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ERROR ESTIMATES FOR BARYCENTRIC FINITE VOLUMES COMBINED WITH NONCONFORMING FINITE ELEMENTS APPLIED TO NONLINEAR CONVECTION-DIFFUSION PROBLEMS*

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Abstract. The subject of the paper is the derivation of error estimates for the combined finite volume-finite element method used for the numerical solution of nonstationary nonlinear convection-diffusion problems. Here we analyze the combination of barycentric finite volumes associated with sides of triangulation with the piecewise linear nonconforming Crouzeix-Raviart finite elements. Under some assumptions on the regularity of the exact solution, the $L^2(L^2)$ and $L^2(H^1)$ error estimates are established. At the end of the paper, some computational results are presented demonstrating the application of the method to the solution of viscous gas flow.

Keywords: nonlinear convection-diffusion problem, compressible Navier-Stokes equations, cascade flow, barycentric finite volumes, Crouzeix-Raviart nonconforming piecewise linear finite elements, monotone finite volume scheme, discrete maximum principle, a priori estimates, error estimates

MSC 2000: 65M12, 65M50, 35K60, 76M10, 76M25

1. Introduction

Many processes in science and technology are described by convection-diffusion equations. We can mention, e.g., processes of fluid dynamics, hydrology and environmental protection. There is an extensive literature on the numerical solution of convection-diffusion problems. Let us mention, e.g., the papers [1], [25], [26], [37],

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[40], [46], [48], [49] and the monographs [36], [39] (and the references therein), devoted mainly to linear problems. The main difficulty which must be overcome is the precise resolution of the so-called boundary layers. If the equation under consideration represents a nonlinear conservation law with a small dissipation, then beside boundary layers also shock waves appear (slightly smeared due to dissipation). This is particularly the case for the system describing viscous gas flow.

In [6], [8], [13], [14], [15] we developed numerical methods for the solution of the high-speed viscous compressible flow in domains with complex geometry. These methods are based on the combination of a finite volume scheme for the discretization of inviscid convective terms and the finite element discretization of viscous terms. The finite element method is one of the most powerful tools for solving partial differential equations, particularly of elliptic and parabolic types (cf. [4], [27], [33], [41]). On the other hand, in Computational Fluid Dynamics, especially for convection dominated flows, the upwind finite volume schemes are very popular. (For an extensive treatment of the finite volume methods, we refer the reader to [9]. See also [11] or [30].) In [6], [8], [13], [14], [15], we have developed combined finite volume-finite element methods, which exploit advantages of both the above methods. Numerical experiments proved the efficiency and robustness of these methods with respect to the precise resolution of boundary layers and shock capturing. Since the complete viscous gas flow problem is rather complex, the theoretical analysis of the combined finite volume-finite element methods has been carried out for the case of a simplified scalar nonlinear conservation law equation with a dissipation term, which is the simplest prototype of the compressible Navier-Stokes equation. Papers [16], [17], [18] are concerned with the convergence and error estimates for the method using dual finite volumes over a triangular mesh combined with conforming piecewise linear triangular finite elements.

Another possibility is the combination of the so-called barycentric finite volumes constructed over a triangular grid with the well-known Crouzeix-Raviart nonconforming piecewise linear finite elements used for the numerical solution of incompressible viscous flows ([5], [45]). The upwind version of the Crouzeix-Raviart finite element method was developed and analyzed in [37] for a linear stationary convection-diffusion equation. This was the inspiration for Schieweck and Tobiska who investigated in [40] upwind schemes for steady Navier-Stokes equations. In [2] the convergence analysis of the combined barycentric finite volume-nonconforming finite element method applied to a nonlinear convection-diffusion problem is given. In [6] and [13] this method was applied with success to the numerical solution of a compressible viscous flow. A similar approach was proposed in [3].

Here we will be concerned with the continuation of results from [2]. We will present the analysis of the error estimates of the finite volume-finite element method combining barycentric finite volumes with nonconforming Crouzeix-Raviart finite elements applied to an initial-boundary value problem for a scalar nonlinear conservation law with a diffusion term. The basic tools used in the investigation of error estimates presented here are the discrete maximum principle, a priori error estimates and analysis of the discretization and truncation errors, carried out under some assumptions on the regularity of the exact solution. As a result, error estimates are obtained in discrete analogy of $L^2(L^2)$ and $L^2(H^1)$ norms. At the end we present application of the method analyzed to a technically relevant flow problem.

2. Continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with a Lipschitz-continuous boundary $\partial\Omega$. In the space-time cylinder $Q_{\boldsymbol{T}} = \Omega \times (0, \boldsymbol{T})$ $(0 < \boldsymbol{T} < \infty)$ we consider the following initial-boundary value problem:

Find $u\colon Q_{\boldsymbol{T}} \to \mathbb{R}$, $u = u(x,t)$, $x \in \Omega$, $t \in (0, \boldsymbol{T})$, such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} - \nu \Delta u = g \quad \text{in } Q_{\boldsymbol{T}}, \tag{2.1}$$

$$u|_{\partial\Omega \times (0,\boldsymbol{T})} = 0, \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega, \tag{2.3}$$

where $\nu > 0$ is a given real constant and $f_s\colon \mathbb{R} \to \mathbb{R}$, $s = 1, 2$, $g\colon Q_{\boldsymbol{T}} \to \mathbb{R}$, $u^0\colon \Omega \to \mathbb{R}$ are given functions. Precise assumptions on these functions will be given later.

In what follows we will work with the Lebesgue spaces $L^p(\Omega)$, the Sobolev spaces $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$, the subspace $H_0^1(\Omega) \subset H^1(\Omega)$ of functions with zero traces on $\partial\Omega$ and Bochner spaces $L^q(0, \boldsymbol{T}; X)$, $C([0, \boldsymbol{T}], X)$, where X is a Banach space. For their definitions and properties see, e.g., [34].

We set

$$V = H_0^1(\Omega). \tag{2.4}$$

In the space $H^1(\Omega)$ beside its norm we will often work with the seminorm

$$|u|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2}, \tag{2.5}$$

which is an equivalent norm on V. We can write $|u|_{H^1(\Omega)} = ((u,u))^{1/2}$, where

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \quad u, v \in H^1(\Omega), \tag{2.6}$$

is a scalar product on V. Further we set

$$(u,v)=\int_\Omega uv\,\mathrm{d}x,\quad u,v\in L^2(\Omega). \tag{2.7}$$

We will assume that

$$f_s\in C^2(\mathbb{R}),\quad f_s(0)=0,\quad s=1,2, \tag{2.8}$$

$$g\in C([0,\boldsymbol{T}];W^{1,q}(\Omega))\ \text{ for some }\ q>2, \tag{2.9}$$

$$u^0\in H_0^1(\Omega)\cap C(\overline{\Omega}). \tag{2.10}$$

Now we derive the weak formulation of problem (2.1)–(2.3). Multiplying (2.1) by an arbitrary $v\in V$, integrating over Ω and using Green's theorem, we obtain the identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t)v\,\mathrm{d}x-\int_\Omega\sum_{s=1}^2 f_s(u(t))\frac{\partial v}{\partial x_s}\,\mathrm{d}x+\nu\int_\Omega\nabla u(t)\cdot\nabla v\,\mathrm{d}x&\\ =\int_\Omega g(t)v\,\mathrm{d}x,\ \ \forall\,v\in V,\ \ \forall\,t\in[0,\boldsymbol{T}].&\end{aligned} \tag{2.11}$$

Here, for $t\in[0,\boldsymbol{T}], u(t)$ means the function "$x\in\Omega\mapsto u(t)(x)=u(x,t)$". Let us set

$$b(\varphi,v)=-\int_\Omega\sum_{s=1}^2 f_s(\varphi)\frac{\partial v}{\partial x_s}\,\mathrm{d}x\quad\text{for }\ \varphi\in L^\infty(\Omega),\ \ v\in V. \tag{2.12}$$

Definition 1.

We say that a function u is a *weak solution* of problem (2.1)–(2.3), if it satisfies the conditions

$$u\in L^2(0,\boldsymbol{T};V)\cap L^\infty(Q_{\boldsymbol{T}}), \tag{2.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),v)+b(u(t),v)+\nu(\!(u(t),v)\!)=(g(t),v)\ \ \forall\,v\in V, \tag{2.14}$$

$$\text{in the sense of distributions on }\ (0,\boldsymbol{T}),$$

$$u(0)=u^0. \tag{2.15}$$

It follows from [16] that the solution of problem (2.13)–(2.15) exists and is unique.

3. Discrete problem

By $\mathcal{T}_h$ we will denote a triangulation of Ω with standard properties (see e.g. [4]): $T \in \mathcal{T}_h$ are closed triangles and

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T, \tag{3.1}$$

$$\begin{aligned} &\text{if } T_1, T_2 \in \mathcal{T}_h,\ T_1 \neq T_2, \text{ then } T_1 \cap T_2 = \emptyset, \\ &\text{or } T_1 \cap T_2 \text{ is a common side of } T_1 \text{ and } T_2, \\ &\text{or } T_1 \cap T_2 \text{ is a common vertex of } T_1 \text{ and } T_2. \end{aligned} \tag{3.2}$$

By $\mathbb{S}_h$ we denote the set of all sides of all triangles $T \in \mathcal{T}_h$. We introduce a numbering of triangles $T \in \mathcal{T}_h$ and their sides $S \in \mathbb{S}_h$ in such a way that

$$\mathcal{T}_h = \{T_i;\ i \in I\}, \quad \mathbb{S}_h = \{S_j;\ j \in J\}, \tag{3.3}$$

where I and J are suitable index sets of positives integers. By Q_j we denote the centre of a side $S_j \in \mathbb{S}_h$ and put $\mathcal{P}_h = \{Q_j;\ j \in J\}$. Moreover, we set

$$J^\circ = \{i \in J;\ Q_i \in \Omega\}. \tag{3.4}$$

Sometimes we will use the local notation S_T^i and Q_T^i, $i = 1, 2, 3$, for the sides of a triangle $T \in \mathcal{T}_h$ and their centres, respectively. Then

$$\begin{aligned} \mathcal{S}_h &= \{S_T^i;\ i = 1, 2, 3,\ T \in \mathcal{T}_h\}, \\ \mathcal{P}_h &= \{Q_T^i;\ i = 1, 2, 3,\ T \in \mathcal{T}_h\}. \end{aligned} \tag{3.5}$$

By $h(T)$ and $\theta(T)$ we denote the length of the longest side and the magnitude of the smallest angle, respectively, of the triangle $T \in \mathcal{T}_h$, and put

$$h = \max_{T \in \mathcal{T}_h} h(T), \quad \theta_h = \min_{T \in \mathcal{T}_h} \theta(T). \tag{3.6}$$

Now let us construct the *barycentric mesh* $\mathcal{D}_h = \{D_i;\ i \in J\}$ over the basic mesh $\mathcal{T}_h$. The *barycentric finite volumes* D_i are closed polygons defined in the following way: We join the barycentre of each triangle $T \in \mathcal{T}_h$ with its vertices. Then around each side S_i, $i \in J^\circ$, we obtain a closed quadrilateral D_i containing S_i. If $S_j \subset \partial\Omega$ is a side with vertices P_1, P_2 of a triangle $T \in \mathcal{T}_h$ adjacent to $\partial\Omega$, then by D_j we denote the triangle with the sides S_j and segments connecting the barycentre of T with P_1 and P_2. (See Figs. 1, 2.) Obviously,

$$\overline{\Omega} = \bigcup_{i \in J} D_i. \tag{3.7}$$

If $D_i \neq D_j$ and the set $\partial D_i \cap \partial D_j$ contains more than one point, we call D_i and D_j *neighbours* and set $\Gamma_{ij} = \partial D_i \cap \partial D_j$ (= the common side of D_i and D_j). Further,

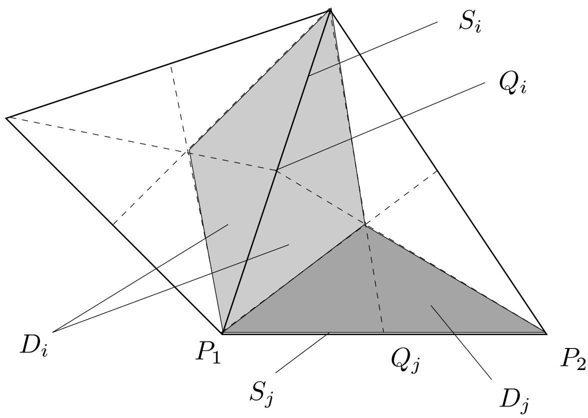

Figure 1. Barycentric finite volume.

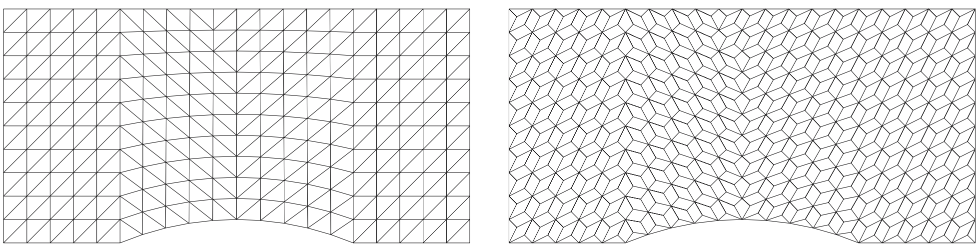

Figure 2. Triangular mesh and associated barycentric finite volume mesh.

we define the set $s(i) = \{j \in J\,;\ D_j$ is a neighbour of $D_i\}$. If $Q_i \in \partial\Omega$ then we set $S(i) = s(i) \cup \{-1\}$ and $\Gamma_{i,-1} = S_i \subset \partial\Omega$, otherwise we put $S(i) = s(i)$. Then we can write

$$\partial D_i = \bigcup_{j \in S(i)} \Gamma_{ij}. \tag{3.8}$$

In the sequel we use the following notation: $|T|$ = area of $T \in \mathcal{T}_h$, $|D_i|$ = area of $D_i \in \mathcal{D}_h$ (i.e., $i \in J$), ℓ_{ij} = length of the segment Γ_{ij}, $|\partial D_i|$ = length of ∂D_i, $\boldsymbol{n}_{ij} = (n_{ij1}, n_{ij2})$ = unit outer normal to ∂D_i on Γ_{ij} (i.e., $\boldsymbol{n}_{ij}$ points from D_i to D_j). Moreover, let us consider a partition $0 = t_0 < t_1 < \ldots$ of the interval $(0, \boldsymbol{T})$ and set $\tau_k = t_{k+1} - t_k$ for $k = 0, 1, \ldots$

Let us define the following spaces over grids $\mathcal{T}_h$ and $\mathcal{D}_h$:

$$\begin{aligned} \text{(3.9)}\quad & X_h = \{v_h \in L^2(\Omega);\ v_h|_T \text{ is linear } \forall\, T \in \mathcal{T}_h,\ v_h \text{ is continuous at } Q_j\ \ \forall\, j \in J\},\\ & V_h = \{v_h \in X_h\,;\ v_h(Q_i) = 0\ \ \forall\, i \in J - J^\circ\},\\ & Z_h = \{w_h \in L^2(\Omega);\ w_h|_{D_i} = \text{const.}\ \ \forall\, i \in J\},\\ & Y_h = \{w_h \in Z_h\,;\ w_h = 0\ \text{ on }\ D_i \in \mathcal{D}_h\ \ \forall\, i \in J - J^\circ\}. \end{aligned}$$

We can notice that $X_h \not\subset H^1(\Omega)$ and $V_h \not\subset V = H_0^1(\Omega)$. Therefore, we speak about *nonconforming, piecewise linear finite elements.* (By G. Strang, the use of nonconforming finite elements belongs to one of the basic *finite element variational crimes*, see [43]).

In the spaces from (3.9) we easily construct *simple bases*: The system $\{w_i\,;\ i \in J\}$ of functions $w_i \in X_h$ such that $w_i(Q_j) = \delta_{ij} =$ Kronecker's delta, $i, j \in J$, forms a basis in X_h. The system $\{w_i,\ i \in J^\circ\}$ is a basis in V_h. Furthermore, denoting by $d_i = \chi_{D_i}$ the characteristic function of $D_i \in \mathcal{D}_h$, we have bases in Z_h and Y_h as the systems $\{d_i\,;\ i \in J\}$ and $\{d_i\,;\ i \in J^\circ\}$, respectively.

By I_h we denote the interpolation operator for nonconforming finite elements (see [11], 8.9.79). If $v\colon\ H^1(\Omega) \oplus X_h = \{v + v_h\,;\ v \in H^1(\Omega),\ \ v_h \in X_h\} \to \mathbb{R}$, then

$$I_h v \in X_h, \quad (I_h v)(Q_i) = \frac{1}{|S_i|}\int_{S_i} v \,\mathrm{d}S, \quad i \in J. \tag{3.10}$$

This integral exists due to the imbedding $L^2(S) \subset L^1(S)$ and the theorem on traces in the space $H^1(T)$:

$$\|\varphi\|_{L^2(\partial T)} \leqslant c\|\varphi\|_{H^1(T)}, \quad \varphi \in H^1(T) \quad (c = c(T)). \tag{3.11}$$

By L_h we denote the so-called *lumping operator* which can be applied to all functions v defined at the points Q_i, $i \in J$:

$$L_h v = \sum_{i \in J} v(Q_i) d_i \in Z_h. \tag{3.12}$$

Obviously, $L_h(V_h) = Y_h$.

In order to define the discrete problem to (2.13)–(2.15), we put

$$\begin{aligned}
&\text{a)} \quad (u,v)_h = \int_\Omega (I_h u)(I_h v)\,\mathrm{d}x,\\
&\qquad\qquad u, v \in H^1(\Omega) \oplus X_h,\\
&\text{b)} \quad (\!(u,v)\!)_h = \sum_{T \in \mathcal{T}_h} \int_T \nabla u \cdot \nabla v \,\mathrm{d}x,\\
&\qquad\qquad u, v \in L^2(\Omega),\ \ u|_T, v|_T \in H^1(T)\ \ \forall\, T \in \mathcal{T}_h,\\
&\text{c)} \quad \tilde{b}_h(u,v) = \sum_{T \in \mathcal{T}_h} \int_T \sum_{s=1}^2 \frac{\partial f_s(u)}{\partial x_s}\, v \,\mathrm{d}x,\\
&\qquad\qquad u \in L^\infty(\Omega),\ \ v \in L^2(\Omega),\ \ u|_T \in H^1(T)\ \ \forall\, T \in \mathcal{T}_h.
\end{aligned} \tag{3.13}$$

By $\|\cdot\|_h$ we denote the discrete L^2-norm induced by $(\cdot,\cdot)_h$. For $u_h, v_h \in X_h$ we have $I_h u_h = u_h$, $I_h v_h = v_h$ and, hence,

$$(u_h, v_h)_h = (u_h, v_h), \quad \|v_h\|_h = \|v_h\|_{L^2(\Omega)}. \tag{3.14}$$

Furthermore,

$$(3.15)\qquad \begin{aligned} &(\!(u,v)\!)_h = (\!(u,v)\!), && u,v \in H^1(\Omega),\\ &\tilde{b}_h(u,v) = b(u,v), && u \in H^1(\Omega)\cap L^\infty(\Omega),\ \ v \in L^2(\Omega). \end{aligned}$$

The bilinear form $(\!(\cdot,\cdot)\!)_h$ induces in $X_h \oplus H^1(\Omega)$ the seminorm

$$(3.16)\qquad \|u_h\|_{X_h} = \left(\sum_{T\in\mathcal{T}} \int_T |\nabla u_h|^2 \, \mathrm{d}x\right)^{1/2}, \quad u_h \in X_h \oplus H^1(\Omega).$$

Under the notation

$$(3.17)\qquad \|u_h\|_{X_h(T)} = \left(\int_T |\nabla u_h|^2 \, \mathrm{d}x\right)^{1/2}, \quad T \in \mathcal{T}_h,\ \ u_h \in X_h \oplus H^1(\Omega),$$

we have

$$(3.18)\qquad \|u_h\|_{X_h}^2 = \sum_{T\in\mathcal{T}_h} \|u_h\|_{X_h(T)}^2, \quad u_h \in X_h \oplus H^1(\Omega).$$

Of course, for $u \in H^1(\Omega)$ we have $\|u\|_{X_h} = |u|_{H^1(\Omega)}$. The following Cauchy inequality holds:

$$(3.19)\qquad (\!(u_h,v_h)\!)_h \leqslant \|u_h\|_{X_h}\|v_h\|_{X_h}, \quad u_h, v_h \in X_h \oplus H^1(\Omega).$$

In the case when the diffusion ν is small, it is suitable to modify the "convection" form $\tilde{b}_h$ with the aid of the *finite volume approach*. Let $u \in H^1(\Omega)\cap C(\overline{\Omega})$, $v \in V_h$. Then we have by (3.12) and Green's formula that

$$\begin{aligned} \int_\Omega \sum_{s=1}^2 \frac{\partial f_s(u)}{\partial x_s} v \, \mathrm{d}x &\approx \int_\Omega \sum_{s=1}^2 \frac{\partial f_s(u)}{\partial x_s} L_h v \, \mathrm{d}x\\ &= \sum_{i\in J} v(Q_i) \int_{D_i} \sum_{s=1}^2 \frac{\partial f_s(u)}{\partial x_s} \, \mathrm{d}x\\ &= \sum_{i\in J} v(Q_i) \int_{\partial D_i} \sum_{s=1}^2 f_s(u) n_s \, \mathrm{d}S\\ &= \sum_{i\in J} v(Q_i) \sum_{j\in S(i)} \int_{\Gamma_{ij}} \sum_{s=1}^2 f_s(u) n_s \, \mathrm{d}S\\ &= \sum_{i\in J} v(Q_i) \sum_{j\in s(i)} \int_{\Gamma_{ij}} \sum_{s=1}^2 f_s(u) n_s \, \mathrm{d}S\\ &\approx \sum_{i\in J} v(Q_i) \sum_{j\in s(i)} H\big(u(Q_i), u(Q_j), \boldsymbol{n}_{ij}\big)\ell_{ij}. \end{aligned}$$

The function H defined on $\mathbb{R}^2 \times \mathcal{S}$, where $\mathcal{S} = \{\boldsymbol{n} \in \mathbb{R}^2;\ |\boldsymbol{n}| = 1\}$, is called a *numerical flux.* The form

$$b_h(u,v) = \sum_{i\in J} v(Q_i) \sum_{j\in s(i)} H\big(u(Q_i), u(Q_j), \boldsymbol{n}_{ij}\big)\ell_{ij} \tag{3.20}$$

obtained above has sense for all $u, v \in X_h$. We will use it as an approximation of the forms b and $\tilde{b}_h$.

Definition 2. We define the *approximate solution* of problem (2.1)–(2.3) as functions u_h^k, $t_k \in [0, \boldsymbol{T}]$, given by the conditions

$$u_h^0 = I_h u^0, \tag{3.21}$$

$$u_h^{k+1} \in V_h, \quad t_k \in [0, \boldsymbol{T}), \tag{3.22}$$

$$\frac{1}{\tau_k}(u_h^{k+1} - u_h^k, v_h) + b_h(u_h^k, v_h) + \nu(\!(u_h^{k+1}, v_h)\!)_h = (g^{k+1}, v_h)_h, \quad \forall\, v_h \in V_h,\ \ t_k \in [0, \boldsymbol{T}) \tag{3.23}$$

where $g^k = g(\cdot, t_k)$. The function u_h^k is the approximate solution at time t_k.

As we see, the scheme defined above is *semiimplicit.* The diffusion linear term is treated in an implicit way, whereas the nonlinear convective terms are discretized explicitly in order to obtain an easily solvable system of algebraic equations on every time level.

Properties of the numerical flux. In what follows we use the following assumptions:

1. $H = H(y, z, \boldsymbol{n})$ is *locally Lipschitz-continuous* with respect to y, z: for any $M > 0$ there exists a constant $c(M) > 0$ such that

$$|H(y,z,\boldsymbol{n}) - H(y^*, z^*, \boldsymbol{n})| \leqslant c(M)(|y - y^*| + |z - z^*|) \quad \forall\, y, y^*, z, z^* \in [-M, M],\ \ \forall\, \boldsymbol{n} \in \mathcal{S}. \tag{3.24}$$

2. H is *consistent*:

$$H(u,u,\boldsymbol{n}) = \sum_{s=1}^{2} f_s(u) n_s \ \ \forall\, u \in \mathbb{R},\ \ \forall\, \boldsymbol{n} = (n_1, n_2) \in \mathcal{S}. \tag{3.25}$$

3. H is *conservative*:

$$H(y,z,\boldsymbol{n}) = -H(z,y,-\boldsymbol{n}) \ \ \forall\, y, z \in \mathbb{R},\ \ \forall\, \boldsymbol{n} \in \mathcal{S}. \tag{3.26}$$

4. H is *monotone* in the following sense: For a given fixed number $M > 0$ the function $H(y, z, \boldsymbol{n})$ is nonincreasing with respect to the second variable z on the set

$$\mathcal{M}_M = \{(y, z, \boldsymbol{n});\ y, z \in [-M, M],\ \ \boldsymbol{n} \in \mathcal{S}\}. \tag{3.27}$$

In [2] the following results are proved:

Lemma 1. *Problem* (3.21)–(3.23) *has the following properties:*

1. *The bilinear forms* $(\cdot,\cdot)_h$ *and* $(\!(\cdot,\cdot)\!)_h$ *are scalar products on* V_h.
2. *For each* $u_h \in X_h$, $b_h(u_h,\cdot)$ *is a linear continuous form on* V_h.
3. *If* $i \in J$ *and* $T \in \mathcal{T}_h$ *is a triangle for which the midpoint* $Q_i \in T$, *then*

$$|T \cap D_i| = \frac{1}{3}|T|. \tag{3.28}$$

4. *The scalar product* $(\cdot,\cdot)_h$ *can be expressed with the aid of numerical integration using the centres* Q_T^i *of sides of triangles* $T \in \mathcal{T}_h$ *as integration points:*

$$(u,v)_h = \frac{1}{3}\sum_{T\in\mathcal{T}_h}|T|\sum_{j=1}^{3} u(Q_T^j)v(Q_T^j) = (L_h u, L_h v), \quad u,v \in X_h. \tag{3.29}$$

5. *We have*

$$\|v_h\|_{L^2(\Omega)} = \|L_h v_h\|_{L^2(\Omega)}, \quad v_h \in X_h, \tag{3.30}$$

$$(u_h,v_h) = (u_h,v_h)_h, \quad u_h,v_h \in X_h, \tag{3.31}$$

6. *Problem* (3.22)–(3.23) *has a unique solution* u_h^{k+1}.

4. Stability and consistency

Our aim will be to investigate the behaviour of the error $e_h^k = u(t_k) - u_h^k$. To this end, let us consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ $(h_0 > 0)$ of triangulations of the domain Ω, set $\tau = \boldsymbol{T}/r$ for an integer $r > 1$ and define the partition of the interval $[0,\boldsymbol{T}]$ formed by time instants $t_k = k\tau$, $k = 0,1,\ldots,r$. In what follows, the symbols $c, c_1, c_2, \ldots, \tilde{c}, \hat{c}, \ldots$ will denote constants independent of h, τ, ν, whereas $C, C_1, \ldots$ are independent of h, τ, but dependent on ν.

We introduce the following assumptions:

1. Let the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ be *regular*, i.e., there exists $\vartheta_0 > 0$ such that

$$\theta_h \geqslant \vartheta_0 > 0 \ \ \forall\, h \in (0,h_0). \tag{4.1}$$

2. The triangulations $\mathcal{T}_h$, $h \in (0,h_0)$, are of *weakly acute type*:

$$\text{the magnitude of all angles of all } T \in \mathcal{T}_h,\ \ h \in (0,h_0), \text{ is less than or equal to } \pi/2. \tag{4.2}$$

3. The *inverse assumption* is satisfied: There exists $c_1 > 0$ such that

$$\frac{h}{h(T)} \leqslant c_1 \ \ \forall\, T \in \mathcal{T}_h,\ \ \forall\, h \in (0,h_0). \tag{4.3}$$

In view of [4], Remark 3.1.3, assumptions (4.1) and (4.3) imply the existence of a constant $c_2 > 0$ such that

$$h^2 \leqslant c_2 |T| \ \ \forall\, T \in \mathcal{T}_h \ \ \forall\, h \in (0, h_0). \tag{4.4}$$

We summarize some results from [2] and derive some important estimates.

4.1. L^∞-stability.

In virtue of (2.9) and (2.10), $u^0 \in C(\overline{\Omega})$ and $g \in C(\overline{Q}_{\boldsymbol{T}})$. Hence, there exist constants $\widetilde{M}$ and $\widetilde{K}$ such that

$$\widetilde{M} := \|u^0\|_{L^\infty(\Omega)}, \quad \widetilde{K} := \|g\|_{L^\infty(Q_{\boldsymbol{T}})} < \infty. \tag{4.5}$$

Let us put

$$M^* = \widetilde{M} + \boldsymbol{T}\widetilde{K}, \quad M = 3M^*. \tag{4.6}$$

Theorem 1. *If $\tau > 0$ and $h \in (0, h_0)$ satisfy the stability condition*

$$\tau c(M^*)|\partial D_i| \leqslant |D_i|, \quad i \in J, \tag{4.7}$$

where $c(M^*)$ is the constant from (3.24), *then*

$$\|u_h^k\|_{L^\infty(\Omega)} \leqslant M, \quad t_k \in [0, \boldsymbol{T}]. \tag{4.8}$$

P r o o f. See [2], Theorem 2. □

Lemma 2. *Assumptions* (4.1), (4.3) *and the consequence* (4.4) *imply that there exists a constant $c_3 > 0$ such that*

$$|D_i|/|\partial D_i| \geqslant c_3 h \ \ \forall\, i \in J, \ \ \forall\, h \in (0, h_0). \tag{4.9}$$

P r o o f. See [2], Lemma 3. □

R e m a r k 1. Let us note that the condition

$$0 \leqslant \tau \leqslant c_3 c(M^*)^{-1} h \tag{4.10}$$

together with (4.9) imply (4.7). Hence, the stability condition (4.7) can be replaced by condition (4.10), which means that $\tau = O(h)$.

4.2. Consistency.

Lemma 3 (Discrete Friedrich's inequality). *There exists a constant $\hat{c}_1$ independent of h such that*

$$\|u_h\|_{L^2(\Omega)} \leqslant \hat{c}_1 \|u_h\|_{X_h}, \quad u_h \in V_h, \ \ h \in (0, h_0). \tag{4.11}$$

Proof. In [45], Chap. I, § 4, Proposition 4.13 or [11], Lemma 8.9.92, this lemma is proved provided Ω is convex. For the case of a general polygonal domain, see [10]. □

Lemma 4. *The interpolation operator I_h defined by (3.10) has the following properties:*

$$\text{If } \varphi \in V \ \text{ then } \ I_h\varphi \in V_h. \tag{4.12}$$

Let $\varphi \in H^{k+1}(\Omega)$, where $k = 0$ or 1. Then for $h \in (0, h_0)$ we have

$$\|\varphi - I_h\varphi\|_{X_h} \leqslant c_6 h^k \|\varphi\|_{H^{k+1}(\Omega)}, \tag{4.13}$$

$$\|\varphi - I_h\varphi\|_{L^2(\Omega)} \leqslant c_7 h^{k+1} \|\varphi\|_{H^{k+1}(\Omega)}, \tag{4.14}$$

$$\|I_h\varphi\|_{X_h} \leqslant c_8 \|\varphi\|_{H^1(\Omega)}, \tag{4.15}$$

$$\varphi \in H^1(\Omega) \Rightarrow \|\varphi - I_h\varphi\|_{X_h} \to 0 \ \text{ as } \ h \to 0, \tag{4.16}$$

with $c_6 > 0$, $c_7 > 0$, $c_8 > 0$ independent of φ and h.

Proof. See [11], Lemma 8.9.81. □

Lemma 5. *There exist constants $c_{10} > 0$ and $c_{11} > 0$ such that for any $h \in (0, h_0)$ we have*

$$\|v_h - L_h v_h\|_{L^2(\Omega)} \leqslant c_{10} h \|v_h\|_{X_h}, \quad v_h \in X_h, \tag{4.17}$$

$$|(g^k, v_h) - (g^k, v_h)_h| \leqslant c_{11} h \|g^k\|_{W^{1,q}(\Omega)} \|v_h\|_{X_h}, \quad v_h \in V_h. \tag{4.18}$$

If $M > 0$ and $\kappa \in (0, 1)$, then there exists a constant $\tilde{c} = \tilde{c}(M, \kappa)$ such that

$$\begin{aligned} &|\tilde{b}_h(u_h, v_h) - b_h(u_h, v_h)| \leqslant \tilde{c} h^{1-\kappa} (\|u_h\|^2_{X_h} + \|u_h\|_{X_h}) \|v_h\|_{X_h} \\ &\forall\, u_h \in V_h \cap L^\infty(\Omega), \ \ \|u_h\|_{L^\infty(\Omega)} \leqslant M \ \ \forall\, v_h \in V_h, \ \ h \in (0, h_0), \end{aligned} \tag{4.19}$$

where the forms $\tilde{b}_h$ and b_h are defined by (3.13) and (3.20), respectively.

Proof. See [2], Lemma 6. □

Lemma 6. *If $M > 0$, then there exist constants $c^* = c^*(M)$ and $c_1^* = c_1^*(M)$ such that*

$$
\begin{aligned}
&(4.20) && |b_h(u_h, v_h)| \leqslant c^* \|u_h\|_{L^\infty(\Omega)} \|v_h\|_{X_h},\\
&(4.21) && |b_h(u_h, v_h)| \leqslant c_1^* \|u_h\|_{X_h} \, \|v_h\|_{L^2(\Omega)},\\
& && u_h \in X_h,\ \ \|u_h\|_{L^\infty(\Omega)} \leqslant M,\ \ v_h \in V_h,\ \ h \in (0, h_0).
\end{aligned}
$$

Proof. For the proof of (4.20), see [2], Lemma 7. Here we prove (4.21). By (3.20), (3.25) and the relation $\sum_{j\in s(i)} \boldsymbol{n}_{ij}\ell_{ij} = 0$ valid for $i \in J^\circ$, for $u \in X_h$ such that $\|u\|_{L^\infty(\Omega)} \leqslant M$ and $v \in V_h$, we have

$$
\begin{aligned}
(4.22)\qquad b_h(u, v) = &\sum_{i\in J} v(Q_i) \sum_{j\in s(i)} H\big(u(Q_i), u(Q_j), \boldsymbol{n}_{ij}\big)\ell_{ij}\\
&- \sum_{i\in J} v(Q_i) \sum_{j\in s(i)} H\big(u(Q_i),\, u(Q_i), \boldsymbol{n}_{ij}\big)\ell_{ij}.
\end{aligned}
$$

If $i \in J$ and $j \in s(i)$, then we denote by T^{ij} the triangle from $\mathcal{T}_h$ such that $\Gamma_{ij} \subset T^{ij}$. It is easy to see that

$$
\begin{aligned}
(4.23)\qquad &|Q_i - Q_j| \leqslant \frac{h}{2}, \quad \ell_{ij} \leqslant \frac{2}{3}\, h,\\
&|u(Q_i) - u(Q_j)| \leqslant \frac{h}{2} |(\nabla u|_{T^{ij}})|.
\end{aligned}
$$

From (3.24), (4.22), (4.23) and (4.4) we find that

$$
\begin{aligned}
(4.24)\qquad |b_h(u, v)| &\leqslant c(M) \sum_{i\in J} |v(Q_i)| \sum_{j\in s(i)} |u(Q_i) - u(Q_j)|\ell_{ij}\\
&\leqslant c(M) \sum_{i\in J} |v(Q_i)| \sum_{j\in s(i)} |(\nabla u|_{T^{ij}})| \frac{h^2}{3}\\
&\leqslant \frac{c_2}{3}\, c(M) \sum_{i\in J} |v(Q_i)| \sum_{j\in s(i)} |T^{ij}|\, |(\nabla u|_{T^{ij}})|.
\end{aligned}
$$

Since each $T \in \mathcal{T}_h$ appears in (4.24) as some T^{ij} at most six times and $v(Q_i) = L_h v|_{D_i}$, we have

$$
|b_h(u, v)| \leqslant 2c_2 c(M) \sum_{T\in\mathcal{T}_h} \int_T |\nabla u|\, |L_h v| \,\mathrm{d}x.
$$

Using the Cauchy inequality, (3.14), (3.29) and (3.30), we finally conclude that

$$|b_h(u,v)| \leqslant 2c_2c(M)\Big(\sum_{T\in\mathcal{T}_h}\int_T |\nabla u|^2\,\mathrm{d}x\Big)^{1/2}\|L_hv\|_{L^2(\Omega)} = 2c_2c(M)\|u\|_{X_h}\|v\|_{L^2(\Omega)},$$

which we wanted to prove. □

4.3. A priori estimates.

Theorem 2. *There exist constants $\hat{c}>0$ and $\hat{c}_0>0$ independent of h, τ, m and ν such that*

$$\max_{t_k\in[0,\boldsymbol{T}]}\|u_h^k\|_{L^2(\Omega)} \leqslant \hat{c}, \tag{4.25}$$

$$\tau\sum_{k=0}^{m}\|u_h^k\|^2_{X_h} \leqslant \hat{c}_0(\nu^{-2}+\nu^{-1}),\quad m\in\{0,\dots,r\}, \tag{4.26}$$

for all $\tau,h>0$ satisfying the conditions $h\in(0,h_0)$ and (4.7).

Proof. Estimate (4.25) is a consequence of Theorem 1 and the inequality $\|u_h^k\|_{L^2(\Omega)}\leqslant|\Omega|^{1/2}\|u_h^k\|_{L^\infty(\Omega)}$, where $|\Omega|$ is the area of Ω. Estimate (4.26) is obtained in the same way as in the proof of Theorem 4 from [2]. □

Theorem 3. *There exists a constant $C_1>0$, $C_1=O\big(\nu^{-\frac{3}{2}}\big)$, independent of h and τ such that*

$$\|u_h^k\|_{X_h}\leqslant C_1,\quad t_k\in[0,\boldsymbol{T}], \tag{4.27}$$

for $h\in(0,h_0)$ and $\tau>0$ satisfying (4.7).

Proof. Let $\tau>0$ and $h\in(0,h_0)$ satisfy condition (4.7). Since $(\cdot,\cdot)_h$ and $(\!(\cdot,\cdot)\!)_h$ are scalar products on V_h, we can define a mapping $A_h\colon V_h\to V_h$ such that

$$(A_h\varphi_h,v_h)_h = (\!(\varphi_h,v_h)\!)_h,\quad v_h\in V_h. \tag{4.28}$$

Substituting $v_h:=A_hu_h^k$ in (3.23) with $k:=k-1$ and using (4.28), we find that

$$(\!(u_h^k-u_h^{k-1},u_h^k)\!)_h+\tau b_h(u_h^{k-1},A_hu_h^k)+\tau\nu(A_hu_h^k,A_hu_h^k)_h=\tau(g^k,A_hu_h^k)_h. \tag{4.29}$$

Now, the relations

$$2(\!(z-v,z)\!)_h=\|z\|^2_{X_h}-\|v\|^2_{X_h}+\|z-v\|^2_{X_h},$$

(3.29) and (2.9) imply that

$$\begin{aligned}(4.30)\qquad &\|u_h^k\|_{X_h}^2 - \|u_h^{k-1}\|_{X_h}^2 + \|u_h^k - u_h^{k-1}\|_{X_h}^2 + 2\tau\nu\|A_h u_h^k\|_{L^2(\Omega)}^2 \\ &= 2\tau(g^k, A_h u_h^k)_h - 2\tau b_h(u_h^{k-1}, A_h u_h^k) \\ &\leqslant 2c_{12}\tau\|g\|_{C([0,\boldsymbol{T}],W^{1,q}(\Omega))}\|A_h u_h^k\|_{L^2(\Omega)} - 2\tau b_h(u_h^{k-1}, A_h u_h^k).\end{aligned}$$

By (4.21) and Theorem 1 we have

$$(4.31)\qquad |b_h(u_h^{k-1}, A_h u_h^k)| \leqslant c_1^*\|u_h^{k-1}\|_{X_h}\|A_h u_h^k\|_{L^2(\Omega)}.$$

Substituting this estimate into (4.30) and using Young's inequality, we find that

$$\begin{aligned}(4.32)\qquad &\|u_h^k\|_{X_h}^2 - \|u_h^{k-1}\|_{X_h}^2 + \|u_h^k - u_h^{k-1}\|_{X_h}^2 + 2\tau\nu\|A_h u_h^k\|_{L^2(\Omega)}^2 \\ &\leqslant 2c_{12}\tau\|g\|_{C([0,\boldsymbol{T}],W^{1,q}(\Omega))}\|A_h u_h^k\|_{L^2(\Omega)} + 2\tau c_1^*\|u_h^{k-1}\|_{X_h}\,\|A_h u_h^k\|_{L^2(\Omega)} \\ &\leqslant 2\tau\nu\|A_h u_h^k\|_{L^2(\Omega)}^2 + \frac{\tau c_{13}}{\nu}\big(\|g\|_{C([0,\boldsymbol{T}],W^{1,q}(\Omega))}^2 + \|u_h^{k-1}\|_{X_h}^2\big),\end{aligned}$$

where $c_{13} = \max(c_{12}^2, (c_1^*)^2)$. Hence,

$$(4.33)\qquad \|u_h^k\|_{X_h}^2 - \|u_h^{k-1}\|_{X_h}^2 + \|u_h^k - u_h^{k-1}\|_{X_h}^2 \leqslant \frac{\tau c_{13}}{\nu}\big(\|g\|_{C([0,\boldsymbol{T}],W^{1,q}(\Omega))}^2 + \|u_h^{k-1}\|_{X_h}^2\big).$$

The summation of (4.33) over $k = 1, \dots, m$, $t_m \in (0, \boldsymbol{T}]$, and estimate (4.26) yield

$$\begin{aligned}(4.34)\qquad &\|u_h^m\|_{X_h}^2 - \|u_h^0\|_{X_h}^2 + \sum_{k=1}^m \|u_h^k - u_h^{k-1}\|_{X_h}^2 \\ &\leqslant \frac{c_{13}\boldsymbol{T}}{\nu}\|g\|_{C([0,\boldsymbol{T}],W^{1,q}(\Omega))}^2 + \frac{c_{13}}{\nu}\tau\sum_{k=1}^m \|u_h^{k-1}\|_{X_h}^2 \\ &\leqslant c_{14}\left(\frac{1}{\nu} + \frac{1}{\nu^2} + \frac{1}{\nu^3}\right), \\ &\quad c_{14} = \max(\boldsymbol{T}c_{13}\|g\|_{C([0,\boldsymbol{T}],W^{1,q}(\Omega))}^2, c_{13}\widehat{c}_0).\end{aligned}$$

From this and the estimate

$$(4.35)\qquad \|u_h^0\|_{X_h}^2 = \|I_h u_0\|_{X_h}^2 \leqslant c_8\|u^0\|_{H^1(\Omega)}^2$$

(cf. (4.15)) we finally obtain (4.27) with C_1 such that

$$(4.36)\qquad C_1^2 = c_8\|u^0\|_{H^1(\Omega)}^2 + c_{14}\left(\frac{1}{\nu} + \frac{1}{\nu^2} + \frac{1}{\nu^3}\right) \leqslant \bar{c}_{14}\frac{1}{\nu^3}.$$

$\square$

5. Truncation error

Let us suppose that the exact solution $u\colon (0,\boldsymbol{T}) \to V$ of problem (2.13)–(2.15) satisfies the conditions

$$
\begin{aligned}
&\text{(5.1)}\quad \text{a)} && u \in L^\infty(0,\boldsymbol{T}; H^2(\Omega)\cap W^{1,\infty}(\Omega)),\\
&\qquad\quad\ \ \text{b)} && u' \in L^\infty(0,\boldsymbol{T}; L^2(\Omega)),\\
&\qquad\quad\ \ \text{c)} && u'' \in L^\infty(0,\boldsymbol{T}; L^2(\Omega)).
\end{aligned}
$$

By u' and u'' we denote the first and second derivatives of the mapping $u\colon (0,\boldsymbol{T}) \to V$. The above assumptions imply that $u \in C^1([0,\boldsymbol{T}]; L^2(\Omega)) \cap C(\overline{Q}_{\boldsymbol{T}})$. We set $\widehat{M} = \|u\|_{L^\infty(Q_{\boldsymbol{T}})} < \infty$. In what follows we write $u^k = u(t_k) = u(\cdot, t_k)$. For simplicity we put

$$
\begin{aligned}
c_{26} &= \|u\|_{L^\infty(0,\boldsymbol{T};H^2(\Omega))},\\
c_{27} &= \|u'\|_{L^\infty(0,\boldsymbol{T};L^2(\Omega))},\\
c_{28} &= \|u''\|_{L^\infty(0,\boldsymbol{T};L^2(\Omega))}.
\end{aligned} \tag{5.2}
$$

Let us investigate the truncation error.

Lemma 7. *The form*

$$
\hat{b}(u,v) = \sum_{T\in\mathcal{T}_h} \int_T \sum_s f_s(u)\frac{\partial v}{\partial x_s}\,\mathrm{d}x, \quad u \in L^\infty(\Omega),\ \ v \in X_h, \tag{5.3}
$$

is locally Lipschitz-continuous: For $\widehat{M} > 0$ there exists a constant $\tilde{c}_4 = \tilde{c}_4(\widehat{M})$ such that

$$
\begin{gathered}
|\hat{b}(z, v_h) - \hat{b}(\widetilde{z}, v_h)| \leqslant \tilde{c}_4 \|z - \widetilde{z}\|_{L^2(\Omega)} \|v_h\|_{X_h}\\
\forall\, z,\ \ \widetilde{z} \in L^\infty(\Omega),\ \ \|z\|_{L^\infty(\Omega)},\ \ \|\widetilde{z}\|_{L^\infty(\Omega)} \leqslant \widehat{M}\ \ \forall\, v_h \in X_h.
\end{gathered} \tag{5.4}
$$

P r o o f. By the definition of $\hat{b}$, (2.8) and the Cauchy inequality, we find that for z, $\widetilde{z}$, v_h with the above properties we have

$$
\begin{aligned}
|\hat{b}(z, v_h) - \hat{b}(\widetilde{z}, v_h)| &= \Bigl|\sum_{T\in\mathcal{T}_h} \int_T \Bigl(\int_0^1 \sum_{s=1}^2 f_s'\bigl(\widetilde{z} + t(z-\widetilde{z})\bigr)\,\mathrm{d}t\Bigr)(z-\widetilde{z})\frac{\partial v_h}{\partial x_s}\,\mathrm{d}x\Bigr|\\
&\leqslant \sqrt{2} \max_{\xi\in[-\widehat{M},\widehat{M}],\, s=1,2} |f_s'(\xi)|\, \|z-\widetilde{z}\|_{L^2(\Omega)} \|v_h\|_{X_h},
\end{aligned}
$$

which is (5.4), where

$$
\tilde{c}_4 = \sqrt{2} \max_{\xi\in[-\widehat{M},\widehat{M}],\, s=1,2} |f_s'(\xi)|. \tag{5.5}
$$

$\square$

Lemma 8. *Under assumptions* (5.1), *for* $t_k \in [0, \boldsymbol{T})$ *we have*

$$|(u^{k+1} - u^k, v_h) - \tau(u'(t_{k+1}), v_h)| \leqslant c_{15}\tau^2 \|v_h\|_{X_h}, \quad v_h \in V_h, \tag{5.6}$$

$$\|u^{k+1} - u^k\|_{L^2(\Omega)} \leqslant c_{16}\tau, \tag{5.7}$$

$$|\tilde{b}_h(u^{k+1}, v_h) - \tilde{b}_h(u^k, v_h)| \leqslant c_{17}(\tau + h)\|v_h\|_{X_h}, \quad v_h \in V_h, \tag{5.8}$$

with $c_{15} = c_{15}(u)$, $c_{16} = c_{16}(u)$ *and* $c_{17} = c_{17}(u)$.

P r o o f. a) The proof of (5.6) is based on the following result (see [11], § 8.2, or [24]): If $\eta\colon (0, \boldsymbol{T}) \to L^2(\Omega)$ is such that $\eta, \eta' \in L^1(0, \boldsymbol{T}; L^2(\Omega))$ and $v \in L^2(\Omega)$, then $(\eta', v) \in L^1(0, \boldsymbol{T})$ and

$$\int_{t_1}^{t_2} (\eta'(t), v)\,\mathrm{d}t = \bigl(\eta(t_2) - \eta(t_1), v\bigr), \quad t_1, t_2 \in [0, \boldsymbol{T}]. \tag{5.9}$$

This and (5.1) imply that

$$\bigl(u(t_{k+1}) - u(t_k), v\bigr) = \int_{t_k}^{t_{k+1}} (u'(t), v)\,\mathrm{d}t, \tag{5.10}$$

and hence,

$$(u^{k+1} - u^k, v) - \tau\bigl(u'(t_{k+1}), v\bigr) = \int_{t_k}^{t_{k+1}} \bigl(u'(t) - u'(t_{k+1}), v\bigr)\,\mathrm{d}t. \tag{5.11}$$

Since $u'' \in L^\infty(0, \boldsymbol{T}; L^2(\Omega))$, we have

$$(u'(t) - u'(t_{k+1}), v) = \int_{t_{k+1}}^{t} (u''(\vartheta), v)\,\mathrm{d}\vartheta \tag{5.12}$$

and thus,

$$\int_{t_k}^{t_{k+1}} \bigl(u'(t) - u'(t_{k+1}), v\bigr)\,\mathrm{d}t = \int_{t_k}^{t_{k+1}} \left(\int_{t_{k+1}}^{t} (u''(\vartheta), v)\,\mathrm{d}\vartheta\right) \mathrm{d}t. \tag{5.13}$$

This, (5.10)–(5.12), the Cauchy inequality, assumption (5.1) c) and (5.2) imply that

$$\begin{aligned} |(u^{k+1} - u^k, v) - \tau(u'(t_{k+1}), v)| &\leqslant \tau^2 \|u''\|_{L^\infty(0,\boldsymbol{T};L^2(\Omega))} \|v\|_{L^2(\Omega)} \\ &= \tau^2 c_{28} \|v\|_{L^2(\Omega)}. \end{aligned} \tag{5.14}$$

Now, we substitute $v := v_h \in V_h$, use (4.11) and obtain (5.6) with $c_{15} = c_{28}\hat{c}_1$.

b) Since $u' \in L^\infty(0,\boldsymbol{T};L^2(\Omega))$, we can write

$$\|u^{k+1}-u^k\|_{L^2(\Omega)} = \left\|\int_{t_k}^{t_{k+1}} u'(t)\,\mathrm{d}t\right\|_{L^2(\Omega)} \leqslant \tau\|u'\|_{L^\infty(0,\boldsymbol{T};L^2(\Omega))} = \tau c_{27},$$

which yields (5.7) with $c_{16}=c_{27}$.

c) In view of the definition of $\tilde{b}_h$ in (3.13), we can write

$$\begin{aligned}(5.15)\quad |\tilde{b}_h(u^{k+1},v_h)-\tilde{b}_h(u^k,v_h)| &= \left|\sum_{T\in\mathcal{T}_h}\int_T\sum_{s=1}^2\Big(\frac{\partial f_s(u^{k+1})}{\partial x_s}-\frac{\partial f_s(u^k)}{\partial x_s}\Big)v_h\,\mathrm{d}x\right|\\
&= \left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}\sum_{s=1}^2\big(f_s(u^{k+1})-f_s(u^k)\big)n_s v_h\,\mathrm{d}S\right.\\
&\quad\left.-\sum_{T\in\mathcal{T}_h}\int_T\sum_{s=1}^2\big(f_s(u^{k+1})-f_s(u^k)\big)\frac{\partial v_h}{\partial x_s}\,\mathrm{d}x\right|\\
&\leqslant \left|\sum_{T\in\mathcal{T}_h}\int_{\partial T}\sum_{s=1}^2\big(f_s(u^{k+1})-f_s(u^k)\big)n_s v_h\,\mathrm{d}S\right|\\
&\quad+|\hat{b}(u^{k+1},v_h)-\hat{b}(u^k,v_h)|.\end{aligned}$$

The first part of the right-hand side in inequality (5.15) can be written in the form

$$(5.16)\qquad R_1 := \left|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^3\int_{S_T^j}\sum_{s=1}^2\big(f_s(u^{k+1})-f_s(u^k)\big)(n_T^j)_s v_h\,\mathrm{d}S\right|,$$

where $S_T^j\subset\partial T$ are sides of T, $j=1,2,3$, and $(n_T^j)_s$ is the s-th component of the unit outer normal to ∂T on S_T^j. Now we use the assertion of Lemma 8.9.85 from [11]:

$$\begin{aligned}(5.17)\qquad &\left|\sum_{T\in\mathcal{T}_h}\sum_{j=1}^3\int_{S_T^j}(n_T^j)_s v_h\varphi\,\mathrm{d}S\right| \leqslant c_{18}h\|\varphi\|_{H^1(\Omega)}\|v_h\|_{X_h},\\
&\qquad s=1,2\ \ \forall\varphi\in H^1(\Omega)\ \ \forall v_h\in V_h,\ \ h\in(0,h_0).\end{aligned}$$

This and (5.1) yield the estimate of (5.16):

$$\begin{aligned}(5.18)\qquad R_1 &\leqslant c_{18}h\sum_{s=1}^2\|f_s(u^{k+1})-f_s(u^k)\|_{H^1(\Omega)}\|v_h\|_{X_h}\\
&\leqslant c_{18}h\max_{\substack{\xi\in[-\widehat{M},\widehat{M}]\\ s=1,2}}|f_s'(\xi)|\big(\|u^k\|_{H^1(\Omega)}+\|u^{k+1}\|_{H^1(\Omega)}\big)\|v_h\|_{X_h}\\
&\leqslant c_{19}\,h\|v_h\|_{X_h},\end{aligned}$$

$$(5.19)\qquad c_{19} = c_{18}\max_{\substack{\xi\in[-\widehat{M},\widehat{M}]\\ s=1,2}}|f_s'(\xi)|\,2c_{26}.$$

The second term on the right-hand side of (5.15) is estimated with the aid of (5.4) and (5.7):

$$|\hat{b}(u^{k+1}, v_h) - \hat{b}(u^k, v^h)| \leqslant \tilde{c}_4 c_{16} \tau \|v_h\|_{X_h}.$$

This and (5.18) already yield (5.8) with $c_{17} = \max(c_{19}, \tilde{c}_4 c_{16})$. □

Using the above results, we get an estimate of the truncation error.

Theorem 4. *Under assumptions* (5.1) *we have*

$$\begin{aligned}
(5.20) \qquad & (u_h^{k+1} - u^{k+1}, v_h) - (u_h^k - u^k, v_h) \\
& \quad + \tau[\tilde{b}_h(u_h^k, v_h) - \tilde{b}_h(u^k, v_h)] + \tau\nu((u_h^{k+1} - u^{k+1}, v_h))_h \\
& = -\tau\varepsilon_1(h, u^{k+1}, v_h) - \tau\varepsilon_2(\tau, h, u^k, u^{k+1}, v_h) \\
& \quad + \tau\varepsilon_3(h, u_h^k, v_h) + \tau\varepsilon_4(h, v_h), \qquad v_h \in V_h,\ \ t_k \in [0, \boldsymbol{T}),
\end{aligned}$$

where $\varepsilon_1, \ldots, \varepsilon_4$ *(defined in the proof) satisfy the estimates*

$$\begin{aligned}
(5.21) \qquad & |\varepsilon_1(h, u^{k+1}, v_h)| \leqslant c_{20} h \, \|v_h\|_{X_h}, \\
(5.22) \qquad & |\varepsilon_2(\tau, h, u^k, u^{k+1}, v_h)| \leqslant c_{22}(\tau + h)\, h \|u^{k+1}\|_{H^2(\Omega)} \|v_h\|_{X_h}, \\
(5.23) \qquad & |\varepsilon_3(h, u_h^k, v_h)| \leqslant \tilde{c} h^{1-\kappa}(\|u_h^k\|_{X_h}^2 + \|u_h^k\|_{X_h})\|v_h\|_{X_h}, \\
(5.24) \qquad & |\varepsilon_4(h, v_h)| \leqslant c_{11} h \|g^k\|_{W^{1,q}(\Omega)} \|v_h\|_{X_h},
\end{aligned}$$

where $\kappa \in (0, 1)$ *follows from Lemma* 5.

P r o o f. In virtue of (5.1), equation (2.1) is satisfied a.e. in Ω for each $t \in (0, \boldsymbol{T})$. We multiply (2.1) by $v_h \in V_h$ and integrate over Ω at the time level t_{k+1}. In this way we obtain the relation

$$(5.25) \qquad (u'(t_{k+1}), v_h) + \tilde{b}_h(u^{k+1}, v_h) + \nu(\Delta u^{k+1}, v_h) = (g^{k+1}, v_h).$$

Further,

$$\begin{aligned}
(5.26) \quad (\Delta u^{k+1}, v_h) &= \sum_{T \in \mathcal{T}_h} \int_T \Delta u^{k+1} v_h \,\mathrm{d}x \\
&= \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\nabla u^{k+1} \cdot \boldsymbol{n}) v_h \,\mathrm{d}S - \sum_{T \in \mathcal{T}_h} \int_T \nabla u^{k+1} \cdot \nabla v_h \,\mathrm{d}x \\
&= \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\nabla u^{k+1} \cdot \boldsymbol{n}) v_h \,\mathrm{d}S - ((u^{k+1}, v_h))_h.
\end{aligned}$$

Due to (5.17), for

$$\varepsilon_1(h, u^{k+1}, v_h) := \nu \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\nabla u^{k+1} \cdot \boldsymbol{n}) v_h \, \mathrm{d}S$$

we find that

$$\begin{aligned} |\varepsilon_1(h, u^{k+1}, v_h)| &\leqslant \nu c_{18} h \sum_{s=1}^{2} \left\| \frac{\partial u^{k+1}}{\partial x_s} \right\|_{H^1(\Omega)} \|v_h\|_{X_h} \\ &\leqslant \nu \sqrt{2} c_{18} h \|u^{k+1}\|_{H^2(\Omega)} \|v_h\|_{X_h}. \end{aligned} \tag{5.27}$$

We put $c_{20} = \sqrt{2} c_{18} c_{26}$.

Setting

$$\begin{aligned} \varepsilon_2(\tau, h, u^k, u^{k+1}, v_h) := &\frac{1}{\tau}(u^{k+1} - u^k, v_h) - (u'(t_{k+1}), v_h) \\ &+ [\tilde{b}_h(u^k, v_h) - \tilde{b}_h(u^{k+1}, v_h)], \end{aligned} \tag{5.28}$$

we can write relation (5.25) in the form

$$\begin{aligned} &(u^{k+1} - u^k, v_h) + \tau \tilde{b}_h(u^k, v_h) + \tau\nu (\!(u^{k+1}, v_h)\!)_h \\ &\quad = \tau(g^{k+1}, v_h) + \tau\varepsilon_2(\tau, h, u^k, u^{k+1}, v_h) + \tau\varepsilon_1(h, u^{k+1}, v_h). \end{aligned} \tag{5.29}$$

The estimate of $\varepsilon_2(\tau, h, u^k, u^{k+1}, v_h)$ follows from (5.6) and (5.8):

$$|\varepsilon_2(\tau, h, u^k, u^{k+1}, v_h)| \leqslant c_{22}(\tau + h)\|v_h\|_{X_h}, \quad c_{22} = c_{15} + c_{17}. \tag{5.30}$$

By (3.23), for the approximate solution we have

$$(u_h^{k+1} - u_h^k, v_h) + \tau b_h(u_h^k, v_h) + \tau\nu (\!(u_h^{k+1}, v_h)\!)_h = \tau(g^{k+1}, v_h)_h, \quad v_h \in V_h, \tag{5.31}$$

which can be rewritten as

$$\begin{aligned} &(u_h^{k+1} - u_h^k, v_h) + \tau \tilde{b}_h(u_h^k, v_h) + \tau\nu (\!(u_h^{k+1}, v_h)\!)_h \\ &\quad = \tau(g^{k+1}, v_h) + \tau \left[\tilde{b}_h(u_h^k, v_h) - b_h(u_h^k, v_h) \right] \\ &\qquad + \tau \left[(g^{k+1}, v_h)_h - (g^{k+1}, v_h) \right]. \end{aligned} \tag{5.32}$$

We set

$$\varepsilon_3(h, u_h^k, v_h) = \tilde{b}_h(u_h^k, v_h) - b_h(u_h^k, v_h) \tag{5.33}$$

and

$$\varepsilon_4(h, v_h) = (g^{k+1}, v_h)_h - (g^{k+1}, v_h). \tag{5.34}$$

It is seen from (4.19) and (4.18) that

$$|\varepsilon_3(h, v_h^k, v_h)| \leqslant \tilde{c}\, h^{1-\kappa}(\|u_h^k\|_{X_h}^2 + \|u_h^k\|_{X_h})\|v_h\|_{X_h} \tag{5.35}$$

and

$$|\varepsilon_4(h, v_h)| \leqslant c_{11} h \|g^k\|_{W^{1,q}(\Omega)} \|v_h\|_{X_h}. \tag{5.36}$$

Now we subtract (5.29) from (5.32) and obtain (5.20). From (5.27), (5.30), (5.35) and (5.36) we conclude that (5.21)–(5.24) hold. □

6. Error estimates

We denote by

$$e_h^k = u_h^k - u^k \tag{6.1}$$

the error of the method at time $t = t_k$. Obviously, $e_h^k \in V_h \oplus V = \{v_h + v;\ v_h \in V_h, v \in V\} \subset X_h \oplus H^1(\Omega)$. Our goal is to estimate e_h^k in a suitable norm in terms of h and τ. The error in the space-time cylinder $Q_{\boldsymbol{T}}$ can be characterized by a continuous piecewise linear function $e\colon\ [0, \boldsymbol{T}] \to V_h \oplus V$ such that

$$e(t_k) = e_h^k \quad \text{for} \quad t_k \in [0, \boldsymbol{T}]. \tag{6.2}$$

In Lemma 4 some properties of the interpolation operator I_h, defined by (3.10), were formulated. They can be generalized in the following way:

Lemma 9. *For each $v \in X_h \oplus H^1(\Omega)$ the following inequalities hold:*

$$\begin{aligned} &\text{a)} \quad \|v - I_h v\|_{L^2(\Omega)} \leqslant c_{23} h \|v\|_{X_h}, \\ &\text{b)} \quad \|v - I_h v\|_{X_h} \leqslant c_{24} \|v\|_{X_h}, \\ &\text{c)} \quad \|I_h v\|_{X_h} \leqslant c_{25} \|v\|_{X_h}, \end{aligned} \tag{6.3}$$

where c_{23}, c_{24}, c_{25} are constants independent of h and v.

Proof. Since $v|_T \in H^1(T)$ for each T, it follows from the general approximation finite element properties ([4], Theorem 3.1.4) and assumption (3.1) that

$$\|v - I_h v\|_{L^2(T)} \leqslant ch|v|_{H^1(T)},$$
$$\|v - I_h v\|_{H^1(T)} \leqslant c|v|_{H^1(T)}$$

with c independent of v, T and h. This immediately yields (6.3). □

For our further considerations, because of the control of some terms, we introduce the "*inverse stability assumption*"

$$h \leqslant \tilde{\tilde{c}}\tau \tag{6.4}$$

with a constant $\tilde{c}$ independent of h and τ. Hence, $h = O(\tau)$. This condition seems to be non-standard, but we can meet it also in other works concerned with the numerical solution of evolution problems, as e.g. [39], § 4.2, 5.1 or [31].

Now we come to the fundamental result.

Theorem 5. *Let assumptions* (2.8)–(2.10), (3.1), (3.2), (3.24)–(3.27), (4.1)–(4.3) *be satisfied. Further, let* $\{u_h^k\}_{t_k=k\tau\in[0,\boldsymbol{T}]}$ *be the approximate solution of problem* (2.13)–(2.15) *obtained with the aid of the discrete problem* (3.21)–(3.23). *Let the exact solution* u *of* (2.13)–(2.15) *satisfy conditions* (5.1). *Moreover, we assume that* $u^0 \in H^2(\Omega)$ *and set*

$$\begin{aligned}\|e\|^2_{h,\tau,L^2} &:= \tau\sum_{k=0}^{r}\|e_h^k\|^2_{L^2(\Omega)}, \quad \tau = \boldsymbol{T}/r,\\ \|e\|^2_{h,\tau,\nu,X_h} &:= \tau\nu\sum_{k=0}^{r-1}\|e_h^{k+1}\|^2_{X_h}.\end{aligned} \tag{6.5}$$

Let $\kappa \in (0,1/2)$. *Then there exist constants* $C_2 = O(\nu^{-6}\exp(2\boldsymbol{T}\mathbf{c}/\nu))$ *and* $C_3 = O(\nu^{-7}\exp(2\boldsymbol{T}\mathbf{c}/\nu))$ *such that*

$$\begin{aligned}&\text{a)} \quad \|e\|^2_{h,\tau,L^2} \leqslant C_2 h^{2(1-\kappa)},\\ &\text{b)} \quad \|e\|^2_{h,\tau,\nu,X_h} \leqslant C_3\, h^{1-2\kappa}\end{aligned} \tag{6.6}$$

for all $h \in (0, h_0)$ *and* $\tau > 0$ *satisfying conditions* (4.10), (6.4) *and* $2\mathbf{c}\tau \leqslant \nu$, *where* $\mathbf{c} > 0$ *is the constant appearing in the proof.*

Proof. Let $h \in (0, h_0)$ and $\tau > 0$ satisfy conditions (4.10) and (6.4). Then condition (4.7) is satisfied. From (6.1) and (5.20) we obtain the relation

$$\begin{aligned}&(e_h^{k+1}, v_h) - (e_h^k, v_h) + \tau\nu((e_h^{k+1}, v_h))_h\\ &\quad = -\tau[\tilde{b}_h(u_h^k, v_h) - \tilde{b}_h(u^k, v_h)] - \varepsilon_1(\tau, u^k, u^{k+1}, v_n)\\ &\qquad - \tau\varepsilon_2(h, u^{k+1}, v_h) + \tau\varepsilon_3(h, u_h^k, v_h) + \tau\varepsilon_4(h, v_h).\end{aligned} \tag{6.7}$$

Let us set $v_h := I_h e_h^{k+1}$. Denoting by I the identity operator ($I\varphi = \varphi$), we get

$$
\begin{aligned}
(6.8)\qquad & (e_h^{k+1}, e_h^{k+1}) - (e_h^k, e_h^{k+1}) + \tau\nu (\!(e_h^{k+1}, e_h^{k+1})\!)_h \\
&= -\tau[\tilde{b}_h(u_h^k, I_h e_h^{k+1}) - \tilde{b}_h(u^k, I_h e_h^{k+1})] \\
&\quad - \tau\varepsilon_1(h, u^{k+1}, I_h e_h^{k+1}) - \tau\varepsilon_2(\tau, h, u^k, u^{k+1}, I_h e_h^{k+1}) \\
&\quad - \tau\varepsilon_3(h, u_h^k, I_h e_h^{k+1}) + \tau\varepsilon_4(h, I_h e_h^{k+1}) + (e_h^{k+1}, (I - I_h)\, e_h^{k+1}) \\
&\quad - (e_h^k, (I - I_h)\, e_h^{k+1}) + \tau\nu (\!(e_h^{k+1}, (I - I_h)\, e_h^{k+1})\!)_h .
\end{aligned}
$$

From (6.1) it follows that $(I - I_h) e_h^{k+1} = I_h u^{k+1} - u^{k+1}$. Hence, by assumption (5.1) and Lemma 4 we have

$$
\begin{aligned}
(6.9)\qquad & \|(I - I_h) e_h^{k+1}\|_{L^2(\Omega)} \leqslant c_7 h^2 \|u^{k+1}\|_{H^2(\Omega)}, \\
(6.10)\qquad & \|(I - I_h) e_h^{k+1}\|_{X_h} \leqslant c_6 h \|u^{k+1}\|_{H^2(\Omega)}.
\end{aligned}
$$

Now we can write (6.8) in the form

$$
\begin{aligned}
(6.11)\qquad & \|e_h^{k+1}\|_{L^2(\Omega)}^2 - \|e_h^k\|_{L^2(\Omega)}^2 + \|e_h^{k+1} - e_h^k\|_{L^2(\Omega)}^2 + 2\tau\nu\|e_h^{k+1}\|_{X_h}^2 \\
&\leqslant \sigma(1) + \sigma(2) + \ldots + \sigma(6) + \tau\sigma(7),
\end{aligned}
$$

where

$$
\begin{aligned}
(6.12)\qquad & \sigma(1) = |\tau\varepsilon_1(h, u^{k+1}, I_h e_h^{k+1})|, \\
& \sigma(2) = |\tau\varepsilon_2(\tau, u^k, u^{k+1}, I_h e_h^{k+1})|, \\
& \sigma(3) = |\tau\varepsilon_3(h, u_h^k, I_h e_h^{k+1})|, \\
& \sigma(4) = |\tau\varepsilon_4(h, I_h e_h^{k+1})|, \\
& \sigma(5) = |(e_h^{k+1}, (I - I_h)\, e_h^{k+1}) - (e_h^k, (I - I_h)\, e_h^{k+1})|, \\
& \sigma(6) = |\tau\nu (\!(e_h^{k+1}, (I - I_h)\, e_h^{k+1})\!)|, \\
& \sigma(7) = |\tilde{b}_h(u_h^k, I_h e_h^{k+1}) - \tilde{b}_h(u^k, I_h e_h^{k+1})|.
\end{aligned}
$$

Let us estimate these terms. From (4.10), (5.1), (5.21)–(5.24), (6.1), (6.3) and (6.4) we find that

$$
\begin{aligned}
(6.13)\qquad & \sigma(1) \leqslant \nu c_{20} c_{25} \tau h \|e_h^{k+1}\|_{X_h}, \\
& \sigma(2) \leqslant c_{22} c_{25} \tau(\tau + h) \|u^{k+1}\|_{H^2(\Omega)} \|e_h^{k+1}\|_{X_h} \leqslant c_{29} \tau h \|e_h^{k+1}\|_{X_h}, \\
& \sigma(3) \leqslant c_{25} \tilde{c} \tau h^{1-\kappa} (\|u_h^k\|_{X_h}^2 + \|u_h^k\|_{X_h}) \|e_h^{k+1}\|_{X_h}, \\
& \sigma(4) \leqslant c_{11} c_{25} \tau h \|g^k\|_{W^{1,q}(\Omega)} \|e_h^{k+1}\|_{X_h} \leqslant c_{30} \tau h \|e_h^{k+1}\|_{X_h}, \\
& \qquad c_{29} = c_{22} c_{25} (1 + c_3 c(M^*)^{-1}) c_{26}, \\
& \qquad c_{30} = c_{11} c_{25} \|g^k\|_{W^{1,q}(\Omega)}.
\end{aligned}
$$

Furthermore, the Cauchy inequality, (6.9), (6.10), Young's inequality and (6.4) imply that

$$\begin{aligned}
\sigma(5) &\leqslant \|e_h^{k+1} - e_h^k\|_{L^2(\Omega)} \|(I - I_h)e_h^{k+1}\|_{L^2(\Omega)} \\
&\leqslant c_7 h^2 \|e_h^{k+1} - e_h^k\|_{L^2(\Omega)} \|u^{k+1}\|_{H^2(\Omega)} \\
&\leqslant c_7 c_{26} h^2 \|e_h^{k+1} - e_h^k\|_{L^2(\Omega)},
\end{aligned} \tag{6.14}$$

$$\begin{aligned}
\sigma(6) &\leqslant c_6 \tau \nu h \|e_h^{k+1}\|_{X_h} \|u^{k+1}\|_{H^2(\Omega)} \\
&\leqslant c_{31} \tau h \nu \|e_h^{k+1}\|_{X_h}, \quad c_{31} = c_6 c_{26}.
\end{aligned} \tag{6.15}$$

By (3.13),

$$\begin{aligned}
\sigma(7) &= \left| \sum_{T \in \mathcal{T}_h} \int_T \sum_{s=1}^{2} \left(f_s'(u_h^k) \frac{\partial u_h^k}{\partial x_s} - f_s'(u^k) \frac{\partial u^k}{\partial x_s} \right) I_h e_h^{k+1} \, \mathrm{d}x \right| \\
&\leqslant \left| \sum_{T \in \mathcal{T}_h} \int_T \sum_{s=1}^{2} (f_s'(u_h^k) - f_s'(u^k)) \frac{\partial u^k}{\partial x_s} I_h e_h^{k+1} \, \mathrm{d}x \right| \\
&\quad + \left| \sum_{T \in \mathcal{T}_h} \int_T \sum_{s=1}^{2} f_s'(u_h^k) \left(\frac{\partial u_h^k}{\partial x_s} - \frac{\partial u^k}{\partial x_s} \right) I_h e_h^{k+1} \, \mathrm{d}s \right|.
\end{aligned}$$

Using the bound (4.8), assumption (5.1), and a similar process as in the proof of Lemma 8, we find that

$$\begin{aligned}
&\sigma(7) \leqslant c_{32} (\|e_h^k\|_{L^2(\Omega)} + \|e_h^k\|_{X_h}) \, \|I_h e_h^{k+1}\|_{L^2(\Omega)}, \\
&c_{32} = \max(\|u\|_{L^\infty(0,\boldsymbol{T};W^{1,\infty}(\Omega))} \max_{\substack{\xi \in [-\widehat{M}, \widehat{M}] \\ s=1,2}} |f_s''(\xi)|, \max_{\substack{\xi \in [-\widehat{M}, \widehat{M}] \\ s=1,2}} |f_s'(\xi)|).
\end{aligned} \tag{6.16}$$

This and (6.3) a) imply that

$$\begin{aligned}
\sigma(7) &\leqslant c_{33} (\|e_h^k\|_{L^2(\Omega)} + \|e_h^k\|_{X_h}) (\|e_h^{k+1}\|_{L^2(\Omega)} + h \|e_h^{k+1}\|_{X_h}), \\
c_{33} &= c_{32} \max(c_{23}, 1).
\end{aligned} \tag{6.17}$$

By (4.25), (4.27) and properties (5.1) of the solution of the continuous problem,

$$\|e_h^k\|_{L^2(\Omega)} = \|u^k - u_h^k\|_{L^2(\Omega)} \leqslant \|u^k\|_{L^2(\Omega)} + \|u_h^k\|_{L^2(\Omega)} \leqslant c_{26} + \hat{c} =: c_{34} \tag{6.18}$$

and

$$\|e_h^k\|_{X_h} = \|u^k - u_h^k\|_{X_h} \leqslant |u^k|_{H^1(\Omega)} + \|u_h^k\|_{X_h} \leqslant c_{26} + C_1(\nu) =: \widehat{C}(\nu) \tag{6.19}$$

where, in view of (4.36),

$$C_1(\nu) \leqslant \sqrt{\bar{c}_{14}} \frac{1}{\nu^{3/2}} \Rightarrow \widehat{C}(\nu) = O(\nu^{-3/2}). \tag{6.20}$$

Therefore,

$$\begin{aligned}(6.21)\qquad \sigma(7) &\leqslant c_{35}(\|e_h^k\|_{L^2(\Omega)}\|e_h^{k+1}\|_{L^2(\Omega)} + \|e_h^k\|_{X_h}\|e_h^{k+1}\|_{L^2(\Omega)} \\ &\quad + (h + h\widehat{C}(\nu))\|e_h^{k+1}\|_{X_h}), \quad c_{35} = c_{33}\max(1, c_{34}).\end{aligned}$$

Now, using Young's inequality in (6.21), we have

$$\begin{aligned}(6.22)\qquad \tau\sigma(7) &\leqslant \tau c_{35}\nu\|e_h^k\|^2_{L^2(\Omega)} + \frac{\tau c_{35}}{4\nu}\|e_h^{k+1}\|^2_{L^2(\Omega)} + \frac{\tau c_{35}^2}{4\nu}\|e_h^{k+1}\|^2_{L^2(\Omega)} \\ &\quad + \tau\nu\|e_h^k\|^2_{X_h} + \frac{\tau\nu}{4}\|e_h^{k+1}\|^2_{X_h} + \frac{\tau}{\nu}C(\nu)h^2 \\ &\leqslant \tau\mathbf{c}\nu\|e_h^k\|^2_{L^2(\Omega)} + \frac{\tau\mathbf{c}}{\nu}\|e_h^{k+1}\|^2_{L^2(\Omega)} \\ &\quad + \tau\nu\|e_h^k\|^2_{X_h} + \frac{\tau\nu}{4}\|e_h^{k+1}\|^2_{X_h} + \frac{\tau}{\nu}C(\nu)h^2\end{aligned}$$

with $\mathbf{c} = \max(c_{35}, (c_{35} + c_{35}^2)/4)$ and $C(\nu) = c_{35}^2(1 + \widehat{C}(\nu))^2$. Further, for $\sigma(1), \ldots, \sigma(6)$ we again use Young's inequality, (6.15), (6.4) and (4.27). Then we obtain estimates

$$\begin{aligned}(6.23)\qquad \sigma(1) &\leqslant c_{36}\tau h^2 + \frac{\tau\nu}{8}\|e_h^{k+1}\|^2_{X_h}, \quad c_{36} = 2(c_{20}c_{25})^2, \\ \sigma(2) + \sigma(4) &\leqslant \frac{c_{37}\tau h^2}{\nu} + \frac{\tau\nu}{8}\|e_h^{k+1}\|^2_{X_h}, \quad c_{37} = 2(c_{29} + c_{30})^2, \\ \sigma(3) &\leqslant c_{25}^2\frac{\tilde{c}^2\tau}{\nu}h^{2(1-\kappa)}(\|u_h^k\|^2_{X_h} + \|u_h^k\|_{X_h})^2 + \frac{\tau\nu}{4}\|e_h^{k+1}\|^2_{X_h} \\ &\leqslant c_{40}\frac{\tau}{\nu}h^{2(1-\kappa)}(\nu^{-6} + \nu^{-4.5} + \nu^{-3}) + \frac{\tau\nu}{4}\|e_h^{k+1}\|^2_{X_h}, \\ &\qquad c_{40} := c_{25}^2\tilde{c}^2\max(\bar{c}_{14}^2, 2\bar{c}_{14}^3, \bar{c}_{14}^2), \\ \sigma(5) &\leqslant \frac{c_7^2c_{26}^2}{4}h^4 + \|e_h^{k+1} - e_h^k\|^2_{L^2(\Omega)} \leqslant c_{41}\tau h^3 + \|e_h^{k+1} - e_h^k\|^2_{L^2(\Omega)}, \\ &\qquad c_{41} := \frac{c_7^2c_{26}^2}{4}\tilde{\tilde{c}}, \\ \sigma(6) &\leqslant c_{31}^2\tau\nu h^2 + \frac{\tau\nu}{4}\|e_h^{k+1}\|^2_{X_h}.\end{aligned}$$

Now, estimates (6.11), (6.22) and (6.23) imply that

$$\begin{aligned}(6.24)\qquad &\|e_h^{k+1}\|^2_{L^2(\Omega)} - \|e_h^k\|^2_{L^2(\Omega)} + \tau\nu\|e_h^{k+1}\|^2_{X_h} \\ &\leqslant \tau\mathbf{c}\nu\|e_h^k\|^2_{L^2(\Omega)} + \frac{\tau\mathbf{c}}{\nu}\|e_h^{k+1}\|^2_{L^2(\Omega)} + \frac{\tau}{\nu}C(\nu)h^2 \\ &\quad + \tau\nu\|e_h^k\|^2_{X_h} + c_{36}\tau h^2 + \frac{c_{37}\tau h^2}{\nu} \\ &\quad + \frac{c_{40}\tau}{\nu}h^{2(1-\kappa)}(\nu^{-6} + \nu^{-4.5} + \nu^{-3}) + c_{41}\tau h^3 + c_{31}^2\tau\nu h^2,\end{aligned}$$

which gives

$$
\begin{aligned}
(6.25)\quad &\Big(1-\frac{\tau\mathbf{c}}{\nu}\Big)\|e_h^{k+1}\|^2_{L^2(\Omega)}+\tau\nu\|e_h^{k+1}\|^2_{X_h}\\
&\leqslant (1+\tau\mathbf{c}\nu)\|e_h^{k}\|^2_{L^2(\Omega)}+\tau\nu\|e_h^{k}\|^2_{X_h}+\frac{\tau}{\nu}C(\nu)h^2+c_{36}\tau h^2+\frac{c_{37}\tau h^2}{\nu}\\
&\quad+\frac{c_{40}\tau h^{2(1-\kappa)}}{\nu}(\nu^{-6}+\nu^{-4.5}+\nu^{-3})+c_{41}\tau h^3+c_{31}^2\tau\nu h^2.
\end{aligned}
$$

Using the inverse stability condition (6.4), we obtain

$$
\begin{aligned}
(6.26)\quad \Big(1-\frac{\tau\mathbf{c}}{\nu}\Big)\|e_h^{k+1}\|^2_{L^2(\Omega)}+\tau\nu\|e_h^{k+1}\|^2_{X_h}&\leqslant (1+\tau\mathbf{c}\nu)\|e_h^{k}\|^2_{L^2(\Omega)}\\
&\quad+\tau\nu\|e_h^{k}\|^2_{X_h}+\tau^2 q(\nu,h),
\end{aligned}
$$

where

$$
\begin{aligned}
(6.27)\quad q(\nu,h):=&\Big[\frac{\tilde{\tilde{c}}C(\nu)}{\nu}h+\frac{c_{37}\tilde{\tilde{c}}h}{\nu}+c_{36}\tilde{\tilde{c}}h+\frac{c_{40}\tilde{\tilde{c}}}{\nu}h^{1-2\kappa}(\nu^{-6}+\nu^{-4.5}+\nu^{-3})\\
&+c_{41}\tilde{\tilde{c}}h^2+c_{31}^2\tilde{\tilde{c}}\nu h\Big]\\
\leqslant\ & h^{1-2\kappa}\Big[\frac{C(\nu)\tilde{\tilde{c}}}{\nu}h_0^{2\kappa}+\frac{c_{37}\tilde{\tilde{c}}h_0^{2\kappa}}{\nu}+c_{36}\tilde{\tilde{c}}h_0^{2\kappa}+\frac{c_{40}\tilde{\tilde{c}}}{\nu}(\nu^{-6}+\nu^{-4.5}+\nu^{-3})\\
&+c_{41}\tilde{\tilde{c}}h_0^{1+2\kappa}+c_{31}^2\tilde{\tilde{c}}\nu h_0^{2-\kappa}\Big]
\end{aligned}
$$

for $h\in(0,h_0)$.

Now, we sum (6.26) over $k=0,\dots,m$ ($t_m\in[0,\boldsymbol{T})$), which results in

$$
\begin{aligned}
(6.28)\quad &\Big(1-\frac{\tau\mathbf{c}}{\nu}\Big)\sum_{k=0}^{m}\|e_h^{k+1}\|^2_{L^2(\Omega)}+\tau\nu\sum_{k=0}^{m}\|e_h^{k+1}\|^2_{X_h}\\
&\leqslant (1+\tau\mathbf{c}\nu)\sum_{k=0}^{m}\|e_h^{k}\|^2_{L^2(\Omega)}+\tau\nu\sum_{k=0}^{m}\|e_h^{k}\|^2_{X_h}+\tau\boldsymbol{T}q(\nu,h).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(6.29)\quad &\Big(1-\frac{\tau\mathbf{c}}{\nu}\Big)\sum_{k=0}^{m}\|e_h^{k+1}\|^2_{L^2(\Omega)}+\tau\nu\|e_h^{m+1}\|^2_{X_h}\\
&\leqslant (1+\tau\mathbf{c}\nu)\sum_{k=0}^{m}\|e_h^{k}\|^2_{L^2(\Omega)}+\tau\nu\|e_h^{0}\|^2_{X_h}+\tau\boldsymbol{T}q(\nu,h).
\end{aligned}
$$

Let us denote

$$\xi_m = \sum_{k=0}^{m} \|e_h^k\|^2_{L^2(\Omega)}. \tag{6.30}$$

Then

$$\begin{aligned}\Big(1-\frac{\tau\mathbf{c}}{\nu}\Big)(\xi_{m+1}-\|e_h^0\|^2_{L^2(\Omega)})+\tau\nu\|e_h^{m+1}\|^2_{X_h}\\ \leqslant (1+\tau\mathbf{c}\nu)\xi_m+\tau\nu\|e_h^0\|^2_{X_h}+\tau\boldsymbol{T}q(\nu,h).\end{aligned} \tag{6.31}$$

Using the estimates

$$\|e_h^0\|_{X_h} = \|u^0 - I_h u^0\|_{X_h} \leqslant c_6\|u^0\|_{H^2(\Omega)}h = \sqrt{c_{37}}h, \tag{6.32}$$

$$\begin{aligned}\|e_h^0\|_{L_2} = \|u^0 - I_h u^0\|_{L_2} \leqslant c_7\|u^0\|_{H^2(\Omega)}h^2\\ \leqslant c_7(\tilde{\tilde{c}}\tau)^{1/2}h^{3/2}\|u^0\|_{H^2(\Omega)} = \sqrt{c_{38}}\tau^{1/2}h^{3/2}\end{aligned} \tag{6.33}$$

and assuming that $\mathbf{c}\tau/\nu \leqslant 1/2$ (see the assumptions of the theorem), we get from (6.31) and (6.30) that

$$\xi_{m+1} \leqslant \frac{1+\tau\mathbf{c}\nu}{1-\tau\mathbf{c}/\nu}\xi_m + \frac{\tau\boldsymbol{T}q(\nu,h)}{1-\tau\mathbf{c}/\nu} + c_{38}\tau h^3 + \frac{c_{37}\tau\nu h^2}{1-\tau\mathbf{c}/\nu}. \tag{6.34}$$

If we set

$$A = \frac{1+\tau\mathbf{c}\nu}{1-\tau\mathbf{c}/\nu}, \tag{6.35}$$

the relation (6.34) can be written as

$$\xi_{m+1} \leqslant A\xi_m + \tau\bigg[\frac{q(\nu,h)\boldsymbol{T}}{1-\tau\mathbf{c}/\nu} + c_{38}h^3 + \frac{c_{37}\nu h^2}{1-\tau\mathbf{c}/\nu}\bigg]. \tag{6.36}$$

From this we obtain

$$\xi_m \leqslant A^m\xi_0 + \frac{A^m-1}{A-1}\tau\bigg[\frac{q(\nu,h)\boldsymbol{T}}{1-\tau\mathbf{c}/\nu} + c_{38}h + \frac{c_{37}\nu h^2}{1-\tau\mathbf{c}/\nu}\bigg]. \tag{6.37}$$

In virtue of the inequality $\tau\mathbf{c}/\nu \leqslant 1/2$, we have

$$A \leqslant 1 + 2\tau\mathbf{c}\Big(\nu + 1/\nu\Big) \leqslant \exp\Big(2\tau\mathbf{c}\Big(\nu+\frac{1}{\nu}\Big)\Big).$$

Hence,

$$(6.38)\quad \xi_m \leqslant \exp\Big(2m\mathbf{c}\tau\Big(\nu + \frac{1}{\nu}\Big)\Big)\xi_0 + \frac{\exp\Big(2m\mathbf{c}\tau(\nu + 1/\nu)\Big) - 1}{\mathbf{c}(\nu^2+1)(\nu - \mathbf{c}\tau)^{-1}}\,[q(\nu,h)\boldsymbol{T} + c_{38}h + c_{37}\nu h^2]$$
$$\leqslant \xi_0 \exp\Big(2\boldsymbol{T}\boldsymbol{c}\Big(\nu + \frac{1}{\nu}\Big)\Big) + \frac{\nu}{\mathbf{c}(\nu^2+1)}\Big(\exp\Big(2\boldsymbol{T}\boldsymbol{c}\Big(\nu + \frac{1}{\nu}\Big)\Big) - 1\Big)[q(\nu,h)\boldsymbol{T} + c_{38}h + c_{37}\nu h^2].$$

Further, due to the relation (6.33) we have

$$\begin{aligned}\xi_0 \exp\Big(2\boldsymbol{T}\boldsymbol{c}\Big(\nu + \frac{1}{\nu}\Big)\Big) &= \|e_h^0\|^2_{L^2(\Omega)} \exp\Big(2\boldsymbol{T}\mathbf{c}\Big(\nu + \frac{1}{\nu}\Big)\Big)\\ &\leqslant c_{38} \exp\Big(2\boldsymbol{T}\boldsymbol{c}\Big(\nu + \frac{1}{\nu}\Big)\Big)\tau h^3\\ &=: C^*(\nu)\tau h^3\end{aligned}$$

and we conclude from (6.38) that

$$(6.39)\qquad \xi_m \leqslant C^*(\nu)\tau h^3 + \widetilde{C}(\nu)[q(\nu,h)\boldsymbol{T} + c_{38}h^3 + c_{37}\nu h^2],$$

where

$$(6.40)\qquad \widetilde{C}(\nu) = \frac{\nu}{\mathbf{c}(\nu^2+1)}\Big(\exp\Big(2\boldsymbol{T}\boldsymbol{c}\Big(\nu + \frac{1}{\nu}\Big)\Big) - 1\Big).$$

Due to (6.5) and (6.30),

$$(6.41)\qquad \|e\|^2_{h,\tau,L^2(\Omega)} = \tau\xi_r.$$

From this and (6.39) we find that

$$\|e\|^2_{h,\tau,L^2(\Omega)} \leqslant C^*(\nu)\tau^2 h^3 + \tau\widetilde{C}(\nu)[q(\nu,h)\boldsymbol{T} + c_{38}h^3 + c_{37}\nu h^2].$$

Then, in view of (4.10), (6.20), (6.27) and the fact that $\widehat{C}(\nu) = O(\nu^{-3/2})$, $\widetilde{C}(\nu) = O(\nu \exp(2\boldsymbol{T}\mathbf{c}/\nu))$ and $q(\nu,h) = O(\nu^{-7})$, we find that

$$\|e\|^2_{h,\tau,L^2(\Omega)} \leqslant C_2 h^{2(1-\kappa)}, \quad C_2 = O(\nu^{-6}\exp(2\boldsymbol{T}\mathbf{c}/\nu)),$$

which yields estimate (6.6) a).

Now we establish estimate (6.6) b). From (6.28) we obtain

$$(6.42)\quad \Big(1-\frac{\tau\mathbf{c}}{\nu}\Big)\|e_h^{m+1}\|^2_{L^2(\Omega)}+\tau\nu\|e_h^{m+1}\|^2_{X_h} \leqslant \mathbf{c}\tau\Big(\frac{1}{\nu}+\nu\Big)\sum_{k=0}^{m}\|e_h^k\|^2_{L^2(\Omega)}+\|e_k^0\|^2_{L^2(\Omega)}+\tau\nu\|e_h^0\|^2_{X_h}+\tau\boldsymbol{T}q(\nu,h).$$

As above we assume that $2\mathbf{c}\tau\leqslant\nu$. Then (6.42) implies that

$$\tau\nu\|e_h^{m+1}\|^2_{X_h}\leqslant\mathbf{c}\tau\Big(\frac{1}{\nu}+\nu\Big)\sum_{\nu=0}^{M}\|e_h^k\|^2_{L^2(\Omega)}+\|e_h^0\|^2_{L^2(\Omega)}+\tau\nu\|e_h^0\|_{X_h}+\tau\boldsymbol{T}q(\nu,h).$$

Now the summation of this inequality over $m=0,\dots,r-1$, estimates of $\|e_h^0\|_{L^2(\Omega)}$ and $\|e_h^0\|_{X_h}$, (6.6) a) and (6.27) immediately yield (6.6) b). □

R e m a r k 2. a) The above results can be extended to the case when $\Omega\subset\mathbb{R}^3$ is a bounded polyhedral domain and q from (2.9) is greater than three. The maximum principle can be applied in this case on the basis of the results from [32].

b) There are some open questions and problems: the proof of error estimates for other combined finite volume-finite element schemes (fully explicit or implicit schemes, the method of fractional steps), the study of higher order schemes, the derivation of efficient a posteriori error estimates, and generalization to systems of equations.

c) Particularly interesting, but rather difficult, would be the investigation of the behaviour of the error in dependence on the coefficient ν. The behaviour of the constants C_2 and C_3 from the error estimates in Theorem 5 is rather pessimistic for small ν. It would be desirable to develop error estimates uniform with respect to ν. However, this has been obtained only in very few works analyzing simple problems under rather special assumptions when complete analytic behaviour of solutions is known ([1], [35] and citations in [39]).

7. Applications to viscous compressible flow

The main motivation for developing the combined finite volume-finite element schemes was the numerical simulation of viscous compressible high-speed flow. The goal was to construct a sufficiently efficient, robust and reliable method for the computation of complicated flow fields with shock waves, boundary layers and their interaction.

In what follows we describe a method combining barycentric finite volumes with nonconforming piecewise linear finite elements, applied to the solution of a high-speed flow past a cascade of profiles modeling the flow in steam and gas turbines or compressors.

We consider gas flow in a space-time cylinder $Q_{\boldsymbol{T}} = \Omega \times (0, \boldsymbol{T})$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain representing the region occupied by the fluid and $\boldsymbol{T} > 0$.

The complete system of a viscous compressible flow consisting of the continuity equation, Navier-Stokes equations and the energy equation can be written in the dimensionless form

$$\frac{\partial w}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(w)}{\partial x_s} = \sum_{s=1}^{2} \frac{\partial R_s(w, \nabla w)}{\partial x_s} \quad \text{in } Q_{\boldsymbol{T}}. \tag{7.1}$$

Here

$$\begin{aligned} w &= (w_1,\, w_2,\, w_3,\, w_4)^{\mathrm{T}} = (\varrho, \varrho v_1, \varrho v_2, e)^{\mathrm{T}}, \\ w &= w(x,t), \quad x \in \Omega,\ t \in (0, \boldsymbol{T}), \\ f_s(w) &= (\varrho v_s, \varrho v_s v_1 + \delta_{s1} p, \varrho v_s v_2 + \delta_{s2} p, (e+p) v_s)^{\mathrm{T}}, \\ R_s(w, \nabla w) &= \left(0, \tau_{s1}, \tau_{s2}, \tau_{s1} v_1 + \tau_{s2} v_2 + \frac{\gamma}{\mathrm{Re}\ \mathrm{Pr}} \frac{\partial \theta}{\partial x_s}\right)^{\mathrm{T}}, \\ \tau_{sr} &= \frac{1}{\mathrm{Re}} \left[\left(\frac{\partial v_s}{\partial x_r} + \frac{\partial v_r}{\partial x_s}\right) - \frac{2}{3} \operatorname{div} \boldsymbol{v} \delta_{sr}\right], \quad s, r = 1, 2. \end{aligned} \tag{7.2}$$

From thermodynamics we have

$$p = (\gamma - 1)(e - \varrho |\boldsymbol{v}|^2/2), \quad e = \varrho(\theta + |\boldsymbol{v}|^2/2). \tag{7.3}$$

We use the standard notation for dimensionless quantities: t—time, x_1, x_2—Cartesian coordinates in $\mathbb{R}^2$, ϱ—density, $\boldsymbol{v} = (v_1, v_2)$—velocity vector with components v_s in the directions x_s, $s = 1, 2$, p—pressure, θ—absolute temperature, e—total energy, τ_{sr}—components of the viscous part of the stress tensor, δ_{sr}—Kronecker delta, $\gamma > 1$—Poisson adiabatic constant, Re—Reynolds number, Pr—Prandtl number. We neglect the outer volume force. The functions f_s, called inviscid (Euler) fluxes, are defined in the set $D = \{(w_1, \ldots, w_4) \in \mathbb{R}^4;\ w_1 > 0\}$. The viscous terms R_s are defined in $D \times \mathbb{R}^8$. (Due to physical reasons it is also suitable to require $p > 0$.)

System (7.1), (7.3) is equipped with an initial condition

$$w(x, 0) = w^0(x), \quad x \in \Omega \tag{7.4}$$

(which means that at time $t = 0$ we prescribe, e.g., ϱ, v_1, v_2 and θ) and boundary conditions. In the simulation of the flow past a cascade of profiles the region occupied by the fluid is represented by an infinitely connected plane domain $\widetilde{\Omega}$, bounded in one space direction (say x_1) and unbounded but periodic in the other direction (x_2).

Assuming also the periodicity of the flow field, we can choose the computational domain Ω in the form of one period of the original domain $\widetilde{\Omega}$ (see Fig. 3). The boundary $\partial\Omega$ is formed by disjoint parts Γ_I, Γ_O, Γ_W, Γ^+ and Γ^-. On Γ_I, Γ_O and Γ_W, representing the inlet, outlet and impermeable profile, respectively, we prescribe conditions

$$\text{(7.5)} \qquad \begin{aligned} &\text{(i)} \quad \varrho = \varrho^*, \quad v_s = v_s^*, \quad s = 1,2, \quad \theta = \theta^* && \text{on } \Gamma_I, \\ &\text{(ii)} \quad v_s = 0, \quad s = 1,2, \quad \frac{\partial\theta}{\partial n} = 0 && \text{on } \Gamma_W, \\ &\text{(iii)} \quad \sum_{s=1}^{2} \tau_{sr} n_s = 0, \quad r = 1,2, \quad \frac{\partial\theta}{\partial n} = 0 && \text{on } \Gamma_O. \end{aligned}$$

Here $\partial/\partial n$ denotes the derivative in the direction of the unit outer normal $\boldsymbol{n} = (n_1, n_2)^{\mathrm{T}}$ to $\partial\Omega$; w^0, ϱ^*, v_s^* and θ^* are given functions.

Moreover, the arcs Γ^- and Γ^+ are piecewise linear artificial cuts such that

$$\text{(7.6)} \qquad \Gamma^+ = \{(x_1, x_2 + \tau);\ (x_1, x_2) \in \Gamma^-\},$$

where $\tau > 0$ is the width of one period of the cascade in the direction x_2. On $\Gamma^\pm$ we consider the periodicity condition

$$\text{(7.7)} \qquad w(x_1, x_2 + \tau, t) = w(x_1, x_2, t), \quad (x_1, x_2) \in \Gamma^-.$$

The same condition is imposed on the first-order derivatives of the vector function w.

Let us note that equations (7.1) and (7.3) are of hyperbolic-parabolic type and that nothing is known about the existence and uniqueness of the solution of problem (7.1), (7.3)–(7.5) and (7.7).

We carry out the discretization of system (7.1) similarly as in Section 2. Assuming that Ω is a polygonal domain, we denote by $\mathcal{T}_h$ a triangulation of Ω and by Q_i, $i \in J$, the midpoints of the sides of all triangles $T \in \mathcal{T}_h$. We use nonconforming piecewise linear finite elements. This means that the components of the state vector are approximated by functions from the finite dimensional space X_h defined in (3.9). Further, we set $\boldsymbol{X}_h = [X_h]^4$ and

(7.8) a) $\boldsymbol{V}_h = \{\varphi_h = (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \in \boldsymbol{X}_h;\ \varphi_i(Q_j) = 0$ for midpoints Q_j lying on the part of $\partial\Omega$ where w_i satisfies the Dirichlet condition and φ_h satisfies the periodicity condition (7.7)$\}$,

b) $\boldsymbol{W}_h = \{w_h \in \boldsymbol{X}_h$; its components satisfy the Dirichlet boundary conditions following from (7.5) and periodicity condition (7.7)$\}$.

Moreover, we consider a partition $0 = t_0 < t_1 < \dots$ of the interval $(0, \boldsymbol{T})$ and set $\tau_k = t_{k+1} - t_k$.

Multiplying (7.1) considered on a time level t_k by any $\varphi_h \in \boldsymbol{V}_h$, integrating over Ω, using Green's theorem, taking into account the boundary conditions (7.5) and the periodicity conditions (7.7) for w, φ_h and for the derivatives of w, we obtain the following integral identity:

$$\int_\Omega \frac{\partial w}{\partial t}\varphi_h \,\mathrm{d}x + \int_\Omega \sum_{s=1}^{2} \frac{\partial f_s(w)}{\partial x_s}\varphi_h \,\mathrm{d}x + \int_\Omega \sum_{s=1}^{2} R_s(w, \nabla w)\frac{\partial \varphi_h}{\partial x_s}\,\mathrm{d}x = 0. \tag{7.9}$$

Now approximating the time derivative by the difference and the convective terms with fluxes f_s by a form b_h defined similarly as in (3.20) with the aid of the finite volume approach and evaluating the integrals with the aid of the quadrature formula using midpoints of sides as integration points, i.e.,

$$\int_T F \,\mathrm{d}x \approx \frac{1}{3}|T| \sum_{i=1}^{3} F(Q_T^i) \tag{7.10}$$

for $F \in C(T)$ and a triangle T with midpoints of sides Q_T^i, $i = 1, 2, 3$, we arrive at the following scheme for the calculation of an approximate solution w_h^{k+1} on the $(k+1)$-st time level:

$$\begin{aligned}
&\text{a)} && w_h^{k+1} \in \boldsymbol{W}_h,\\
&\text{b)} && (w_h^{k+1}, \varphi_h)_h = (w_h^k, \varphi_h)_h - \tau_k\{b_h(w_h^k, \varphi_h) + a_h(w_h^k, \varphi_h)\}\\
& && \forall\, \varphi_h \in \boldsymbol{V}_h.
\end{aligned} \tag{7.11}$$

Here

$$(w_h, \varphi_h)_h = \frac{1}{3} \sum_{T \in \mathcal{T}_h} |T| \sum_{i=1}^{3} w_h(Q_T^i)\varphi_h(Q_T^i), \quad w_h, \varphi_h \in \boldsymbol{X}_h \tag{7.12}$$

and $a_h(w_h^k, \varphi_h)$ approximates the viscous terms of the form

$$\int_{\Omega_h} \sum_{s=1}^{2} R_s(w_h^k, \nabla w_h^k)\frac{\partial \varphi_h}{\partial x_s}\,\mathrm{d}x.$$

Namely,

(7.13)

$$
\begin{aligned}
a_h(w_h,\varphi_h) &= a_h^1(w_h,\varphi_h) + \ldots + a_h^4(w_h,\varphi_h), \qquad a_h^1 \equiv 0,\\
a_h^2(w_h,\varphi_h) &= \sum_{T\in\mathcal{T}_h} |T| \biggl\{ 2\frac{\partial v_{h,1}}{\partial x_1}\biggr|_T \frac{\partial \varphi_{h,2}}{\partial x_1}\biggr|_T - \frac{2}{3}(\operatorname{div}\boldsymbol{v}_h)|_T \frac{\partial \varphi_{h,2}}{\partial x_1}\biggr|_T\\
&\quad + \biggl(\frac{\partial v_{h,2}}{\partial x_1}\biggr|_T + \frac{\partial v_{h,1}}{\partial x_2}\biggr|_T\biggr)\frac{\partial \varphi_{h,2}}{\partial x_2}\biggr|_T\biggr\}/\mathrm{Re},\\
a_h^3(w_h,\varphi_h) &= \sum_{T\in\mathcal{T}_h} |T| \biggl\{ \biggl(\frac{\partial v_{h,2}}{\partial x_1}\biggr|_T + \frac{\partial v_{h,1}}{\partial x_2}\biggr|_T\biggr)\frac{\partial \varphi_{h,3}}{\partial x_1}\biggr|_T\\
&\quad + 2\frac{\partial v_{h,2}}{\partial x_2}\biggr|_T \frac{\partial \varphi_{h,3}}{\partial x_2}\biggr|_T - \frac{2}{3}(\operatorname{div}\boldsymbol{v}_h)|_T \frac{\partial \varphi_{h,3}}{\partial x_2}\biggr|_T\biggr\}/\mathrm{Re},\\
a_h^4(w_h,\varphi_h) &= \sum_{T\in\mathcal{T}_h} \biggl\{ \frac{1}{3}|T| \biggl(\tau_{h,11}|_T \sum_{i=1}^3 v_{h,1}(Q_T^i) + \tau_{h,12}|_T \sum_{i=1}^3 v_{h,2}(Q_T^i)\biggr)\frac{\partial \varphi_{h,4}}{\partial x_1}\biggr|_T\\
&\quad + \frac{1}{3}|T| \biggl(\tau_{h,21}|_T \sum_{i=1}^3 v_{h,1}(Q_T^i) + \tau_{h,22}|_T \sum_{i=1}^3 v_{h,2}(Q_T^i)\biggr)\frac{\partial \varphi_{h,4}}{\partial x_2}\biggr|_T\\
&\quad + \frac{\gamma}{\mathrm{Re}\,\mathrm{Pr}}|T| \sum_{j=1}^2 \frac{\partial \theta_h}{\partial x_j}\biggr|_T \frac{\partial \varphi_{h,4}}{\partial x_j}\biggr|_T\biggr\},\\
\tau_{h,rs}|_T &= \frac{1}{\mathrm{Re}}\biggl(\frac{\partial v_{h,r}}{\partial x_s} + \frac{\partial v_{h,s}}{\partial x_r} - \frac{2}{3}\operatorname{div}\boldsymbol{v}_h \delta_{rs}\biggr)\biggr|_T = \text{const.}
\end{aligned}
$$

By $v_{h,s}$ and θ_h we denote the functions from the space X_h approximating the velocity components and temperature. Moreover, b_h representing the approximation of the convective terms is expressed as

$$
b_h(w_h,\varphi_h) = \sum_{i\in J} \varphi_h(Q_i) \sum_{j\in S(i)} \sum_{\alpha=1}^{\beta_{ij}} H\bigl(w_h(Q_i), w_h(Q_j), \boldsymbol{n}_{ij}^\alpha\bigr)\,\ell_{ij}^\alpha, \qquad w_h,\varphi_h \in \boldsymbol{X}_h. \tag{7.14}
$$

As H we use here the well-known *Osher-Solomon numerical flux* (cf. [38], [42], [15]).

From (7.14) we see that the used scheme is fully explicit. The reason is its simple algorithmization. However, its application is conditioned by the use of a suitable *stability condition*. Namely, the following condition has been used in practical computations:

$$
\begin{aligned}
\max\biggl\{ &\max_{i\in J} \frac{\tau_k}{|D_i|}|\partial D_i| \Bigl(\max_{j\in S(i),\alpha=1,\ldots,\beta_{ij}} \varrho(\mathbb{P}(w_i^k, \boldsymbol{n}_{ij}^\alpha))\Bigr),\\
&\max_{T\in\mathcal{T}_h} \frac{3}{4}\frac{h(T)}{\sigma(T)}\frac{\tau_k}{|T|}\frac{1}{\mathrm{Re}}\biggr\} \leqslant \mathrm{CFL} \approx 0.85,
\end{aligned}
\tag{7.15}
$$

where $\mathbb{P}(w, \boldsymbol{n}) = \sum_{s=1}^{2} (Df_s(w)/Dw)n_s$, $\varrho(\mathbb{P})$ = spectral radius of the matrix $\mathbb{P}$, $h(T)$ is the length of the maximal side of $T \in \mathcal{T}_h$ and $\sigma(T)$ is the radius of the largest circle inscribed into T. Condition (7.15) is obtained on the basis of linearization and in analogy with the scalar problem (for details see [28]).

The use of the semiimplicit (or implicit) version of scheme (7.11) would require the solution of a nonlinear algebraic system on each time level.

Another possible time discretization which we have applied with success is the *inviscid-viscous operator splitting* described, e.g., in [6], [13], [14], [15].

In order to get sufficiently accurate computational results with a good resolution of shock waves and boundary layers, it is suitable to apply an *adaptive mesh refinement* strategy. We have developed several adaptive techniques based on a shock indicator and error indicators, leading to satisfactory results (see, e.g., [6], [7], [12], [21], [29]).

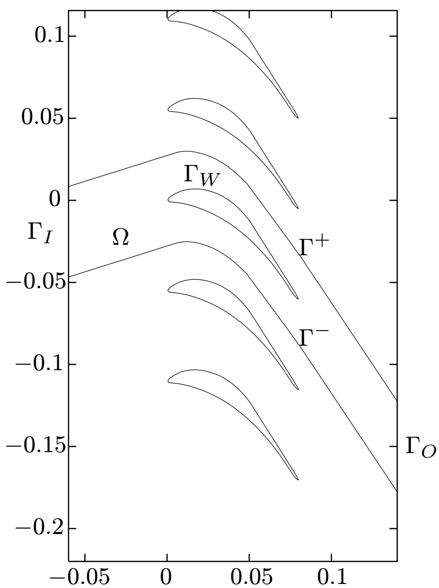

Figure 3. Cascade of profiles with the computational domain Ω and the boundary parts Γ_I, Γ_O, Γ_W and the artificial periodical cuts Γ^+ and Γ^-.

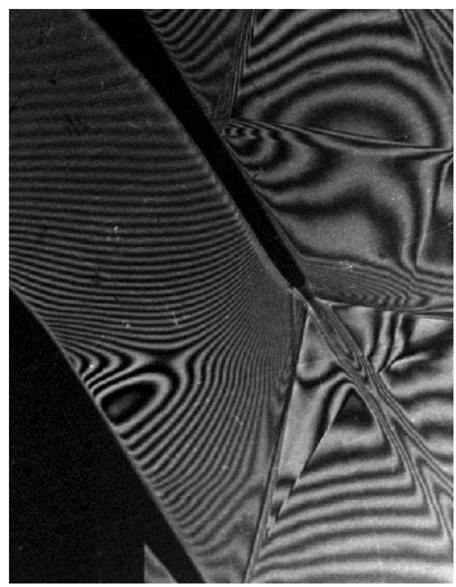

Figure 4. The wind tunnel interferogram showing density isolines (Courtesy of the Institute of Thermomechanics, Academy of Science of Czech Republic, Prague).

E x a m p l e 1. The method described was applied to the numerical simulation of the flow past a turbine cascade shown in Fig. 3. The goal was to obtain the steady state solution with the aid of the time stabilization for $t \to \infty$. The computational results are compared with a wind tunnel experiment (by courtesy of the Institute

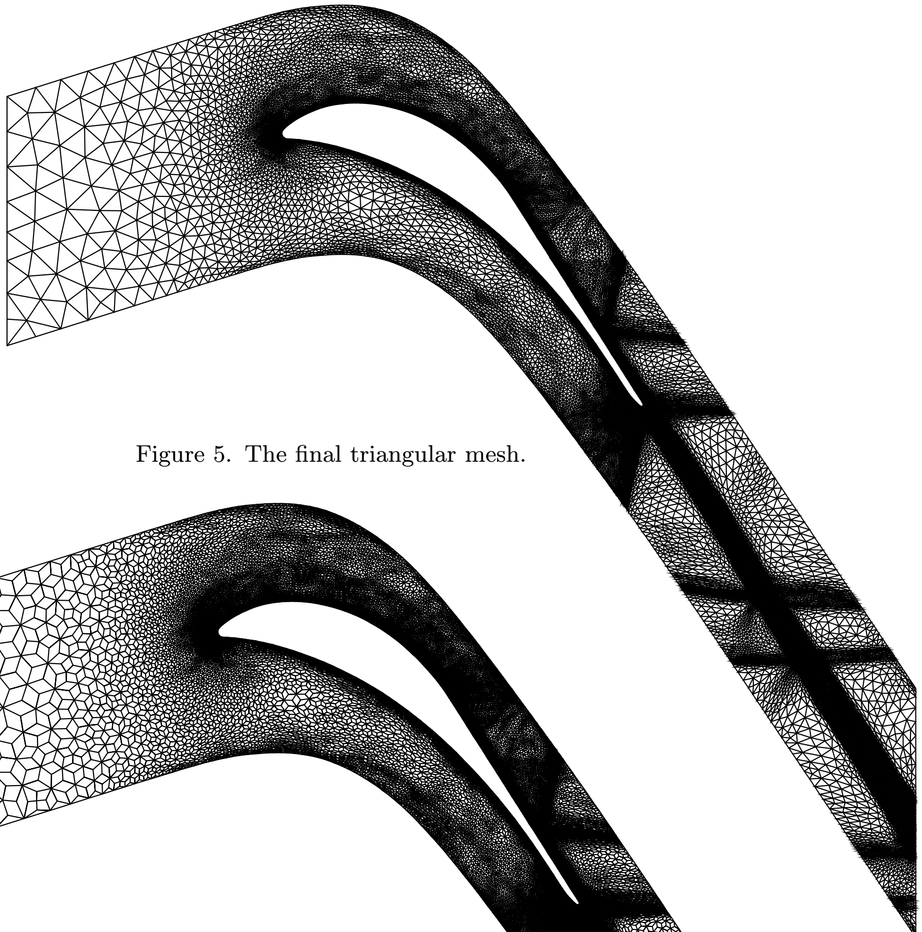

Figure 5. The final triangular mesh.

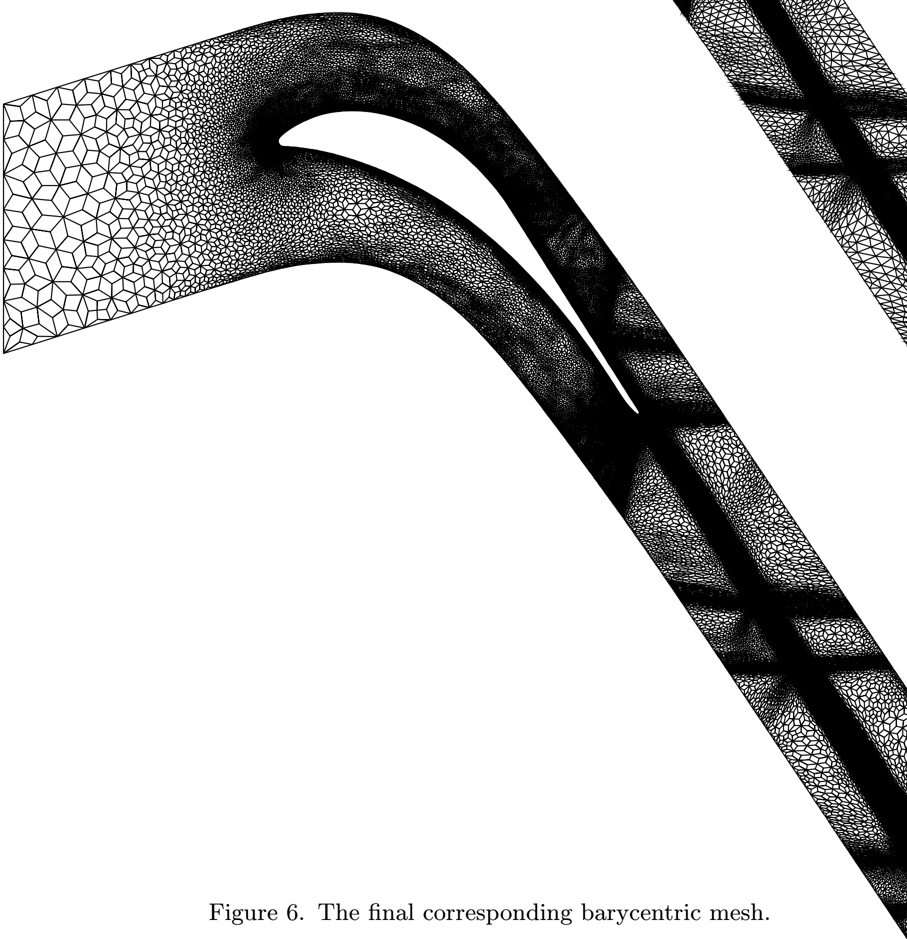

Figure 6. The final corresponding barycentric mesh.

of Thermomechanics of the Academy of Sciences of the Czech Republic in Prague, see [44]). The experiment and computations were performed for the following data: angle of attack $= 19^\circ\,18'$, inlet Mach number $= 0.32$, outlet Mach number $= 1.18$, $\gamma = 1.4$, Reynolds number $\mathrm{Re} = 1.5 \cdot 10^6$, Prandtl number $\mathrm{Pr} = 0.72$.

Fig. 4 represents the wind tunnel interferogram showing density isolines (see [44]). In Fig. 5 and Fig. 6 the final triangular and the corresponding barycentric mesh obtained with the aid of anisotropic mesh refinement ([6], [7]) are plotted, respectively. Fig. 7 shows the pressure distribution along the profile compared with the measurement. Further, Fig. 8 shows the computed density isolines. We see that a good agreement of computational results with experiment was achieved. Let us note that the inviscid-viscous operator splitting method ([6], [13], [14], [15]) gives nearly identical results.

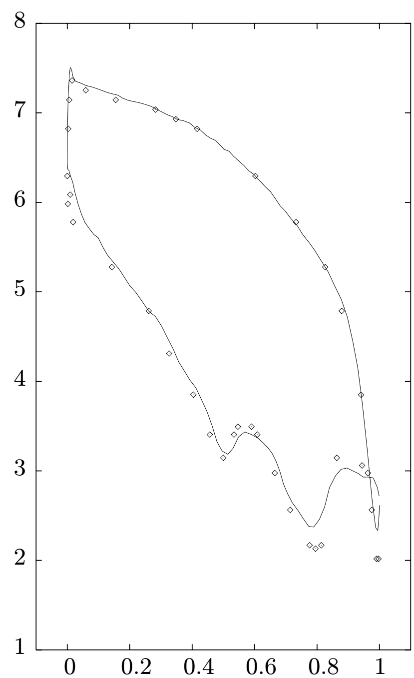

Figure 7. Pressure distribution along the profile compared with measurement.

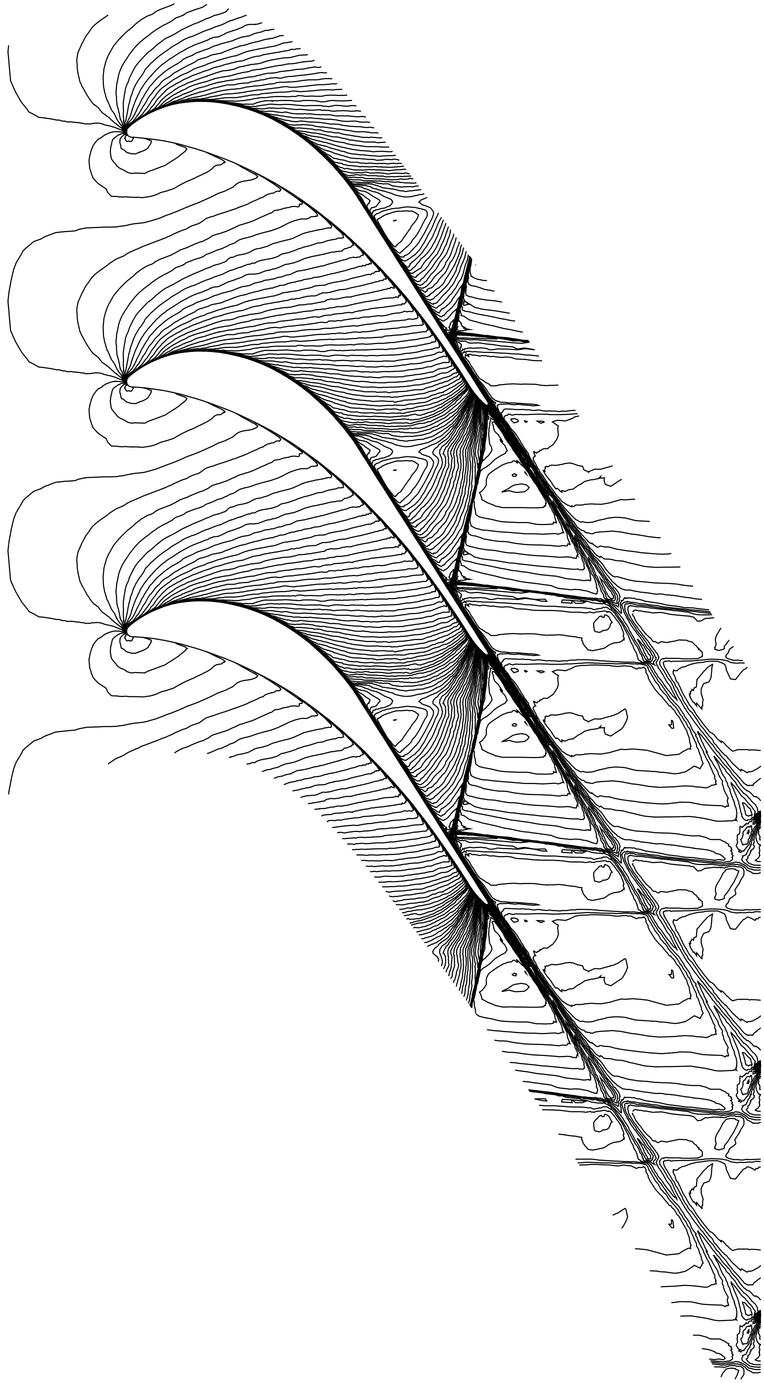

Figure 8. Density isolines.

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