 2 \sum_{i=1}^{n-1} \left[\frac{i+1}{2} \right] g_{n-i}(1) & \\ = -\sum_{i=1}^{N-1} d_{ni} g_i(1), & \quad n = 1, 2, \ldots, N - 1. \end{aligned} \tag{15}$$

P r o o f. To prove (15) we assume that n is odd (for n even the proof is similar). Then we have

$$\begin{aligned} P_n := & -\frac{n+1}{N}\sum_{i=1}^{N-1} i g_{N-i}(1) + 2\sum_{i=1}^{n-1}\left[\frac{i+1}{2}\right] g_{n-i}(1) \\ = & -\frac{n+1}{N}\sum_{i=1}^{N-n} i g_{N-i}(1) - \frac{n+1}{N}\sum_{i=N-n+1}^{N-1} i g_{N-i}(1) \\ & + 2\sum_{i=1}^{n-1}\left[\frac{i+1}{2}\right] g_{n-i}(1). \end{aligned}$$

Furthermore, by changing the sum index in the second sum, we get

$$\begin{aligned} P_n &= -\frac{n+1}{N}\sum_{i=1}^{N-n} i g_{N-i}(1) - \frac{n+1}{N}\sum_{i=1}^{n-1}(N-n+i) g_{n-i}(1) + 2\sum_{i=1}^{n-1}\left[\frac{i+1}{2}\right] g_{n-i}(1) \\ &= -\frac{n+1}{N}\sum_{i=1}^{N-n} i g_{N-i}(1) - \frac{1}{N}\sum_{i=1}^{n-1}\bigl(i(N-n-1)+s_i N\bigr) g_i(1) \\ &= -\frac{n+1}{N}\sum_{i=n}^{N-1}(N-i) g_i(1) - \frac{1}{N}\sum_{i=1}^{n-1}\bigl(i(N-n-1)+s_i N\bigr) g_i(1), \end{aligned}$$

and hence we have (15). The proof is complete. □

If we use the results of Lemmas 2 and 3 and Remark 3, then expression (12) takes an equivalent form

$$y^h(t_n+rh) = \begin{cases} \dfrac{n+1}{N}\,\psi_h(b) + \left(2-\dfrac{n+1}{N}\right)\psi_h(a) - y^h(t_1-rh) + Q_n(r), \\ \qquad n=1,3,\dots,N-1-s_N, \quad r\in[0,1], \\ \dfrac{n}{N}\,\psi_h(b) - \dfrac{n}{N}\,\psi_h(a) + y^h(a+rh) + Q_n(r), \\ \qquad n=2,4,\dots,N-2+s_N, \quad r\in[0,1], \end{cases} \tag{16}$$

with

$$Q_n(r) := -\sum_{i=1}^{N-1} d_{ni} g_i(1) + \sum_{i=1}^{n}(-1)^{2n+1-i} g_{n+1-i}\bigl(s_i r + (1-r)(1-s_i)\bigr).$$

From (16), by taking $n=1$ and $r=0$, we can simply compute $y^h(t_1)$, namely

$$y^h(t_1) = \frac{1}{N}\,\psi_h(b) + \left(1-\frac{1}{N}\right)\psi_h(a) - \frac{1}{2}\sum_{i=1}^{N-1} d_{1i} g_i(1).$$

Notice that (16) is well defined if y^h is extra given on $(a, a+h)$ and thus y^h of form (16) is a solution of (11).

In our considerations, we will need to have also some properties for z^h defined by (7). Using (16), we can easily write

$$
\begin{aligned}
(17)\quad & y^h(t_{n+1}+rh) - y^h(t_{n-1}+rh) \\
&= \frac{2}{N}\,[\psi_h(b)-\psi_h(a)] + Q_{n+1}(r) - Q_{n-1}(r) \\
&= \frac{2}{N}[\psi_h(b)-\psi_h(a)] \\
&\quad - \sum_{i=1}^{N-1}(d_{n+1,i}-d_{n-1,i})g_i(1) + g_{n+1}(r) - g_n(1-r), \\
&\qquad\qquad n = 1,2,\dots,N-2,\ \ r\in[0,1]
\end{aligned}
$$

with $Q_0(r)=\theta$, $r\in[0,1]$ and $d_{0i}=0$, $i=1,2,\dots,N-1$. In the next lemma, some properties of Q_n will be given.

Lemma 4. *The following equalities hold:*

(i) $Q_n(0) + (1/N)s_n g^{(N-1)}(1) = -\sum_{i=1}^{N-1} d^*_{ni} g_i(1)$,

(ii) $Q_{n+1}(0) + (2/N)(1-s_n)g^{(N-1)}(1) = Q_n(1)$,

(iii $Q_n(r) - 2Q_n(0) + Q_{n-1}(1-r) - (2/N)s_n g^{(N-1)}(1) = g_n(r), \quad r\in[0,1]$,

(iv) $Q_{N-1}(1) - (1/N)s_N g^{(N-1)}(1) = \theta$

for $n=1,2,\dots,N$, where

$$
d^*_{ni} := \begin{cases} i - \dfrac{in}{N} & \text{if} \quad i \leqslant n-1, \\[2ex] n - \dfrac{in}{N} & \text{if} \quad n \leqslant i. \end{cases}
$$

P r o o f. Assume that n is odd. Then $s_n = 1$, and because of Remark 3 we get

$$
\begin{aligned}
Q_n(0) + \frac{1}{N}\,g^{(N-1)}(1) = &-\sum_{i=1}^{n-1} d_{ni}g_i(1) - \sum_{i=n}^{N-1} d_{ni}g_i(1) + \frac{1}{N}\sum_{i=1}^{n-1}(N-i)g_i(1) \\
&+ \frac{1}{N}\sum_{i=n}^{N-1}(N-i)g_i(1) + \sum_{i=1}^{n-1}(s_i-1)g_i(1) \\
= &\sum_{i=1}^{n-1}\left(\frac{in}{N}-i\right)g_i(1) + \sum_{i=n}^{N-1}\left(\frac{in}{N}-n\right)g_i(1),
\end{aligned}
$$

which proves that (i) holds. In the same manner, we can prove (i) for n even.

By the same argument we can prove (ii), (iii) and (iv). □

R e m a r k 4. The difference system (11) can also be solved for a fixed number of $r \in (0,1]$. If $r = 1$, then it is known that the solution of (11) has the form

$$y^h(t_n) = \frac{n}{N}\,\psi_h(b) + \left(1 - \frac{n}{N}\right)\psi_h(a) - \sum_{i=1}^{N-1} d^*_{ni} g_i(1), \quad n = 0, 1, \ldots, N. \tag{18}$$

Thus (18) follows immediately from (16), (11) and (i) of Lemma 4. If r is fixed and $r \in (0,1)$, then we need two extra values $y^h(a + rh)$ and $y^h(t_1 - rh)$ to solve (11) (see (16)); for $r = \frac{1}{2}$ only the value $y^h\bigl(a + \frac{1}{2}h\bigr)$ is needed.

Lemma 5. *The following results are true:*

$$\sum_{i=1}^{N-1} d_{ni} = \begin{cases} \dfrac{n+1}{2}\,(N-n), & n = 1, 3, \ldots, N-1-s_N, \\[2ex] \dfrac{n}{2}\,(N-n-1), & n = 2, 4, \ldots, N-2+s_N, \end{cases} \tag{19}$$

$$\max_n \sum_{i=1}^{N-1} d_{ni} \leqslant \frac{(N+1)^2}{8}, \tag{20}$$

$$\max_n \max_i d_{ni} \leqslant \frac{N}{4} + \frac{1}{N}. \tag{21}$$

P r o o f. First, we prove (19). Assume that n is odd. Indeed, (19) is true for $n = 1$. Consequently, from the definition of d_{ni} we obtain

$$\begin{aligned}
\sum_{i=1}^{N-1} d_{ni} &= \sum_{i=1}^{n-1} d_{ni} + \sum_{i=n}^{N-1} d_{ni} \\
&= \sum_{i=1}^{n-1} \left(i + s_i - \frac{i(n+1)}{N}\right) + \sum_{i=n}^{N-1} \left(1 - \frac{i}{N}\right)(n+1) \\
&= \frac{n-1}{2} + \left(1 - \frac{n+1}{N}\right) \sum_{i=1}^{n-1} i + (n+1) \sum_{i=n}^{N-1} \left(1 - \frac{i}{N}\right).
\end{aligned}$$

Hence we get (19) when n is odd. In the same manner we can prove (19) if n is even.

Notice that inequality (20) results from the relation

$$\begin{aligned}
\max_{n=1,2,\ldots,N-1} \sum_{i=1}^{N-1} d_{ni} &= \max\left(\max_{n \text{ odd}} \frac{n+1}{2}\,(N-n),\ \max_{n \text{ even}} \frac{n}{2}\,(N-n-1)\right) \\
&\leqslant \max\left(\frac{(N+1)^2}{8}, \frac{(N-1)^2}{8}\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\max_n \max_i d_{ni} &= \max\Big(\max_{n \text{ odd}} \max_i d_{ni}, \max_{n \text{ even}} \max_i d_{ni}\Big) \\
&= \max\left(\max_{n \text{ odd}}\left(1+(n-1)\left(1-\frac{n+1}{N}\right)\right), \max_{n \text{ even}} n\left(1-\frac{n}{N}\right)\right) \\
&\leqslant \max\left(\frac{N}{4}+\frac{1}{N}, \frac{N}{4}\right) = \frac{N}{4}+\frac{1}{N},
\end{aligned}$$

so (21) holds. This completes the proof. □

Lemma 6. *The following statements hold:*

$$\sum_{i=1}^{N-1} |d_{n+1,i} - d_{n-1,i}| = \frac{2n(n+1)}{N} + N - 2n - 1, \quad n = 1, 2, \ldots, N-1, \tag{22}$$

$$\max_n \max_i |d_{n+1,i} - d_{n-1,i}| \leqslant 2. \tag{23}$$

P r o o f. Assume that n is even, hence $n+1$ and $n-1$ are odd. Observe that

$$D_n := \sum_{i=1}^{N-1} |d_{n+1,i} - d_{n-1,i}| = \left(\sum_{i=1}^{n-2} + \sum_{i=n-1}^{n} + \sum_{i=n+1}^{N-1}\right)|d_{n+1,i} - d_{n-1,i}|.$$

According to the definition of d_{ni} we have

$$\begin{aligned}
D_n = &\sum_{i=1}^{n-2}\left|i+s_i-\frac{i(n+2)}{N}-i-s_i+\frac{in}{N}\right| \\
&+\sum_{i=n-1}^{n}\left|i+s_i-\frac{i(n+2)}{N}-n+\frac{in}{N}\right| \\
&+\sum_{i=n+1}^{N-1}\left|n+2-\frac{i(n+2)}{N}-n+\frac{in}{N}\right|,
\end{aligned}$$

so, after some calculations, (22) holds for n odd. Formula (22), when n is even, can be proved in the same manner. Condition (23) is obviously satisfied. □

R e m a r k 5. It is simple to see that $\max_n D_n = N-1$.

3. Consistency

Convergence and consistency are important notions in numerical analysis. Before we formulate some theorems about them, let us introduce

Definition 1. The method (6)–(7) is called convergent if

$$\lim_{h\to 0}\sup_{t\in\tilde J}\|y^h(t)-\varphi(t)\|=0 \quad\text{and}\quad \lim_{h\to 0}\sup_{t\in\tilde J}\|z^h(t)-\varphi'(t)\|=0.$$

The order of convergence is p if

$$\sup_{t\in\tilde J}\|y^h(t)-\varphi(t)\|=O(h^p) \quad\text{and}\quad \sup_{t\in\tilde J}\|z^h(t)-\varphi'(t)\|=O(h^p)\ \text{ as }\ h\to 0.$$

Definition 2. The method (6)–(7) is called consistent with (1) if there exists a function $\varepsilon\colon J_h\times H\to\mathbb{R}_+=[0,\infty)$, $J_h=[a+h,b-h]$, such that the conditions

1° $\|\varphi(t+rh)-2\varphi(t)+\varphi(t-rh)-h^2F_{(h,r)}(T_0\varphi,T_1\varphi')(t)\|\leqslant\varepsilon(t,h)$, $r\in[0,1]$,

2° $\lim\limits_{h\to 0}h^{-1}\sum\limits_{i=1}^{N-1}\varepsilon(t_i,h)=0$

hold. The order of consistency is p if

$$h^{-1}\sum_{i=1}^{N-1}\varepsilon(t_i,h)=O(h^p)\quad\text{as}\quad h\to 0.$$

The problem of consistency of method (6)–(7) is considered in

Theorem 1. *Suppose that*

1° $f\colon \mathcal{C}(\tilde J,\mathbb{R}^q)\times\mathcal{C}^1(\tilde J,\mathbb{R}^q)\to L^\infty(J,\mathbb{R}^q)$, $\psi\in C^1(\bar J,\mathbb{R}^q)$, $F_{(h,r)}\colon\mathcal{C}(\tilde J,\mathbb{R}^q)\times\tilde{\mathcal{C}}(\tilde J,\mathbb{R}^q)\to L^\infty(J,\mathbb{R}^q)$ *are defined for* $(h,r)\in H\times[0,1]$; $F_{(h,\cdot)}(x,y)(t)$ *is continuous and*

$$F_{(h,0)}(T_0^hx,T_1^hy)(t)=\theta,$$

2° *there exists a unique solution* φ *of* (1),

3° φ'' *is a Riemann integrable function,*

then (6)–(7) *is consistent with* (1) *provided*

$$\lim_{h\to 0}h\sum_{i=1}^{N-1}\|r^2f(T_0\varphi,T_1\varphi')(t_i)-F_{(h,r)}(T_0\varphi,T_1\varphi')(t_i)\|=0,\quad r\in[0,1]. \tag{24}$$

P r o o f. The proof is similar to the corresponding one given in [16] and therefore it is omitted (see also [9]). □

R e m a r k 6. If

$$h\sum_{i=1}^{N-1}\|r^2 f(T_0\varphi, T_1\varphi')(t_i) - F_{(h,r)}(T_0\varphi, T_1\varphi')(t_i)\| = O(h^p)$$

uniformly in r as $h\to 0$, then (6)–(7) is consistent with (1) of order $\min(1,p)$ or $\min(\gamma,p)$ if φ'' is of bounded variation or φ'' satisfies the Hölder condition with the exponent $\gamma\in(0,1]$, respectively (see also [9], [16]).

R e m a r k 7. If $F_{(\cdot,r)}(x,y)(t)$ is continuous, then (24) remains true provided

$$F_{(0,r)}(T_0\varphi, T_1\varphi')(t) = r^2 f(T_0\varphi, T_1\varphi')(t).$$

4. Convergence

In this section we show that method (6)–(7) is convergent. The result is obtained under the assumption that the method is consistent with (1) and the operator F satisfies the Lipschitz condition with respect to the last two arguments with the corresponding constants. Error estimates are given, too. But first, we put

$$\begin{aligned}
A(h) &:= \frac{h}{8}(4K_2 + 10K_1(b-a) + hK_1),\\
B(h) &:= K_1\left(\frac{b-a}{4h} + \frac{h}{b-a} + 1\right) + \frac{3}{2h}K_2,\\
\tilde\eta(h) &:= (K_1+K_2)\eta(h) + \left(2K_1 + K_2 + \frac{2}{b-a}K_2\right)\bar\eta(h),\\
\bar\varepsilon(t,h) &:= h^2\delta(t,h) + \varepsilon(t,h),
\end{aligned}$$

where constants K_1, K_2 and functions δ, η and $\bar\eta$ will be defined later.

We formulate the following convergence result.

Theorem 2. *Suppose that*

1° *condition* 1° *of Theorem* 1 *holds,*

2° *there exist constants* $K_1, K_2 \geqslant 0$ *and a function* $\delta\colon J\times H\to\mathbb{R}_+$ *such that* $\lim\limits_{h\to 0} h\sum\limits_{i=1}^{N-1}\delta(t_i,h) = 0$ *and the condition*

$$\begin{aligned}
&\|F_{(h,r)}(x,y)(t) - F_{(h,r)}(\bar x,\bar y)(t)\|\\
&\qquad\leqslant K_1\sup_{\tau\in\tilde J}\|x(\tau)-\bar x(\tau)\| + K_2\sup_{\tau\in\tilde J}\|y(\tau)-\bar y(\tau)\| + \delta(t,h)
\end{aligned}$$

holds for $t\in J$, $h\in H$, $r\in[0,1]$, $x,\bar x\in C(\tilde J,\mathbb{R}^q)$, $y,\bar y\in\tilde C(\tilde J,\mathbb{R}^q)$, 3° *problem* (1) *has a unique solution* φ, *and* φ'' *is bounded,*

4° *method* (6)–(7) *is consistent with* (1),

5° $\varrho := K_1(b-a)^2/8 + K_2(b-a)/2 < 1$,

6° $\psi_h^{(i)}$ *are continuous approximations of* $\psi^{(i)}$, $i = 0, 1$, *such that*

$$\sup_{t \in \bar{J}} \|\psi_h^{(i)}(t) - \psi^{(i)}(t)\| \leqslant \bar{\eta}(h) \quad \text{and} \quad \bar{\eta}(h) \to 0 \ \text{ as } \ h \to 0,$$

7° y^h *is defined and continuous on* $(a, a+h)$, $y^h(a+) = \psi_h(a)$ *and there exists* $\eta\colon H \to \mathbb{R}_+$, $\eta(h) \to 0$ *as* $h \to 0$, *such that*

$$\sup_{(a,a+h)} \|y^h(\tau) - \varphi(\tau)\| \leqslant \eta(h),$$

8° z^h *is defined and continuous on* $J_0 = [a, a+h) \cup (b-h, b]$, *and in addition*

$$\sup_{\tau \in J_0} \|z^h(\tau) - \varphi'(\tau)\| \leqslant \eta(h).$$

Then method (6)–(7) *is convergent; for sufficiently small* h, *the estimates*

$$\left\{\begin{aligned} &\sup_{[a-a_0,\, b+b_0]} \|(\tilde{T}_0^h(y^h - \varphi))(t)\| \leqslant K_1^{-1}\zeta(h) \qquad \text{if } K_1 > 0, \\ &\sup_{[a-a_0\ , b+b_0]} \|(\tilde{T}_1^h(z^h - \varphi'))(t)\| \leqslant K_2^{-1}\zeta(h) \qquad \text{if } K_2 > 0 \end{aligned}\right. \tag{25}$$

hold with

$$\zeta(h) := (1 - \varrho - A(h))^{-1} \Big[B(h) \sum_{i=1}^{N-1} \bar{\varepsilon}(t_i, h) + \tilde{\eta}(h) + O(h^2) \Big].$$

Here, the operator $\tilde{T}_i^h y$ *is defined as* $T_i^h y$ *with* $\omega_h^{(i)}$ *instead of* $\psi^{(i)}$, $i = 0, 1$ ($\omega_h^{(i)}$ *is defined in the proof).*

P r o o f. Put

$$\begin{aligned} v^h(t) &:= y^h(t) - \varphi(t), \\ \Delta v^h(t) &:= v^h(t+h) - v^h(t-h), \\ u^h(t) &:= z^h(t) - \varphi'(t), \\ V^h &:= \sup_{t \in \tilde{J}} \|(\tilde{T}_0^h v^h)(t)\|, \\ U^h &:= \sup_{t \in \tilde{J}} \|(\tilde{T}_1^h u^h)(t)\|, \\ \omega_h^{(i)}(t) &:= \psi_h^{(i)} - \psi^{(i)}(t), \quad i = 0, 1, \quad t \in \bar{J}, \\ \bar{\varphi}(t) &:= \frac{1}{2h}[\varphi(t+h) - \varphi(t-h)] - \varphi'(t), \quad t \in J_h, \\ G(t, h, r) &:= h^2 F_{(h,r)}(T_0^h y^h, T_1^h z^h)(t) - \varphi(t + rh) + 2\varphi(t) - \varphi(t - rh). \end{aligned}$$

First of all, we shall find a bound for $\|v^h\|$. It is easy to see that v^h satisfies the difference problem

$$
(26)\quad \begin{cases} v^h(t) = \omega_h(t) \quad \text{for} \quad t = a \ \text{or} \ t = b, \\ v^h(t_n + rh) - 2v^h(t_n) + v^h(t_n - rh) = G(t_n, h, r), \quad n = 1, 2, \dots, N-1, \\ \qquad r \in [0,1]. \end{cases}
$$

Indeed, $G(t, h, 0) = \theta$. Furthermore,

$$
(27)\quad u^h(t_n + rh) = \frac{1}{2h}\,\Delta v^h(t_n + rh) + \overline{\varphi}(t_n + rh), \quad n = 1, 2, \dots, N-2, \ \ r \in [0,1].
$$

By Lemma 1, (16) and (26), we see that for $r \in [0,1]$

$$
(28)\quad v^h(t_n + rh) = \begin{cases} \sigma_h(n) - v^h(t_1 - rh) + Q_n(r), & n = 1, 3, \dots, N - 1 - s_N, \\ \sigma_h(n) + v^h(a + rh) + Q_n(r), & n = 2, 4, \dots, N - 2 + s_N, \end{cases}
$$

with $G(t_n, h, r)$ instead of $g_n(r)$ in the definition of Q_n, and

$$
\sigma_h(n) = \begin{cases} \dfrac{n+1}{N}\,\omega_h(b) + \left(2 - \dfrac{n+1}{N}\right)\omega_h(a), & n = 1, 3, \dots, N - 1 - s_N, \\[2ex] \dfrac{n}{N}\,\omega_h(b) - \dfrac{n}{N}\omega_h(a), & n = 2, 4, \dots, N - 2 + s_N. \end{cases}
$$

Moreover, (17) and (28) yield

$$
\begin{aligned}
(29)\quad \Delta v^h(t_n + rh) = {} & -\sum_{i=1}^{N-1} (d_{n+1,i} - d_{n-1,i}) G(t_i, h, 1) + G(t_{n+1}, h, r) \\
& - G(t_n, h, 1 - r) + \frac{2}{N}[\omega_h(b) - \omega_h(a)], \\
& \qquad\qquad n = 1, 2, \dots, N-2, \ \ r \in [0,1].
\end{aligned}
$$

It results from the above that an estimate for the operator G is needed. By conditions 2° and 4° we obtain

$$
\begin{aligned}
(30)\quad \|G(t_n, h, r)\| & \leqslant h^2 \|F_{(h,r)}(T_0^h y^h, T_1^h z^h)(t_n) \\
& \qquad - F_{(h,r)}(T_0\varphi, T_1\varphi')(t_n)\| + \varepsilon(t_n, h) \\
& \leqslant h^2 K_1 V^h + h^2 K_2 U^h + \bar{\varepsilon}(t_n, h).
\end{aligned}
$$

This, (28), (20)–(21) and 7° yield a bound for $\|v^h\|$, namely

$$\begin{aligned}
\|v^h(t_n+rh)\| &\leqslant \|Q_n(r)\| + S_h(n) \\
&\leqslant \sum_{i=1}^{N-1} d_{ni}\|G(t_i,h,1)\| \\
&\quad + \sum_{i=1}^{n} \|G(t_{n+1-i},h,s_i r + (1-r)(1-s_i))\| + S_h(n) \\
&\leqslant \sum_{i=1}^{N-1} d_{ni}\{h^2K_1V^h + h^2K_2U^h + \bar\varepsilon(t_i,h)\} \\
&\quad + \sum_{i=1}^{n}\{h^2K_1V^h + h^2K_2U^h + \bar\varepsilon(t_{n+1-i},h)\} + S_h(n) \\
&\leqslant h^2\left\{\frac{(N+1)^2}{8} + N\right\}[K_1V^h + K_2U^h] \\
&\quad + \left(\frac{N}{4} + \frac{1}{N} + 1\right)\sum_{i=1}^{N-1} \bar\varepsilon(t_i,h) + S_h(n) \\
&= \left\{\frac{(b-a+h)^2}{8} + h(b-a)\right\}[K_1V^h + K_2U^h] \\
&\quad + \left(\frac{b-a}{4h} + \frac{h}{b-a} + 1\right)\sum_{i=1}^{N-1} \bar\varepsilon(t_i,h) + 2\bar\eta(h) + \eta(h) \\
&=: \zeta_1^h, \qquad n = 1,2,\dots,N-1, \quad r \in (0,1),
\end{aligned}$$

where

$$S_h(n) := \frac{n+1}{N}\,\|\omega_h(b)\| + \frac{2N-n-1}{N}\,\|\omega_h(a)\| + \eta(h) \leqslant 2\bar\eta(h) + \eta(h)$$

(here $\omega_h(t) = \omega_h^{(0)}(t)$ for $t=a$ or $t=b$). Furthermore, (18) leads to

$$\begin{aligned}
\|v^h(t_n)\| &\leqslant \frac{n}{N}\,\|\omega_h(b)\| + \left(1-\frac{n}{N}\right)\|\omega_h(a)\| + \sum_{i=1}^{N-1} d^*_{ni}\|G(t_i,h,1)\| \\
&\leqslant \frac{n}{N}\,\|\omega_h(b)\| + \left(1-\frac{n}{N}\right)\|\omega_h(a)\| \\
&\quad + \sum_{i=1}^{N-1} d^*_{ni}[h^2K_1V^h + h^2K_2U^h + \bar\varepsilon(t_i,h)] \\
&\leqslant \bar\eta(h) + \frac{(b-a)^2}{8}[K_1V^h + K_2U^h] \\
&\quad + \frac{b-a}{4h}\sum_{i=1}^{N-1} \bar\varepsilon(t_i,h), \quad n = 0,1,\dots,N.
\end{aligned}$$

This and the previous estimates for $\|v^h(t_n+rh)\|$ give the relations

$$\sup_{[t_n,t_{n+1}]} \|v^h(\tau)\| \leqslant \zeta_1^h, \quad n=0,1,\dots,N-1,$$
$$\sup_{t\in J} \|v^h(t)\| \leqslant \zeta_1^h,$$

and from that and 6° we finally obtain

$$V^h = \sup_{t\in \tilde{J}} \|(\tilde{T}_0^h v^h)(t)\| \leqslant \zeta_1^h. \tag{31}$$

By the same argument, using (27), (29)–(30), (23), 6° and Remark 5, we have

$$\begin{aligned}
\|u^h(t_n+rh)\| &\leqslant \frac{1}{2h}\sum_{i=1}^{N-1} |d_{n+1,i}-d_{n-1,i}|\,\|G(t_i,h,1)\| \\
&\quad + \frac{1}{2h}\Big\{\|G(t_{n+1},h,r)\| + \|G(t_n,h,1-r)\| + \frac{4}{N}\bar{\eta}(h)\Big\} \\
&\quad + \|\bar{\varphi}(t_n+rh)\| \\
&\leqslant \frac{h}{2}(N+1)[K_1V^h + K_2U^h] \\
&\quad + \frac{3}{2h}\sum_{i=1}^{N-1}\bar{\varepsilon}(t_i,h) + \frac{2}{b-a}\,\bar{\eta}(h) + O(h^2) \\
&= \frac{b-a+h}{2}[K_1V^h+K_2U^h] + \frac{3}{2h}\sum_{i=1}^{N-1}\bar{\varepsilon}(t_i,h) + \frac{2}{b-a}\,\bar{\eta}(h) + O(h^2) \\
&=: \zeta_2^h, \quad n=1,2,\dots,N-2, \quad r\in[0,1];
\end{aligned}$$

hence

$$\sup_{[a+h,b-h]} \|u^h(t)\| \leqslant \zeta_2^h. \tag{32}$$

This, 6° and 8° yield

$$U^h \leqslant \zeta_2^h + \eta(h) + \bar{\eta}(h). \tag{33}$$

In order to achieve (25), we first introduce the notation

$$\|v^h\|_* := K_1V^h + K_2U^h.$$

Combining this with (31) and (33), we get

$$\begin{aligned}\|v^h\|_* &\leqslant K_1\zeta_1^h + K_2[\zeta_2^h + \eta(h) + \bar{\eta}(h)] \\ &= (\varrho + A(h))\|v^h\|_* + B(h)\sum_{i=1}^{N-1} \bar{\varepsilon}(t_i, h) + \tilde{\eta}(h) + O(h^2).\end{aligned}$$

Because $\varrho < 1$ and $A(h) \to 0$ as $h \to 0$, there exists an $\bar{h}$ such that $\varrho + A(h) < 1$ for $h \leqslant \bar{h}$; hence we have (25).

The proof is complete. □

Remark 8. Assume that method (6)–(7) is consistent with (1) of order p and

$$\eta(h) = O(h^p), \quad \bar{\eta}(h) = O(h^p) \quad \text{and} \quad h\sum_{i=1}^{N-1} \delta(t_i, h) = O(h^p) \ \text{ as } \ h \to 0.$$

Let $K_1, K_2 > 0$. Then the order of convergence is $\min(2, p)$. If f does not depend on y' then $K_2 = 0$ and, in this case, only the first formula of (25) remains true with

$$\zeta(h) = (1 - \varrho - A(h))^{-1}\Bigg[B(h)\sum_{i=1}^{N-1} \bar{\varepsilon}(t_i, h) + \tilde{\eta}(h)\Bigg].$$

The order of convergence is now p.

Remark 9. It results from the proof that condition 2° of Theorem 2 is needed only to get estimate (30). It means that $\bar{x}$ and $\bar{y}$ appearing in 2° can be replaced by $T_0\varphi$ and $T_1\varphi'$, respectively.

Remark 10. According to 7° and 8°, the approximations y^h and z^h have to be defined in advance on the corresponding sets. We will distinguish only two types of approximations. One of them is the linear approximation for y^h,

$$y^h(a + rh) = (1 - r)\psi_h(a) + ry^h(t_1), \quad r \in [0, 1], \tag{34}$$

and then

$$\begin{cases} z^h(a + rh) = \dfrac{1}{h}[y^h(t_1 + rh) - y^h(a + rh)], \\[2ex] z^h(b - rh) = \dfrac{1 - r}{h}[\psi_h(b) - y^h(b - h)] + rz^h(b - h) \end{cases} \tag{35}$$

for $r \in [0, 1)$. Indeed, we also can use the quadratic approximation for y^h, namely

$$\begin{aligned} y^h(a + rh) = {}& \frac{1}{2}(r^2 - 3r + 2)\psi_h(a) - (r^2 - 2r)y^h(t_1) \\ &+ \frac{1}{2}(r^2 - r)y^h(t_2), \quad r \in [0, 1], \end{aligned} \tag{36}$$

and then

$$
(37) \qquad \begin{cases} z^h(a+rh) = & \dfrac{1}{2h}[-3y^h(a+rh)+4y^h(t_1+rh)-y^h(t_2+rh)], \\ z^h(b-rh) = & \dfrac{1}{4h}(r^2-3r+2)[3\psi_h(b)-4y^h(b-h)+y^h(b-2h)] \\ & -(r^2-2r)z^h(b-h)+\dfrac{1}{2}(r^2-r)z^h(b-2h) \end{cases}
$$

for $r \in [0,1)$.

Below, we consider method (6)–(7) with (34)–(35). Our objective is to verify the estimates (31) and (33) for this method. We will use the notation introduced in the proof of Theorem 2. First of all, a corresponding estimate of $\|v^h(t)\|$ for $t \in (a, a+h)$ and also of $\|u^h(\tau)\|$ for $\tau \in J_0$ will be stated. Notice that the following results hold:

$$
\begin{aligned}
\varphi_1 &:= (1-r)\varphi(a) + r\varphi(t_1) - \varphi(a+rh) = O(h^2), \\
\varphi_2 &:= \frac{1}{h}[\varphi(t_1+rh) - \varphi(a+rh)] - \varphi'(a+rh) = O(h), \\
\varphi_3 &:= \varphi(a) + \varphi(a+h) - \varphi(a+(1-r)h) - \varphi(a+rh) = O(h^2), \\
\varphi_4 &:= \frac{1-r}{h}[\varphi(b) - \varphi(b-h)] + r\varphi'(b-h) - \varphi'(b-rh) = O(h)
\end{aligned}
$$

as $h \to 0$.

By this, (34) and (18), we can write

$$
\begin{aligned}
v^h(a+rh) &= y^h(a+rh) - \varphi(a+rh) \\
&= (1-r)\omega_h(a) + rv^h(t_1) + \varphi_1 \\
&= \frac{r}{N}\,\omega_h(b) + \left(1-\frac{r}{N}\right)\omega_h(a) - r\sum_{i=1}^{N-1} d^*_{1i}G(t_i,h,1) + O(h^2).
\end{aligned}
$$

Using estimate (30), this implies

$$
\begin{aligned}
\|v^h(a+rh)\| &\leqslant \bar{\eta}(h) + r\sum_{i=1}^{N-1}\left(1-\frac{i}{N}\right)[h^2K_1V^h + h^2K_2U^h + \bar{\varepsilon}(t_i,h)] + O(h^2) \\
&\leqslant \bar{\eta}(h) + \frac{N-1}{2}\,h^2[K_1V^h + K_2U^h] + \left(1-\frac{1}{N}\right)\sum_{i=1}^{N-1}\bar{\varepsilon}(t_i,h) + O(h^2) \\
&\leqslant \bar{\eta}(h) + \frac{h(b-a)}{2}\,[K_1V^h + K_2U^h] + \sum_{i=1}^{N-1}\bar{\varepsilon}(t_i,h) + O(h^2), \quad r \in [0,1).
\end{aligned}
$$

Combining this with the previous estimates of $\|v^h(t)\|$ obtained in the proof of Theorem 2, we see that (31) remains true with $\bar{\zeta}_1^h$ instead of ζ_1^h, where $\bar{\zeta}_1^h$ is defined as ζ_1^h with $\eta(h) = O(h^2)$.

Our next objective is to have an estimate for $\|u^h(a+rh)\|$. By (34) we obtain

$$v^h(a+(1-r)h) + v^h(a+rh) = \omega_h(a) + v^h(t_1) + \varphi_3.$$

This, (35) and (28) yield the result

$$\begin{aligned}
u^h(a+rh) &= \frac{1}{h}\left\{\frac{2}{N}\,\omega_h(b) + \left(2 - \frac{2}{N}\right)\omega_h(a) + Q_1(r)\right. \\
&\qquad \left. - v^h(a+(1-r)h) - v^h(a+rh)\right\} + O(h) \\
&= \frac{1}{b-a}\,[\omega_h(b) - \omega_h(a)] \\
&\qquad + \frac{1}{h}\left\{\sum_{i=1}^{N-1}[d_{1i}^* - d_{1i}]G(t_i,h,1) + G(t_1,h,r)\right\} + O(h)
\end{aligned}$$

and hence we get

$$\begin{aligned}
(38)\qquad \|u^h(a+rh) &\leqslant \frac{2}{b-a}\,\bar{\eta}(h)\| \\
&\qquad + \frac{1}{h}\left\{\sum_{i=1}^{N-1}|d_{1i}^* - d_{1i}|\,[h^2K_1V^h + h^2K_2U^h + \bar{\varepsilon}(t_i,h)]\right\} \\
&\qquad + \frac{1}{h}[h^2K_1V^h + h^2K_2U^h + \bar{\varepsilon}(t_1,h)] + O(h) \\
&= \frac{2}{b-a}\,\bar{\eta}(h) + \frac{b-a+h}{2}\,[K_1V^h + K_2U^h] \\
&\qquad + \left(\frac{2}{h} - \frac{1}{b-a}\right)\sum_{i=1}^{N-1}\bar{\varepsilon}(t_i,h) + O(h)
\end{aligned}$$

for $r \in [0,1)$. Similarly as above, by (35), (18), (27), (29), we easily obtain

$$\begin{aligned}
u^h(b-rh) &= \frac{1-r}{h}\,[\omega_h(b) - v^h(b-h)] + ru^h(b-h) + \varphi_4 \\
&\leqslant \frac{1}{b-a}\,[\omega_h(b) - \omega_h(a)] + \frac{1-r}{h}\sum_{i=1}^{N-1} d_{N-1,i}^*G(t_i,h,1) \\
&\qquad - \frac{r}{2h}\sum_{i=1}^{N-1}(d_{N-1,i} - d_{N-3,i})G(t_i,h,1) \\
&\qquad + \frac{r}{2h}\,G(b-h,h,1) + O(h), \qquad r \in [0,1).
\end{aligned}$$

Because of (30), (23) and Remark 5, this yields

$$
\begin{aligned}
(39)\quad & \|u^h(b-rh)\| \\
& \leqslant \frac{2}{b-a}\,\bar{\eta}(h) + \frac{1-r}{h}\Big(1-\frac{2}{N}\Big)\sum_{i=1}^{N-2}\bar{\varepsilon}(t_i,h) + h^2[K_1V^h + K_2U^h] \\
& \quad \times \Big\{\frac{1-r}{hN}\sum_{i=1}^{N-2} i + \frac{1-r}{h}\Big(1-\frac{1}{N}\Big) + \frac{r}{2h}\sum_{i=1}^{N-1}|d_{N-1,i}-d_{N-3,i}| + \frac{r}{2h}\Big\} \\
& \quad + \frac{1-r}{h}\Big(1-\frac{1}{N}\Big)\bar{\varepsilon}(t_{N-1},h) + \frac{r}{h}\sum_{i=1}^{N-1}\bar{\varepsilon}(t_i,h) + \frac{r}{2h}\bar{\varepsilon}(t_{N-1},h) + O(h) \\
& \leqslant \frac{2}{b-a}\,\bar{\eta}(h) + \frac{b-a}{2}[K_1V^h + K_2U^h] \\
& \quad + \Big(\frac{2}{h} - \frac{1}{b-a}\Big)\sum_{i=1}^{N-1}\bar{\varepsilon}(t_i,h) + O(h^2), \quad r \in [0,1),
\end{aligned}
$$

which, combined with (32), (38), (39), obviously reduces to

$$
U^h \leqslant \bar{\zeta}_2^h + \bar{\eta}(h),
$$

where

$$
\bar{\zeta}_2^h := \frac{b-a+h}{2}\,[K_1V^h + K_2U^h] + \frac{2}{h}\sum_{i=1}^{N-1}\bar{\varepsilon}(t_i,h) + \frac{2}{b-a}\,\bar{\eta}(h) + O(h).
$$

Based on the above we can say that estimate (25), with the corresponding form of ζ, remains true for method (6)–(7), (34)–(35). Under the assumptions of Remark 8, the order of convergence equals $\min(1,p)$.

The next method (6)–(7), (36)–(37) is a little more complicated one. Similarly as for (34)–(35), we can prove that method (6)–(7), (36)–(37) is convergent and its order of convergence is equal to $\min(2,p)$.

It is also possible to define y^h and z^h on the "initial" sets in the following way:

$$
y^h(t) = \psi_h(a), \qquad t \in (a, a+h),
$$

and

$$
\begin{cases}
z^h(a+rh) = \dfrac{1}{h}\,[y^h(t_1) - \psi_h(a)], \\[2ex]
z^h(b-rh) = \dfrac{1}{h}\,[\psi_h(b) - y^h(b-h)],
\end{cases}
$$

or

$$\begin{cases} z^h(a+rh) = \dfrac{1}{2h}\,[-3\psi_h(a) + 4y^h(t_1) - y^h(t_2)], \\ z^h(b-rh) = \dfrac{1}{2h}\,[3\psi_h(b) - 4y^h(b-h) + y^h(b-2h)] \end{cases}$$

for $r \in [0,1)$. Notice that y^h will be now only a piecewise continuous function, while before it was continuous.

5. Some comments

In [5], problem (1) is considered for

$$f(y,y')(t) = f_0(t, y(t), y(\tau(t, y(t)))) =: f(y)(t), \quad t \in J, \tag{40}$$

with $\tau\colon J \times \mathbb{R} \to \mathbb{R}$ of advanced type and $f_0\colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For finding the approximate solution y^h, the following procedure is proposed:

$$\begin{cases} y^h(t_{n+1}) - 2y^h(t_n) + y^h(t_{n-1}) = h^2 f_0^h(t_n, y^h(t_n), y^h(\tau(t_n, y^h(t_n)))), \\ y^h(t_n + rh) = \sum\limits_{j=0}^{3} P_j(r) y^h(t_{n-1+j}), \quad r \in (0,1], \end{cases} \tag{41}$$

where

$$P_0(r) := -\frac{1}{6}(r^3 - 3r^2 + 2r), \qquad P_1(r) := \frac{1}{2}(r^3 - 2r^2 - r + 2),$$
$$P_2(r) := -\frac{1}{2}(r^3 - r^2 - 2r), \qquad P_3(r) := \frac{1}{6}(r^3 - r)$$

are the Lagrange fundamental polynomials of cubic interpolation (in [5], f_0^h is replaced by f_0). According to the results of paper [5], method (41) is convergent if condition 5° of Theorem 2 is replaced by

$$\left[L_1 + L_2\left(\frac{10}{3}\,QP + D\right)\right]\frac{(b-a)^2}{8} < 1 \tag{42}$$

with

$$Q := \max\left\{L_\psi, S(b-a) + \frac{|\psi(b) - \psi(a)|}{b-a}\right\}, \quad D := \sup\left\{\sum_{j=0}^{3} |P_j(r)|\colon\ r \in [0,1]\right\},$$
$$|f_0(t,x,y)| \leqslant S.$$

Here L_1 and L_2 are Lipschitz constants of f_0 with respect to the second and third variables, respectively. P is a Lipschitz constant of τ with respect to the second variable, while L_ψ denotes a Lipschitz constant of ψ.

Let us introduce the abbreviation

$$f_n^h := f_0^h(t_n, y^h(t_n), (T_0^h y^h)(\tau(t_n, y^h(t_n)))),$$

where f_0^h denotes an approximation of f_0. After some calculations using (41) and (18), the equality

$$\sum_{j=0}^{3} P_j(1-r) y^h(t_{n-2+j}) = \sum_{j=0}^{3} P_j(-r) y^h(t_{n-1+j}) + \frac{r^3 - r}{6}\, h^2 [f_{n-1}^h - 2f_n^h + f_{n+1}^h]$$

can be proved. Now, it is simple to see that

$$\begin{aligned} & y^h(t_n + rh) - 2y^h(t_n) + y^h(t_n - rh) \\ &\quad = \sum_{j=0}^{3} [P_j(r) + P_j(-r)] y^h(t_{n-1+j}) - 2y^h(t_n) + \frac{r^3 - r}{6}\, h^2 [f_{n-1}^h - 2f_n^h + f_{n+1}^h] \\ &\quad = r^2 [y^h(t_{n+1}) - 2y^h(t_n) + y^h(t_{n-1})] + \frac{r^3 - r}{6}\, h^2 [f_{n-1}^h - 2f_n^h + f_{n+1}^h] \\ &\quad = r^2 h^2 f_n^h + \frac{r^3 - r}{6}\, h^2 [f_{n-1}^h - 2f_n^h + f_{n+1}^h]. \end{aligned}$$

It means that if we take

$$F_{(h,r)}(T_0^h y^h)(t_n) = r^2 f_n^h + \frac{r^3 - r}{6} [f_{n-1}^h - 2f_n^h + f_{n+1}^h],$$

then y^h defined by (41) satisfies (6), too.

If φ is the solution of (1), then, after some calculations, we obtain

$$|\varphi'(t)| \leqslant \frac{|\psi(b) - \psi(a)|}{b - a} + \frac{S(b-a)}{2} =: \bar{L}, \quad t \in J.$$

Hence, for $\overline{Q} := \max(L_\psi, \bar{L})$ we have

$$\begin{aligned} & |f_0(t_n, y^h(t_n), y^h(\tau(t_n, y^h(t_n)))) - f_0(t_n, \varphi(t_n), \varphi(\tau(t_n, \varphi(t_n))))| \\ &\qquad \leqslant L_1 |y^h(t_n) - \varphi(t_n)| \\ &\qquad\quad + L_2 |y^h(\tau(t_n, y^h(t_n))) - \varphi(\tau(t_n, y^h(t_n))) \\ &\qquad\quad + \varphi(\tau(t_n, y^h(t_n))) - \varphi(\tau(t_n, \varphi(t_n)))| \\ &\qquad \leqslant L_1 |v^h(t_n)| + L_2 |v^h(\tau(t_n, y^h(t_n)))| + L_2 \overline{Q} P |v^h(t_n)|. \end{aligned}$$

Assume that

$$|f(y)(t) - f^h(y)(t)| \leqslant \delta_1(t, h) \quad \text{with} \quad \lim_{h \to 0} h \sum_{i=1}^{N-1} \delta_1(t_i, h) = 0,$$

and put $f_n := f(y^h)(t_n)$. Then we can write

$$
\begin{aligned}
&|F_{(h,r)}(T_0^h y^h)(t_n) - F_{(h,r)}(T_0\varphi)(t_n)| \\
&\quad = \Big| \frac{r^3 - r}{6}[f_{n-1}^h + f_{n+1}^h - f_{n-1} - f_{n+1}] + \left(r^2 - \frac{r^3 - r}{3}\right)[f_n^h - f_n] \\
&\qquad + \frac{r^3 - r}{6}[f_{n-1} + f_{n+1} - f(\varphi)(t_{n-1}) - f(\varphi)(t_{n+1})] \\
&\qquad + \left(r^2 - \frac{r^3 - r}{3}\right)[f_n - f(\varphi)(t_n)] \Big| \\
&\quad \leqslant \frac{r - r^3}{6}\,[\delta_1(t_{n-1}, h) + \delta_1(t_{n+1}, h)] + \frac{-r^3 + 3r^2 + 2r}{3}\,\delta_1(t_n, h) \\
&\qquad + \left(\frac{r - r^3}{3} + \frac{-r^3 + 3r^2 + 2r}{3}\right)[L_1 + L_2(\overline{Q}P + 1)]V^h.
\end{aligned}
$$

According to Remark 9, we see that condition 2° of Theorem 2 remains true with

$$K_1 = L_1 + L_2(\overline{Q}P + 1), \quad K_2 = 0 \text{ and } \delta(t_n, h) = \delta_1(t_{n-1}, h) + \delta_1(t_n, h) + \delta_1(t_{n+1}, h).$$

Moreover,

$$
\begin{aligned}
&|\varphi(t + rh) - 2\varphi(t) + \varphi(t - rh) - h^2 F_{(h,r)}(T_0\varphi, T_1\varphi')(t)| \\
&\quad = \Big| \varphi(t + rh) - 2\varphi(t) + \varphi(t - rh) \\
&\qquad - h^2\left[r^2\varphi''(t) + \frac{r^3 - r}{6}[\varphi''(t - h) - 2\varphi''(t) + \varphi''(t + h)]\right] \\
&\qquad + h^2\Big\{ r^2[f(\varphi)(t) - f^h(\varphi)(t)] + \frac{r^3 - r}{6}[(f(\varphi)(t - h) - f^h(\varphi)(t - h))] \\
&\qquad + \frac{r^3 - r}{6}[-2(f(\varphi)(t) - f^h(\varphi)(t)) + (f(\varphi)(t + h) - f^h(\varphi)(t + h))] \Big\} \Big| \\
&\quad = O(h^4) + h^2\left\{ \frac{-r^3 + 3r^2 + 2r}{3}\,\delta_1(t, h) + \frac{r - r^3}{6}\,[\delta_1(t - h, h) + \delta_1(t + h, h)] \right\},
\end{aligned}
$$

which proves that method (41) is consistent with (1), and now ε appearing in Definition 2 is of the form

$$\varepsilon(t, h) = O(h^4) + h^2[\delta_1(t - h, h) + \delta_1(t, h) + \delta_1(t + h, h)]$$

(if $\delta_1(t, h) = O(h^\nu)$ uniformly in t as $h \to 0$ for some $\nu > 0$, then (41) has the order of consistency equal to $\min(\nu, 2)$).

Now, condition 5° of Theorem 2 yields

$$[L_1 + L_2(\overline{Q}P + 1)]\frac{(b-a)^2}{8} < 1, \tag{43}$$

which is superior to (42) because $D = \frac{5}{4}$ and $\overline{Q} \leqslant Q$. If τ does not depend on the second variable, then $P = 0$, and thus (43) reduces to the corresponding result obtained in [15]. If f_0 does not depend on the last variable, then $L_2 = 0$ and thus we have the problem considered, for example, in [9], [16].

Theorem 2 gives also sufficient conditions which ensure the convergence of method (6)–(7) for the case when the mapping f is of the form

$$\begin{aligned} &f(y, y')(t) \\ &\quad = f_1(t, y(t), y(\tau_1(t, y(t))), y(\tau_2(t, y'(y))), y'(t), y'(\tau_3(t, y(t))), y'(\tau_4(t, y'(t)))), \\ &\qquad t \in J, \end{aligned}$$

for $f_1\colon\ J \times \mathbb{R}^6 \to \mathbb{R}$, $\tau_i\colon\ J \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3, 4$. If we assume that the conditions

$$\begin{gathered} |f_1(t, y_1, y_2, y_3, y_4, y_5, y_6) - f_1(t, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4, \bar{y}_5, \bar{y}_6)| \leqslant \sum_{i=1}^{6} L_i |y_i - \bar{y}_i|, \quad L_i \geqslant 0, \\ |\tau_i(t, y_1) - \tau_i(t, \bar{y}_1)| \leqslant P_i |y_1 - \bar{y}_1|, \quad P_i \geqslant 0, \quad i = 1, 2, 3, 4, \\ |\psi'(t_1) - \psi'(t_2)| \leqslant \overline{L}_\psi |t_1 - t_2|, \quad t_1, t_2 \in \overline{J}, \quad \overline{S} := \max(S, \overline{L}_\psi) \end{gathered}$$

hold for some nonnegative L_i, P_j and $\overline{L}_\psi$, then, for $F = f_1$, condition 2° of Theorem 2 is obviously satisfied for

$$\begin{aligned} K_1 &= L_1 + L_2(1 + \overline{Q}P_1) + L_3 + L_5\overline{S}P_3, \\ K_2 &= L_3\overline{Q}P_2 + L_4 + L_5 + L_6(1 + \overline{S}P_4). \end{aligned}$$

Notice that Theorem 2 is quite general. It gives sufficient conditions which ensure that the difference method is convergent when, for example, the operator f is of the Volterra-Fredholm type of the form

$$\begin{aligned} &f(y, y')(t) = \\ &\quad f_2(t, y(\alpha_1(t)), y'(\alpha_2(t)), \int_a^{\gamma(t)} k_1(\tau, t, y(\tau), y'(\tau)\,\mathrm{d}\tau, \int_a^b k_2\big(\tau, t, y(\tau), y'(\tau)\big)\,\mathrm{d}\tau), \\ &\qquad t \in J \end{aligned}$$

for $f_2\colon\ J \times (\mathbb{R}^m)^4 \to \mathbb{R}^m$, $k_i\colon\ J \times J \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $\alpha_i\colon\ J \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, 2$, and $\gamma\colon\ J \to \mathbb{R}$, $\gamma(t) \leqslant t$.

6. Construction of methods of arbitrary order

In this section, a scheme for constructing methods of arbitrary order will be given. This technique can be applied for methods of type (6)–(7) for which the operator F is defined by

$$F_{(h,r)}(T_0^h y^h, T_1^h z^h)(t) := \sum_{i=0}^{k} b_i(t,h,r) f^h\left(t + \left(\frac{2}{k}\,i - 1\right)h\right), \tag{44}$$

$$f^h(t) := f^h(T_0^h y^h, T_1^h z^h)(t), \tag{45}$$

where an integer $k > 0$ is given, $b_i\colon\ J \times H \times [0,1] \to \mathbb{R}$ and $b_i(t,h,0) = 0$. Denote by $\overline{F}$ the operator obtained from F by replacing f^h by f on the right-hand side of (44)–(45). Indeed,

$$F_{(h,r)}(\ldots) = F_{(h,r)}(\ldots) - \overline{F}_{(h,r)}(\ldots) + \overline{F}_{(h,r)}(\ldots).$$

We say that the method (6)–(7), (44)–(45) is consistent with (1) if Definition 2 remains true with $\overline{F}$ instead of F. To give some conditions on the consistency and convergence of (6)–(7), (44)–(45) we introduce

Definition 3 (see [10], [16]). We say $\varphi\colon\ J \to \mathbb{R}^q$ is in class $S_p^R(J)$, $p \geqslant 1$, if φ is $p-1$ times differentiable on J and there exists a bounded function which we will denote by $\varphi^{(p)}\colon\ J \to \mathbb{R}^q$, such that the $(p-1)$st derivative $\varphi^{(p-1)}$ is the Riemann integral of $\varphi^{(p)}$. We write $\varphi \in S_p^B(J)$ if $\varphi \in S_p^R(J)$ and $\varphi^{(p)}$ is of bounded variation, i.e. there exists a constant V such that, for any partition $a \leqslant s_0 < s_1 < \ldots < s_q \leqslant b$, we have

$$\sum_{i=1}^{q} \|\varphi^{(p)}(s_i) - \varphi^{(p)}(s_{i-1})\| \leqslant V.$$

We write $\varphi \in S_p^H(J)$ if $\varphi \in S_p^R(J)$ and $\varphi^{(p)}$ satisfies the Hölder condition with an exponent $\gamma \in (0,1]$.

Put

$$C_j^k(t,h,r) := \frac{r^j}{j!}[1 + (-1)^j] - \frac{1}{(j-2)!}\sum_{i=0}^{k} b_i(t,h,r)\left(\frac{2}{k}\,i - 1\right)^{j-2}, \quad j = 2,3,\ldots.$$

The order of consistency of method (6)–(7), (44)–(45) will be defined under the assumptions given in the next lemma.

Lemma 7. *If*

1° *problem* (1) *has a unique solution* φ,

2° $b_i\colon J \times H \times [0,1] \to \mathbb{R}$ *are continuous with respect to the last variable;* b_i *are bounded and* $b_i(t,h,0) = 0$, $i = 0,1,\ldots,k$,

3° $\begin{cases} C_j^k(t,h,r) = 0 \quad \text{for } t \in J,\ h \in H,\ r \in [0,1] \text{ and } j = 2,3,\ldots,p-1, \\ C_p^k(t,h,r) \not\equiv 0, \end{cases}$

then the method (6)–(7), (44)–(45) *is consistent with* (1) *of order* $p-2$ *if* $\varphi \in S_p^R(J)$, *of order* $p-1$ *if* $\varphi \in S_p^B(J)$ *and of order* $p-2+\gamma$ *if* $\varphi \in S_p^H(J)$.

P r o o f. The Taylor formula for $\varphi \in S_p^R(J)$ yields

$$\begin{aligned} Q(t,h,r) &= \varphi(t+rh) - 2\varphi(t) + \varphi(t-rh) \\ &\quad - h^2 \sum_{i=0}^{k} b_i(t,h,r)\varphi''\Big(t + \Big(\frac{2}{k}\,i - 1\Big)h\Big) \\ &= \sum_{i=2}^{p} h^i \varphi^{(i)}(t) C_i^k(t,h,r) + T(t,h,r) \\ &= O(h^p) + T(t,h,r), \end{aligned}$$

where

$$\begin{aligned} T(t,h,r) := {}& \frac{1}{(p-1)!}\int_t^{t+rh} (t+rh-s)^{p-1}[\varphi^{(p)}(s) - \varphi^{(p)}(t)]\,\mathrm{d}s \\ &+ \frac{1}{(p-1)!}\int_t^{t-rh} (t-rh-s)^{p-1}[\varphi^{(p)}(s) - \varphi^{(p)}(t)]\,\mathrm{d}s \\ &- \frac{h^2}{(p-3)!}\sum_{i=0}^{k} b_i(t,h,r)\int_t^{t+(\frac{2}{k}\,i-1)h} \Big(t + \Big(\frac{2}{k}\,i - 1\Big)h - s\Big)^{p-3} \\ &\times [\varphi^{(p)}(s) - \varphi^{(p)}(t)]\,\mathrm{d}s. \end{aligned}$$

Changing the intervals of integration and putting $\bar{k} = \big[\frac{1}{2}k\big]$, we have

$$\begin{aligned} T(t,h,r) = {}& \frac{1}{(p-1)!}\int_0^{rh} (rh-u)^{p-1}[\varphi^{(p)}(t+u) - \varphi^{(p)}(t)]\,\mathrm{d}u \\ &+ \frac{(-1)^{p-1}}{(p-1)!}\int_0^{-rh} (rh+u)^{p-1}[\varphi^{(p)}(t+u) - \varphi^{(p)}(t)]\,\mathrm{d}u \\ &- \frac{h^2}{(p-3)!}\sum_{i=0}^{\bar{k}} b_i(t,h,r)\int_0^{-(1-\frac{2}{k}\,i)h} \Big(\Big(\frac{2}{k}\,i - 1\Big)h - u\Big)^{p-3} \\ &\times [\varphi^{(p)}(t+u) - \varphi^{(p)}(t)]\,\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&- \frac{h^2}{(p-3)!} \sum_{i=\bar{k}+1}^{k} b_i(t,h,r) \int_0^{\left(\frac{2}{k}\, i-1\right)h} \left(\left(\frac{2}{k}\, i-1\right)h-u\right)^{p-3} \\
&\quad \times [\varphi^{(p)}(t+u)-\varphi^{(p)}(t)]\,\mathrm{d}u \\
&= h^{p-1}\biggl\{ \frac{r^{p-1}}{(p-1)!} \int_0^{rh} \left(1-\frac{u}{rh}\right)^{p-1} [\varphi^{(p)}(t+u)-\varphi^{(p)}(t)]\,\mathrm{d}u \\
&\quad + \frac{r^{p-1}(-1)^p}{(p-1)!} \int_0^{rh} \left(1-\frac{u}{rh}\right)^{p-1} [\varphi^{(p)}(t-u)-\varphi^{(p)}(t)]\,\mathrm{d}u \\
&\quad + \frac{(-1)^{p-3}}{(p-3)!} \sum_{i=0}^{\bar{k}} \left(1-\frac{2}{k}\, i\right)^{p-3} b_i(t,h,r) \\
&\quad \times \int_0^{\left(1-\frac{2}{k}\, i\right)h} \left(1-\frac{u}{(1-\frac{2}{k}\, i)h}\right)^{p-3} [\varphi^{(p)}(t-u)-\varphi^{(p)}(t)]\,\mathrm{d}u \\
&\quad - \frac{1}{(p-3)!} \sum_{i=\bar{k}+1}^{k} \left(\frac{2}{k}\, i-1\right)^{p-3} b_i(t,h,r) \\
&\quad \times \int_0^{\left(\frac{2}{k}\, i-1\right)h} \left(1-\frac{u}{(\frac{2}{k}\, i-1)h}\right)^{p-3} [\varphi^{(p)}(t+u)-\varphi^{(p)}(t)]\,\mathrm{d}u \biggr\}.
\end{aligned}
$$

Now, after some calculations, we can prove the assertion of the lemma (see also [14]). The proof is complete. □

7. Examples

We assume that $b_i(t,h,r) = b_i(r)$, $(t,h,r) \in J \times H \times [0,1]$, $i = 0,1,\ldots,k$. Notice that condition 3° of Lemma 8 influences the order of consistency of the method (6)–(7), (44)–(45). The integer p appearing in 3° depends on both k and b_i. Below, the integer p will be found for given values of k and b_i.

I. Put $k=1$. Then, for $b_0(r) = b_1(r) = \frac{1}{2}r^2$, we find $p=4$, while if $b_0(r) = r^2$, $b_1 = 0$, then $p=3$.

II. Put $k=2$ and $b_0 = b_2$. Then, for $b_1(r) = r^2 - 2b_2(r)$, we have $p=4$ (compare F from Section 5, where $b_0 = b_2 = \frac{1}{6}(r^3 - r)$). The method is simplest if $b_0 = b_2 = 0$. If $b_2(r) = \frac{1}{12}r^4$, then $b_1(r) = r^2\bigl(1-\frac{1}{6}r^2\bigr)$ and thus $p=6$. It is very popular and widely used for the case when $r=1$, i.e. for $b_0(1) = b_2(1) = \frac{1}{12}$, $b_1(1) = \frac{10}{12}$.

III. Put $k=3$. For

$$
b_0(r) = b_3(r) = -\frac{1}{16}r^2 + \frac{3}{32}r^4, \qquad b_1(r) = b_2(r) = \frac{9}{16}r^2 - \frac{3}{32}r^4
$$

we obtain only $p = 6$. If

$$b_0(r) = \frac{3}{16}r^4 - \frac{1}{8}r^2, \quad b_1(r) = -\frac{3}{8}r^4 + \frac{3}{4}r^2, \quad b_2(r) = \frac{3}{16}r^4 + \frac{3}{8}r^2, \quad b_3 = 0,$$

then $p = 5$.

IV. Put $k = 4$. For

$$\begin{aligned} b_0(r) = b_4(r) &= \frac{2}{45}r^6 - \frac{1}{36}r^4, \\ b_1(r) = b_3(r) &= -\frac{8}{45}r^6 + \frac{4}{9}r^4, \\ b_2(r) &= \frac{12}{45}r^6 - \frac{15}{18}r^4 + r^2 \end{aligned}$$

we have $p = 8$. If

$$b_0 = b_4 = 0, \qquad b_1(r) = b_3(r) = \frac{r^4}{3}, \qquad b_2(r) = -\frac{2}{3}r^4 + r^2,$$

then $p = 6$.

V. In numerical considerations only the methods from Example II are known, so when $p = 4$ or $p = 6$. The above examples show that choosing in the corresponding manner the integer k and the coefficients b_i, we can always construct the method to be of a fixed order p. Such methods are new.

Concluding this paper, we apply the results of Theorem 2 and Lemma 8 to formulate the main theorem of this section. This theorem gives the order of convergence of the method (6)–(7), (44)–(45).

Theorem 3. *If conditions 2°–3° of Lemma 7 are satisfied, and*

1° *$f, f^h \colon \mathcal{C}(\tilde{J}, \mathbb{R}^q) \times \tilde{\mathcal{C}}(\tilde{J}, \mathbb{R}^q) \to L^\infty(J, \mathbb{R}^q)$ and the assumptions 1°, 3°, 5°–8° of Theorem 2 are satisfied with F of the form (44) and $\eta(h) = O(h^s)$, $\bar{\eta}(h) = O(h^s)$ as $h \to 0$ for some $s > 0$,*

2° *there exist constants $A_1, A_2 \geqslant 0$ such that the condition*

$$\|f(x,y)(t) - f(\bar{x},\bar{y})(t)\| \leqslant A_1 \sup_{[a-a_0,t+b_0]} \|x(\tau) - \bar{x}(\tau)\| + A_2 \sup_{[a-a_0,t+b_0]} \|y(\tau) - \bar{y}(\tau)\|$$

holds for $x, \bar{x} \in \mathcal{C}(\tilde{J}, \mathbb{R}^q)$, $y, \bar{y} \in \tilde{\mathcal{C}}(\tilde{J}, \mathbb{R}^q)$,

3° *f^h is an approximation of f, i.e.*

$$\|f^h(x,y)(t) - f(x,y)(t)\| = O(h^s)$$

uniformly in t as $h \to 0$,

then the method (6)–(7), (44)–(45) *has convergence of order* $m(2,s,p-2)$ *if* $\varphi \in S_p^R(J)$, *of order* $m(2,s,p-1)$ *if* $\varphi \in S_p^B(J)$ *and of order* $m(2,s,p-2+\gamma)$ *if* $\varphi \in S_p^H(J)$. *Here*

$$m(2,s,p) := \begin{cases} \min(2,s,p) & \textit{if} \quad A_2 > 0, \\ \min(s,p) & \textit{if} \quad A_2 = 0. \end{cases}$$

Proof. It is simple to show that the assumptions of Theorem 2 remain true for $\delta(t,h) \equiv 0$ and

$$K_1 = A_1 \sum_{i=0}^{k} B_i, \quad K_2 = A_2 \sum_{i=0}^{k} B_i, \quad |b_i(t,h,r)| \leqslant B_i, \quad i = 0,1,\ldots,k.$$

Now, by Theorem 2, Remark 8 and Lemma 7, the assertion of the theorem is obviously satisfied. This completes the proof. □

Remark 11. It results from Theorem 2 (or Remark 8) that the order of convergence of our method may be greater than two only if f does not depend on y', and it is reasonable to construct the method (6)–(7), (44)–(45) in such a way, to guarantee its maximal order of consistency.

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