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QUASISTATIC FRICTIONAL PROBLEMS FOR ELASTIC AND VISCOELASTIC MATERIALS

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Abstract. We consider two quasistatic problems which describe the frictional contact between a deformable body and an obstacle, the so-called foundation. In the first problem the body is assumed to have a viscoelastic behavior, while in the other it is assumed to be elastic. The frictional contact is modeled by a general velocity dependent dissipation functional. We derive weak formulations for the models and prove existence and uniqueness results. The proofs are based on the theory of evolution variational inequalities and fixed-point arguments. We also prove that the solution of the viscoelastic problem converges to the solution of the corresponding elastic problem, as the viscosity tensor converges to zero. Finally, we describe a number of concrete contact and friction conditions to which our results apply.

Keywords: elastic material, viscoelastic material, frictional contact, evolution variational inequality, fixed point, weak solution, approach to elasticity, subdifferential boundary conditions

MSC 2000: 74M10, 74D05, 74B99, 58E35, 49J40

1. Introduction

Frictional contact between deformable bodies can be frequently found in industry and everyday life. The contact of the braking pads with the wheels, the tire with the road, the piston with the shirt, a shoe with the floor, are just four simple examples. For this reason, considerable progress has been made with the modeling and analysis of contact problems, and the engineering literature concerning this topic is rather extensive.

An early attempt at the study of frictional contact problems within the framework of variational inequalities was made in [9]. Excellent references on analysis and numerical approximation of variational inequalities arising from contact problems are [14] and [15]. The mathematical, mechanical and numerical state of the art can be found in the proceedings [23] and in the special issue [26]. Only recently, however, have the quasistatic and dynamic problems been considered. The reason lies in the considerable difficulties that the process of frictional contact presents in the modeling and analysis, because of the complicated surface phenomena involved.

Quasistatic processes arise when the forces applied to a system vary slowly in time so that acceleration is negligible. Quasistatic contact problems with normal compliance and friction have been considered in [3], [4], [19] and more recently in [20], within linearized elasticity. The existence of a weak solution to the quasistatic Signorini's contact problem with friction for elastic materials has been established in [8]. The variational analysis of some quasistatic frictional contact problems can be found for instance in [24], [25], [27], within nonlinear viscoelasticity. There, the problems were formulated as evolution variational inequalities for which existence and uniqueness results were obtained. Numerical analysis including error estimates for semi-discrete and fully discrete schemes in the study of quasistatic contact problems with viscoelastic materials can be found in [7], [12]. Existence and uniqueness results for dynamic frictional contact problems involving viscoelastic materials have been obtained in [16], [17]. There, one of the major mathematical difficulties of the problems consists in Signorini's boundary conditions formulated in terms of displacements. Dynamic contact problems with unilateral contact conditions formulated in velocities were analyzed in [18] and [10] in the study of viscoelastic and thermoviscoelastic materials, respectively.

In this paper we investigate two mathematical models for the quasistatic process of frictional contact between a deformable body and an obstacle, the so-called foundation. In the first problem we assume that the body has a viscoelastic behavior. In the other, we assume that the body is elastic. We denote by $\boldsymbol{u} = (u_i)$ the displacement field, by $\boldsymbol{\sigma} = (\sigma_{ij})$ the stress field and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ will represent the small strain tensor. For the viscoelastic body we use a linear Kelvin-Voigt constitutive law, i.e.

$$\sigma_{ij} = a_{ijkl}\,\varepsilon_{kl}(\boldsymbol{u}) + b_{ijkl}\,\varepsilon_{kl}(\dot{\boldsymbol{u}}), \tag{1.1}$$

where $\mathcal{A} = (a_{ijkl})$ is the elasticity tensor and $\mathcal{B} = (b_{ijkl})$ is the viscosity tensor. Here and everywhere in the sequel the dot above represents the time derivative.

We also model the behavior of the elastic body with the linear elastic constitutive law

$$\sigma_{ij} = a_{ijkl}\,\varepsilon_{kl}(\boldsymbol{u}). \tag{1.2}$$

Finally, we model the frictional contact with a subdifferential boundary condition of the form

$$\boldsymbol{u} \in U, \quad \varphi(\boldsymbol{v}) - \varphi(\dot{\boldsymbol{u}}) \geqslant -\boldsymbol{\sigma\nu}\,(\boldsymbol{v} - \dot{\boldsymbol{u}}) \ \ \forall \boldsymbol{v} \in U \tag{1.3}$$

in which U represents the set of contact admissible test functions, $\boldsymbol{\sigma\nu}$ denotes the Cauchy stress vector on the contact boundary, and φ is a given convex function. The inequality in (1.3) holds almost everywhere on the contact surface. Examples and detailed explanations of inequality problems in contact mechanics which lead to boundary conditions of this form can be found in the monograph [22] and more recently in [28]. We note however that, in a variational form, the frictional problems we study here include a linear subspace as the set of admissible displacement fields. Therefore, our results do not apply to the study of frictional problems involving unilateral contact conditions.

The present paper serves two purposes. The first purpose is to provide the variational analysis of the mechanical problems and to show the existence of a unique solution to each model. The other is to study the behavior of the solution of the viscoelastic problem when the viscosity operator converges to zero, and to establish the link with the corresponding solution of the elastic problem.

The paper is organized as follows. In Section 2 we introduce some notation and preliminaries. In Section 3 we state the mechanical problem, i.e. the quasistatic problem for viscoelastic materials (1.1) with the general frictional contact condition (1.3), and the quasistatic problem for elastic materials (1.2) with the same frictional contact condition (1.3). Then, we list the assumptions imposed on the problem data and derive variational formulations to the problems. In Section 4 we prove our main existence and uniqueness results, Theorem 4.1 and Theorem 4.2. The proof of these theorems are based on results concerning evolutionary variational inequalities and fixed point arguments. In Section 5 we prove a convergence result which states that the solution of the viscoelastic problem converges to the solution of the elastic problem, as the viscosity tensor converges to zero. Finally, in Section 6, we present a number of concrete examples of frictional contact boundary conditions which may be cast in the abstract form (1.3) and to which our result apply.

2. Notation and preliminaries

In this short section, we present the notation we will use and some preliminary materials. For furthur details we refer the reader to [9], [13], [22].

We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$; "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and S_d, respectively. Thus,

$$\boldsymbol{u}\cdot\boldsymbol{v}=u_iv_i,\qquad |\boldsymbol{v}|=(\boldsymbol{v}\cdot\boldsymbol{v})^{1/2},\quad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij},\quad |\boldsymbol{\tau}|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2},\quad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in S_d.$$

Here and below, the indices i and j run between 1 and d, the summation convention over repeated indices is used and the index that follows a comma indicates the partial derivative with respect to the corresponding component of the independent variable.

Let $\Omega\subset\mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary Γ. We also use the following notation:

$$\begin{aligned}
H&=\{\boldsymbol{u}=(u_i)\ \mid u_i\in L^2(\Omega)\},\\
\mathcal{H}&=\{\boldsymbol{\sigma}=(\sigma_{ij})\mid \sigma_{ij}=\sigma_{ji}\in L^2(\Omega)\},\\
H_1&=\{\boldsymbol{u}=(u_i)\ \mid u_i\in H^1(\Omega)\},\\
\mathcal{H}_1&=\{\boldsymbol{\sigma}\in\mathcal{H}\qquad \mid \sigma_{ij,j}\in H\}.
\end{aligned}$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with inner products given by

$$\begin{aligned}
(\boldsymbol{u},\boldsymbol{v})_H&=\int_\Omega u_iv_i\,\mathrm{d}x,\\
(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}&=\int_\Omega \sigma_{ij}\tau_{ij}\,\mathrm{d}x,\\
(\boldsymbol{u},\boldsymbol{v})_{H_1}&=(\boldsymbol{u},\boldsymbol{v})_H+(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},\\
(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1}&=(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}}+(\operatorname{Div}\boldsymbol{\sigma},\operatorname{Div}\boldsymbol{\tau})_H,
\end{aligned}$$

respectively, where $\boldsymbol{\varepsilon}\colon H_1\to\mathcal{H}$ and $\operatorname{Div}\colon\mathcal{H}_1\to H$ are the *deformation* and the *divergence* operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad \operatorname{Div}\boldsymbol{\sigma}=(\sigma_{ij,j}).$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively.

Since the boundary Γ is Lipschitz continuous, the unit outward normal vector $\boldsymbol{\nu}$ on the boundary is defined a.e. For every vector field $\boldsymbol{v}\in H_1$ we use the notation $\boldsymbol{v}$ to denote the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ, and we denote by v_ν and $\boldsymbol{v}_\tau$ the *normal* and the *tangential* components of $\boldsymbol{v}$ on the boundary given by

$$v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu},\quad \boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}. \tag{2.1}$$

For a regular (say C^1) stress field $\boldsymbol{\sigma}$, the application of its trace on the boundary to $\boldsymbol{\nu}$ is the Cauchy stress vector $\boldsymbol{\sigma\nu}$. We define, similarly, the *normal* and *tangential* components of the stress on the boundary by the formulas

$$\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and we recall that the following Green's formula holds:

$$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v} \, \mathrm{d}a \ \ \forall \boldsymbol{v} \in H_1. \tag{2.3}$$

Finally, for every real Hilbert space X we use the classical notation for the spaces $L^p(0,T,X)$ and $W^{k,p}(0,T,X)$, $1 \leqslant p \leqslant +\infty$, $k = 1, 2, \ldots$

3. Problems statement and variational formulations

In this section we describe the contact problems, list the assumptions imposed on the data and derive variational formulations.

The physical setting is as follows. A deformable body occupies a domain Ω and is acted upon by given forces and tractions and, as a result, its mechanical state evolves over the time interval $[0,T]$, $T > 0$. We assume that the boundary Γ of Ω is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $\operatorname{meas} \Gamma_1 > 0$. The body is clamped on $\Gamma_1 \times (0,T)$ and surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0,T)$. The solid is in frictional contact with a rigid obstacle on $\Gamma_3 \times (0,T)$ and this is where our main interest lies. Moreover, a volume force of density $\boldsymbol{f}_0$ acts on the body in $\Omega \times (0,T)$. We assume a quasistatic process and use (1.3) as boundary contact conditions. With these assumptions, we have

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{3.2}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.3}$$

$$\boldsymbol{u} \in U, \quad \varphi(\boldsymbol{v}) - \varphi(\dot{\boldsymbol{u}}) \geqslant -\boldsymbol{\sigma\nu}(\boldsymbol{v} - \dot{\boldsymbol{u}}) \ \ \forall \boldsymbol{v} \in U \ \text{ on } \Gamma_3 \times (0,T). \tag{3.4}$$

To complete the model we assume first that the body is viscoelastic, i.e. we choose (1.1) as the constitutive law and prescribe initial displacement field. Thus

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\boldsymbol{u}) + \mathcal{B}\varepsilon(\dot{\boldsymbol{u}}) \quad \text{in } \Omega \times (0,T), \tag{3.5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{3.6}$$

To conclude, the mechanical problem of frictional contact of a viscoelastic body may be formulated classically as follows.

Problem $\mathbf{P}_1$. *Find a displacement field* $\boldsymbol{u}\colon \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}\colon \Omega \times [0,T] \to S_d$ *which satisfy* (3.1)–(3.6).

Next, we assume that the body is linear elastic, i.e. we choose (1.2) as the constitutive law:

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega \times (0,T). \tag{3.7}$$

The classical formulation of the mechanical problem of frictional contact of an elastic body is the following.

Problem $\mathbf{P}_2$. *Find a displacement field* $\boldsymbol{u}\colon \Omega \times [0,T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma}\colon \Omega \times [0,T] \to S_d$ *which satisfy* (3.1)–(3.4), (3.6) *and* (3.7).

To obtain variational formulations of the contact problems P_1 and P_2, we suppose that $U \subset H_1$, $U + \mathcal{D}(\Omega)^d \subset U$, $\varphi\colon \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}$ and, to accommodate (3.2) and (3.4), we define

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_1\} \cap U. \tag{3.8}$$

Let $j\colon V \to (-\infty, +\infty]$ be the functional

$$j(\boldsymbol{v}) = \begin{cases} \displaystyle\int_{\Gamma_3} \varphi(\boldsymbol{v})\,\mathrm{d}a & \text{if } \varphi(\boldsymbol{v}) \in L^1(\Gamma_3), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.9}$$

We suppose everywhere in the sequel that

(3.10) V is a closed subspace of H_1,

(3.11) j is a convex lower semicontinuous function on V such that $j \not\equiv +\infty$.

We also assume that the elasticity and viscosity tensors possess the usual properties of ellipticity and symmetry, i.e.

$$\begin{cases} \text{(a) } \mathcal{A}\colon \Omega \times S_d \to S_d; \\ \text{(b) } a_{ijkh} \in L^\infty(\Omega); \\ \text{(c) } \mathcal{A}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{A}\boldsymbol{\tau} \ \ \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d, \text{ a.e. in } \Omega; \\ \text{(d) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \qquad \mathcal{A}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geqslant m_{\mathcal{A}} |\boldsymbol{\tau}|^2 \ \ \forall \boldsymbol{\tau} \in S_d, \text{ a.e. in } \Omega, \end{cases} \tag{3.12}$$

$$
(3.13) \qquad \begin{cases} \text{(a)}\ \mathcal{B}\colon\ \Omega \times S_d \to S_d; \\ \text{(b)}\ b_{ijkh} \in L^\infty(\Omega); \\ \text{(c)}\ \mathcal{B}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \boldsymbol{\sigma}\cdot\mathcal{B}\boldsymbol{\tau}\ \ \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in S_d,\ \text{ a.e. in } \Omega; \\ \text{(d)}\ \text{there exists } m_{\mathcal{B}} > 0 \text{ such that} \\ \qquad \mathcal{B}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geqslant m_{\mathcal{B}}\,|\boldsymbol{\tau}|^2\ \ \forall \boldsymbol{\tau} \in S_d,\ \text{ a.e. in } \Omega. \end{cases}
$$

The body forces and surface tractions satisfy

$$
(3.14) \qquad \boldsymbol{f}_0 \in W^{1,2}(0,T;H), \qquad \boldsymbol{f}_2 \in W^{1,2}(0,T;L^2(\Gamma_2)^d),
$$

and the initial displacement

$$
(3.15) \qquad \boldsymbol{u}_0 \in V.
$$

Since $\operatorname{meas}\Gamma_1 > 0$, Korn's inequality holds and thus there exists a positive constant C_K which depends only on Ω and Γ_1, such that

$$
(3.16) \qquad |\boldsymbol{\varepsilon}(\boldsymbol{u})|_{\mathcal{H}} \geqslant C_K\,|\boldsymbol{u}|_{H_1}\ \ \forall \boldsymbol{u} \in V.
$$

A proof of Korn's inequality may be found in Nečas and Hlaváček [21], p. 79. For $\boldsymbol{u},\boldsymbol{v} \in V$ let

$$
(3.17) \qquad (\boldsymbol{u},\boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad |\boldsymbol{u}|_V = (\boldsymbol{u},\boldsymbol{u})_V^{1/2}.
$$

Using now (3.12) and (3.16) we obtain that $(\cdot,\cdot)_V$ is an inner product on V and, moreover, $|\cdot|_V$ and $|\cdot|_{H_1}$ are equivalent norms on V. Therefore, (3.10) implies that $(V,|\cdot|_V)$ is a real Hilbert space.

Next, using Riesz's representation theorem, we denote by $\boldsymbol{f}(t)$ the element of V given by

$$
(3.18) \qquad (\boldsymbol{f}(t),\boldsymbol{v})_V = (\boldsymbol{f}_0(t),\boldsymbol{v})_H + (\boldsymbol{f}_2(t),\boldsymbol{v})_{L^2(\Gamma_2)^d}\ \ \forall \boldsymbol{v} \in V,\ t \in [0,T],
$$

and we note that conditions (3.14) imply

$$
(3.19) \qquad \boldsymbol{f} \in W^{1,2}(0,T;V).
$$

Finally, in the study of the elastic problem P_2 we use an additional assumption on the initial displacement $\boldsymbol{u}_0 \in V$, that is

$$
(3.20) \qquad (\boldsymbol{u}_0,\boldsymbol{v})_V + j(\boldsymbol{v}) \geqslant (\boldsymbol{f}(0),\boldsymbol{v})_V\ \ \forall \boldsymbol{v} \in V.
$$

Let $t \in [0,T]$. Using (2.1)–(2.3), it is straightforward to show that if $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ are two regular functions satisfying (3.1)–(3.4), then $\boldsymbol{u}(t) \in V$, $\boldsymbol{\sigma}(t) \in \mathcal{H}_1$ and

$$(3.21)\qquad \big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\big)_{\mathcal{H}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \ \ \forall \boldsymbol{v} \in V.$$

Keeping in mind (3.21), we obtain the following variational formulations of the mechanical problems P_1 and P_2, denoted P_1' and P_2', respectively.

Problem P_1'. *Find a displacement field* $\boldsymbol{u}\colon \Omega \times [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma}\colon \Omega \times [0,T] \to \mathcal{H}_1$ *such that*

$$(3.22)\qquad \boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) \ \ \forall t \in [0,T],$$

$$(3.23)\qquad \big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\big)_{\mathcal{H}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V, \ \ t \in [0,T],$$

$$(3.24)\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

Problem P_2'. *Find a displacement field* $\boldsymbol{u}\colon \Omega \times [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma}\colon \Omega \times [0,T] \to \mathcal{H}_1$ *such that*

$$(3.25)\qquad \boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \ \ \forall t \in [0,T],$$

$$(3.26)\qquad \big(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\big)_{\mathcal{H}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V, \quad \text{a.e. } t \in (0,T),$$

$$(3.27)\qquad \boldsymbol{u}(0) = \boldsymbol{u}_0.$$

The well-posedness of the variational problems P_1' and P_2' is discussed in the next section, where existence and uniqueness results in the study of these problems are established.

4. Existence and uniqueness results

The unique solvability of problems P_1' and P_2' follows from the followings results.

Theorem 4.1. *Assume that* (3.10)–(3.15) *hold. Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *of the problem* P_1'. *Moreover, the solution satisfies*

$$(4.1)\qquad \boldsymbol{u} \in W^{2,2}(0,T;V), \quad \boldsymbol{\sigma} \in W^{1,2}(0,T;\mathcal{H}_1).$$

Theorem 4.2. *Assume that* (3.10)–(3.12), (3.14), (3.15) *and* (3.20) *hold. Then there exists a unique solution* $\{\boldsymbol{u},\boldsymbol{\sigma}\}$ *of the problem* P_2'. *Moreover, the solution satisfies*

$$\boldsymbol{u}\in W^{1,2}(0,T;V),\quad \boldsymbol{\sigma}\in W^{1,2}(0,T;\mathcal{H}_1). \tag{4.2}$$

We conclude that, under the assumptions (3.10)–(3.15), the viscoelastic contact problem P_1 has a unique weak solution which satisfies (4.1), and under the assumptions (3.10)–(3.12), (3.14), (3.15) and (3.20), the elastic contact problem P_2 has a unique weak solution which satisfies (4.2).

The proof of Theorem 4.1 is based on fixed point arguments, similar to those used in [24], [28], and is carried out in several steps. We suppose in the sequel that (3.10)–(3.15) hold and everywhere in this section C will denote a positive constant whose value may change from line to line. In the first step, we assume that the elastic part of the stress is given and solve the corresponding variational problem for the velocity field. More precisely, we have the following result.

Lemma 4.3. *For all* $\boldsymbol{\eta}\in W^{1,2}(0,T;\mathcal{H})$ *there exists a unique solution* $\boldsymbol{v}_\eta\in W^{1,2}(0,T;V)$ *such that*

$$\begin{aligned}\big(\mathcal{B}\,\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t)),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t))\big)_{\mathcal{H}}+\big(\boldsymbol{\eta}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t))\big)_{\mathcal{H}}+j(\boldsymbol{v})-j(\boldsymbol{v}_\eta(t))&\\ \geqslant(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{v}_\eta(t))_V\ \ \forall\boldsymbol{v}\in V,\ t\in[0,T].&\end{aligned}\tag{4.3}$$

Proof. Let $\boldsymbol{\eta}\in W^{1,2}(0,T;\mathcal{H})$ and $t\in[0,T]$. It follows from classical results for elliptic variational inequalities (see e.g. [5]) that there exists a unique element $\boldsymbol{v}_\eta(t)\in V$ such that

$$\begin{aligned}b(\boldsymbol{v}_\eta(t),\boldsymbol{v}-\boldsymbol{v}_\eta(t))+\big(\boldsymbol{\eta}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{v}_\eta(t))\big)_{\mathcal{H}}+j(\boldsymbol{v})-j(\boldsymbol{v}_\eta(t))&\\ \geqslant(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{v}_\eta(t))_V\ \ \forall\boldsymbol{v}\in V,&\end{aligned}\tag{4.4}$$

where $b\colon V\times V\to\mathbb{R}$ is the bilinear form on $V\times V$ given by

$$b(\boldsymbol{u},\boldsymbol{v})=(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}\ \ \forall\boldsymbol{u},\boldsymbol{v}\in V.\tag{4.5}$$

Since from (4.4) and (4.5) we obtain (4.3), it remains to prove the regularity $\boldsymbol{v}_\eta\in W^{1,2}(0,T;V)$ of the solution. Let $t_1,t_2\in[0,T]$. For the sake of simplicity in writing, we denote $\boldsymbol{v}_\eta(t_i)=\boldsymbol{v}_i$, $\boldsymbol{\eta}(t_i)=\boldsymbol{\eta}_i$, $\boldsymbol{f}(t_i)=\boldsymbol{f}_i$, for $i=1,2$. Using (4.3), we derive the relation

$$\begin{aligned}(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{v}_1)-\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{v}_2),\boldsymbol{\varepsilon}(\boldsymbol{v}_1)-\boldsymbol{\varepsilon}(\boldsymbol{v}_2))_{\mathcal{H}}\leqslant(\boldsymbol{f}_1-\boldsymbol{f}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_V&\\ +(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2,\boldsymbol{\varepsilon}(\boldsymbol{v}_1)-\boldsymbol{\varepsilon}(\boldsymbol{v}_2))_{\mathcal{H}}.&\end{aligned}$$

Then we use the assumptions (3.13) and (3.16) to derive

$$|\boldsymbol{v}_1 - \boldsymbol{v}_2|_V \leqslant C(|\boldsymbol{f}_1 - \boldsymbol{f}_2|_V + |\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2|_{\mathcal{H}}). \tag{4.6}$$

Keeping in mind the regularities $\boldsymbol{f} \in W^{1,2}(0,T;V)$ and $\boldsymbol{\eta} \in W^{1,2}(0,T;\mathcal{H})$, we deduce from (4.6) that $\boldsymbol{v}_\eta \in W^{1,2}(0,T;V)$. □

Consider now the closed subset of $W^{1,2}(0,T;\mathcal{H})$ defined by

$$\mathcal{W} = \{\boldsymbol{\eta} \in W^{1,2}(0,T;\mathcal{H}) \mid \boldsymbol{\eta}(0) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_0)\}.$$

Lemma 4.3, (3.12) and (3.15) allow us to consider the operator $\Lambda\colon \mathcal{W} \to \mathcal{W}$ defined by

$$\Lambda\boldsymbol{\eta}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \tag{4.7}$$

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)\,\mathrm{d}s + \boldsymbol{u}_0 \tag{4.8}$$

for all $t \in [0,T]$ where, for every $\boldsymbol{\eta} \in \mathcal{W}$, $\boldsymbol{v}_\eta \in W^{1,2}(0,T;V)$ denotes the solution of the variational inequality (4.3).

We have the following result.

Lemma 4.4. *The operator Λ has a unique fixed point $\boldsymbol{\eta}^* \in \mathcal{W}$.*

Proof. Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in \mathcal{W}$ and let $t \in [0,T]$. From the definitions (4.7), (4.8) and (3.12) we have

$$|\Lambda\,\boldsymbol{\eta}_1(t) - \Lambda\,\boldsymbol{\eta}_2(t)|_{\mathcal{H}} \leqslant C\int_0^t |\boldsymbol{v}_{\eta_1}(s) - \boldsymbol{v}_{\eta_2}(s)|_V\,\mathrm{d}s, \tag{4.9}$$

$$\left|\frac{\mathrm{d}}{\mathrm{dt}}(\Lambda\,\boldsymbol{\eta}_1(t) - \Lambda\,\boldsymbol{\eta}_2(t))\right|_{\mathcal{H}} \leqslant C|\boldsymbol{v}_{\eta_1}(t) - \boldsymbol{v}_{\eta_2}(t)|_V. \tag{4.10}$$

An argument similar to that in the proof of (4.6) shows that

$$|\boldsymbol{v}_{\eta_1}(t) - \boldsymbol{v}_{\eta_2}(t)|_V \leqslant C|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)|_{\mathcal{H}} \leqslant C\int_0^t |\dot{\boldsymbol{\eta}}_1(s) - \dot{\boldsymbol{\eta}}_2(s)|_{\mathcal{H}}\,\mathrm{d}s. \tag{4.11}$$

Combining (4.9)–(4.11) we find

$$|\Lambda\,\boldsymbol{\eta}_1(t) - \Lambda\,\boldsymbol{\eta}_2(t)|^2_{\mathcal{H}} \leqslant C\int_0^t |\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)|^2_{\mathcal{H}}\,\mathrm{d}s,$$

$$\left|\frac{\mathrm{d}}{\mathrm{dt}}(\Lambda\,\boldsymbol{\eta}_1(t) - \Lambda\,\boldsymbol{\eta}_2(t))\right|^2_{\mathcal{H}} \leqslant C\int_0^t |\dot{\boldsymbol{\eta}}_1(s) - \dot{\boldsymbol{\eta}}_2(s)|^2_{\mathcal{H}}\,\mathrm{d}s.$$

Iterating the last two inequalities n times we infer

$$|\Lambda^n \boldsymbol{\eta}_1 - \Lambda^n \boldsymbol{\eta}_2|^2_{W^{1,2}(0,T;\mathcal{H})} \leqslant \frac{C^n\, T^n}{n!} |\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2|^2_{W^{1,2}(0,T;\mathcal{H})}.$$

Since $\lim\limits_{n\to\infty} C^n T^n/n! = 0$, the previous inequality implies that for n large enough, the power Λ^n of Λ is a contraction in $\mathcal{W}$. It follows now from Banach's fixed point theorem that there exists a unique element $\boldsymbol{\eta}^* \in \mathcal{W}$ such that $\Lambda^n \boldsymbol{\eta}^* = \boldsymbol{\eta}^*$. Moreover, since $\Lambda^n(\Lambda\boldsymbol{\eta}^*) = \Lambda(\Lambda^n\boldsymbol{\eta}^*) = \Lambda\boldsymbol{\eta}^*$, we deduce that $\Lambda\boldsymbol{\eta}^*$ is also a fixed point ot the operator Λ^n. We conclude by the uniqueness of the fixed point that $\Lambda\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$, which shows that $\boldsymbol{\eta}^*$ is a fixed point of Λ. The uniqueness of the fixed point of the operator Λ results straigtforward from the uniqness of the fixed point of the operator Λ^n. □

We now have all the ingredients to prove Theorem 4.1.

P r o o f of Theorem 4.1. *Existence.* Let $\boldsymbol{\eta}^* \in \mathcal{W}$ be the fixed point of Λ and let $\boldsymbol{u}_{\eta^*}$ be the function defined by (4.8) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. We define a function $\boldsymbol{\sigma}_{\eta^*}$ by

$$\boldsymbol{\sigma}_{\eta^*}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta^*}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\eta^*}(t)) \ \ \forall t \in [0,T]. \tag{4.12}$$

Since $\boldsymbol{\eta}^*(t) = \Lambda\, \boldsymbol{\eta}^*(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_{\eta^*}(t))$ and $\boldsymbol{v}_{\eta^*}(t) = \dot{\boldsymbol{u}}_{\eta^*}(t)$ for all $t \in [0,T]$, it follows from (4.3) and (4.12) that

$$(\boldsymbol{\sigma}_{\eta^*}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\eta^*}(t))_{\mathcal{H}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}_{\eta^*}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{\eta^*}(t))_V \quad \forall \boldsymbol{v} \in V,\ t \in [0,T]. \tag{4.13}$$

Taking $\boldsymbol{v} = \dot{\boldsymbol{u}}_{\eta^*}(t) \pm \boldsymbol{\varphi}$ in (4.13) where $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^d$ and using (3.18) we find

$$\operatorname{Div} \boldsymbol{\sigma}_{\eta^*}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \ \ \forall t \in [0,T]. \tag{4.14}$$

It follows from Lemma 4.3, (4.8) and (3.15) that $\boldsymbol{u}_{\eta^*} \in W^{2,2}(0,T;V)$. Moreover, (4.12), (3.12), (3.13), (4.14) and (3.14) yield $\boldsymbol{\sigma}_{\eta^*} \in W^{1,2}(0,T;\mathcal{H}_1)$. Keeping in mind (4.12), (4.13) and (4.8), we deduce that $\{\boldsymbol{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}\}$ is a solution to problem P_1' which satisfies (4.1).

Uniqueness. To prove the uniqueness part, let $\{\boldsymbol{u}_i, \boldsymbol{\sigma}_i\}$ be two solutions of (3.24)–(3.26) with regularity (4.1), $i = 1,2$. We have

$$\big(\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_i(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_i(t))\big)_{\mathcal{H}} + \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_i(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_i(t))\big)_{\mathcal{H}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}_i(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_i(t))_V \ \ \forall \boldsymbol{v} \in V,\ t \in [0,T],$$

which implies

$$\begin{aligned}\big(\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1(t)) - \mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2(t)), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2(t))\big)_{\mathcal{H}} \\ \leqslant \big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_1(t)) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_2(t)), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2(t))\big)_{\mathcal{H}} \ \ \forall t \in [0,T].\end{aligned}$$

Since $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) = \boldsymbol{u}_0$, using (3.17), (3.13) and the previous inequality, we deduce that

$$|\dot{\boldsymbol{u}}_1(t) - \dot{\boldsymbol{u}}_2(t)|_V \leqslant C \int_0^t |\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)|_V \, \mathrm{d}s \ \ \forall t \in [0,T]. \tag{4.15}$$

The uniqueness part in Theorem 4.1 is now a consequence of (4.15), (3.24) and (3.22). □

We turn now to the proof of Theorem 4.2. To this end, we recall the following existence and uniqueness result.

Theorem 4.5. *Let $(V, (\cdot,\cdot)_V)$ be a real Hilbert space and let $j\colon V \to (-\infty, +\infty]$ be a convex lower semicontinuous functional. Assume that $j \not\equiv \infty$, i.e. the effective domain of j, defined by $D(j) = \{\boldsymbol{v} \in V \mid j(\boldsymbol{v}) \leqslant +\infty\}$, is not empty. Let $\boldsymbol{f} \in W^{1,2}(0,T;V)$ and $\boldsymbol{u}_0 \in V$ be such that*

$$\sup_{\boldsymbol{v} \in D(j)} \{(\boldsymbol{f}(0), \boldsymbol{v})_V - (\boldsymbol{u}_0, \boldsymbol{v})_V - j(\boldsymbol{v})\} < +\infty. \tag{4.16}$$

Then there exists a unique element $\boldsymbol{u} \in W^{1,2}(0,T;V)$ which satisfies

$$(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \tag{4.17}$$

for all $\boldsymbol{v} \in V$, a.e. $t \in (0,T)$, and

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{4.18}$$

Theorem 4.5 has been proved in [6], p. 117, using arguments of the theory of evolution equations with maximal monotone operators. A version of this theorem is considered in [11]. There, the proof is based on a time-discretization method. A generalization of Theorem 4.3 in the case when j depends on the solution, i.e. in the case when $j = j(\boldsymbol{u}, \boldsymbol{v})$ has been established recently in [20].

We use Theorem 4.5 to prove Theorem 4.2.

P r o o f of Theorem 4.2. Note that (3.20) implies (4.16) and the assumptions (3.10)–(3.12), (3.14) and (3.15) allow us to use Theorem 4.3. Using (4.17), (4.18) and (3.17), we deduce the existence of a unique function $\boldsymbol{u} \in W^{1,2}(0,T;V)$ which satisfies (3.27) and

$$(4.19)\qquad \big(\mathcal{A}\,\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\big)_{\mathcal{H}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V,\ \text{a.e. } t \in (0,T).$$

Let $\boldsymbol{\sigma}$ be given by (3.25). Using (4.19) we obtain (3.26) and from (3.12) it follows that $\boldsymbol{\sigma} \in W^{1,2}(0,T;\mathcal{H})$. We use now in (3.26) an argument similar to that used in the proof of Theorem 4.1 to obtain $\boldsymbol{\sigma} \in W^{1,2}(0,T;\mathcal{H}_1)$. This concludes the existence part of Theorem 4.2. The uniqueness part results from the uniqueness of the element $\boldsymbol{u} \in W^{1,2}(0,T;V)$ which solves (4.19), guaranteed by Theorem 4.5. □

5. A convergence result

In this section we investigate the behavior of the solution to problem P_1' when the viscosity operator converges to zero. To this end, we assume in the sequel that (3.10)–(3.15) and (3.20) hold and $\theta > 0$. We replace in (3.22) the operator $\mathcal{B}$ by $\theta\mathcal{B}$ to obtain the following viscoelastic problem.

Problem $\mathbf{P}_{1\theta}'$. *Find a displacement field $\boldsymbol{u}_\theta\colon \Omega \times [0,T] \longrightarrow V$ and a stress field $\boldsymbol{\sigma}_\theta\colon \Omega \times [0,T] \longrightarrow \mathcal{H}_1$ such that*

$$(5.1)\qquad \boldsymbol{\sigma}_\theta(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\theta) + \theta\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta) \quad \forall t \in [0,T],$$

$$(5.2)\qquad \big(\boldsymbol{\sigma}_\theta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta(t))\big)_{\mathcal{H}} + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}_\theta(t)) \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_\theta(t))_V \quad \forall \boldsymbol{v} \in V,\ t \in [0,T],$$

$$(5.3)\qquad \boldsymbol{u}_\theta(0) = \boldsymbol{u}_0.$$

By virtue of Theorem 4.1, it follows that problem $\mathrm{P}_{1\theta}'$ has a unique solution $\{\boldsymbol{u}_\theta, \boldsymbol{\sigma}_\theta\}$ which satisfies (4.1). We denote in the sequel by $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ the solution of the elastic problem P_2' given by Theorem 4.2. Here and below, C denotes various positive constants which are independent of θ.

The main result of this section is the following.

Theorem 5.1. *The following estimates hold:*

$$(5.4)\qquad |\boldsymbol{u}_\theta(t) - \boldsymbol{u}(t)|_V \leqslant C\sqrt{\theta}\,|\dot{\boldsymbol{u}}|_{L^2(0,T;V)} \quad \forall t \in [0,T],$$

$$(5.5)\qquad |\boldsymbol{\sigma}_\theta(t) - \boldsymbol{\sigma}(t)|_{\mathcal{H}_1} \leqslant C\sqrt{\theta}\,(|\dot{\boldsymbol{u}}|_{L^2(0,T;V)} + \sqrt{\theta}|\dot{\boldsymbol{u}}(t)|_V), \quad \text{a.e. on } (0,T).$$

P r o o f. Let $t \in [0,T]$. To simplify the notation, we will not indicate explicitly the dependence of various functions on time. Using (5.1) and (5.2), we obtain

$$\text{(5.6)}\qquad \theta(\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta))_{\mathcal{H}}+(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_\theta),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta))_{\mathcal{H}}+j(\dot{\boldsymbol{u}})-j(\dot{\boldsymbol{u}}_\theta)$$
$$\geqslant(\boldsymbol{f},\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_\theta)_V\ \text{ a.e. on } (0,T),$$

and, using (3.25) and (3.26), we deduce

$$\text{(5.7)}\qquad (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}}+j(\dot{\boldsymbol{u}}_\theta)-j(\dot{\boldsymbol{u}})\geqslant(\boldsymbol{f},\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_\theta)_V\ \text{ a.e. on } (0,T).$$

It follows from (5.6), (5.7) and (3.17) that

$$\theta(\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta))_{\mathcal{H}}+(\boldsymbol{u}_\theta-\boldsymbol{u},\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_\theta)_V\geqslant 0\ \text{ a.e. on } (0,T),$$

which implies

$$\text{(5.8)}\qquad \theta(\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta)-\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}))_{\mathcal{H}}+(\boldsymbol{u}_\theta-\boldsymbol{u},\dot{\boldsymbol{u}}_\theta-\dot{\boldsymbol{u}})_V$$
$$\leqslant\theta(\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta))_{\mathcal{H}}\ \text{ a.e. on } (0,T).$$

Integrating (5.8) over $[0,t]$ and using (5.3), (3.27) and (3.13) yields

$$\text{(5.9)}\qquad \theta m_{\mathcal{B}}\int_0^t|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})|_{\mathcal{H}}^2\,\mathrm{d}s+\frac{1}{2}|\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V^2$$
$$\leqslant\theta\int_0^t|\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})|_{\mathcal{H}}|\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})|_{\mathcal{H}}\,\mathrm{d}s.$$

Using now the inequality

$$ab\leqslant\frac{a^2}{2m_{\mathcal{B}}}+\frac{m_{\mathcal{B}}b^2}{2},$$

from (5.9) we find

$$\text{(5.10)}\qquad \theta m_{\mathcal{B}}\int_0^t|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta)-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})|_{\mathcal{H}}^2\,\mathrm{d}s+|\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V^2\leqslant\frac{\theta}{m_{\mathcal{B}}}\int_0^t|\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})|_{\mathcal{H}}^2\,\mathrm{d}s.$$

The last inequality implies

$$\text{(5.11)}\qquad |\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V\leqslant\frac{\sqrt{\theta}}{\sqrt{m_{\mathcal{B}}}}|\mathcal{B}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})|_{L^2(0,T;\mathcal{H})}.$$

The inequality (5.4) is now a consequence of (5.11).

To prove (5.5), we note that (5.1) and (3.25) imply that

$$|\boldsymbol{\sigma}_\theta(t)-\boldsymbol{\sigma}(t)|_{\mathcal{H}} \leqslant C\theta|\dot{\boldsymbol{u}}_\theta(t)|_V + C|\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V,$$

and, keeping in mind that

$$\operatorname{Div}\boldsymbol{\sigma}_\theta(t)=\operatorname{Div}\boldsymbol{\sigma}(t)=-\boldsymbol{f}_0(t),$$

we find

$$|\boldsymbol{\sigma}_\theta(t)-\boldsymbol{\sigma}(t)|_{\mathcal{H}_1} \leqslant C(\theta|\dot{\boldsymbol{u}}_\theta(t)|_V + |\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V). \tag{5.12}$$

Using now (5.8) and (3.13) we have

$$\theta|\dot{\boldsymbol{u}}_\theta(t)-\dot{\boldsymbol{u}}(t)|_V \leqslant C(|\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V + \theta|\dot{\boldsymbol{u}}(t)|_V)$$

and, therefore, we obtain

$$\theta|\dot{\boldsymbol{u}}_\theta(t)|_V \leqslant C(|\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V + \theta|\dot{\boldsymbol{u}}(t)|_V). \tag{5.13}$$

Combining (5.12) and (5.13) we infer

$$|\boldsymbol{\sigma}_\theta(t)-\boldsymbol{\sigma}(t)|_{\mathcal{H}_1} \leqslant C(|\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V + \theta|\dot{\boldsymbol{u}}(t)|_V). \tag{5.14}$$

The inequality (5.5) is now a consequence of (5.14) and (5.4). □

As a consequence of the estimates (5.4)–(5.5), we obtain the convergence of the solution $\{\boldsymbol{u}_\theta,\boldsymbol{\sigma}_\theta\}$ of problem $\mathrm{P}'_{1\theta}$ to the solution $\{\boldsymbol{u},\boldsymbol{\sigma}\}$ of problem P'_2:

Corollary 5.2. *The following convergences hold:*

$$\max_{t\in[0,T]}|\boldsymbol{u}_\theta(t)-\boldsymbol{u}(t)|_V \to 0, \tag{5.15}$$

$$|\boldsymbol{\sigma}_\theta-\boldsymbol{\sigma}|_{L^2(0,T;\mathcal{H}_1)} \to 0, \tag{5.16}$$

as $\theta\to 0+$.

We conclude, by Corollary 5.2, that the weak solution of the frictional elastic problem P_2 may be approached by the weak solution of the frictional viscoelastic problem P_1 provided the coefficient of viscosity is small enough. In addition to the mathematical interest in the convergence result (5.15), (5.16), it is of importance from the mechanical point of view, as it indicates that the elasticity with friction may be considered a limit case of viscoelasticity with friction.

6. Examples of subdifferential conditions with friction

In this section we present examples of contact and dry friction laws which lead to an inequality of the form (3.4) and such that (3.10) and (3.11) hold. We conclude, by Theorems 4.1 and 4.2, that the relevant boundary value problem for each example has a unique weak solution. Moreover, we note that Theorem 5.1 as well as the convergence result presented in Corollary 5.2 hold in all the examples below.

Example 6.1. *Bilateral contact with Tresca's friction law.* This contact condition can be found in [9], [22], and more recently in [1], [2] (see references therein for further details). It is in the form of the following boundary condition:

$$\begin{cases} u_\nu = 0, \quad |\boldsymbol{\sigma}_\tau| \leqslant g, \\ |\boldsymbol{\sigma}_\tau| < g \Longrightarrow \dot{\boldsymbol{u}}_\tau = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau| = g \Longrightarrow \dot{\boldsymbol{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau, \ \lambda \geqslant 0 \end{cases} \quad \text{on } \Gamma_3 \times (0,T). \tag{6.1}$$

Here $g \geqslant 0$ represents the friction bound, i.e., the magnitude of the limiting friction traction at which slip begins. The contact is assumed to be bilateral, i.e., there is no loss of contact during the process.

The set of admissible test functions U consists of those elements of H_1 whose normal component vanishes on Γ_3. We deduce from (3.8) that

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \ \text{on } \Gamma_1, \quad u_\nu = 0 \ \text{on } \Gamma_3\}.$$

Moreover, it is straightforward to show that if $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ is a pair of regular functions satisfying (6.1) then

$$\boldsymbol{\sigma\nu}(\boldsymbol{v} - \dot{\boldsymbol{u}}) \geqslant g|\dot{\boldsymbol{u}}_\tau| - g|\boldsymbol{v}_\tau| \ \ \forall \boldsymbol{v} \in U,$$

a.e. on $\Gamma_3 \times (0,T)$, and so, (3.4) holds with the choice $\varphi(\boldsymbol{v}) = g|\boldsymbol{v}_\tau|$. If $g \in L^\infty(\Gamma_3)$, then we obtain from (3.9) that

$$j(\boldsymbol{v}) = \int_{\Gamma_3} g|\boldsymbol{v}_\tau| \, \mathrm{d}a \ \ \forall \boldsymbol{v} \in V.$$

In this case assumptions (3.10) and (3.11) hold. We conclude that our abstract results can be applied to the viscoelastic problem (3.1)–(3.3), (3.5), (3.6), (6.1) as well as to the elastic problem (3.1)–(3.3), (3.6), (3.7), (6.1) which have, both of them, a unique weak solution. Moreover, the solution of the viscoelastic problem converges to the solution of the elastic problem when the viscosity converges to zero, as follows from Theorem 5.1.

E x a m p l e 6.2. *Bilateral contact with viscoelastic friction condition.* We consider problems with the boundary conditions

$$u_\nu = 0, \quad \boldsymbol{\sigma}_\tau = -\mu|\dot{\boldsymbol{u}}_\tau|^{p-1}\dot{\boldsymbol{u}}_\tau \quad \text{on } \Gamma_3 \times (0,T), \tag{6.2}$$

where $\mu \geqslant 0$ is the coefficient of friction and $0 < p \leqslant 1$. Here, the tangential shear is proportional to the power p of the tangential speed, which is the case when the contact surface is lubricated with a thin layer of non-Newtonian fluid.

It is straightforward to show that if $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ is a pair of regular functions satisfying (6.2), then (3.4) holds with

$$U = \{\boldsymbol{v} \in H_1 \mid u_\nu = 0 \text{ on } \Gamma_3\},$$

and

$$\varphi(\boldsymbol{v}) = \frac{\mu}{p+1}\,|\boldsymbol{v}_\tau|^{p+1}.$$

Using (3.8) we obtain $V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\ u_\nu = 0 \text{ on } \Gamma_3\}$. Assuming $\mu \in L^\infty(\Gamma_3)$, we deduce from (3.9)

$$j(\boldsymbol{v}) = \frac{1}{p+1}\int_{\Gamma_3} \mu|\boldsymbol{v}_\tau|^{p+1}\,\mathrm{d}a \quad \forall \boldsymbol{v} \in V.$$

Since in this case the assumptions (3.10) and (3.11) are satisfied, we may apply Theorem 4.1 to the mechanical problem (3.1)–(3.3), (3.5), (3.6), (6.2) and conclude that it has a unique weak solution. Also, we may apply Theorem 4.2 to the mechanical problem (3.1)–(3.3), (3.6), (3.7), (6.2) and conclude that it has a unique weak solution.

E x a m p l e 6.3. *Viscoelastic contact with Tresca's friction law.* We consider the contact problem with the boundary conditions

$$\begin{cases} -\sigma_\nu = k|\dot{u}_\nu|^{q-1}\dot{u}_\nu, \quad |\boldsymbol{\sigma}_\tau| \leqslant g, \\ |\boldsymbol{\sigma}_\tau| < g \Longrightarrow \dot{\boldsymbol{u}}_\tau = 0, \\ |\boldsymbol{\sigma}_\tau| = g \Longrightarrow \dot{\boldsymbol{u}}_\tau = -\lambda\boldsymbol{\sigma}_\tau, \quad \lambda \geqslant 0 \end{cases} \quad \text{on } \Gamma_3 \times (0,T). \tag{6.3}$$

Here $g, k \geqslant 0$ and the normal contact stress depends on a power of the normal speed (this condition may describe the motion of a body, say a wheel, on a fine granular material, say the sand on a beach, see e.g. [27], [28]. We have $U = H_1$, $0 < q \leqslant 1$, $V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}$ and

$$\varphi(\boldsymbol{v}) = \frac{k}{q+1}\,|v_\nu|^{q+1} + g|\boldsymbol{v}_\tau|.$$

If $g \in L^\infty(\Gamma_3)$ and $k \in L^\infty(\Gamma_3)$ then we deduce from (3.9) that

$$j(\boldsymbol{v}) = \int_{\Gamma_3} \left(\frac{1}{q+1}\, k|v_\nu|^{q+1} + g|\boldsymbol{v}_\tau| \right) \mathrm{d}a \quad \forall \boldsymbol{v} \in V.$$

Assumptions (3.10) and (3.11) hold and we conclude, by our abstract existence and uniqueness results, the existence of the unique weak solution of problems (3.1)–(3.3), (3.5), (3.6), (6.3) and (3.1)–(3.3), (3.6), (3.7), (6.3), respectively.

E x a m p l e 6.4. *Viscoelastic contact with friction.* Here, the body is moving on sand or a granular material and the normal stress is proportional to a power of the normal speed, while the tangential shear is proportional to a power of the tangential speed. We choose the following boundary conditions:

$$-\sigma_\nu = k|\dot{u}_\nu|^{q-1}\dot{u}_\nu, \quad \boldsymbol{\sigma}_\tau = -\mu|\dot{\boldsymbol{u}}_\tau|^{p-1}\dot{\boldsymbol{u}}_\tau \quad \text{on } \Gamma_3 \times (0,T). \tag{6.4}$$

Here $\mu \in L^\infty(\Gamma_3)$ and $k \in L^\infty(\Gamma_3)$ are positive functions and $0 < p,\ q \leqslant 1$. We choose $U = H_1$, $V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}$ and

$$\varphi(\boldsymbol{v}) = \frac{k}{q+1}\,|v_\nu|^{q+1} + \frac{\mu}{p+1}\,|\boldsymbol{v}_\tau|^{p+1}.$$

We may apply Theorem 4.1 to the viscoelastic problem (3.1)–(3.3), (3.5), (3.6), (6.4), since assumptions (3.10) and (3.11) are satisfied. Thus, the problem has a unique weak solution. Also, we may apply Theorem 4.2 to the viscoelastic problem (3.1)–(3.3), (3.6), (3.7), (6.4) and conclude that it has a unique solution, too.

E x a m p l e 6.5. *Contact with normal damped response and Tresca's friction law.* In this problem the contact condition describes the normal damped response of a thin lubricant layer, and has recently been considered in [24], [25]. The contact pressure depends on the velocity, but only under compression, and the contact conditions are

$$\begin{cases} -\sigma_\nu = k(\dot{u}_\nu)_+ + p_0, \quad |\boldsymbol{\sigma}_\tau| \leqslant g, \\ |\boldsymbol{\sigma}_\tau| < g \Longrightarrow \dot{\boldsymbol{u}}_\tau = 0, \qquad\qquad \text{on } \Gamma_3 \times (0,T). \\ |\boldsymbol{\sigma}_\tau| = g \Longrightarrow \dot{\boldsymbol{u}}_\tau = -\lambda\boldsymbol{\sigma}_\tau, \quad \lambda \geqslant 0 \end{cases} \tag{6.5}$$

Here $g \in L^\infty(\Gamma_3)$ and $k \in L^\infty(\Gamma_3)$ are given positive functions, $p_0 \in L^\infty(\Gamma_3)$ and $r_+ = \max\{r, 0\}$. We choose $U = H_1$, $V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}$ and

$$\varphi(\boldsymbol{v}) = \frac{k}{2}((v_\nu)_+)^2 + p_0 v_\nu + g|\boldsymbol{v}_\tau|.$$

The viscoelastic problem (3.1)–(3.3), (3.5), (3.6), (6.5) has a unique weak solution by Theorem 4.1, since assumptions (3.10) and (3.11) are satisfied. We also conclude by Theorem 4.2 that the elastic problem (3.1)–(3.3), (3.6), (3.7), (6.5) has a unique weak solution.

E x a m p l e 6.6. *Contact with normal damped response and viscoelastic friction law.* This is a version of Examples 6.2 and 6.5 above and the contact conditions are

$$-\sigma_\nu = k(\dot{u}_\nu)_+ + p_0, \quad \boldsymbol{\sigma}_\tau = -\mu|\dot{\boldsymbol{u}}_\tau|^{p-1}\dot{\boldsymbol{u}}_\tau \quad \text{on } \Gamma_3 \times (0,T). \tag{6.6}$$

Here $k \in L^\infty(\Gamma_3)$ and $\mu \in L^\infty(\Gamma_3)$ are positive functions, $p_0 \in L^\infty(\Gamma_3)$ and $0 < p \leqslant 1$. We choose $U = H_1$, $V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}$ and

$$\varphi(\boldsymbol{v}) = \frac{k}{2}((v_\nu)_+)^2 + p_0 v_\nu + \frac{\mu}{p+1}\,|\boldsymbol{v}_\tau|^{p+1}.$$

Since assumptions (3.10) and (3.11) hold, Theorem 4.1 guarantees the existence of the unique solution of problem (3.1)–(3.3), (3.5), (3.6) (6.6), while Theorem 4.2 guarantees the existence of a unique solution of problem (3.1)–(3.3), (3.6), (3.7), (6.6).

R e m a r k 6.7. In the examples above the normal pressure and tangential stress are related to powers of the normal and tangential speeds. This is dictated by the structure of the functional φ which depends only on the surface velocity. An important extension of the theorems would allow for the additional dependence on the displacements, such as in the normal compliance contact condition.

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