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GLOBAL EXISTENCE FOR A NUCLEAR FLUID IN ONE DIMENSION: THE $T > 0$ CASE

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Abstract. We consider a simplified one-dimensional thermal model of nuclear matter, described by a system of Navier-Stokes-Poisson type, with a non monotone equation of state due to an effective nuclear interaction.

We prove the existence of globally defined (large) solutions of the corresponding free boundary problem, with an exterior pressure P which is not required to be positive, provided sufficient thermal dissipation is present. We give also a partial description of the asymptotic behaviour of the system, in the two cases $P > 0$ and $P < 0$.

Keywords: Navier-Stokes equations, compressible fluid

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1. Introduction

A lot of theoretical work has been devoted in recent years to hydrodynamical description of low energy nuclear collisions between heavy ions, for which physicists expect phase transition phenomena [1], analogous to condensation-nucleation processes in classical fluids.

As nuclei are only finite systems, one realizes that a hydrodynamical regime for them can hold only for large ones. Although such nuclei are near to unstability, they have been produced in recent experiments.

Another physical context where this fluid approximation is valid is astrophysics of neutron stars, which can be seen as "giant" nuclei.

So we consider a fluid limit for nuclear matter, which is a homogeneous mixture of two kinds of nuclear constituents: protons (positively charged) and neutrons (not charged), taking into account the short range nuclear interaction (we consider a simplified version of the so-called Skyrme interaction [8]) acting on both kinds of particles, together with the long range Coulomb interaction acting only between charged particles (protons).

A simple attractive situation to begin with is the one dimension geometry, which has been largely considered by nuclear physicists to test more realistic situations [2], [3].

In [5] a 3d simplified hydrodynamical model of nuclear matter has been presented, leading to a compressible Navier-Stokes system with some special features due to quantum effects, and in [6] we proved global existence and large time behavior for a related zero-temperature model.

However, a well known difficulty in the hydrodynamical limit procedures is the completely coherent derivation of the dissipative process from the microscopic theory. In the present work, we put "by hand" the viscous and thermal effects in our model, in the spirit of [4].

In what follows, we extend [6] to the $T > 0$ case, describing hot nuclear matter, and we show that, despite a possible destabilizing influence of the pressure, which is non monotone and not always positive, the presence of viscous and thermal dissipation is sufficient to prevent the solution from developing singularities in finite time.

Finally, we also give some partial information on the large time behaviour of the system, postponing a more complete asymptotic study to another work [7].

If we denote by $u(x,t) = \frac{1}{n(x,t)}$ the specific volume, $v(x,t)$ the velocity, $e(x,t)$ the internal energy, $\theta(x,t)$ the temperature, $Q(x,t)$ the thermal flux, $\sigma(x,t)$ the stress, Φ the (Coulomb) potential of electrostatic interaction between protons, the (lagrangian) Navier-Stokes-Poisson system to be solved is (see [5])

$$\tag{1} \begin{cases} u_t = v_x, \\ v_t = \sigma_x - \dfrac{\Phi_x}{u}, \\ e_t + pv_x = -Q_x + \nu\, \dfrac{v_x^2}{u}, \\ \Big(\dfrac{\Phi_x}{u}\Big)_x = -g_C, \end{cases}$$

for $t \geqslant 0$ and $x \in [0, M]$, where M is the (conserved) mass of the slab and $\nu > 0$ is the constant viscosity coefficient.

One computes easily the following formula for the Coulomb contribution:

$$\tag{2} \frac{\Phi_x}{u} = g_C\Big(\frac{M}{2} - x\Big),$$

where g_C is a positive coupling constant.

In this formula, we suppose explicitly that the potential is zero at the center of the slab.

The stress is now $\sigma(u,v,\theta) = -p(u,\theta) + \nu \frac{v_x}{u}$, where p is the pressure given in the low order Thomas-Fermi approximation by

$$p(u,\theta) = \frac{3}{8}\,t_0 u^{-2} + \frac{1}{8}\,t_3 u^{-3} + \frac{\hbar^2}{5m}\Big(\frac{3\pi^2}{2}\Big)^{2/3} u^{-5/3} + \frac{R\theta}{u}. \tag{3}$$

The constants $t_0 < 0$ and $t_3 > 0$ are the parameters of the Skyrme interaction for nuclear matter in the so-called "spin isospin-saturated isoscalar" approximation [2], and the last term is the well known high temperature contribution for a degenerate Fermi gas with constant R.

The internal energy is

$$e(u,\theta) = \varepsilon(u,\theta) + e_m. \tag{4}$$

In this expression, the first contribution is the "proper" internal energy, given by

$$\varepsilon(u,\theta) = \frac{3}{8}\,t_0 u^{-1} + \frac{1}{16}\,t_3 u^{-2} + \frac{3\hbar^2}{10m}\Big(\frac{3\pi^2}{2}\Big)^{2/3} u^{-2/3} + C_v\theta,$$

where $C_v = \frac{3}{2}R$ and e_m is a "zero point" energy, which is a positive constant suitably chosen to ensure that $e(u,\theta) \geqslant 0$ for any $0 < u < \infty$ and $0 \leqslant \theta < \infty$.

The thermal flux is $Q(u,\theta) = -\kappa(u,\theta)\frac{\theta_x}{u}$, where κ satisfies the growth conditions

$$\begin{cases} \kappa_1(1+\theta^q) \leqslant \kappa(u,\theta) \leqslant \kappa_2(1+\theta^q), \\ |\kappa_u(u,\theta)| + |\kappa_\theta(u,\theta)| + |\kappa_{uu}(u,\theta)| \leqslant \kappa_2(1+\theta^q), \end{cases} \tag{5}$$

in which the coefficients κ_1, κ_2 and q are positive.

At this point, we suspect that the exponent q in (5) plays a major role in the a priori estimates.

Recently, several authors [10], [14], [11] have considered analogous problems for general fluids or solids, under various growth constraints of the same type for the state functions $p(u,\theta)$, $e(u,\theta)$ and $\kappa(u,\theta)$, with respect to their arguments.

Unfortunately, these constraints are not satisfied in our model, and we have to adapt their methods.

To get a well-posed problem, we consider, for each x in $[0,M]$, the initial conditions

$$(u,v,\theta)(x,0) = (u_0,v_0,\theta_0)(x) \tag{6}$$

together with, for each $t \geqslant 0$, dynamical conditions for the stress on the (free) boundaries:

$$\sigma(0,t) = \sigma(M,t) = -P, \tag{7}$$

where P is a real constant (not required to be positive).

Finally, we assume Neumann boundary conditions for the flux:

$$Q(0,t) = Q(M,t) = 0. \tag{8}$$

We finally suppose that the following compatibility conditions hold between the data:

$$\begin{cases} \sigma(u_0, v_0, \theta_0)(0,0) = \sigma(u_0, v_0, \theta_0)(M,0) = -P, \\ Q(u_0, \theta_0)(0,0) = Q(u_0, \theta_0)(M,0) = 0. \end{cases} \tag{9}$$

We emphasize that the equation of state we consider presents some analogy with Van der Waals theory of liquid-gas transition, which is not surprising, due to the similarities of the interaction between particles.

An elementary computation shows that there exists a critical temperature θ_c such that, if $\theta > \theta_c$, the fluid is always in the gaseous phase, no matter how much it is compressed.

Here, the critical temperature θ_c is obtained by solving the system

$$\left.\frac{\partial p}{\partial u}\right|_{\theta} = \left.\frac{\partial^2 p}{\partial u^2}\right|_{\theta} = 0. \tag{10}$$

One can check that this system has a unique positive solution.

One also checks that, if $\theta < \theta_c$, the small u region corresponds to the liquid phase (where $p \sim \frac{2a_3}{u^3}$), and that the large u region corresponds to the gaseous phase (where $p \sim \frac{R\theta}{u}$), these two disconnected regions being stable under small perturbations.

As in the Van der Waals theory, there also exists an unstable subdomain in the low density region (spinodal region), corresponding to $\left.\frac{\partial p}{\partial u}\right|_{\theta} > 0$, where small perturbations can grow.

Namely, there exist exactly two points $u_m(\theta)$ and $u_M(\theta)$ satisfying $\frac{\partial p}{\partial u}(u_m, \theta) = \frac{\partial p}{\partial u}(u_M, \theta) = 0$, such that the range $0 < u < u_m$ corresponds to the dense region, $u_m < u < u_M$ corresponds to the unstable region, and $u_M < u$ corresponds to the dilute phase. The two phases are separated by the spinodal curve defined as the set of pairs $(u_m, p(u_m, \theta))$ and $(u_M, p(u_M, \theta)$.

We have

$$\begin{cases} p_u(u,\theta) < 0 \ \text{ if } \ 0 < u < u_m(\theta) \ \text{ or } \ u > u_M(\theta), \\ p_u(u,\theta) > 0 \ \text{ if } \ u_m(\theta) < u < u_M(\theta). \end{cases} \tag{11}$$

We see that there also exists a temperature $\bar{\theta}$ such that

1. if $\theta > \bar{\theta}$, the pressure $p(u,\theta)$ is always positive for $0 < u < \infty$,
2. if $0 < \theta < \bar{\theta}$, there exist exactly two numbers $0 < u^- < u^+ < \infty$, depending on θ, such that the pressure $p(u,\theta)$ behaves as follows:

$$\begin{cases} p(u,\theta) > 0 \ \text{ if } \ 0 < u < u^- \ \text{ or } \ u > u^+, \\ p(u,\theta) < 0 \ \text{ if } \ u^- < u < u^+. \end{cases} \tag{12}$$

As $\theta \to p(u,\theta)$ is increasing, one checks easily the following global bound:

$$\forall u > 0, \ \theta > 0\colon \ p(u,\theta) \geqslant p_m, \tag{13}$$

where $-\infty < p_m < 0$.

Our purpose in the sequel is to prove that, provided the above viscous and thermal dissipative processes are present, our model is globally defined in time.

More precisely, we are going to show that the problem (1), (6), (7), (8), (9) with the pressure law (3) and the energy (4) can be investigated by using some ideas developed recently to study thermoviscoelastic materials [14] modulo some additional modifications.

We use the notation $(C^r[0,M], \|\cdot\|_r)$ and $(C^{r,r/2}(Q_T), |\!|\!|\cdot|\!|\!|_r)$ with $Q_T = [0,M] \times [0,T]$ for the usual Hölder spaces (see [9]).

Our main result is the following.

Theorem 1. *Assume that u_0, u_{0x}, v_0, v_{0x}, v_{0xx}, θ_0, θ_{0x}, θ_{0xx} are in $C^r[0,M]$ for some $0 < r < 1$.*

Suppose that u_0, θ_0 are positive on $[0,M]$ and that the compatibility conditions hold between boundary conditions and initial data.

Then if $q \geqslant 2$, there exists a unique solution $\big(u(x,t), v(x,t), \theta(x,t)\big)$ to the problem (1)–(9) such that $u(x,t) > 0$, $\theta(x,t) > 0$ on $[0,M] \times [0,\infty)$.

Moreover, for any $T > 0$ we have

$$(u, u_x, u_t, u_{xt}, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}) \in (C^{r,r/2}(Q_T))^{12},$$

and

$$(u_{tt}, v_{xt}, \theta_{xt}) \in (L^2(Q_T))^3.$$

2. Proof of theorem 1

The proof of Theorem 1 is based on the Leray-Schauder fixed point theorem, relying on suitable a priori estimates. In fact, one shows, by using a straightforward extension of a result of Kawohl [10], that the proof of Theorem 1 is completed by establishing the following result.

Theorem 2. *Suppose that the problem* (1)–(9) *has at least a classical solution* $\big(u(x,t), v(x,t), \theta(x,t)\big)$.

Then the functions $(u, v, \theta, v_x, \theta_x)$ *are bounded on* $C^{r,r/2}(Q_T)$ *with* $r = 1/3$*:*

$$\|u\|_{1/3} + \|v\|_{1/3} + \|\theta\|_{1/3} + \|v_x\|_{1/3} + \|\theta_x\|_{1/3} \leqslant C(T),$$

where $C(T)$ *depends only on* T*, the physical parameters of the problem, and the data.*

As usual, we begin with some energy estimates.

Lemma 1. *Let* $T > 0$ *be an arbitrary positive number.*

The following estimates hold for any $t \leqslant T$*:*

$$\int_0^M \Big[\frac{1}{2}\, v^2 + e + u\Big]\, \mathrm{d}x \leqslant C(T), \tag{14}$$

$$\Phi(t) + \int_0^t \Psi(t)\, \mathrm{d}t \leqslant C(T), \tag{15}$$

where $\Phi(t) = \int_0^M [R(u - \log u - 1) + C_v(\theta - \log\theta - 1)]\, \mathrm{d}x$, $\Psi(t) = \int_0^M \big(\frac{v_x^2}{u\theta} + \kappa\frac{\theta_x^2}{u\theta^2}\big)\, \mathrm{d}x$, *and* $C(T)$ *is a positive constant depending only on* T*, the physical parameters of the problem and the data.*

P r o o f. Multiplying by v the second relation in (1) and adding the third relation (1), we find the conservation law

$$\Big(\frac{1}{2}\, v^2 + e + g_C\Big(\frac{1}{2}\, M - x\Big) r\Big)_t = (\sigma v - Q)_x, \tag{16}$$

where $r = r(x,t)$ is the Lagrangian position defined by $\frac{\partial}{\partial t} r(x,t) = v(x,t)$.

Using the first equation (1), one sees, by integrating by parts, that

$$\Big(\int_0^M g_C\Big(\frac{1}{2}\, M - x\Big) r(x,t)\, \mathrm{d}x\Big)_t = \Big(\int_0^M \frac{1}{2}\, g_C x(x-M) u(x,t)\, \mathrm{d}x\Big)_t.$$

Now, by integrating (16) on $[0, M]$ and using boundary conditions, we obtain

$$\int_0^M \Big[\frac{1}{2}\,v^2 + e + f(x)u\Big]\,\mathrm{d}x = E_0, \tag{17}$$

where $f(x) \equiv P - \frac{1}{2}\,g_C\,x(M-x)$ and $E_0 = \int_0^M \big[\frac{1}{2}\,v_0^2 + e_0 + f(x)u_0\big]\,\mathrm{d}x$.

As the quantity $f(x)$ has not a definite sign, we need some extra information to get (14). For this purpose, we use an argument of Nagasawa [18].

First, we remark that, by superposing if necessary a rigid motion, one can always suppose that

$$\int_0^M v(x,t)\,\mathrm{d}x = 0.$$

Now, as the function $\Omega(x,t) = \int_0^x v(y,t)\,\mathrm{d}y$ is clearly the unique solution of the Dirichlet problem

$$\begin{cases} \Omega_{xx} = u_t, \\ \Omega(0) = \Omega(M) = 0, \end{cases} \tag{18}$$

we get the explicit formula

$$\Omega(x,t) = \int_0^M G(x,y)\,u_t(y,t)\,\mathrm{d}y, \tag{19}$$

where $G(x,y)$ is the Green function of the problem (18), given by $G(x,y) = \frac{1}{M}[xy - M\cdot\min(x,y)]$ for $(x,y) \in [0,M]\times[0,M]$.

By multiplying (19) by u, integrating on $[0,M]$, integrating by parts the right-hand sides and using the symmetry of G, we get

$$\int_0^M u(x,t)\int_0^x v(y,t)\,\mathrm{d}y = \frac{1}{2}\,\frac{\partial}{\partial t}\int_0^M\!\!\int_0^M G(x,y)\,u(x,t)\,u(y,t)\,\mathrm{d}x\,\mathrm{d}y. \tag{20}$$

By integrating on $[0,x]$ the second relation in (1), multiplying by u, and integrating on $[0,M]\times[0,t]$, we find

$$\begin{aligned} \int_0^t\!\!\int_0^M \big(v^2 + (p(u,\theta) - f(x))u\big)\,\mathrm{d}x\,\mathrm{d}s + \nu\int_0^M u_0(x)\,\mathrm{d}x \\ - \int_0^M u_0(x)\int_0^x v_0(y)\,\mathrm{d}y\,\mathrm{d}x \\ = \nu\int_0^M u(x,t)\,\mathrm{d}x - \int_0^M u(x,t)\int_0^x v(y,t)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{21}$$

By integrating one more in t we obtain

$$\begin{aligned}
(22)\qquad & \int_0^t \int_0^s \int_0^M \big(v^2 + (p(u,\theta) - f(x))u\big)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s \\
&= \nu \int_0^t \int_0^M u(x,s)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t \int_0^M u(x,s) \int_0^x v(y,s)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \nu\, t \int_0^M u_0(x)\,\mathrm{d}x + t \int_0^M u_0(x) \int_0^x v_0(y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

After (20), the second term in the right-hand side is

$$-\frac{1}{2}\Big(\int_0^M\!\!\int_0^M G(x,y)\,u(x,t)\,u(y,t)\,\mathrm{d}x\,\mathrm{d}y - \int_0^M\!\!\int_0^M G(x,y)\,u_0(x)\,u_0(y)\,\mathrm{d}x\,\mathrm{d}y\Big),$$

so, by putting it into (22), we obtain

$$\begin{aligned}
(23)\qquad & \int_0^t \int_0^s \int_0^M \big(v^2 + (p(u,\theta) - f(x))u\big)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s \\
&= \nu \int_0^t \int_0^M u(x,s)\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{2}\int_0^M\!\!\int_0^M G(x,y)\,u(x,t)\,u(y,t)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \frac{1}{2}\int_0^M\!\!\int_0^M G(x,y)\,u_0(x)\,u_0(y)\,\mathrm{d}x\,\mathrm{d}y - \nu t \int_0^M u_0(x)\,\mathrm{d}x \\
&\quad + t \int_0^M u_0(x) \int_0^x v_0(y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

As $G(x,y) < 0$, for $(x,y) \in [0,M] \times [0,M]$, we get the integrated bound:

$$(24)\qquad \int_0^t \int_0^M u(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant C_3(T) + \int_0^t \int_0^s \int_0^M \big(v^2 + (p(u,\theta) - f(x))u\big)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s.$$

Now, if we denote by p_1 the "increasing" part of p, defined by

$$p(u,\theta) \equiv p_1(u,\theta) - \frac{3}{8}\,|t_0| u^{-2},$$

and if we denote by e_1 the "increasing" part of e, defined by

$$e(u,\theta) \equiv e_1(u,\theta) - \frac{3}{8}\,|t_0| u^{-1},$$

we can check easily that

$$(25)\qquad \exists C > 0\colon\ \forall u > 0,\ \forall \theta > 0\colon\ u\ p_1(u,\theta) \leqslant C e_1(u,\theta),$$

where $C = \max(2, C_v/R)$.

So

$$(26)\qquad \int_0^t\int_0^M u(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant C(T)+\int_0^t\int_0^s\int_0^M\Big(\frac{1}{2}\,v^2+e_1(u,\theta)+u\Big)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s.$$

Now, using (17) we obtain

$$\int_0^M\Big(\frac{1}{2}\,v^2+e_1\Big)\,\mathrm{d}x\leqslant C_4\Big(1+\int_0^M u\,\mathrm{d}x+\int_0^M u^{-1}\,\mathrm{d}x\Big),$$

where $C_4=\max(|E_0|,\frac{3}{8}|t_0|,|P|+\frac{1}{8}g_C M^2)$.

By virtue of the Cauchy-Schwarz inequality we have

$$\int_0^M u^{-1}\,\mathrm{d}x\leqslant C_\varepsilon+\varepsilon\int_0^M u^{-2}\,\mathrm{d}x\leqslant C_\varepsilon+\varepsilon\int_0^M e_1\,\mathrm{d}x,$$

so finally

$$(27)\qquad \int_0^M\Big[\frac{1}{2}\,v^2+e_1\Big]\,\mathrm{d}x\leqslant C_5\Big(1+\int_0^M u\,\mathrm{d}x\Big).$$

By putting (27) into (26) we arrive at

$$\int_0^t\int_0^M u(x,s)\,\mathrm{d}x\,\mathrm{d}s\leqslant C_6(T)\Big(1+\int_0^t\int_0^s\int_0^M u\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}t\Big).$$

By using Gronwall's lemma, we get

$$(28)\qquad \int_0^t\int_0^M u(x,s)\,\mathrm{d}x\,\mathrm{d}s\leqslant C_7(T),$$

which gives an integrated bound for the energy:

$$(29)\qquad \int_0^t\int_0^M\Big[\frac{1}{2}\,v^2+C_v\theta+u\Big]\,\mathrm{d}x\,\mathrm{d}s\leqslant C_8(T).$$

Now, by virtue of (21) and (25)

$$\int_0^M u(x,s)\,\mathrm{d}x\leqslant C_9(T)\Big(1+\int_0^t\int_0^M[v^2+C_v\theta+u]\,\mathrm{d}x\,\mathrm{d}s + \int_0^M u(x,t)\int_0^x v(y,t)\,\mathrm{d}y\,\mathrm{d}x\Big),$$

which gives, in view of (29)

$$\int_0^M u(x,s)\,\mathrm{d}x \leqslant C_{10}(T)\Big(1+\int_0^M u(x,t)\int_0^x v(y,t)\,\mathrm{d}y\,\mathrm{d}x\Big). \tag{30}$$

Integrating by parts in the last integral we have

$$\int_0^M u(x,t)\int_0^x v(y,t)\,\mathrm{d}y\,\mathrm{d}x = -\int_0^M v(x,t)\int_0^x u(y,t)\,\mathrm{d}y\,\mathrm{d}x.$$

Now one checks that there exists a number $a(t)\in[0,M]$ such that

$$\int_0^M \int_{a(t)}^x u(y,t)\,\mathrm{d}y\,\mathrm{d}x = 0.$$

Consequently,

$$\int_0^M u(x,t)\int_0^x v(y,t)\,\mathrm{d}y\,\mathrm{d}x = -\int_0^M v(x,t)\int_{a(t)}^x u(y,t)\,\mathrm{d}y\,\mathrm{d}x.$$

Inserting (30) and using the Cauchy-Schwarz inequality one has

$$\int_0^M u(x,s)\,\mathrm{d}x \leqslant C_{10}(T)\Big(1+\varepsilon\int_0^M v^2\,\mathrm{d}x + C_\varepsilon\int_0^M\Big(\int_{a(t)}^x u(y,t)\,\mathrm{d}y\Big)^2\mathrm{d}x\Big). \tag{31}$$

Now, using (23), (28) and (29), we infer

$$-\frac{1}{2}\int_0^M\int_0^M G(x,y)\,u(x,t)\,u(y,t)\,\mathrm{d}x\,\mathrm{d}y \leqslant C_{11}(T). \tag{32}$$

By Fourier expanding G, we get the formula

$$G(x,y) = -2\sum_{n>0}\frac{M^2}{n^2\pi^2}\sin\frac{n\pi x}{M}\sin\frac{n\pi y}{M},$$

which gives

$$-\frac{1}{2}\int_0^M\int_0^M G(x,y)\,u(x,t)\,u(y,t)\,\mathrm{d}x\,\mathrm{d}y = \sum_{n>0}\Big(\int_0^M\frac{M}{n\pi}\sin\frac{n\pi x}{M}\,u(x,t)\,\mathrm{d}x\Big)^2.$$

Integrating by parts the right-hand sides leads to

$$-\frac{1}{2}\int_0^M\int_0^M G(x,y)\,u(x,t)\,u(y,t)\,\mathrm{d}x\,\mathrm{d}y = \sum_{n>0}\Big(\int_0^M\Big(\int_{a(t)}^x u(\xi,t)\,\mathrm{d}\xi\Big)\cos\frac{n\pi x}{M}\,\mathrm{d}x\Big)^2.$$

By Fourier expanding $x \to \int_{a(t)}^x u(\xi,t)\,\mathrm{d}\xi$ and using Plancherel's theorem, we obtain finally

$$-\frac{1}{2}\int_0^M\int_0^M G(x,y)\,u(x,t)\,u(y,t)\,\mathrm{d}x\,\mathrm{d}y = \int_0^M \left(\int_{a(t)}^x u(\xi,t)\,\mathrm{d}\xi\right)^2 \mathrm{d}x.$$

Then, from (32) we have

$$\int_0^M \left(\int_{a(t)}^x u(\xi,t)\,\mathrm{d}\xi\right)^2 \mathrm{d}x \leqslant C_{12}(T),$$

and inserting it into (31) yields

$$\int_0^M u(x,s)\,\mathrm{d}x \leqslant C_{10}(T)\left(1+\varepsilon\int_0^M v^2\,\mathrm{d}x\right). \tag{33}$$

Substituting it into (17), we have

$$\int_0^M \left[\frac{1}{2}v^2+e\right] = \mathrm{d}x \leqslant |E_0| + \left(P+\frac{1}{8}\,g_C M^2\right)\int_0^M u\,\mathrm{d}x + \varepsilon\int_0^M v^2\,\mathrm{d}x.$$

So, for ε small enough we have

$$\int_0^M v^2\,\mathrm{d}x \leqslant C_{13}(T),$$

which, by (33), implies

$$\int_0^M u\,\mathrm{d}x \leqslant C_{14}(T),$$

which gives (14) by virtue of (17).

To get (15), we rewrite the third equation (1) as follows:

$$e_\theta\theta_t + \theta p_\theta v_x - \frac{\nu}{u}\,v_x^2 = \left(\frac{\kappa}{u}\,\theta_x\right)_x. \tag{34}$$

If we multiply (34) by θ^{-1}, we get

$$e_\theta\frac{\theta_t}{\theta} + p_\theta u_t = \frac{\nu}{u\theta}\,v_x^2 + \frac{\kappa}{u\theta^2}\,{\theta_x}^2 + \left(\frac{\kappa}{u\theta}\,\theta_x\right)_x. \tag{35}$$

By integrating (35) on $[0,M]\times[0,t]$ and using the above estimates, we obtain finally

$$\int_0^t\int_0^M \left(\frac{\nu}{u\theta}\,v_x^2+\frac{\kappa}{u\theta^2}\,{\theta_x}^2\right)\mathrm{d}x\,\mathrm{d}t + \int_0^M \big(R(u-\log u-1)+C_v(\theta-\log\theta-1)\big)\,\mathrm{d}x \leqslant C(T).$$

$\square$

We get now the positivity of θ in a standard manner, by applying the maximum principle to the parabolic equation (34) and using the boundary conditions and the positivity of θ_0:

Proposition 1. *One has*

$$\theta(x,t) > 0 \ \text{ on } \ [0,M]\times[0,\infty). \tag{36}$$

We have now the following estimates for temperature and for powers of the specific volume.

Lemma 2. *There exists a positive constant $C(T)$ such that*

$$\left(\int_0^M u\,\mathrm{d}x, \int_0^M \frac{\mathrm{d}x}{u^\alpha}, \int_0^M \theta\,\mathrm{d}x\right) \leqslant C(T) \tag{37}$$

for any $2/3 \leqslant \alpha \leqslant 2$.

Proof. By using Lemma 1, one gets directly $\int_0^M u\,\mathrm{d}x \leqslant C(T)$ and $\int_0^M \theta\,\mathrm{d}x \leqslant C(T)$.

As the coefficient t_0 is negative, we write

$$\int_0^M \left(\frac{1}{16}\,t_3 u^{-2} + \frac{3\hbar^2}{10m}\Big(\frac{3\pi^2}{2}\Big)^{2/3} u^{-2/3} + e_m + C_v\theta\right)\mathrm{d}x \leqslant E_0 + \int_0^M \frac{3}{8}\,|t_0|u^{-1}\,\mathrm{d}x.$$

By using Schwarz's inequality, one has, for $0<\varepsilon<1$ and $C_1>0$

$$\int_0^M (u^{-2}+u^{-2/3}+\theta)\,\mathrm{d}x \leqslant C_1(\varepsilon) + \varepsilon\int_0^M u^{-2}\,\mathrm{d}x.$$

By absorbing the last term on the right-hand side, we get the estimate for $\int\theta$ and for $\int u^{-2/3}$ and $\int u^{-2}$, and we conclude by interpolation for $\int u^{-\alpha}$. □

Lemma 3. *If $q \geqslant 2$, we have*

$$\int_0^t \max_{x\in[0,M]} \theta^\alpha(x,s)\,\mathrm{d}s \leqslant C(T) \tag{38}$$

for $0 \leqslant \alpha \leqslant 2$.

Proof. An elementary argument gives the inequality

$$\theta^r(x,s) \leqslant \frac{1}{M}\int_0^M \theta^r(x,s)\,\mathrm{d}x + r\int_0^M \theta^{r-1}|\theta_x|\,\mathrm{d}x,$$

where the first integral is bounded provided that $0 \leqslant r \leqslant 1$, by interpolation according to Lemma 1.

By using the Cauchy-Schwarz inequality we obtain

$$\theta^r(x,s) \leqslant C + C \int_0^M \frac{\kappa^{1/2}}{u^{1/2}\theta} |\theta_x| \frac{u^{1/2}\theta^r}{\kappa^{1/2}} \, \mathrm{d}x \, \mathrm{d}s,$$

hence

$$\theta^r(x,s) \leqslant C + C\Big(\int_0^M \frac{\kappa}{u\theta^2} \theta_x^2 \, \mathrm{d}x\Big)^{1/2} \Big(\int_0^M u \frac{\theta^{2r}}{1+\theta^q} \, \mathrm{d}x\Big)^{1/2}.$$

By virtue of Lemma 1, the last integral is bounded if $q \geqslant 2r$.

So, by taking the square and integrating in t, we get finally

$$\int_0^t \max_{x\in[0,M]} \theta^{2r}(x,s) \, \mathrm{d}s \leqslant C + C \int_0^t \int_0^M \frac{\kappa}{u\theta^2} \theta_x^2 \, \mathrm{d}x \, \mathrm{d}s, \tag{39}$$

where the right-hand side is bounded due to Lemma 1. $\square$

Lemma 4. *There is a pair (u_m, u_M) of positive numbers, depending only on T and the data, such that*

$$\forall (x,t) \in [0,M] \times [0,T]\colon\ u_m \leqslant u(x,t) \leqslant u_M. \tag{40}$$

P r o o f. In the following, C_j denote various positive T-dependent constants.

Let us consider first the lower bound. From (1) we get

$$v_t + p_x = \nu(\log u)_{tx} + g_C\Big(x - \frac{M}{2}\Big). \tag{41}$$

Integrating on $[0,x] \times [0,t]$, we find

$$\begin{aligned} &-\nu \log u(x,t) + \int_0^t p(x,s) \, \mathrm{d}s \\ &\quad = -\int_0^x \big(v(z,t) - v_0(z)\big) \, \mathrm{d}z + \int_0^t p(0,s) \, \mathrm{d}s - \nu \log u(0,t) + \nu \log u_0(0) \\ &\qquad - \nu \log u_0(x) + t \, \frac{g_C}{2} \, x(x-M). \end{aligned}$$

Integrating on $[0,t]$ the boundary condition $\sigma(0,t) = -P$, we have

$$\int_0^t p(0,s) \, \mathrm{d}s - \nu \log u(0,t) + \nu \log u_0(0) = Pt,$$

so we obtain the identity

$$\nu \log \frac{u(x,t)}{u_0(x)} = \int_0^t (p(x,s) - P)\,\mathrm{d}s + \int_0^x \big(v(z,t) - v_0(z)\big)\,\mathrm{d}z - tf(x), \tag{42}$$

which gives the lower bound

$$u(x,t) \geqslant u_m$$

with $u_m = \min\limits_{[0,M]} u_0(x) \exp(\frac{1}{\nu}[-(8ME_0)^{1/2} + T(p_m - P)])$.

By using (41) and Lemma 1 we find

$$\log u(x,t) \leqslant C_3 + C_4 \bigg(\int_0^M \big(\nu(\log u(y,t))_y - v\big)^2\,\mathrm{d}y \bigg)^{1/2}.$$

So, we just need an upper bound for the right-hand side.

For that purpose, we multiply (41) by $\nu(\log u(x,s))_x - v$ and integrate on $[0,M] \times [0,t]$:

$$\begin{aligned} \int_0^M \big(\nu(\log u(x,t))_x - v\big)^2\,\mathrm{d}x &- \int_0^M \big(\nu(\log u_0(x))_x - v_0\big)^2\,\mathrm{d}x \\ = \int_0^t \int_0^M &(p_u u_x + p_\theta \theta_x)\big(\nu(\log u(x,s))_x - v\big)\,\mathrm{d}x\,\mathrm{d}s \\ &- \int_0^t \int_0^M g_C\Big(x - \frac{M}{2}\Big)\big(\nu(\log u(x,s))_x - v\big)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

A simple computation gives

$$u p_u = -\varphi(u) - \frac{R\theta}{u} + \frac{C_5}{u^2},$$

where φ is a positive function such that $0 \leqslant \varphi(u) \leqslant C_6$.

So we get

$$\begin{aligned} \int_0^M \big(\nu(\log u(x,t))_x - v\big)^2\,\mathrm{d}x &- \int_0^M \big(\nu(\log u_0(x))_x - v_0\big)^2\,\mathrm{d}x \\ + \nu \int_0^t \int_0^M &\Big(\varphi(u) + \frac{R\theta}{u}\Big)\big((\log u(x,s))_x\big)^2\,\mathrm{d}x\,\mathrm{d}s \\ = C_5 \int_0^t \int_0^M &\Big(\nu \frac{\nu}{u^2}\big((\log u(x,s))_x\big)^2\,\mathrm{d}x\,\mathrm{d}s \\ &- \int_0^t \int_0^M u p_u v (\log u(x,s))_x\Big)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t \int_0^M p_\theta \theta_x \big(\nu(\log u(x,s))_x - v\big)\,\mathrm{d}x\,\mathrm{d}s \\ &- \int_0^t \int_0^M g_C\Big(x - \frac{M}{2}\Big)\big(\nu(\log u(x,s))_x - v\big)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{43}$$

We first bound the first term on the right-hand side by using Lemma 1:

$$C_5 \int_0^t \int_0^M \frac{\nu}{u^2} \big((\log u(x,s))_x\big)^2 \,\mathrm{d}x\,\mathrm{d}s \\ \leqslant C_7 + C_8 \int_0^t \int_0^M (1+\theta)\big(\nu(\log u(x,s))_x - v\big)^2 \,\mathrm{d}x\,\mathrm{d}s.$$

The second term gives

$$\int_0^t \int_0^M |up_u|\,|v|\,|(\log u(x,s))_x| \,\mathrm{d}x\,\mathrm{d}s \\ \leqslant \frac{1}{2} \int_0^t \int_0^M (1+\theta)\big((\log u(x,s))_x\big)^2 \,\mathrm{d}x\,\mathrm{d}s \\ + \frac{1}{2} \int_0^t \max_{x\in[0,M]} (1+\theta(x,s)) \Big(\int_0^M v^2 \,\mathrm{d}x \Big) \,\mathrm{d}s.$$

By using Lemma 1 and 3 we have

$$\left| \int_0^t \int_0^M up_u v (\log u(x,s))_x \,\mathrm{d}x\,\mathrm{d}s \right| \leqslant C_9 \bigg(1 + \int_0^t \int_0^M (1+\theta)\big((\log u(x,s))_x\big)^2 \,\mathrm{d}x\,\mathrm{d}s \bigg).$$

The third term gives

$$\left| \int_0^t \int_0^M \frac{R}{u} \theta_x \big(\nu(\log u(x,s))_x - v\big) \,\mathrm{d}x\,\mathrm{d}s \right|, \\ \leqslant \int_0^t \int_0^M \frac{R}{u} \frac{\theta u^{1/2}}{\kappa^{1/2}} \big|\nu(\log u(x,s))_x - v\big| \frac{\kappa^{1/2}|\theta_x|}{u^{1/2}\theta} \,\mathrm{d}x\,\mathrm{d}s,$$

where, by virtue of Lemma 1, the right-hand side is bounded by

$$C_{10} \bigg(1 + \int_0^t \int_0^M \big(\nu(\log u(x,s))_x - v\big)^2 \,\mathrm{d}x\,\mathrm{d}s \bigg).$$

Finally, the last term gives

$$\left| \int_0^t \int_0^M g_C \Big(x - \frac{M}{2} \Big) \big(\nu(\log u(x,s))_x - v\big) \,\mathrm{d}x\,\mathrm{d}s \right| \\ \leqslant C_{11} \bigg(1 + \int_0^t \int_0^M \big(\nu(\log u(x,s))_x - v\big)^2 \,\mathrm{d}x\,\mathrm{d}s \bigg).$$

By collecting all of these estimates and putting $F(t) = \int_0^M \big(\nu(\log u(x,t))_x - v\big)^2 \,\mathrm{d}x$ and $K(t) = \max_{x\in[0,M]} (1+\theta)(x,t)$ we get

$$F(t) \leqslant C_{12} \bigg(1 + \int_0^t K(s)F(s) \,\mathrm{d}s \bigg).$$

Now, by applying Gronwall's lemma, we obtain finally

$$F(t) \leqslant C_{13} \exp\left(\int_0^t K(s)\,\mathrm{d}s\right)\left[1 + \int_0^t \exp\left(-\int_0^s K(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right].$$

As the right-hand side is bounded by Lemma 3, this gives an upper bound for $\int_0^M \bigl(\nu(\log u(x,t))_x - v\bigr)^2\,\mathrm{d}x$, and consequently for u. □

Lemma 5. *One has the inequality*

$$\int_0^T\int_0^M v_x^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C(T). \tag{44}$$

P r o o f. By multiplying the second equation (1) by v and integrating on $[0,M]\times[0,T]$ we have

$$\begin{aligned}\frac{1}{2}\int_0^M v(x,T)^2\,\mathrm{d}x &+ \int_0^T\int_0^M \nu\,\frac{v_x^2}{u}\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^M v_0(x)^2\,\mathrm{d}x + P\int_0^T \bigl(v(M,t) - v(0,t)\bigr)\,\mathrm{d}t\\ &\quad + \int_0^T\int_0^M p v_x\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_0^M g_C\Bigl(x - \frac{M}{2}\Bigr)v\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By virtue of Lemma 1, the right-hand side is bounded by

$$C + C\int_0^T\int_0^M p^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_0^M \nu\,\frac{v_x^2}{u}\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$\int_0^T\int_0^M p^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\int_0^T\int_0^M (1+\theta^2)\,\mathrm{d}x\,\mathrm{d}t,$$

where the right-hand side is bounded in view of Lemma 3. □

Lemma 6. *One has the inequality*

$$\int_0^T\int_0^M \theta^{q+2}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \max_{[0,M]} \theta^{q+1}\,\mathrm{d}t \leqslant C(T). \tag{45}$$

P r o o f. We have by Lemma 1

$$\int_0^T\int_0^M \theta^{q+2}\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T \left(\max_{x\in[0,M]} \theta^{q+1}\int_0^M \theta\,\mathrm{d}x\right)\mathrm{d}t \leqslant C(T)\int_0^T \max_{x\in[0,M]} \theta^{q+1}\,\mathrm{d}t.$$

But we also have, by using the Sobolev imbedding theorem, Lemma 1 and 4 and the Minkowski inequality:

$$\begin{aligned}\max_{x\in[0,M]}\theta^{q+1} &\leqslant C(T)+(q+1)\varepsilon\int_0^M \frac{u\theta^{2q+2}}{\kappa}\,\mathrm{d}x + C_\varepsilon\int_0^M \kappa\frac{\theta_x^2}{u\theta^2}\,\mathrm{d}x\\ &\leqslant C(T)+(q+1)\varepsilon u_M\cdot\max_{x\in[0,M]}\theta^{q+1}\int_0^M\theta\,\mathrm{d}x + C_\varepsilon\int_0^M\kappa\frac{\theta_x^2}{u\theta^2}\,\mathrm{d}x.\end{aligned}$$

Then, by using Lemma 1 and choosing ε small enough, we get

$$\max_{x\in[0,M]}\theta^{q+1}\leqslant C(T)+\frac{1}{2}\max_{x\in[0,M]}\theta^{q+1},$$

which implies (45). □

Lemma 7. *One has the inequality*

$$\max_{t\in[0,T]}\int_0^M u_x^2\,\mathrm{d}x\leqslant C(T). \tag{46}$$

P r o o f. As the proof of Lemma 4 tells us that $\int_0^M\big(\nu(\log u(x,t))_x - v\big)^2\,\mathrm{d}x\leqslant C$, we deduce

$$\nu^2\int_0^M\frac{u_x^2}{u^2}\,\mathrm{d}x+\int_0^M v^2\,\mathrm{d}x\leqslant 2\nu\int_0^M|v|\frac{|u_x|}{u}\,\mathrm{d}x.$$

The result follows by using the Cauchy-Schwarz inequality. □

Now we need to improve (44).

Lemma 8. *One has the bound*

$$\int_0^T\int_0^M|v_x|^3\,\mathrm{d}x\,\mathrm{d}t\leqslant C(T). \tag{47}$$

P r o o f. By using an argument of Dafermos and Hsiao [14] and the function Ω introduced in (18), we check that Ω satisfies the following parabolic problem:

$$\begin{cases}\Omega_t-\dfrac{\nu}{u}\Omega_{xx}=-p(x,t)+f(x),\\ \Omega(0,t)=0,\\ \Omega(M,t)=\displaystyle\int_0^M v(y,t)\,\mathrm{d}y=0,\\ \Omega(x,0)=\displaystyle\int_0^x v_0(y)\,\mathrm{d}y.\end{cases} \tag{48}$$

Then linear parabolic L^p estimates give in particular $\|\Omega_{xx}\|_{L^1([0,T];L^3(0,M))} \leqslant C + \|p\|_{L^1([0,T];L^3(0,M))}$, and we have just to verify that the right-hand side is finite. We have

$$\int_0^T\int_0^M p^3(x,t)\,\mathrm{d}x\,\mathrm{d}t \leqslant C\int_0^T\int_0^M (1+\theta+\theta^2+\theta^3)\,\mathrm{d}x\,\mathrm{d}t.$$

By Lemma 3, the first three terms on the right-hand side are bounded and, for the last one, we get

$$\int_0^T\int_0^M \theta^3\,\mathrm{d}x\,\mathrm{d}t \leqslant C\int_0^T \max_{x\in[0,M]}\theta^2(x,t)\Big(\int_0^M \theta(x,t)\,\mathrm{d}x\Big)\,\mathrm{d}t,$$

which is also bounded by virtue of Lemma 3. □

Following [10], [11] we consider three quantities:

$$\mathbf{X}(t) = \int_0^t\int_0^M (1+\theta^q)\theta_t^2\,\mathrm{d}x,\ \ \mathbf{Y} = \max_{t\in[0,T]}\int_0^M (1+\theta^{2q})\theta_x^2\,\mathrm{d}x,\ \ \mathbf{Z} = \max_{t\in[0,T]}\int_0^M v_{xx}^2\,\mathrm{d}x.$$

By Schwarz inequality we have

$$\begin{aligned}\max_{x\in[0,M]}\theta^{2q+2}(x,t) &\leqslant C + C\int_0^M \theta^{2q+1}|\theta_x|\,\mathrm{d}x\\ &\leqslant C + C\Big(\int_0^M \theta^{2q}\theta_x^2\,\mathrm{d}x\Big)^{1/2}\max_{x\in[0,M]}\theta^{q+1/2}.\end{aligned}$$

So

$$\max_{x\in[0,M]}\theta^{2q+2}(x,t) \leqslant C + C\mathbf{Y}^{1/2}\max_{x\in[0,M]}\theta^{q+1/2},$$

and we obtain

$$\max_{Q_T}\theta \leqslant C(1+\mathbf{Y}^{\frac{1}{2q+3}}).$$

Now, using the interpolation inequality

$$\|v_x\|^2_{L^2(0,M)} \leqslant C\big(\|v\|^2_{L^2(0,M)} + \|v\|_{L^2(0,M)}\|v_{xx}\|_{L^2(0,M)}\big)$$

we have

$$\max_{[0,T]}\int_0^M v_x^2\,\mathrm{d}x \leqslant C(1+\mathbf{Z}^{\frac{1}{2}}).$$

The Sobolev imbedding theorem gives

$$v_x^2(x,t) \leqslant C\|v_x\|^2_{L^2} + C\int_0^M |v_x(x,t)|\,|v_{xx}(x,t)|\,\mathrm{d}x.$$

By combining the last two estimates we obtain

$$\max_{Q_T} |v_x| \leqslant C(1+\mathbf{Z}^{\frac{3}{8}}),$$

where $Q_T = [0,T]\times[0,M]$.

Lemma 9. *One has the following inequalities:*

$$\mathbf{Y} \leqslant C(1+\mathbf{Z}^{\frac{2q+1}{2q+2}}), \quad \mathbf{X} \leqslant C(1+\mathbf{Z}^{\frac{2q+1}{2q+2}}). \tag{49}$$

Proof. As before, we denote by C, C_j various positive constants.

We consider the function

$$K(u,\theta) = \int_0^\theta \frac{\kappa(u,\xi)}{u}\,\mathrm{d}\xi,$$

which satisfies

$$|K_u|, |K_{uu}| \leqslant C(1+\theta^{q+1}).$$

We multiply (34) by K_t, and integrate on Q_t with $t \leqslant T$

$$\int_0^t\int_0^M \Big(C_v\theta_t + \frac{R\theta}{u}\,v_x - \frac{\nu}{u}\,v_x^2\Big)K_t\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_0^M \frac{\kappa}{u}\,\theta_x K_{tx}\,\mathrm{d}x\,\mathrm{d}s = 0. \tag{50}$$

One computes easily $K_t = K_u v_x + \frac{\kappa}{u}\,\theta_t$ and $K_{tx} = \big(\frac{\kappa}{u}\,\theta_x\big)_t + K_u v_{xx} + K_{uu}v_x u_x + \big(\frac{\kappa}{u}\big)_u u_x\theta_t$.

So we have

$$\begin{aligned} \int_0^t\int_0^M C_v\frac{\kappa}{u}\,\theta_t^2\,\mathrm{d}x\,\mathrm{d}s = &-\int_0^t\int_0^M C_v K_u v_x\theta_t\,\mathrm{d}x\,\mathrm{d}s \\ &-\int_0^t\int_0^M \Big(\frac{R\theta}{u}\,v_x - \frac{\nu}{u}\,v_x^2\Big)K_u\,\mathrm{d}x\,\mathrm{d}s \\ &-\int_0^t\int_0^M \Big(C_v\theta_t + \frac{R\theta}{u}\,v_x - \frac{\nu}{u}\,v_x^2\Big)\frac{\kappa}{u}\,\theta_t\,\mathrm{d}x\,\mathrm{d}s \\ &-\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_x K_{tx}\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{51}$$

To bound each term in (51), one first gets

$$\begin{aligned} &\int_0^t\int_0^M C_v\frac{\kappa}{u}\,\theta_t^2\,\mathrm{d}x\,\mathrm{d}s \\ &\qquad \geqslant \frac{C_v\kappa_1}{u_m}\int_0^t\int_0^M (1+\theta^q)\theta_t^2\,\mathrm{d}x\,\mathrm{d}s - \Big|\int_0^t\int_0^M C_v K_u v_x\theta_t\,\mathrm{d}x\,\mathrm{d}s\Big|. \end{aligned} \tag{52}$$

Now, the Minkowski inequality yields

$$\begin{aligned}\left|\int_0^t\int_0^M C_v K_u v_x\theta_t\,\mathrm{d}x\,\mathrm{d}s\right| &\leqslant C_v\kappa_2\int_0^t\int_0^M (1+\theta^{q+1})|\theta_t|\,|v_x|\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \frac{1}{2}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X}+C\int_0^t\int_0^M(1+\theta^{q+2})v_x^2\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \frac{1}{2}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X}+C\max_{Q_T}v_x^2\int_0^t\int_0^M(1+\theta^{q+2})\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

So, by virtue of Lemma 6 we conclude

$$\left|\int_0^t\int_0^M C_v K_u v_x\theta_t\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant \frac{1}{2}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X}+C_1\mathbf{Z}^{3/4}. \tag{53}$$

Next

$$\begin{aligned}&\left|\int_0^t\int_0^M\Big(\frac{R\theta}{u}\,v_x-\frac{\nu}{u}\,v_x^2\Big)K_u v_x\,\mathrm{d}x\,\mathrm{d}s\right|\\ &\quad\leqslant C\int_0^t\int_0^M\big((1+\theta^{q+2})v_x^2+(1+\theta^{q+1})|v_x|^3\big)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad\leqslant C\max_{Q_t}v_x^2\int_0^t\int_0^M(1+\theta^{q+3})\,\mathrm{d}x\,\mathrm{d}s+C\max_{Q_t}(1+\theta^{q+1})\int_0^t\int_0^M|v_x|^3\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

So

$$\left|\int_0^t\int_0^M\Big(\frac{R\theta}{u}\,v_x-\frac{\nu}{u}\,v_x^2\Big)K_u v_x\,\mathrm{d}x\,\mathrm{d}s\right|\leqslant C_2+C_2\mathbf{Z}^{3/4}+C_2\mathbf{Y} \tag{54}$$

and consequently

$$\begin{aligned}&\left|\int_0^t\int_0^M\Big(\frac{R\theta}{u}\,v_x-\frac{\nu}{u}\,v_x^2\Big)\frac{\kappa}{u}\,\theta_t\,\mathrm{d}x\,\mathrm{d}s\right|\\ &\qquad\leqslant\frac{1}{4}\frac{C_v\kappa_1}{u_m}\,\mathbf{X}+C\mathbf{Z}^{3/4}+C\max_{Q_T}(1+\theta^q)\max_{Q_T}|v_x|\int_0^t\int_0^M|v_x|^3\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

So, finally, by using Lemmas 4 and 8 we obtain

$$\left|\int_0^t\int_0^M\Big(\frac{R\theta}{u}\,v_x-\frac{\nu}{u}\,v_x^2\Big)\frac{\kappa}{u}\,\theta_t\,\mathrm{d}x\,\mathrm{d}s\right|\leqslant\frac{1}{4}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X}+C_3\mathbf{Z}^{3/4}+C_3+\frac{1}{2}C_2\mathbf{Y}. \tag{55}$$

We have also

$$\int_0^t\int_0^M\frac{\kappa}{u}\,\theta_x\Big(\frac{\kappa}{u}\,\theta_x\Big)_t\,\mathrm{d}x\,\mathrm{d}s=\frac{1}{2}\int_0^M\Big(\frac{\kappa}{u}\,\theta_x\Big)^2(x,t)\,\mathrm{d}x-\frac{1}{2}\int_0^M\Big(\frac{\kappa}{u}\,\theta_x\Big)^2(x,0)\,\mathrm{d}x,$$

so

$$\int_0^t\int_0^M\frac{\kappa}{u}\,\theta_x\Big(\frac{\kappa}{u}\,\theta_x\Big)_t\,\mathrm{d}x\,\mathrm{d}s\geqslant 2C_2\mathbf{Y}-C_4. \tag{56}$$

Further,

$$\left|\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_x K_u v_{xx}\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant \int_0^t\int_0^M (1+\theta^{2q+1})|\theta_x|\,|v_{xx}|\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant C\mathbf{Z}^{1/2}\mathbf{Y}^{q/(2q+1)}$$

$$\leqslant C_5 + C_5\mathbf{Z}^{(2q+1)/(2q+2)} + \frac{1}{4}\,C_2\mathbf{Y},$$

and in the same manner we estimate

$$\left|\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_x K_{uu} v_x u_x\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant \int_0^t\int_0^M (1+\theta^{2q+1})|\theta_x|\,|v_x|\,|u_x|\,\mathrm{d}x\,\mathrm{d}s \tag{57}$$

$$\leqslant C_6 + C_6\mathbf{Z}^{3/4} + \frac{1}{8}\,C_2\mathbf{Y}.$$

Now

$$\left|\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_x\Big(\frac{\kappa}{u}\Big)_u u_x\theta_t\,\mathrm{d}x\,\mathrm{d}s\right|$$

$$\leqslant \frac{1}{16}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X} + C\int_0^t\int_0^M |\theta_x|(1+\theta^q)\Big(\frac{\kappa}{u}\,\theta_x\Big)^2 u_x^2\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant \frac{1}{16}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X} + C\max_{Q_T}(1+\theta^q)\int_0^t \max_{Q_T}\Big(\frac{\kappa}{u}\,\theta_x\Big)^2\,\mathrm{d}s$$

$$\leqslant \frac{1}{16}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X} + C(1+\mathbf{Y}^{q/(2q+3)})$$

$$\times\int_0^t\int_0^M (1+\theta^{2q})\theta_x^2 + \Big|\frac{\kappa}{u}\,\theta_x\Big|\,\Big|\Big(\frac{\kappa}{u}\,\theta_x\Big)_x\Big|\,\mathrm{d}x\,\mathrm{d}s,$$

so the left-hand side of the above inequality is bounded by

$$\leqslant \frac{1}{16}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X}+C(1+\mathbf{Y}^{(2q+1)/(2q+3)})+C(1+\mathbf{Y}^{q/(2q+3)})\int_0^t\int_0^M \Big|\frac{\kappa}{u}\,\theta_x\Big|\,\Big|\Big(\frac{\kappa}{u}\,\theta_x\Big)_x\Big|\,\mathrm{d}x\,\mathrm{d}s.$$

But we have, by virtue of Lemma 1

$$\int_0^t\int_0^M \Big|\frac{\kappa}{u}\,\theta_x\Big|\,\Big|\Big(\frac{\kappa}{u}\,\theta_x\Big)_x\Big|\,\mathrm{d}x\,\mathrm{d}s \leqslant C\bigg(\int_0^t\int_0^M (1+\theta^{q+1})\Big(\frac{\kappa}{u}\,\theta_x\Big)_x^2\,\mathrm{d}x\,\mathrm{d}s\bigg)^{1/2}$$

$$\leqslant C(1+\max_{Q_T}\theta^{1/2})\mathbf{X}^{1/2} + C\mathbf{Z}^{3/8} + C(1+\max_{Q_T}\theta^{(q+3)/2}).$$

By using this estimate, we get finally

$$\left|\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_x\Big(\frac{\kappa}{u}\Big)_u u_x\theta_t\,\mathrm{d}x\,\mathrm{d}s\right| \leqslant C_5 + \frac{1}{8}\,\frac{C_v\kappa_1}{u_m}\,\mathbf{X} + C_5\mathbf{Z}^{3/4} + \frac{1}{16}\,C_2\mathbf{Y}. \tag{58}$$

By collecting all of the above estimates, we conclude

$$\mathbf{Y}+\mathbf{X} \leqslant C(T)(1+\mathbf{Z}^{(2q+1)/(2q+2)}),$$

which completes the proof. $\square$

Lemma 10. *One has the inequality*

$$\max_{[0,T]} \int_0^M v_t^2 \,\mathrm{d}x + \int_0^T\!\!\int_0^M v_{xt}^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C(T)\big(1 + \mathbf{Z}^{(2q+1)/(2q+2)}\big). \tag{59}$$

P r o o f. We sketch a formal proof, which can be made rigorous, by using adapted mollifiers (see [10], [11]).

If we differentiate with respect to t the second equation (1), multiply it by v_t and integrate on Q_T, we get

$$\frac{1}{2}\int_0^M v_t^2(x,t)\,\mathrm{d}x - \frac{1}{2}\int_0^M v_t^2(x,0)\,\mathrm{d}x = \int_0^T\!\!\int_0^M v_t\sigma_{xt}\,\mathrm{d}x\,\mathrm{d}t.$$

By integrating by parts on $[0,M]$, the right-hand side becomes

$$\int_0^T\!\!\int_0^M v_{tx}\Big[p_t - \nu\frac{v_{tx}}{u} + \nu\frac{v_x^2}{u^2}\Big]\,\mathrm{d}x\,\mathrm{d}t,$$

and we get the majorization

$$\frac{1}{2}\int_0^M v_t^2(x,t)\,\mathrm{d}x + \int_0^T\!\!\int_0^M \nu\frac{v_{tx}^2}{u}\,\mathrm{d}x\,\mathrm{d}t \leqslant C + C\int_0^T\!\!\int_0^M (p_t^2 + v_x^4)\,\mathrm{d}x\,\mathrm{d}t.$$

The integral in the right-hand side is bounded by

$$\begin{aligned} &C\int_0^T\!\!\int_0^M \big(\theta_t^2 + (1+\theta^2)v_x^2 + v_x^4\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad\leqslant C + C\mathbf{X} + C\mathbf{Z}^{\frac{3}{4}} + C\max_{Q_T} v_x^2 \int_0^T\!\!\int_0^M v_x^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

which completes the proof, by virtue of Lemma 9. □

Lemma 11. *There exists a constant $C(T)$ such that*

$$\Big(\mathbf{Z},\, \mathbf{X},\, \theta,\, \mathbf{Y},\, \max_{[0,T]}\int_0^M v_x^2\,\mathrm{d}x,\, \max_{Q_T}|v_x|,\, \max_{[0,T]}\int_0^M v_t^2\,\mathrm{d}x,\, \int_0^T\!\!\int_0^M v_{xt}^2\,\mathrm{d}x\,\mathrm{d}t\Big) \leqslant C(T). \tag{60}$$

P r o o f. By the second equation (1)

$$v_{xx} = \frac{u}{\nu}\Big(v_t + p_x + \frac{\nu}{u^2}\,u_x v_x\Big).$$

So

$$\int_0^M v_{xx}^2\,\mathrm{d}x \leqslant C\int_0^M \bigl(v_t^2+\theta_x^2+(1+\theta^2)u_x^2+u_x^2v_x^2\bigr)\,\mathrm{d}x$$
$$\leqslant C\int_0^M \bigl(v_t^2+\theta_x^2+(1+\theta^2)u_x^2+v_x^2\bigr)\,\mathrm{d}x.$$

By virtue of Lemma 10

$$\mathbf{Z}\leqslant C+C(1+\mathbf{Z}^{(2q+1)/(2q+2)})+C\mathbf{Y}+C(1+\mathbf{Y}^{2/(2q+5)})+C\mathbf{Z}^{3/4},$$

so, acording to Lemma 9, we obtain

$$\mathbf{Z}\leqslant C(1+\mathbf{Z}^{(2q+1)/(2q+2)}).$$

This implies that $\mathbf{Z}$ is bounded, so is $\mathbf{Y}$, and all the other quantities. □

Lemma 12. *There exists a constant $C(T)$ such that*

$$\left(\max_{[0,T]}\int_0^M \theta_t^2\,\mathrm{d}x,\ \int_0^T\int_0^M \theta_{xt}^2\,\mathrm{d}x\,\mathrm{d}t,\ \int_0^T\int_0^M \theta_{xx}^2\,\mathrm{d}x\right)\leqslant C(T). \tag{61}$$

P r o o f. By differentiating equation (34) with respect to t, multiplying by θ_t and integrating on $[0,M]$, we get

$$\begin{aligned}
&\frac{1}{2}C_v\int_0^M \theta_t^2\,\mathrm{d}x+\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_{xt}^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\frac{1}{2}C_v\int_0^M \theta_t^2(x,0)\,\mathrm{d}x-\int_0^t\int_0^M\Bigl[\frac{R}{u}\,\theta_t^2v_x-\frac{R\theta}{u^2}\,\theta_tv_x^2+\frac{R\theta}{u}\,\theta_tv_{xt}+\frac{\nu}{u^2}\,\theta_tv_x^3\\
&\qquad-\frac{2\nu}{u}\,\theta_tv_xv_{xt}+\frac{\kappa_u}{u}\,v_x\theta_x\theta_{xt}+\frac{\kappa_\theta}{u}\,\theta_t\theta_x\theta_{xt}-\frac{\kappa}{u^2}\,u_t\theta_x\theta_{xt}\Bigr]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\frac{1}{2}C_v\int_0^M \theta_t^2\,\mathrm{d}x+\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_{xt}^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant C+C\int_0^t\int_0^M\bigl[|v_x|\theta_t^2+|\theta_t|v_x^2+|\theta_tv_{xt}|+|\theta_tv_x^3|+|\theta_tv_xv_{xt}|\\
&\qquad+(1+\theta^q)(|v_x|+|\theta_t|)|\theta_{xt}\theta_x|\bigr]\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

By Lemma 11 one gets

$$\frac{1}{2}C_v\int_0^M \theta_t^2\,\mathrm{d}x+\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_{xt}^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C+\varepsilon\int_0^t\int_0^M \frac{\kappa}{u}\,\theta_{xt}^2\,\mathrm{d}x\,\mathrm{d}t+C(\varepsilon)\int_0^t\int_0^M \theta_t^2\theta_x^2\,\mathrm{d}x\,\mathrm{d}t,$$

where the first term on the right-hand side is absorbed in the left-hand side for ε small enough, and the last term is bounded by virtue of Lemma 11.

To obtain the last estimate in (61), one gets by using the third equation (1)

$$\int_0^t\int_0^M \theta_{xx}^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\int_0^t\int_0^M [u_x^2\theta_x^2+\theta_x^4+\theta_t^2+v_x^2+v_x^4]\,\mathrm{d}x\,\mathrm{d}t,$$

where the right-hand side is bounded according to Lemma 11. □

The Hölder regularity of the solution is now proved in the same manner that in [11], which completes the proof of Theorem 2.

3. Asymptotic behaviour in the neutral case

We suppose from now on that the fluid has only a neutral component, which corresponds physically to the situation of neutron stars.

3.1. A partial asymptotic result in the confining case.

Clearly, the above a-priori estimates are not sufficient to investigate the asymptotic behaviour of the system, and we have to derive more accurate bounds, independent of time.

In fact, the non-monotone character of the pressure makes a precise asymptotic study difficult to carry out: the unstable zone corresponding to the $p_u > 0$ region may produce some high frequency density waves for large time.

In the past, few authors considered asymptotic problems for non monotone fluids or solids: Hsiao and Luo [16] have studied the large time behaviour for a model of thermoviscoelastic material with a non-monotone pressure previously introduced by Dafermos and Hsiao [14], by using results of Andrews and Ball [19].

More recently Shen, Zheng and Zhu [17] have investigated the large time behaviour for another model of thermoviscoelastic material, appearing in the theory of phase transitions for shape memory alloys.

Unfortunately, the constraints proposed in all these various works are not satisfied by our model. Moreover, we suspect that they are not sufficient to prevent the appearance of high-frequency density oscillations, due to the presence of the non monotone contribution, independent of the temperature, in the pressure.

Although these oscillations are not sufficient to alter the existence of a global solution, they seem to destroy the attractor structure of the system.

Theorem 3. *Let $r(x,t)$ be the Lagrangian position $r(x,t)$ of a point (x,t) of the slab, defined by*

$$\frac{\mathrm{d}r(x,t)}{\mathrm{d}t} = v(x,t),$$

and

$$R(t) := r(M,t) - r(0,t),$$

the spatial extension of the domain.

There exist two positive numbers $0 < R_m < R_M < \infty$, independent of time, such that

$$\forall t > 0\colon\ R_m \leqslant R(t) \leqslant R_M. \tag{62}$$

This (weak) result tells us that the fluid remains confined in a fixed slab for any time: the free boundaries neither collapse, nor escape to the infinity.

This simple result is a direct consequence of the two following lemmas.

Lemma 13. *Provided $P > 0$, the following estimates hold for any $t \geqslant 0$:*

$$\int_0^M \left[\frac{1}{2}v^2 + e + Pu\right] \mathrm{d}x = E_0, \tag{63}$$

where $E_0 = \int_0^M \left[\frac{1}{2}v_0^2 + e_0 + Pu_0\right] \mathrm{d}x$;

$$\Phi(t) + \int_0^t \Psi(t)\,\mathrm{d}t \leqslant E_1, \tag{64}$$

where

$$\Phi(t) = \int_0^M [R(u - \log\ u - 1) + C_v(\theta - \log\ \theta - 1)]\,\mathrm{d}x,$$

$$\Psi(t) = \int_0^M \left(\nu\frac{v_x^2}{u\theta} + \kappa(u,\theta)\frac{\theta_x^2}{u\theta^2}\right) \mathrm{d}x,$$

and the constants E_0 and E_1 are positive and independent of t.

P r o o f. The inequality (63) is obtained by integrating the fourth relation (1) on $[0, M] \times (0, t)$ and using the boundary conditions (7), (8).

Multiplying by v the second relation in (1) we find the relation

$$\left(\frac{1}{2}v^2 + e\right)_t = (\sigma v - q)_x. \tag{65}$$

Now, by integrating (65) on $[0, M]$ and using the boundary conditions, we obtain (63).

If we multiply the third equation (1) by θ^{-1}, we get

$$e_\theta \frac{\theta_t}{\theta} + p_\theta u_t = \frac{\nu}{u\theta}\, v_x^2 + \frac{\kappa}{u\theta^2}\, {\theta_x}^2 - \left(\frac{q_x}{\theta}\right)_x. \tag{66}$$

By integrating on $[0, M] \times [0, t]$ and using (5),we get (64). $\square$

Lemma 14. *There is a pair* $(\underline{u}, \overline{u})$ *of positive numbers, depending only on the initial data and the physical constants, such that*

$$\forall (x, t) \in [0, M] \times [0, T]\colon\ \underline{u} \leqslant u(x, t) \leqslant \overline{u}. \tag{67}$$

P r o o f. If we denote by u_-^0 the smallest real zero of $u \to p(u, 0) - P$, we check first easily that

$$\forall 0 < u < u_-^0,\ \forall \theta > 0\colon\ p(u, \theta) > P. \tag{68}$$

Using (68) we arrive at

$$p(u, \theta) > 0, \text{ if } 0 < u < u_0^-.$$

So, provided

$$\underline{u} := \min\Big\{u_0^-, \min_{x\in[0,M]} u_0(x)\Big\} \exp\bigl(-\sqrt{2E_0}\bigr),$$

let us prove that one has the lower bound

$$u(x, t) > \underline{u}. \tag{69}$$

In fact, by integrating the second equation (1), one obtains the formula

$$\begin{aligned} \log u(y, \tau) = {} & \log u(y, s) + \frac{1}{\nu}\int_s^\tau \bigl[p\bigl(u(y, t), \theta(y, t)\bigr) - P\bigr]\,\mathrm{d}t \\ & + \frac{1}{\nu}\int_0^y v(x, \tau)\,\mathrm{d}x - \frac{1}{\nu}\int_0^y v(x, s)\,\mathrm{d}x. \end{aligned}$$

Clearly $u_0(x) > \overline{u}$.

Suppose that the bound $u(x,t) > \underline{u}$ is violated on $[0,M] \times (0,\infty)$.

In that case, there is a number $\tau > 0$ and a $y(\tau) \in [0,M]$ such that $u(x,t) > \underline{u}$ except at (y,τ), where

$$u(y,\tau) = \underline{u}. \tag{70}$$

There are two possibilities:

1. $u(y,\tau) < u^-$ for $0 \leqslant t < \tau$,
2. $u(y,\tau) < u^-$ for $0 \leqslant s < t \leqslant \tau$ but $u(y,s) = u^-$.

In the first case

$$\begin{aligned}\log u(y,\tau) = {} & \log u(y,0) + \frac{1}{\nu}\int_0^\tau \big[p\big(u(y,t),\theta(y,t)\big) - P\big]\,\mathrm{d}t \\ & + \frac{1}{\nu}\int_0^y v(x,\tau)\,\mathrm{d}x - \frac{1}{\nu}\int_0^y v(x,0)\,\mathrm{d}x,\end{aligned} \tag{71}$$

hence

$$\log u(y,\tau) > \log u_0(y) - 2\sqrt{E_0} > \log \underline{u},$$

which contradicts (70).

In the second case we have

$$\log u(y,\tau) > \log u^- - 2\sqrt{E_0},$$

which also contradicts (70), so we have proved (69).

To get the upper bound, we use the following formula, easily obtained from (71):

$$u(x,t) = \frac{1}{B(x,t)Y(t)}\left[u_0(x) + \frac{1}{\nu}\int_0^t B(x,s)Y(s)p(u,\theta)u(x,s)\,\mathrm{d}s\right], \tag{72}$$

where $Y(t) = \exp\frac{Ps}{\nu}$ and $B(x,t) = \exp\big(\frac{1}{\nu}\int_0^x \big(v_0(y) - v(y,t)\big)\,\mathrm{d}y\big)$.

One has, according to Lemma 1:

$$C^{-1} \leqslant B(x,t) \leqslant C.$$

Now

$$p(u,\theta) = p_1(u) + \frac{R\theta}{u},$$

so, by (69) we have

$$|p(u,\theta)| \leqslant C(1+\theta).$$

By putting it into (72) we obtain

$$u(x,t) \leqslant C\bigg(1 + \int_0^t \mathrm{e}^{-(t-s)\frac{P}{\nu}}\,\mathrm{d}s + \int_0^t \theta(x,s)\,\mathrm{e}^{-(t-s)\frac{P}{\nu}}\,\mathrm{d}s\bigg).$$

But, according to the proof of Lemma 3 we conclude

$$\theta(x,s) \leqslant C(1+V(s)),$$

where $V \in L^1(0,\infty)$, so

$$u(x,t) \leqslant C\left(1 + \int_0^t V(s)\,\mathrm{d}s\right),$$

which completes the proof of (67). □

To prove Theorem 3, one just integrates the first equation (1):

$$R(t) = \int_0^M u(x,t)\,\mathrm{d}x.$$

Then, by Lemma 14, one has (62) with $R_m = Mu_m$ and $R_M = Mu_M$.

3.2. Asymptotic behaviour in the "strongly deconfining" case.

When P is sufficiently negative, one can expect that the slab is forced to expand in all the available space.

This conjecture is true for a large class of gases (see [12], [18], [13]), and for $|P|$ large enough (less than p_m, the bottom of the curve $u \to p(u,0)$), we show below that it also holds in the present case.

Theorem 4. *Consider the unique solution* $(u,v,\theta) \in (C^{r,r/2}(Q_T))^3$ *of problem* (1), (6), (7), (8), (9), *given by Theorem* 1.

Moreover, suppose that

$$P < p_m, \tag{73}$$

and that the "proper" total energy $\mathcal{E}_0$ *satisfies*

$$\mathcal{E}_0 = \int_0^M \left[\frac{1}{2}v_0^2 + \varepsilon(u_0,\theta_0) + f(x)u_0\right]\mathrm{d}x > 0. \tag{74}$$

Then there exists T_0 *such that*

$$\forall t > T_0\colon \int_0^M u(x,t)\,\mathrm{d}x \geqslant C\sqrt{t}, \tag{75}$$

where C *is a positive constant independent of* t.

P r o o f. In the following, all the C_j are positive constant independent of t.

First, in view of (17), it is clear that the "proper" total energy is conserved:

$$\int_0^M \Big[\frac{1}{2}v^2 + \varepsilon(u,\theta) + f(x)u\Big]\,\mathrm{d}x = \int_0^M \Big[\frac{1}{2}v_0^2 + \varepsilon(u_0,\theta_0) + f(x)u_0\Big]\,\mathrm{d}x. \tag{76}$$

Due to (42) and by using (73), we have the estimate

$$\nu \log \frac{u(x,t)}{u_0(x)} \geqslant t(p_m - P) - 2(\mathcal{E}_0 + Me_m).$$

So we get the lower bound

$$u(x,t) \geqslant u_m e^{\eta t} \tag{77}$$

with $u_m = \min\limits_{x\in[0,M]} u_0(x)\exp\big(-(2(\mathcal{E}_0 + Me_m))/\nu\big)$, and $\eta = (p_m - P)/\nu$ is a positive number.

Now, from (21) one gets

$$\begin{aligned}\int_0^t\int_0^M \big(v^2 + \big(p(u,\theta) - f(x)\big)u\big)\,\mathrm{d}x\,\mathrm{d}s \\ \leqslant C_1\bigg(1 + \int_0^M u(x,t)\,\mathrm{d}x + \int_0^M u(x,t)\int_0^x |v(y,t)|\,\mathrm{d}y\,\mathrm{d}x\bigg).\end{aligned}$$

So, by using the Cauchy-Schwarz inequality we arrive at

$$\int_0^t\int_0^M up(u,\theta)\,\mathrm{d}x\,\mathrm{d}s \leqslant C_2\bigg(1 + \Big(\int_0^M u(x,t)\,\mathrm{d}x\Big)^2\bigg). \tag{78}$$

Taking into account the decomposition $p = p_1 - \frac{3}{8}|t_0|u^{-2}$ and the bound (25), one checks that there exist two constants C_3 and C_4, such that

$$\begin{aligned}\int_0^t\int_0^M up(u,\theta)\,\mathrm{d}x\,\mathrm{d}s \geqslant C_3\int_0^t\int_0^M \Big(\frac{1}{2}v^2 + \varepsilon + f(x)u\Big)\,\mathrm{d}x\,\mathrm{d}s \\ - C_4\int_0^t\int_0^M \frac{3}{8}|t_0|u^{-1}\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

By using the lower bound (77) we obtain

$$\int_0^t\int_0^M up(u,\theta)\,\mathrm{d}x\,\mathrm{d}s \geqslant C_3\int_0^t\int_0^M \Big(\frac{1}{2}v^2 + \varepsilon + f(x)u\Big)\,\mathrm{d}x\,\mathrm{d}s - C_5\int_0^t \mathrm{e}^{-\eta s}\,\mathrm{d}s.$$

So we have finally

$$\int_0^t\int_0^M up(u,\theta)\,\mathrm{d}x\,\mathrm{d}s \geqslant C_3\mathcal{E}_0 t + C_6(1 - \mathrm{e}^{-\eta t}) \geqslant Ct$$

for t large enough, and, by putting it into (78), we get (75). □

Finally, by integrating the first equation (1) on $[0,M]$, one gets a simple geometrical consequence of Theorem 3:

Corollary 1. *Let $r(x,t)$ be the Eulerian position of the Lagrangian point (x,t), defined by*

$$v(x,t)=\frac{\mathrm{d}}{\mathrm{dt}}r(x,t).$$

Then, under the conditions (73) *and* (74), *there exists a positive constant T_0 such that the thickness of the slab $R(t)=r(M,t)-r(0,t)$ satisfies*

$$\forall t>T_0\colon\ R(t)\geqslant C\sqrt{t},$$

where C is a positive constant independent of t.

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