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ON EXACT RESULTS IN THE FINITE ELEMENT METHOD

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Abstract. We prove that the finite element method for one-dimensional problems yields no discretization error at nodal points provided the shape functions are appropriately chosen. Then we consider a biharmonic problem with mixed boundary conditions and the weak solution u. We show that the Galerkin approximation of u based on the so-called biharmonic finite elements is independent of the values of u in the interior of any subelement.

Keywords: boundary value elliptic problems, finite element method, generalized splines, elastic plate

MSC 2000: 65N30

1. Introduction

In this paper we extend some results of [13] to (bi)harmonic finite elements whose shape functions are (bi)harmonic on each subelement. In Section 2, we describe an interesting phenomenon which arises when using a special finite element shape function for a general one-dimensional boundary value problem described by a system of linear differential equations with variable coefficients. We find that there is no discretization error at nodal points, i.e., the interpolant of the true solution is equal to the Galerkin solution. Such exact results of the finite element method have been reported in [6], [15], [17], [20], [21], [22]. A similar result for the finite difference method was derived already in 1966 (see [2], p. 145). In [11], p. 134, solutions of boundary value problems are approximated by piecewise harmonic functions, which are not required to be continuous across element boundaries.

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In Section 3, we enhance the finite element space by appropriate bubble functions (see [3], [7]). Then we again obtain that the finite element solution equals the true solution at all nodal points. For a higher order differential operator a similar assertion holds for higher order derivatives.

In Section 4, we examine a biharmonic problem with mixed boundary conditions. We consider finite element shape functions which are biharmonic on each subelement. This property is satisfied for several kinds of C^1-finite elements, e.g., the Heindl piecewise quadratic triangular element [9], the piecewise biquadratic rectangular element [14], the composite Hsieh-Clough-Tocher element or its reduced version [5], [8]. We prove that the values of the weak solution u in the interior of any subelement have no influence on the Galerkin approximation of u.

Throughout the paper we use the standard Sobolev space notation (see [8]), i.e., $H^k(\Omega)$ with the norm $\|\cdot\|_{k,\Omega}$ stands for the Sobolev space of functions whose generalized derivatives up to order k are square integrable over the domain Ω, $(\cdot,\cdot)_{0,\Omega}$ is the scalar product in $L^2(\Omega)$. The space of polynomials of degree j is denoted by P_j.

2. Two-point boundary value problems

Let us consider vector-functions $v\colon \overline{\Omega} \to \mathbb{R}^m$, $v = (v_1, \dots, v_m)^\top$, $m \geqslant 1$, $\Omega = (a,b)$, $-\infty < a < b < \infty$. Consider differential operators

$$L_j(v) = \sum_{s=1}^{m} \sum_{\alpha \leqslant \varkappa_s} n_{js\alpha}(x) v_s^{(\alpha)}, \quad j = 1, \dots, k,$$

where $\varkappa_s \geqslant 0$ are integers for $s = 1, \dots, m$; $v_s^{(\alpha)} \equiv d^\alpha v_s / \mathrm{d}x^\alpha$ in Ω and $n_{js\alpha}$ are bounded measurable functions. Let $A_{ij}(x)$ form a symmetric $k \times k$ matrix $A(x)$ satisfying the condition

$$\xi^\top A(x)\xi \geqslant c_0 |\xi|^2 \ \ \forall \xi \in \mathbb{R}^k \ \text{ and for a.a. } x \in \Omega, \tag{2.1}$$

where $|\cdot|$ is the Euclidean norm and $c_0 > 0$ is a constant. Let

$$W = \prod_{s=1}^{m} H^{\varkappa_s}(\Omega)$$

with the standard Sobolev norm $\|\cdot\|_W$. Let $V \subset W$ be a subspace of vector functions satisfying some boundary conditions at the end points a and b such that

$$\sum_{j=1}^{k} \|L_j(v)\|_{0,\Omega}^2 \geqslant c \|v\|_W^2 \ \ \forall v \in V. \tag{2.2}$$

R e m a r k 2.1. Inequality (2.2) may be called the inequality of Korn's type. The original Korn's inequality in the theory of elasticity concerns the case when $L_j(v)$ are identified as the strain tensor components $\varepsilon_{\rm st}$. For sufficient conditions guaranteeing the validity of (2.2) see e.g. [10] or [18], Theorem 11.3.2 and Lemma 11.3.2.

We will consider weak solutions of the following boundary value problem: Find $u \in V$ such that

$$L^*ALu = f, \tag{2.3}$$

where

$$Lu = [L_1(u), \ldots, L_k(u)]^\top, \quad L^*w = \sum_{j=1}^{k} L_j^*(w_j),$$

where $w = (w_1, \ldots, w_k)^\top$, $f \in V'$ and L_j^* are operators formally adjoint to L_j.

E x a m p l e 2.1.

$$-(p(x)u')' + q(x)u = f \quad \text{ in } \Omega,$$
$$u(a) = u(b) = 0.$$

We put $m = 1$, $\varkappa_1 = 1$, $k = 2$, $W = H^1(\Omega)$, $V = H_0^1(\Omega)$,

$$Lv = \begin{bmatrix} v' \\ v \end{bmatrix}, \quad A(x) = \begin{bmatrix} p(x) & 0 \\ 0 & q(x) \end{bmatrix}.$$

Then $L^*w = -w_1' + w_2$. The condition (2.1) is fulfilled if and only if both $p(x)$ and $q(x)$ are positive and bounded away from zero. The condition (2.2) is satisfied with $c = 1$.

E x a m p l e 2.2.

$$(EI(x)u'')'' = f \quad \text{ in } \Omega \equiv (0, \ell),$$
$$u(0) = u'(0) = u(\ell) = u'(\ell) = 0.$$

This is a classical model of bending of an elastic beam clamped at both ends, where $I(x)$ is the moment of inertia and E denotes Young's modulus of elasticity. We set $m = 1$, $\varkappa_1 = 2$, $k = 1$, $W = H^2(\Omega)$, $V = H_0^2(\Omega)$, $Lv = v''$, $A = EI(x)$ and $L^*w = w''$. The condition (2.1) is satisfied if and only if $I(x)$ is positive and bounded away from zero. Condition (2.2) holds as well.

E x a m p l e 2.3.

$$\begin{aligned} (EI(x)u_1')' + \mathcal{K}G\mathcal{A}(x)(u_2' - u_1) &= 0 \quad \text{in } \Omega \equiv (0, \ell), \\ (\mathcal{K}G\mathcal{A}(x)(u_2' - u_1))' &= F \quad \text{in } \Omega, \\ u_1(0) = u_2(0) = u_1(\ell) = u_2(\ell) &= 0. \end{aligned}$$

This is the Timoshenko-Mindlin model for bending of a clamped elastic beam (see e.g. [16]). Here $\mathcal{A}(x)$ is the area, u_1 the rotation and u_2 the deflection of its cross-section, $\mathcal{K}$ denotes the shear correction factor ($\mathcal{K} = \text{const} > 0$), $G = E(1+\sigma)^{-1}/2$ and σ is the Poisson constant. We set $m = 2$, $\varkappa_1 = \varkappa_2 = 1$, $k = 2$, $W = [H^1(\Omega)]^2$, $V = [H_0^1(\Omega)]^2$,

$$Lv = \begin{bmatrix} v_1' \\ v_2' - v_1 \end{bmatrix}, \quad A(x) = \begin{bmatrix} EI(x) & 0 \\ 0 & \mathcal{K}G\mathcal{A}(x) \end{bmatrix},$$
$$F(x) = f_0(x) + \sum_{j=1}^{J} F_j \delta(x - x_j),$$

where $f_0 \in L^1(\Omega)$, F_j are constants, δ is the Dirac function and $x_j \in \Omega$ are given. Since $I(x)$ and $\mathcal{A}(x)$ are positive and bounded away from zero in practice, condition (2.1) is fulfilled. Condition (2.2) follows from the well-known inequality

$$\|v'\|_{0,\Omega} \geqslant C\|v\|_{0,\Omega} \;\; \forall v \in H_0^1(\Omega). \tag{2.4}$$

Indeed, we may write

$$\frac{1}{2}\|v_2'\|_{0,\Omega}^2 \leqslant \|v_2' - v_1\|_{0,\Omega}^2 + \|v_1\|_{0,\Omega}^2 \leqslant \|L_2 v\|_{0,\Omega}^2 + C^{-2}\|L_1 v\|_{0,\Omega}^2 \tag{2.5}$$

and (2.2) follows from (2.5) and (2.4).

A weak formulation of the general problem (2.3) reads: Find $u \in V$ such that

$$a(u, v) = \langle f, v\rangle \;\; \forall v \in V, \tag{2.6}$$

where $\langle f, v\rangle$ denotes the value of a linear continuous functional $f \in V'$ for the function v,

$$a(u, v) = (\!(ALu, Lv)\!)_{0,\Omega}$$

and

$$(\!(w, z)\!)_{0,\Omega} = \sum_{r=1}^{k} (w_r, z_r)_{0,\Omega}.$$

By (2.1) and (2.2), we find that $a(\cdot,\cdot)$ is a symmetric V-elliptic bilinear form, i.e.,

$$\begin{aligned} a(w,v) &= a(v,w), \\ a(v,v) &\geqslant C\|v\|_W^2 \end{aligned} \tag{2.7}$$

holds for all $v, w \in V$, where C is a positive constant. As a consequence, there exists a unique weak solution for any $f \in V'$.

Next, let $\mathcal{T}_h$ be a finite (in general nonuniform) partition $a = x_0 < x_1 < \ldots < x_N < x_{N+1} = b$ of the interval $[a, b]$. Consider a finite element space

$$V_h = \{v_h \in V \mid L^* AL(v_h\big|_K) = 0 \ \ \forall K \in \mathcal{T}_h\},$$

where K stands for a subinterval $[x_e, x_{e+1}]$ for $e = 0, \ldots, N$.

By a finite element solution of problem (2.6) we mean a function $u_h \in V_h$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle \ \ \forall v_h \in V_h. \tag{2.8}$$

Theorem 2.4. *Assume that* $n_{sj\alpha} \in C^{\varkappa_s-1}(\overline{\Omega})$ *for all* s, j, α *under consideration and* $A_{jr} \in C^{\omega}(\overline{\Omega})$ *for all* $j, r \leqslant k$*, where* $\omega = \max\limits_{s\leqslant m} \varkappa_s - 1$*. Let* (2.1) *and* (2.2) *hold. Denote by* ${}_1u$ *and* ${}_2u$ *the weak solutions of* (2.6) *corresponding to* f_1 *and* f_2*, respectively. Let* ${}_1u_h$ *and* ${}_2u_h$ *be the associated finite element solutions. Let*

$${}_1u_s^{(\varkappa_s-j-1)}(x_e) = {}_2u_s^{(\varkappa_s-j-1)}(x_e), \quad s = 1, \ldots, m, \quad j = 0, \ldots, \varkappa_s - 1$$

hold for all nodal points x_e, $e = 0, \ldots, N+1$*. Then*

$${}_1u_h = {}_2u_h$$

holds in the whole interval $[a, b]$*.*

P r o o f. Using (2.6), (2.8) and the definition of V_h, for

$$w = {}_1u - {}_2u, \quad w_h = {}_1u_h - {}_2u_h$$

and any $v_h \in V_h$ we may write

$$\begin{aligned} a(w_h, v_h) = a(w, v_h) &= \sum_{K\in\mathcal{T}_h} ((ALw, Lv_h))_{0,K} \\ = \sum_{K\in\mathcal{T}_h}\sum_{r=1}^{k}\Big(\sum_{j=1}^{k} A_{rj}L_jw, L_rv_h\Big)_{0,K} &= \sum_{e=0}^{N}\int_{x_e}^{x_{e+1}}\sum_{j,r=1}^{k} L_jw(A_{jr}L_rv_h)\,\mathrm{d}x \\ = \sum_{e=0}^{N}\Big[\sum_{s=1}^{m}\sum_{\beta=0}^{\varkappa_s-1} w_s^{(\varkappa_s-\beta-1)}\sum_{j=1}^{k}\sum_{t=0}^{\beta}(-1)^t\Big(n_{js(\varkappa_s-\beta+t)}&\sum_{r=1}^{k} A_{jr}L_rv_h\Big)^{(t)}\Big]_{x_e}^{x_{e+1}} = 0. \end{aligned}$$

Therefore, $a(w_h, w_h) = 0$ and (2.7) implies that $\|w_h\|_W = 0$. □

Corollary 2.5. *Denote by u_I the V_h-spline interpolant of the weak solution u, i.e., $u_I \in V_h$, such that*

$$u_s^{(\varkappa_s-j-1)}(x_e) = u_{Is}^{(\varkappa_s-j-1)}(x_e), \quad s=1,\dots,m, \;\; j=0,\dots,\varkappa_s-1, \;\; e=0,\dots,N+1.$$

Then

$$u_h = u_I \tag{2.9}$$

holds in the whole interval $[a,b]$.

P r o o f. In Theorem 2.4 we put ${}_1u = u$ and ${}_2u = u_I$, $f_1 = f$ and $f_2 = L^*ALu_I$. Since

$$a(u_I - {}_2u_h, v_h) = 0 \qquad \forall v_h \in V_h$$

by (2.6) and (2.8), we have $u_I = {}_2u_h$ due to (2.7). Now Theorem 2.4 yields

$$u_h = {}_1u_h = {}_2u_h = u_I \quad \text{ in } [a,b].$$

□

R e m a r k 2.6. Corollary 2.5 states that there is no discretization error at nodal points when the above-mentioned finite element method is applied.

E x a m p l e 2.7. For simplicity, let us take $p(x) \equiv 1$ and $q(x) = q_0^2 = \text{const}$ in Example 2.1. Then the general solution of the equation

$$L^*ALv \equiv -v'' + q_0^2 v = 0$$

is

$$v(x) = C_1 \cosh q_0 x + C_2 \sinh q_0 x.$$

For any $Y_1, Y_2 \in R$ there exists a unique pair $\{C_1, C_2\}$ such that $v(x_e) = Y_1$ and $v(x_{e+1}) = Y_2$. Indeed, the determinant of the corresponding matrix equals $\sinh q_0(x_{e+1} - x_e) \neq 0$. The finite element space V_h is formed by continuous functions which belong to $\operatorname{span}\{\cosh q_0 x, \sinh q_0 x\}$ on each subinterval $[x_e, x_{e+1}]$.

E x a m p l e 2.8. Let us take $EI(x) \equiv 1$ and $f \in [C^1(\overline{\Omega})]'$ in Example 2.2. The space V_h consists of cubic Hermite splines on $\mathcal{T}_h$, i.e.,

$$V_h = \{v_h \in C^1(\overline{\Omega}) \mid v_h\big|_K \in P_3(K) \;\; \forall K \in \mathcal{T}_h\}.$$

From Theorem 2.4 we know that if

$${}_1u(x_e) = {}_2u(x_e), \quad {}_1u'(x_e) = {}_2u'(x_e), \quad e = 0,\dots,N+1,$$

then the associated finite element solutions coincide, i.e.,

$$ {}_1u_h = {}_2u_h \quad \text{in } [a,b]. $$

Let u_I be the cubic Hermite interpolant of the true solution u. Then by Corollary 2.5 we have $u_h = u_I$ in the whole interval $[a,b]$. This means that the finite element solution produces exact values of the true solution and its first derivatives at all nodal points,

$$ u_h^{(j)}(x_e) = u^{(j)}(x_e), \quad j = 0,1, \quad e = 0,\ldots,N+1. \tag{2.10} $$

Another proof of (2.10) is given in [15], p. 110. (See also [4] and [12].)

Example 2.9. Consider the beam from Example 2.3 of a rectangular cross-section, the height t of which is variable, namely $t \in C(\overline{\Omega})$ (e.g., piecewise linear). Then $I(x) = I_0 t^3(x)$, $\mathcal{A}(x) = \mathcal{A}_0 t(x)$, where I_0 and $\mathcal{A}_0$ are positive constants. The solution of the corresponding homogeneous system has the form

$$ \begin{aligned} u_1 &= C_1 \int_0^x z t^{-3}(z)\,\mathrm{d}z + C_2 \int_0^x t^{-3}(z)\,\mathrm{d}z + C_3, \\ u_2 &= C_1 \left(\int_0^x (x-z) z t^{-3}(z)\,\mathrm{d}z - \frac{2(1+\sigma)I_0}{\mathcal{A}_0\mathcal{K}} \int_0^x z t^{-1}(z)\,\mathrm{d}z \right) \\ &\quad + C_2 \int_0^x (x-z) t^{-3}(z)\,\mathrm{d}z + C_3 x + C_4. \end{aligned} $$

By means of these formulae we can construct basis functions of the space V_h, i.e. $\{\varphi_{ie}\}$, $i = 1,2$, $e = 0,1,\ldots,N+1$, such that

$$ \varphi_{ie}(x_r) = \delta_{er}, \quad e,r = 0,\ldots,N+1. $$

Then we may write the finite element solutions in the form

$$ v_{ih}(x) = \sum_{e=0}^{N+1} v_{ih}(x_e)\varphi_{ie}(x), \quad i = 1,2. $$

By Corollary 2.5, the finite element solution is exact at all nodal points x_e, i.e., $u_h(x_e) = u(x_e)$ for $e = 0,\ldots,N+1$.

3. Exact results for enhanced finite element spaces

Let us consider again the general problem (2.3). We can prove that the same exact results at the nodal points as those in Section 2 are valid, if we enhance the space V_h by the space of "bubble functions", i.e., if we replace V_h by the space

$$\tilde{V}_h = \operatorname{span}\{\varphi_1, \dots, \varphi_n; \psi_{01}, \psi_{02}, \dots, \psi_{Nm}\},$$

where

$$V_h = \operatorname{span}\{\varphi_1, \dots, \varphi_n\}$$

and $\psi_{is} \in V$, $i = 0, \dots, N$, $s = 1, \dots, m$, are vector bubble functions such that their all components are zeros except the sth component,

$$\operatorname{supp} \psi_{i,s} \subset [x_i, x_{i+1}] \quad \text{for } i = 0, \dots, N, \quad s = 1, \dots, m,$$

and

$$\psi_{is}^{(j)}(x_e) = 0 \quad \text{for } j = 0, \dots, \varkappa_s - 1,\ i = 0, \dots, N,\ s = 1, \dots, m,\ e = 0, \dots, N+1.$$

Let

$$U_h^0 = \operatorname{span}\{\psi_{01}, \dots, \psi_{Nm}\}.$$

The subspaces V_h and U_h^0 are a-orthogonal, i.e.,

$$a(v_h, u_h^0) = 0 \ \forall v_h \in V_h \text{ and } \forall u_h^0 \in U_h^0. \tag{3.1}$$

Indeed, arguing as in the proof of Theorem 2.4, we have (cf. also [1])

$$a(v_h, u_h^0) = \sum_{e=0}^{N} \int_{x_e}^{x_{e+1}} \sum_{j,r=1}^{k} L_j u_h^0 (A_{jr} L_r v_h)\,\mathrm{d}x$$

$$= \sum_{e=0}^{N} \left[\sum_{s=1}^{m} \sum_{\beta=0}^{\varkappa_s - 1} (u_{hs}^0)^{(\varkappa_s - \beta - 1)} \sum_{j=1}^{k} \sum_{t=0}^{\beta} (-1)^t \left(n_{js(\varkappa_s - \beta + t)} \sum_{r=1}^{k} A_{jr} L_r v_h \right)^{(t)} \right]_{x_e}^{x_{e+1}} = 0,$$

since

$$\left(u_{hs}^0\right)^{(\varkappa_s - \beta - 1)}(x_e) = 0 \quad \text{for } e = 0, \dots, N+1, \quad \beta = 0, \dots, \varkappa_s - 1, \quad s = 1, \dots, m.$$

Any function $\tilde{v}_h \in \tilde{V}_h$ can be decomposed into the form

$$\tilde{v}_h = v_I + v_h^0,$$

where $v_I \in V_h$ is the V_h-spline interpolant introduced in Corollary 2.5 and $v_h^0 \in U_h^0$ is a bubble function. Therefore, we can write

$$\tilde{V}_h = V_h \oplus U_h^0, \tag{3.2}$$

making use of (3.1).

Let u_I be the V_h-spline interpolant of the weak solution u and let $\tilde{u}_h \in \tilde{V}_h$ be the finite element solution, i.e.,

$$a(\tilde{u}_h, \tilde{v}_h) = \langle f, \tilde{v}_h \rangle \ \ \forall \tilde{v}_h \in \tilde{V}_h.$$

Since $V_h \subset \tilde{V}_h$, the relation

$$a(\tilde{u}_h - u_I, v_h) = a(u - u_I, v_h) = 0 \ \ \forall v_h \in V_h \tag{3.3}$$

can be derived by an argument parallel to that in the proof of Theorem 2.4. Setting $w_h = \tilde{u}_h - u_I \in \tilde{V}_h$, we may use the decomposition (3.2) to obtain

$$w_h = w_{hI} + w_h^0, \quad w_{hI} \in V_h, \quad w_h^0 \in U_h^0.$$

Then (3.3) yields that

$$0 = a(w_{hI} + w_h^0, w_{hI}) = a(w_{hI}, w_{hI})$$

and $w_{hI} = 0$ follows from (2.7). As a consequence,

$$\tilde{u}_h - u_I = w_h^0 \in U_h^0, \tag{3.4}$$

so that

$$\tilde{u}_{hs}^{(\varkappa_s - j - 1)}(x_e) = u_s^{(\varkappa_s - j - 1)}(x_e)$$

holds for $s = 1, \ldots, m$, $j = 0, \ldots, \varkappa_s - 1$ and $e = 0, \ldots, N+1$.

Remark 3.1. Note that at nodal points, we have obtained the same exact results as in Corollary 2.5.

Example 3.2. Consider the Dirichlet boundary value problem for the equation $-u'' = f$, i.e., $m = \varkappa_1 = k = 1$. Let V_h be the space of continuous piecewise linear functions. If U_h^0 consists of piecewise quadratic bubbles, then the Galerkin approximation $\tilde{u}_h$ equals the true solution u at all nodal points x_e, $e = 0, \ldots, N+1$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz-continuous boundary $\partial\Omega$ (see [18]) and the outward unit normal $n = (n_x, n_y)$. Let $\partial\Omega$ be piecewise smooth with a finite number of corners. Consider a model of an elastic plate

$$\Delta^2 w = (w_{xx} + \sigma w_{yy})_{xx} + (w_{yy} + \sigma w_{xx})_{yy} + 2(1-\sigma) w_{xyxy} = f \quad \text{in } \Omega,$$

where $\sigma \in \left[0, \frac{1}{2}\right)$ is the Poisson constant. The associated bilinear form reads

$$a(u,v) = \int_\Omega [(u_{xx} + \sigma u_{yy}) v_{xx} + (u_{yy} + \sigma u_{xx}) v_{yy} + 2(1-\sigma) u_{xy} v_{xy}] \,\mathrm{d}x \,\mathrm{d}y.$$

For all $u \in H^4(\Omega)$ and $v \in H^2(\Omega)$ we have (see [19], Chapt. 23)

$$a(u,v) = (\Delta^2 u, v)_{0,\Omega} + \left\langle \frac{\partial v}{\partial n}, M(u) \right\rangle_{0,\partial\Omega} + \langle v, T(u) \rangle_{0,\partial\Omega} + \sum_{j=1}^{J} v(s_j) [H(u)](s_j), \tag{4.1}$$

where s_j are corner points of the domain Ω, J denotes the number of corners,

$$\begin{aligned} M(u) &= \sigma \Delta u + (1-\sigma) \frac{\partial^2 u}{\partial n^2} && \text{(bending moment)},\\ T(u) &= -\frac{\partial(\Delta u)}{\partial n} + \frac{\partial(H(u))}{\partial s} && \text{(effective force)},\\ H(u) &= (1-\sigma)\big((u_{xx} - u_{yy}) n_x n_y + u_{xy}(n_y^2 - n_x^2)\big) && \text{(twisting moment)}, \end{aligned}$$

and $[H(u)](s_j) = H(s_j+) - H(s_j-)$ is a jump of the twisting moment.

Let us point out that $[H(u)](s_j) = 0$ for all $j = 1, \ldots, J$ if $u = 0$ on $\partial\Omega$.

Consider the so-called intermediate boundary value problem, which corresponds to a simply supported elastic plate, i.e.,

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega,\\ u = M(u) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Introduce the space $V = H^2(\Omega) \cap H_0^1(\Omega)$. We look for a weak solution $u \in V$ satisfying

$$a(u,v) = \langle f, v \rangle \ \ \forall v \in V, \tag{4.2}$$

where $f \in [C(\overline{\Omega})]'$ is a given load. The approximate solution is searched in the finite element space

$$V_h = \{v_h \in C^1(\overline{\Omega}) \mid v_h\big|_{K_i} \in b(K_i) \ \ \forall K_i \in \mathcal{T}_h, \quad v_h = 0 \text{ on } \partial\Omega\},$$

where $v_h \in b(K_i)$ are biharmonic functions on each subelement K_i $(i = 1, \ldots, r)$ of the element $K \in \mathcal{T}_h$, i.e.,

$$v_h \in H^4(K_i), \quad \Delta^2 v_h = 0 \text{ in } K_i. \tag{4.3}$$

Define the finite element solution $u_h \in V_h$ by the equation

$$a(u_h, v_h) = \langle f, v_h \rangle, \quad v_h \in V_h. \tag{4.4}$$

Theorem 4.1. *Let u_1 and u_2 be two solutions of* (4.2) *corresponding to two right-hand sides f_1 and f_2. Denote by u_{1h} and u_{2h} their finite element solutions, i.e., solutions of* (4.4) *for f_1 and f_2, respectively. If*

$$u_1\big|_{B_h} = u_2\big|_{B_h}, \quad \frac{\partial u_1}{\partial n}\Big|_{B_h} = \frac{\partial u_2}{\partial n}\Big|_{B_h},$$

where

$$B_h = \bigcup_{K \in \mathcal{T}_h} \bigcup_{i=1}^{r} \partial K_i,$$

then $u_{1h} = u_{2h}$ on the whole Ω.

Proof. By (4.2) and (4.4), relations (4.1) and (4.3) can be written as

$$\begin{aligned} a(u_{1h} - u_{2h}, v_h) &= a(u_1 - u_2, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{r} a(u_1 - u_2, v_h)_{K_i} \\ &= \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{r} \Big[(u_1 - u_2, \Delta^2 v_h)_{0,K_i} + \Big\langle \frac{\partial}{\partial n}(u_1 - u_2), M(v_h) \Big\rangle_{0,\partial K_i} \\ &\quad + \langle u_1 - u_2, T(v_h) \rangle_{0,\partial K_i} + \sum_{j=1}^{J} (u_1 - u_2)(s_j)[H(v_h)](s_j) \Big] = 0 \end{aligned} \tag{4.5}$$

for any $v_h \in V_h$. Here $a(\cdot,\cdot)_{K_i}$ is defined similarly to $a(\cdot,\cdot)$, where integration is taken over K_i. By [10], the following Korn's inequality holds:

$$a(v, v) \geqslant (1 - \sigma)(\|v_{xx}\|^2_{0,\Omega} + \|v_{yy}\|^2_{0,\Omega} + \|v_{xy}\|^2_{0,\Omega}) \geqslant C\|v\|^2_{2,\Omega} \ \ \forall v \in V, \tag{4.6}$$

where C is a positive constant. Since $V_h \subset V$, from (4.5) and (4.6) we arrive at $C\|w_h\|^2_{2,\Omega} \leqslant a(w_h, w_h) = 0$ for $w_h = u_{1h} - u_{2h}$. Hence, $u_{1h} - u_{2h} = 0$ in Ω. □

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