Zuzana Dimitrovová

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SHAPE OPTIMIZATION OF ELASTO-PLASTIC BODIES

Zuzana Dimitrovová, Lisbon

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Abstract. Existence of an optimal shape of a deformable body made from a physically nonlinear material obeying a specific nonlinear generalized Hooke's law (in fact, the so called deformation theory of plasticity is invoked in this case) is proved. Approximation of the problem by finite elements is also discussed.

Keywords: mixed boundary value problem, deformation theory of plasticity, shape optimization, cost functional, finite elements

MSC 2000: 46N10, 46E35, 49Q10, 65N30, 74C05

1. Introduction

Shape optimization problems have been the subject of considerable research for many years. One of the main monographs [1], which gives the rigorous mathematical base of various questions related to these problems, was published by Haslinger and Neittaanmäki.

In this paper a shape optimization problem for a deformable body obeying a specific nonlinear generalized Hooke's law, is examined. Regarding the constitutive description of the model, results of the book [2], (Chap. 8) are employed. Among related papers, let us mention [3] and a recent study [4].

A deformable body is represented by a two-dimensional domain where part of its boundary is variable. This part of the boundary has to be optimized with respect to a specific cost functional. Moreover, it is assumed that Dirichlet boundary conditions are prescribed on the variable boundary. First, the existence of at least one optimal shape from a given class of admissible domains is proved. Second, the finite element approximation of the problem is discussed. Numerical implementation and results can be found in [5].

2. Setting of the problem

In this section the boundary value problem is stated in its classical as well as variational formulation and the existence of its unique solution is established. In addition, the formulation of the shape optimization problem is given.

Let a plane body be represented by a bounded simply connected domain $\Omega(\alpha) \subset \mathbb{R}^2$ with the boundary $\partial\Omega(\alpha) = \overline{\Gamma_u} \cup \overline{\Gamma_\alpha} \cup \overline{\Gamma_p}$, where Γ_u, Γ_α, Γ_p are nonempty disjoint sets, open in $\partial\Omega(\alpha)$. Γ_α is a variable part of the boundary and it is assumed that its shape is described by the graph of a function $\alpha \equiv \alpha(x_2)$. The conditions that must be satisfied by admissible functions α will be specified later. Γ_u, Γ_p are "fixed" parts of the boundary. In fact, they are fixed in the sense that their shape is given, but they are variable as well due to a motion of Γ_α, see Fig. 1. This fact should be emphasized; therefore, in what follows we write $\Gamma_p(\alpha)$ and $\Gamma_u(\alpha)$ instead of Γ_p and Γ_u, respectively. A possible partition of $\partial\Omega(\alpha)$ is depicted in Fig. 1.

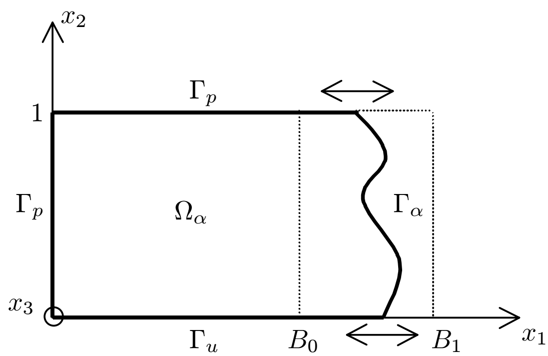

Figure 1. Deformable body $\Omega(\alpha)$ and a partition of its boundary.

Thus

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < \alpha(x_2);\; 0 < x_2 < \gamma\},$$

where γ is a given positive constant. Suppose for simplicity that $\gamma = 1$, consequently

$$\Omega(\alpha) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1 < \alpha(x_2);\; 0 < x_2 < 1\}.$$

The admissible functions α form a set U_{ad} specified as

$$U_{\mathrm{ad}} = \Big\{\alpha \in C^{0,1}([0,1]) \mid 0 < B_0 \leqslant \alpha(x_2) \leqslant B_1 \;\forall x_2 \in [0,1]; \\ \Big|\frac{\mathrm{d}\alpha}{\mathrm{d}x_2}\Big| \leqslant B_2 \text{ a.e. in } [0,1];\; \operatorname{meas}\Omega(\alpha) = B_3\Big\}, \tag{2.1}$$

where B_0, B_1, B_2 and B_3 are given positive constants such that $U_{\mathrm{ad}} \neq \emptyset$.

Let $\alpha \in U_{\mathrm{ad}}$ be arbitrary but fixed.

In the following, a mixed boundary value problem is specified on the domain $\Omega(\alpha)$ corresponding to this particular α. It is supposed that $\Omega(\alpha)$ is subject to body forces $\mathbf{F}(\alpha) = \big(F_1(\alpha), F_2(\alpha)\big)$, surface tractions $\mathbf{P}(\alpha) = \big(P_1(\alpha), P_2(\alpha)\big)$ on $\Gamma_p(\alpha)$ and the Dirichlet boundary conditions are prescribed on $\Gamma_u(\alpha) \cup \Gamma_\alpha$. In order to keep the practical base of the problem, it is assumed that the boundary conditions maintain their character on the part of the boundary where $B_0 \leqslant x_1 \leqslant B_1$. In addition, the plane strain assumption is adopted and the deformation theory for elasto-plastic nonhomogeneous isotropic materials is used. It should be remarked in this context that the deformation theory is suitable just for proportional loading. Only nonreversible processes are investigated and the constitutive law (the stress-strain relation) is assumed to be geometrically linear. In this case, the body can be, in fact, assumed to be elastic and to obey a specific nonlinear generalized Hook's law (see [6], [2]).

The classical formulation of the boundary value problem can be written as follows: find a displacement field $\mathbf{u}(\alpha) = \big(u_1(\alpha), u_2(\alpha)\big)$ (sufficiently smooth) such that

$$\frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{u}(\alpha)) + F_i(\alpha) = 0,$$
$$\varepsilon_{ij}(\mathbf{u}(\alpha)) = \frac{1}{2}\left(\frac{\partial u_j(\alpha)}{\partial x_i} + \frac{\partial u_i(\alpha)}{\partial x_j}\right),$$
$$\tau_{ij}(\mathbf{u}(\alpha)) = \Big(k - \frac{2}{3}\mu\big(\Gamma^2(\mathbf{u}(\alpha))\big)\Big)\vartheta(\mathbf{u}(\alpha))\delta_{ij} + 2\mu\big(\Gamma^2(\mathbf{u}(\alpha))\big)\varepsilon_{ij}(\mathbf{u}(\alpha)),$$
$$\tau_{33}(\mathbf{u}(\alpha)) = \Big(k - \frac{2}{3}\mu\big(\Gamma^2(\mathbf{u}(\alpha))\big)\Big)\vartheta(\mathbf{u}(\alpha)),$$
$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0,\ \ \tau_{13} = \tau_{23} = 0$$

hold in $\Omega(\alpha)$ and the following boundary conditions on $\partial\Omega(\alpha)$ are satisfied:

$$\mathbf{u}(\alpha) = \mathbf{0} \quad \text{on } \Gamma_u(\alpha) \cup \Gamma_\alpha,$$
$$\tau_{ij}(\mathbf{u}(\alpha))\nu_j = P_i(\alpha) \quad \text{on } \Gamma_p(\alpha).$$

The summation convention has been used in the above relations and is used throughout this paper. If not specified differently, the subscripts i and j take the values 1, 2. The symbols $\boldsymbol{\varepsilon}$ and $\boldsymbol{\tau}$ stand for the linearized strain tensor and stress tensor, respectively, δ_{ij} is the Kronecker symbol and ϑ is the first invariant of $\boldsymbol{\varepsilon}$. Furthermore, k and μ denote the bulk and the shear modulus, respectively. Finally, $\Gamma^2 = e_{ij}e_{ij}$, $i, j = 1, 2, 3$, where $\mathbf{e}$ is the deviator of the strain tensor $\boldsymbol{\varepsilon}$. Since the plane strain assumption is adopted we have $e_{ij} = \varepsilon_{ij} - \frac{1}{3}\vartheta\delta_{ij}$, $e_{13} = e_{23} = 0$ and $e_{33} = -\frac{1}{3}\vartheta$ (see [2]).

Next, it is supposed that k and μ satisfy the following conditions (PM):

$$
(\mathrm{PM})\begin{cases}
k = k(\mathbf{x}) \text{ is a continuous function of both variables } x_1,\ x_2;\\
0 < k_0 \leqslant k(\mathbf{x}) \leqslant k_1 < \infty \ \forall \mathbf{x} \in \overline{\Omega(\alpha)},\\
\quad \text{where } k_0 \text{ and } k_1 \text{ are given positive constants;}\\
\mu(\Gamma^2) = \mu(t,\mathbf{x}) \text{ is a continuous function of the variables}\\
\quad t = \Gamma^2,\ x_1,\ x_2 \text{ and continuously differentiable with respect to } t;\\
0 < \mu_0 \leqslant \mu(t,\mathbf{x}) \leqslant \frac{3}{2}k(\mathbf{x}) \text{ and } 0 < \eta_0 \leqslant \mu(t,\mathbf{x}) + 2\frac{\partial\mu(t,\mathbf{x})}{\partial t}t \leqslant \eta_1\\
\quad \forall \mathbf{x} \in \overline{\Omega(\alpha)} \text{ and } t \geqslant 0,\ \text{ where } \mu_0,\ \eta_0,\ \eta_1 \text{ are given positive constants.}
\end{cases}
$$

It should be emphasized that the above introduced constants k_0, k_1, μ_0, η_0 and η_1 are independent of α. The physical background of this model is described in detail in [6] and [2]. The justification and validity of the constitutive relation, specified in this way, is discussed in [6].

In order to present the variational formulation of the boundary value problem, additional notation is given. Let $H^k(\Omega(\alpha))$, k an integer, be the classical Sobolev space of functions, generalized derivatives of which up to the order k are square integrable. It obviously holds that

$$L_2\big(\Omega(\alpha)\big) \equiv H^0\big(\Omega(\alpha)\big).$$

Symbols $\|.\|_{k,\Omega(\alpha)}$ and $(.,.)_{k,\Omega(\alpha)}$ are, as usual, used to denote the norm and the inner product in $H^k(\Omega(\alpha))$, respectively. Furthermore, functional spaces

$$W(\alpha) \equiv \big[H^1\big(\Omega(\alpha)\big)\big]^2 \quad \text{and} \quad K(\alpha) \equiv \big[L_2\big(\Omega(\alpha)\big)\big]^2$$

are introduced and $(\mathbf{f},\mathbf{g})_{s,\Omega(\alpha)}$, $s = 0,1$ will stand for the inner product in $K(\alpha)$ and $W(\alpha)$, respectively, according to the value of s. Its definition is

$$(\mathbf{f},\mathbf{g})_{s,\Omega(\alpha)} = (f_1,g_1)_{s,\Omega(\alpha)} + (f_2,g_2)_{s,\Omega(\alpha)}, \tag{2.2}$$

where $\mathbf{f} = (f_1,f_2)$ & $\mathbf{g} = (g_1,g_2) \in K(\alpha)$ or $W(\alpha)$, respectively. An extension of the definition (2.2) to sets other than $\Omega(\alpha)$ as well as the definition of the norms in $K(\alpha)$ and $W(\alpha)$ is obvious. Now the space of kinematically admissible functions (virtual displacements) can be introduced as

$$V(\alpha) = \{\mathbf{f} \mid \mathbf{f} \in W(\alpha);\ f_i = 0 \text{ on } \Gamma_u(\alpha) \cup \Gamma_\alpha;\ i = 1,2\}.$$

Besides, it is assumed that $\mathbf{F}(\alpha) \in K(\alpha)$ and $\mathbf{P}(\alpha) \in \big[L_2\big(\Gamma_p(\alpha)\big)\big]^2$.

The weak formulation of the boundary value problem reads as follows:

$$(\mathrm{P}(\alpha))\begin{cases}\text{find } \mathbf{u}(\alpha)\in V(\alpha) \text{ such that}\\ \big[\boldsymbol{\tau}(\mathbf{u}(\alpha)),\boldsymbol{\varepsilon}(\mathbf{v})\big]_{\Omega(\alpha)}=\langle\mathbf{L}(\alpha),\mathbf{v}\rangle_{\Omega(\alpha)}\ \forall\mathbf{v}\in V(\alpha),\end{cases}$$

where

$$\big[\boldsymbol{\tau}(\mathbf{u}(\alpha)),\boldsymbol{\varepsilon}(\mathbf{v})\big]_{\Omega(\alpha)}\equiv\int\limits_{\Omega(\alpha)}\tau_{ij}(\mathbf{u}(\alpha))\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\mathbf{x},$$

$$\langle\mathbf{L}(\alpha),\mathbf{v}\rangle_{\Omega(\alpha)}\equiv\big(\mathbf{F}(\alpha),\mathbf{v}\big)_{0,\Omega(\alpha)}+\big(\mathbf{P}(\alpha),\mathbf{v}\big)_{0,\Gamma_p(\alpha)}.$$

Obviously, when the weak solution is sufficiently smooth, then it is also the classical one.

An equivalent formulation of $(\mathrm{P}(\alpha))$ can be given by making use of the energy functional. Namely,

$$(\mathrm{P}(\alpha))\begin{cases}\text{find } \mathbf{u}(\alpha)\in V(\alpha) \text{ such that}\\ \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha))\leqslant\Phi_{\Omega(\alpha)}(\mathbf{v})\ \forall\mathbf{v}\in V(\alpha),\end{cases}$$

where the total potential energy $\Phi_{\Omega(\alpha)}$ can be written in the following way:

$$\Phi_{\Omega(\alpha)}(\mathbf{v})\equiv\int\limits_{\Omega(\alpha)}\left[\frac{1}{2}k\vartheta^2(\mathbf{v})+\frac{1}{2}\int\limits_0^{\Gamma(\mathbf{v},\mathbf{v})}\mu(t)\,\mathrm{d}t\right]\mathrm{d}\mathbf{x}-\langle\mathbf{L}(\alpha),\mathbf{v}\rangle_{\Omega(\alpha)}\tag{2.3}$$

with $(\mathbf{w}\in V(\alpha))$,

$$\Gamma(\mathbf{w},\mathbf{v})=-\frac{2}{3}\vartheta(\mathbf{w})\vartheta(\mathbf{v})+2\varepsilon_{ij}(\mathbf{w})\varepsilon_{ij}(\mathbf{v}).$$

It can be verified that $\Gamma(\mathbf{v},\mathbf{v})=2\Gamma^2(\mathbf{v})$. If the conditions (PM) are met, the functional $\Phi_{\Omega(\alpha)}$ is continuous, coercive and strictly convex (consequently, also weakly lower semi-continuous) on the space $V(\alpha)$. In the sequel, a unique solution $\mathbf{u}(\alpha)$ of $\mathrm{P}(\alpha)$ exists. The detailed analysis of this statement can be found in [2].

Since α was arbitrary, the unique solution $\mathbf{u}(\alpha)$ of $\mathrm{P}(\alpha)$ exists for any $\alpha\in U_{\mathrm{ad}}$.

Finally, the optimization problem can be formulated. First of all it is necessary to introduce a cost functional, the formulation of which depends on the designer. Thus, let $Q\colon(\alpha,\mathbf{v})\to\mathbb{R}^1$, $\alpha\in U_{\mathrm{ad}}$, $\mathbf{v}\in V(\alpha)$ be a cost functional. Then the shape optimization problem reads as follows:

$$(\mathrm{R})\begin{cases}\text{find } \alpha^\star\in U_{\mathrm{ad}} \text{ such that}\\ S(\alpha^\star)\leqslant S(\alpha)\ \forall\alpha\in U_{\mathrm{ad}},\end{cases}$$

where $S(\alpha)=Q\big(\alpha,\mathbf{u}(\alpha)\big)$ with $\mathbf{u}(\alpha)$ being the solution of $(\mathrm{P}(\alpha))$. Assumptions on $S(\alpha)$ guaranteeing the existence of a solution for the problem (R) will be specified in the next section.

3. Existence of an optimal shape

The proof of existence of at least one solution of the optimization problem (R), under some continuity assumptions, is the main result of this section.

Let $\hat{\Omega}=(0,B_1)\times(0,1)$ be given. Then $\hat{\Omega}\supset\Omega(\alpha)\ \forall\alpha\in U_{\mathrm{ad}}$. Suppose that the assumptions (PM) hold on $\overline{\hat{\Omega}}$ and that

$$\mathbf{F}\in\big[L_2(\hat{\Omega})\big]^2 \quad\text{and}\quad \mathbf{P}\in\big[L_2(\hat{\Gamma}_p)\big]^2.$$

Introduce $\hat{W}=\big[H^1(\hat{\Omega})\big]^2$ and

$$\hat{V}=\{\mathbf{v}\in\hat{W}\mid \exists\alpha\in U_{\mathrm{ad}}\colon\ \mathbf{v}=\mathbf{0}\text{ a.e. in }\hat{\Omega}\setminus\Omega(\alpha)\text{ and }\mathbf{v}\big|_{\Omega(\alpha)}\in V(\alpha)\}.$$

If $\mathbf{v}\in V(\alpha)$ then the symbol $\hat{\mathbf{v}}$ will denote a specific extension of $\mathbf{v}$, particularly, $\hat{\mathbf{v}}$ is an element of $\hat{V}$ such that $\hat{\mathbf{v}}\big|_{\Omega(\alpha)}\equiv\mathbf{v}$.

As usual, throughout this paper the arrows $\rightrightarrows$, $\rightarrow$ and $\rightharpoonup$ are used for the uniform, strong and weak convergence, respectively.

In order to ensure the existence of at least one solution of the optimization problem (R), it is assumed that the functional S satisfies the following continuity assumption (PS):

$$(\mathrm{PS})\ \left\{\begin{array}{l} \alpha_n\rightrightarrows\alpha\text{ in }[0,1],\ \alpha_n,\alpha\in U_{\mathrm{ad}},\\ \hat{\mathbf{u}}(\alpha_n)\rightarrow\hat{\mathbf{u}}(\alpha)\text{ in }\hat{V},\\ \mathbf{u}(\alpha_n)\in V(\alpha_n)\text{ and }\mathbf{u}(\alpha)\in V(\alpha)\\ \quad\text{being the solutions of }(\mathrm{P}(\alpha_n))\\ \quad\text{and }(\mathrm{P}(\alpha)),\text{ respectively,}\end{array}\right\}\Rightarrow\liminf_{n\to\infty}S(\alpha_n)\geqslant S(\alpha).$$

It is useful to remark that, for the specification of the strong convergence in $\hat{V}$, (2.2) is adopted.

Before the existence of a solution of the problem (R) is proved, two lemmas are established. For the first, it is also necessary to assume that the partition of the "fixed" part of the boundary $\overline{\Gamma_u(\alpha)}\cup\overline{\Gamma_p(\alpha)}$ is such that $\overline{\Gamma_u(\alpha)}\cap\overline{\Gamma_p(\alpha)}$ contains only a finite number of points. The number of these points does not depend on α, due to the assumption about the character of the boundary conditions specified in Section 2.

Additionally, the usual notation $\mathcal{E}(\overline{\Omega})$ is introduced for the space of infinitely differentiable functions, which can be continuously extended with all their derivatives from Ω onto $\overline{\Omega}$.

Lemma 3.1. *Let* $\alpha_n \rightrightarrows \alpha$ $(n \to \infty)$ *in* $[0,1]$, $\alpha_n, \alpha \in U_{\rm ad}$ *and let* $\mathbf{v}(\alpha) \in V(\alpha)$. *Then there exists a sequence*

$$\{\mathbf{w}_i\} \subset \left[\mathcal{E}\big(\overline{\Omega(\alpha)}\big)\right]^2 \cap V(\alpha)$$

such that for $i \to \infty$

$$\mathbf{w}_i \to \mathbf{v}(\alpha) \quad \textit{in} \quad V(\alpha), \quad \hat{\mathbf{w}}_i \to \hat{\mathbf{v}}(\alpha) \quad \textit{in}\ \hat{V}$$

and for an arbitrary i *there exists an integer* $n_0(i)$ *such that*

$$\Omega(\alpha_n) \supset \operatorname{supp} \mathbf{w}_i \ \ \forall n \geqslant n_0(i), \tag{3.1}$$

yielding

$$\hat{\mathbf{w}}_i\, |_{\Omega(\alpha_n)} \in V(\alpha_n) \ \ \forall n \geqslant n_0(i). \tag{3.2}$$

P r o o f. Let $\mathbf{v}(\alpha) \in V(\alpha)$ be given. Then a sequence

$$\{\mathbf{w}_i\} \subset \left[\mathcal{E}\big(\overline{\Omega(\alpha)}\big)\right]^2 \cap V(\alpha)$$

can be found such that

$$\operatorname{dist}\big(\operatorname{supp} \mathbf{w}_i, \overline{\Gamma_\alpha} \cup \overline{\Gamma_u(\alpha)}\big) > 0 \ \ \forall i \tag{3.3}$$

(at this point the assumption that the set $\overline{\Gamma_u(\alpha)} \cap \overline{\Gamma_p(\alpha)}$ is finite is exploited) and

$$\mathbf{w}_i \to \mathbf{v}(\alpha) \quad \text{in } V(\alpha).$$

Obviously also

$$\hat{\mathbf{w}}_i \to \hat{\mathbf{v}}(\alpha) \quad \text{in } \hat{V}.$$

Let i be a fixed integer. Due to (3.3) there exists $n_0(i)$ such that (3.1) and consequently (3.2) are met. □

Lemma 3.2. *Let* $\alpha_n \rightrightarrows \alpha$ $(n \to \infty)$ *in* $[0,1]$, $\alpha_n, \alpha \in U_{\rm ad}$. *Let* $\mathbf{u}(\alpha_n)$ *be solutions of* $(\mathrm{P}(\alpha_n))$. *Then*

$$\hat{\mathbf{u}}(\alpha_n) \to \tilde{\mathbf{u}} \quad \textit{in}\ \hat{V}$$

where $\tilde{\mathbf{u}}\big|_{\Omega(\alpha)} \equiv \mathbf{u}(\alpha)$ *is the solution of* $(\mathrm{P}(\alpha))$ *and* $\tilde{\mathbf{u}} = \mathbf{0}$ *a.e. in* $\hat{\Omega} \setminus \overline{\Omega(\alpha)}$ *(thus* $\tilde{\mathbf{u}} = \hat{\mathbf{u}}(\alpha)$*).*

P r o o f. Using the assumptions (PM) it can be proved for the solutions $\mathbf{u}(\alpha_n) \in V(\alpha_n)$ of (P(α_n)) that

$$\Phi_{\Omega(\alpha_n)}(\mathbf{u}(\alpha_n)) \geqslant C_1 \int\limits_{\Omega(\alpha_n)} \varepsilon_{ij}(\mathbf{u}(\alpha_n))\varepsilon_{ij}(\mathbf{u}(\alpha_n))\,\mathrm{d}\mathbf{x} - C_2\|\mathbf{u}(\alpha_n)\|_{0,\Omega(\alpha_n)} - C_3\|\mathbf{u}(\alpha_n)\|_{0,\Gamma_p(\alpha_n)}, \tag{3.4}$$

where the constant C_1 depends only on μ_0 and η_0 and the constants C_2 and C_3 depend only on $\|\mathbf{F}\|_{0,\hat{\Omega}}$ and $\|\mathbf{P}\|_{0,\hat{\Gamma}_p}$, respectively. From Korn's inequality it can be shown that the first integral on the right hand side of (3.4) can be estimated using the relation

$$\int\limits_{\Omega(\alpha_n)} \varepsilon_{ij}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})\,\mathrm{d}\mathbf{x} \geqslant C(\Omega(\alpha_n))\|\mathbf{v}\|^2_{1,\Omega(\alpha_n)} \quad \forall \mathbf{v} \in V(\alpha_n), \tag{3.5}$$

where $C(\Omega(\alpha_n))$ is a constant which generally depends on $\Omega(\alpha_n)$. On the other hand, it can be proved that this constant depends solely on B_1 from the definition (2.1) of U_{ad} (see [7], Remark 2.1). Thus $C(\Omega(\alpha_n))$ can be replaced by a constant C that does not depend on $\alpha_n \in U_{\mathrm{ad}}$. Using the trace theorem, the last term on the right hand side of (3.4) can be estimated as

$$\|\mathbf{v}\|_{0,\Gamma_p(\alpha_n)} \leqslant C\|\mathbf{v}\|_{1,\Omega(\alpha_n)}. \tag{3.6}$$

It is easy to understand that the constant C in (3.6) can also be chosen independently of $\alpha_n \in U_{\mathrm{ad}}$. From (3.4)–(3.6) it finally follows for the solutions $\mathbf{u}(\alpha_n) \in V(\alpha_n)$ of (P(α_n)) that

$$\Phi_{\Omega(\alpha_n)}(\mathbf{u}(\alpha_n)) \geqslant C_1\|\mathbf{u}(\alpha_n)\|^2_{1,\Omega(\alpha_n)} - C_2\|\mathbf{u}(\alpha_n)\|_{1,\Omega(\alpha_n)},$$

where C_1 and C_2 do not depend on $\alpha_n \in U_{\mathrm{ad}}$. Consequently, $\Phi_{\Omega(\alpha_n)}$ is coercive, uniformly with respect to $\alpha_n \in U_{\mathrm{ad}}$, since we may choose a fixed $\mathbf{u}^0$ such that

$$\operatorname{supp}\mathbf{u}^0 \subset \Omega(\alpha_n) \ \& \ \mathbf{u}^0 \in V(\alpha_n) \ \ \forall n$$

and write

$$C_3 = \Phi_{\operatorname{supp}\mathbf{u}^0}\mathbf{u}^0 = \Phi_{\Omega(\alpha_n)}\mathbf{u}^0 \geqslant \Phi_{\Omega(\alpha_n)}\mathbf{u}(\alpha_n).$$

Summing up, there exists a constant $C > 0$ that does not depend on $\alpha_n \in U_{\mathrm{ad}}$ and fulfils the inequality

$$\|\hat{\mathbf{u}}(\alpha_n)\|_{1,\hat{\Omega}} = \|\mathbf{u}(\alpha_n)\|_{1,\Omega(\alpha_n)} \leqslant C \ \ \forall n.$$

Consequently, a subsequence $\{\mathbf{u}(\alpha_{n_j})\}$ can be found such that

$$\hat{\mathbf{u}}(\alpha_{n_j}) \rightharpoonup \tilde{\mathbf{u}} \quad \text{in } \hat{V}. \tag{3.7}$$

Denote $\mathbf{u}^\star \equiv \tilde{\mathbf{u}}\big|_{\Omega(\alpha)}$. It will be proved that $\mathbf{u}^\star$ solves $(P(\alpha))$ and that $\tilde{\mathbf{u}} = \mathbf{0}$ a.e. in $\hat{\Omega} \setminus \overline{\Omega(\alpha)}$. First, the latter statement will be treated. Let us introduce a set

$$H_m(\alpha) = \left\{(x_1, x_2) \in \mathbb{R}^2 \;\Big|\; 0 < x_1 < \alpha(x_2) + \frac{1}{m};\ 0 < x_2 < 1\right\},\ m \in \mathbb{N}$$

and let $m \in \mathbb{N}$ be arbitrary but fixed. Then, obviously, there exists $n_{j_0}(m)$ such that

$$H_m(\alpha) \supset \Omega(\alpha_{n_j})\ \ \forall n_j \geqslant n_{j_0}(m).$$

Since

$$\hat{\mathbf{u}}(\alpha_{n_j})\big|_{\hat{\Omega}\setminus\overline{H_m(\alpha)}} = \mathbf{0}\ \ \forall n_j \geqslant n_{j_0}(m),$$

we conclude $\tilde{\mathbf{u}} = \mathbf{0}$ a.e. in $\hat{\Omega} \setminus \overline{\Omega(\alpha)}$ taking into account (3.7) and the fact that m was arbitrary. Obviously, $\mathbf{u}^\star = \mathbf{0}$ on Γ_α.

Similarly as above, one can prove that $\mathbf{u}^\star = \mathbf{0}$ on $\Gamma_u(\alpha)$. In particular, exploiting the assumption about the character of the boundary conditions (specified in Section 2), one may suppose that (in accordance with Fig. 1) the statement has to be proved for $x_2 = 0$. It is enough to prove that

$$\mathbf{u}^\star = \mathbf{0} \text{ on } \{x_1 \mid B_0 < x_1 < \alpha(0)\}. \tag{3.8}$$

Naturally,

$$\mathbf{u}(\alpha_{n_j}) = \mathbf{0} \text{ on } \{x_1 \mid B_0 < x_1 < \alpha_{n_j}(0)\}\ \ \forall n_j.$$

Analogously as above, a set

$$G_m(\alpha) = \left\{x_1 \;\Big|\; B_0 < x_1 < \alpha(0) - \frac{1}{m}\right\},\ m \in \mathbb{N}$$

can be introduced. There exists $n_{j_0}(m)$ such that

$$G_m(\alpha) \subset \{x_1 \mid B_0 < x_1 < \alpha_{nj}(0)\}\ \ \forall n_j \geqslant n_{j_0}(m)$$

and therefore

$$\mathbf{u}(\alpha_{nj}) = \mathbf{0} \quad \text{on } G_m(\alpha)\ \ \forall n_j \geqslant n_{j_0}(m),$$

yielding (3.8) since m was arbitrary.

Summing up, $\mathbf{u}^\star \in V(\alpha)$.

In the following the proof that $\mathbf{u}^\star$ solves $(\mathrm{P}(\alpha))$ is provided. Denote by $\mathbf{u}(\alpha) \in V(\alpha)$ the solution of $(\mathrm{P}(\alpha))$. It has to be proved that

$$\mathbf{u}^\star = \mathbf{u}(\alpha) \quad \text{a.e. in } \Omega(\alpha). \tag{3.9}$$

It follows from the definition of $(\mathrm{P}(\alpha))$ that

$$\Phi_{\Omega(\alpha)}(\mathbf{u}^\star) \geqslant \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)). \tag{3.10}$$

The converse inequality in (3.10) will be proved. Now the statement of Lemma 3.1 is needed, according to which a sequence

$$\{\mathbf{w}_i\} \subset \big[\mathcal{E}\big(\overline{\Omega(\alpha)}\big)\big]^2 \cap V(\alpha)$$

such that

$$\mathbf{w}_i \to \mathbf{u}(\alpha) \quad \text{in } V(\alpha) \tag{3.11}$$

can be chosen with the property that for i fixed there exists $n_{j_0}(i)$ such that (3.1) and (3.2) hold $\forall n_j \geqslant n_{j_0}(i)$. Substituting $\mathbf{w}_i$ into $(\mathrm{P}(\alpha_{n_j}))$, one has

$$\begin{aligned} \Phi_{\Omega(\alpha)}(\mathbf{w}_i) &= \Phi_{\Omega(\alpha_{n_j})}\big(\hat{\mathbf{w}}_i|_{\Omega(\alpha_{n_j})}\big) \geqslant \Phi_{\Omega(\alpha_{n_j})}(\mathbf{u}(\alpha_{n_j})) \\ &= \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha_{n_j})) \ \ \forall n_j \geqslant n_{j_0}(i). \end{aligned} \tag{3.12}$$

Let $n_j \to \infty$. Using (3.7) and the weak lower semi-continuity of $\Phi_{\hat{\Omega}}$ on $\hat{V}$, we can writte

$$\liminf_{n_j \to \infty} \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha_{n_j})) \geqslant \Phi_{\hat{\Omega}}(\tilde{\mathbf{u}}) = \Phi_{\Omega(\alpha)}(\mathbf{u}^\star),$$

which together with (3.12) yields

$$\Phi_{\Omega(\alpha)}(\mathbf{w}_i) \geqslant \Phi_{\Omega(\alpha)}(\mathbf{u}^\star). \tag{3.13}$$

Passing to the limit $(i \to \infty)$ in the above relation, using the continuity of $\Phi_{\Omega(\alpha)}$ on $V(\alpha)$ and (3.11), we obtain

$$\lim_{i \to \infty} \Phi_{\Omega(\alpha)}(\mathbf{w}_i) = \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)) \geqslant \Phi_{\Omega(\alpha)}(\mathbf{u}^\star). \tag{3.14}$$

Then (3.10) and (3.14) imply

$$\Phi_{\Omega(\alpha)}(\mathbf{u}^\star) = \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)).$$

Finally, from this and from the uniqueness of the solution of the problem (P(α)), (3.9) follows.

It remains to show that for a suitable sequence $\{\mathbf{u}(\alpha_{n_j})\}$ one can prove the strong convergence in (3.7). It was already proved that $\mathbf{u}^\star \equiv \tilde{\mathbf{u}}\big|_{\Omega(\alpha)} = \mathbf{u}(\alpha) = \hat{\mathbf{u}}(\alpha)\big|_{\Omega(\alpha)}$ a.e. in $\Omega(\alpha)$. It follows from (3.12) that

$$\Phi_{\hat{\Omega}}(\hat{\mathbf{w}}_i) \geqslant \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha_{n_j}))\ \forall n_j \geqslant n_{j_0}(i). \tag{3.15}$$

Expressing the value of $\Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha_{n_j}))$ by means of the Taylor expansion around $\hat{\mathbf{u}}(\alpha)$ one gets

$$\begin{aligned}
(3.16)\quad & \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha_{n_j})) \\
&= \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha)) + D\Phi_{\hat{\Omega}}\big(\hat{\mathbf{u}}(\alpha); \hat{\mathbf{u}}(\alpha_{n_j}) - \hat{\mathbf{u}}(\alpha)\big) \\
&\quad + \frac{1}{2} D^2\Phi_{\hat{\Omega}}\big(\hat{\mathbf{u}}(\alpha) + t\big(\hat{\mathbf{u}}(\alpha_{n_j}) - \hat{\mathbf{u}}(\alpha)\big); \hat{\mathbf{u}}(\alpha_{n_j}) - \hat{\mathbf{u}}(\alpha), \hat{\mathbf{u}}(\alpha_{n_j}) - \hat{\mathbf{u}}(\alpha)\big) \\
&\geqslant \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha)) + D\Phi_{\hat{\Omega}}\big(\hat{\mathbf{u}}(\alpha); \hat{\mathbf{u}}(\alpha_{n_j}) - \hat{\mathbf{u}}(\alpha)\big) + C\|\hat{\mathbf{u}}(\alpha_{n_j}) - \hat{\mathbf{u}}(\alpha)\|^2_{1,\hat{\Omega}}.
\end{aligned}$$

In the above expression, the fact that the second differential of $\Phi_{\hat{\Omega}}$ is positive definite was used, i.e. that

$$D^2\Phi_{\hat{\Omega}}(\mathbf{u} + t\mathbf{v}; \mathbf{v}, \mathbf{v}) \geqslant C\|\mathbf{v}\|^2_{1,\hat{\Omega}}$$

holds for any $t \in [0,1]$ and $\forall \mathbf{u}, \mathbf{v} \in \hat{V}$; C is a positive constant (for the proof see [2]). Now the strong convergence in (3.7) follows from (3.14)–(3.16).

In addition, since the solution of (P(α)) is unique, the whole sequence $\{\mathbf{u}(\alpha_n)\}$ converges, i.e.

$$\hat{\mathbf{u}}(\alpha_n) \to \hat{\mathbf{u}}(\alpha) \quad \text{in } \hat{V}, \tag{3.17}$$

which completes the proof. □

Lemma 3.2 justifies the possibility of writing down the strong convergence in continuity assumption (PS) without making it too restrictive.

Theorem 3.3. *Let* (PS) *be satisfied. Then there exists at least one solution* $\alpha^\star \in U_{\mathrm{ad}}$ *of the optimization problem* (R).

P r o o f. Denote $q = \inf\limits_{\alpha \in U_{\mathrm{ad}}} \mathrm{S}(\alpha)$. Let $\{\alpha_n\} \subset U_{\mathrm{ad}}$ be a minimizing sequence, i.e.

$$q = \lim_{n\to\infty} S(\alpha_n). \tag{3.18}$$

Since U_{ad} is a compact subset of $C([0,1])$, hence according to the Arzelà-Ascoli theorem there exists a subsequence of $\{\alpha_n\}$ (denoted for the sake of simplicity by the same symbol) such that

$$\alpha_n \rightrightarrows \alpha^\star \text{ in } [0,1] \quad \text{and} \quad \alpha^\star \in U_{\text{ad}}.$$

Taking the solutions $\mathbf{u}(\alpha_n)$ of $(\mathrm{P}(\alpha_n))$ and the solution $\mathbf{u}(\alpha^\star)$ of $(\mathrm{P}(\alpha^\star))$, then, according to Lemma 3.2,

$$\hat{\mathbf{u}}(\alpha_n) \to \hat{\mathbf{u}}(\alpha^\star) \quad \text{in } \hat{V}.$$

Now the assumption (PS) implies

$$\liminf_{n\to\infty} S(\alpha_n) \geqslant S(\alpha^\star)$$

which together with (3.18) completes the proof. □

Two examples of possible cost functionals satisfying (PS) are

$$S_1(\alpha) = \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)),$$
$$S_2(\alpha) = \int_{\Omega(\alpha)} \big(\mathbf{u}(\alpha) - \mathbf{u}_d\big)^2 \,\mathrm{d}\mathbf{x},$$

where $\mathbf{u}(\alpha)$ is the solution of $(\mathrm{P}(\alpha))$, $\Phi_{\Omega(\alpha)}$ is defined by (2.3) and $\mathbf{u}_d$ is a given displacement field such that $\mathbf{u}_d \in [L_2(\hat{\Omega})]^2$.

The verification of (PS) for the funcionals S_1 and S_2 can be done in the following way. Let $\alpha_n \rightrightarrows \alpha$ $(n \to \infty)$ in $[0,1]$, $\alpha_n, \alpha \in U_{\text{ad}}$, moreover, let $\mathbf{u}(\alpha_n)$ and $\mathbf{u}(\alpha)$ be the solutions of $(\mathrm{P}(\alpha_n))$ and $(\mathrm{P}(\alpha))$, respectively, then (compare with Lemma 3.2)

$$\hat{\mathbf{u}}(\alpha_n) \to \hat{\mathbf{u}}(\alpha) \quad \text{in } \hat{V}. \tag{3.19}$$

It will be proved that

$$\liminf_{n\to\infty} S_k(\alpha_n) \geqslant S_k(\alpha), \quad k = 1,2.$$

First, $k=1$. The continuity of $\Phi_{\hat{\Omega}}$ on $\hat{V}$ directly yields that

$$\lim_{n\to\infty} \Phi_{\Omega(\alpha_n)}(\mathbf{u}(\alpha_n)) = \lim_{n\to\infty} \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha_n)) = \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha)) = \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)).$$

For $k=2$, it has to be proved that

$$\liminf_{n\to\infty} \int_{\Omega(\alpha_n)} \big(\mathbf{u}(\alpha_n) - \mathbf{u}_d\big)^2 \,\mathrm{d}\mathbf{x} \geqslant \int_{\Omega(\alpha)} \big(\mathbf{u}(\alpha) - \mathbf{u}_d\big)^2 \,\mathrm{d}\mathbf{x}. \tag{3.20}$$

A set

$$\tilde{G}_m(\alpha) = \left\{(x_1, x_2) \text{ in } \mathbb{R}^2 \;\middle|\; 0 < x_1 < \alpha(x_2) - \frac{1}{m};\; 0 < x_2 < 1\right\},\; m \in \mathbb{N}$$

can be introduced and, obviously, there exists $n_0(m)$ such that

$$\tilde{G}_m(\alpha) \subset \Omega(\alpha_n)\;\; \forall n \geqslant n_0(m).$$

Then

$$\int\limits_{\Omega(\alpha_n)} \big(\mathbf{u}(\alpha_n) - \mathbf{u}_d\big)^2 \,\mathrm{d}\mathbf{x} \geqslant \int\limits_{\tilde{G}_m(\alpha)} \big(\mathbf{u}(\alpha_n) - \mathbf{u}_d\big)^2 \,\mathrm{d}\mathbf{x}\;\; \forall n \geqslant n_0(m),$$

yielding

$$\begin{aligned}
&\liminf_{n\to\infty} \int\limits_{\Omega(\alpha_n)} \big(\mathbf{u}(\alpha_n) - \mathbf{u}_d\big)^2 \,\mathrm{d}\mathbf{x} \\
&\quad\geqslant \liminf_{n\to\infty} \int\limits_{\tilde{G}_m(\alpha)} \big(\mathbf{u}(\alpha_n) - \mathbf{u}_d\big)^2 \,\mathrm{d}\mathbf{x} = \liminf_{n\to\infty} \|\mathbf{u}(\alpha_n) - \mathbf{u}_d\|^2_{0,\tilde{G}_m(\alpha)} \\
&\quad= \lim_{n\to\infty} \|\mathbf{u}(\alpha_n) - \mathbf{u}_d\|^2_{0,\tilde{G}_m(\alpha)} = \|\mathbf{u}(\alpha) - \mathbf{u}_d\|^2_{0,\tilde{G}_m(\alpha)}
\end{aligned}$$

also by virtue of (3.19). Consequently, (3.20) follows from the fact that m is arbitrary.

4. Approximation of (P)

In this section the discretization forms of the boundary value problem as well as of the optimization problem are given. The main contribution of this part of the paper is in the last theorem, which states the relation between the solutions of the discretized and continuous optimization problems.

First, let $h > 0$ be given. Then the set U^h_{ad} can be defined in the following way:

$$U^h_{\mathrm{ad}} = \{\alpha_h \in C([0,1]) \mid \alpha_h\big|_{[a_{i-1},a_i]} \in P_1([a_{i-1}, a_i]),\; i = 1, 2, \ldots, p\} \cap U_{\mathrm{ad}},$$

where $0 \equiv a_0 < a_1 < \ldots < a_p \equiv 1$ is a partition of $[0,1]$ and $P_1([a_{i-1}, a_i])$ denotes the set of all linear functions on an interval $[a_{i-1}, a_i]$, $i = 1, 2, \ldots, p$. Consequently, $\Omega(\alpha_h)$ is a domain with a polygonal boundary.

Let h and $\alpha_h \in U^h_{\mathrm{ad}}$ be fixed. Then $\Omega(\alpha_h)$ can be divided into a finite number of closed triangles T, the collection of which is denoted by $T_h(\alpha_h)$. $T_h(\alpha_h)$ is called a triangulation of $\Omega(\alpha_h)$ if, first, each straight element of Γ_{α_h} composes a full side of a triangle, second,

$$\bigcup_{T \in T_h(\alpha_h)} T = \overline{\Omega(\alpha_h)}$$

and, third,

$$\begin{aligned}
&T, T' \in T_h(\alpha_h) \Longrightarrow T^\circ \cap T'^\circ = \emptyset \\
&\text{and either } T \cap T' = \emptyset \text{ or } T \text{ and } T' \text{ have in common} \\
&\text{one whole edge or one vertex}
\end{aligned}$$

(T° stands for the interior of T).

Let only h be fixed. It will be assumed that the following conditions are satisfied. First, the procedure for the construction of the triangle vertices (nodes) is uniquely associated with each $\alpha_h \in U^h_{\mathrm{ad}}$. Second,

$$\max_{T \in T_h(\alpha_h),\ \alpha_h \in U^h_{\mathrm{ad}}} \operatorname{diam} T \leqslant \beta h,$$

where β is a suitable positive constant, which does not depend on h. Third, $T_h(\alpha_h)$ are topologically equivalent with continuous dependence on α_h. This means that $T_h(\alpha_h)$ contain the same number of nodes and the nodes have the same neighbours for any $\alpha_h \in U^h_{\mathrm{ad}}$.

Furthermore, the usual assumption of uniform regularity of $T_h(\alpha_h)$ with respect to h and α_h will be adopted (see [1]), i.e. there exists, independently of h and α_h, a constant $\delta > 0$, such that each internal angle of each triangle from $T_h(\alpha_h)$ is greater than δ.

The following finite dimensional space can be associated with an arbitrary triangulation $T_h(\alpha_h)$:

$$V_h(\alpha_h) = \Big\{ \mathbf{v}_h \in \big[C\big(\overline{\Omega(\alpha_h)}\big)\big]^2 \mid \mathbf{v}_h \in [P_1(T)]^2 \ \ \forall T \in T_h(\alpha_h) \Big\} \cap V(\alpha_h),$$

where $P_1(T)$ denotes the space of linear functions on the triangle T.

The discretization of the boundary value problem ($\mathrm{P}(\alpha_h)$) is defined as follows:

$$(\mathrm{P}_h(\alpha_h)) \begin{cases} \text{find } \mathbf{u}_h(\alpha_h) \in V_h(\alpha_h) \text{ such that} \\ [\boldsymbol{\tau}(\mathbf{u}_h(\alpha_h)),\ \boldsymbol{\varepsilon}(\mathbf{v}_h)]_{\Omega(\alpha_h)} = \langle \mathbf{L}(\alpha_h), \mathbf{v}_h \rangle_{\Omega(\alpha_h)} \ \ \forall \mathbf{v}_h \in V_h(\alpha_h) \end{cases}$$

or equivalently,

$$(\mathrm{P}_h(\alpha_h)) \begin{cases} \text{find } \mathbf{u}_h(\alpha_h) \in V_h(\alpha_h) \text{ such that} \\ \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)) \leqslant \Phi_{\Omega(\alpha_h)}(\mathbf{v}_h) \ \ \forall \mathbf{v}_h \in V_h(\alpha_h). \end{cases}$$

In a usual way, with each internal node and a node on $\partial\Omega(\alpha_h) \setminus (\overline{\Gamma_u} \cup \overline{\Gamma_{\alpha_h}})$ a piecewise linear and continuous on $\overline{\Omega(\alpha_h)}$ shape function $\varphi^j_{\alpha_h}$, $j = 1, \ldots, Z^h$ will be associated, where Z^h is the total number of nodes. As a consequence of the above specified properties, shape functions are uniquely associated with each α_h and the solution $\mathbf{u}_h(\alpha_h)$ can be expressed as their linear combination.

Analogously, the approximation of (R) can be written as

$$(\mathrm{R}_h) \begin{cases} \text{find } \alpha_h^\star \in U_{\text{ad}}^h \text{ such that} \\ S_h(\alpha_h^\star) \leqslant S_h(\alpha_h) \ \ \forall \alpha_h \in U_{\text{ad}}^h, \end{cases}$$

where $S_h(\alpha_h) = Q\big(\alpha_h, \mathbf{u}_h(\alpha_h)\big)$ with $\mathbf{u}_h(\alpha_h)$ being the solution of $(\mathrm{P}_h(\alpha_h))$.

Similarly to the continuous case two theorems can be established.

Theorem 4.1. *The problem* $(\mathrm{P}_h(\alpha_h))$ *has a unique solution.*

Proof. Since $\Phi_{\Omega(\alpha_h)}$ is continuous on $V(\alpha_h)$ and $V_h(\alpha_h)$ is a finite dimensional subspace of $V(\alpha_h)$, the existence of the solution of $(\mathrm{P}_h(\alpha_h))$ follows directly from the theorem about the existence of the minimum of a continuous function on a compact set. The uniqueness is then a simple consequence of the fact that the solution of $(P(\alpha_h))$ is unique. $\square$

Theorem 4.2. *Let assumption* (PS) *hold. Then there exists at least one solution* $\alpha_h^\star \in U_{\text{ad}}^h$ *of the problem* (R_h).

Proof. Denote $q = \inf\limits_{\alpha_h \in U_{\text{ad}}^h} S_h(\alpha_h)$. A minimizing sequence $\{\alpha_h^n\} \subset U_{\text{ad}}^h$ thus satisfies

$$q = \lim_{n \to \infty} S_h(\alpha_h^n). \tag{4.1}$$

Since U_{ad}^h is a compact subset of $C([0,1])$, according to the Arzelà-Ascoli theorem there exists a subsequence of $\{\alpha_h^n\}$ (denoted for the sake of simplicity by the same symbol) such that

$$\alpha_h^n \rightrightarrows \alpha_h^\star \text{ in } [0,1] \text{ as } n \to \infty, \text{ and } \alpha_h^\star \in U_{\text{ad}}^h.$$

The continuous dependence of $\{T_h(\alpha_h)\}$ on α_h and the properties of the shape functions specified at the beginning of this section yield

$$\hat{\varphi}^j_{\alpha_h^n} \to \hat{\varphi}^j_{\alpha_h^\star} \text{ (as } n \to \infty) \quad \text{in } \hat{V},\ j = 1, \ldots, Z^h.$$

Consequently, the solutions $\hat{\mathbf{u}}_h(\alpha_h^n)$ of $(\mathrm{P}_h(\alpha_h^n))$ also satisfy

$$\hat{\mathbf{u}}_h(\alpha_h^n) \to \hat{\mathbf{u}}_h(\alpha_h^\star) \quad \text{in } \hat{V}.$$

Exploiting the assumption (PS) and (4.1) one can write

$$q \geqslant S_h(\alpha_h^\star),$$

which completes the proof. $\square$

In what follows, the relationship between the solutions of the optimization problems (R_h) and (R) when $h \to 0^+$ will be discussed. At this point, the following stronger continuity assumption on the cost functional S is needed:

$$(\mathrm{PZ}) \left\{ \begin{array}{l} \alpha_h \rightrightarrows \alpha \text{ in } [0,1],\ \alpha_h \in U_{\mathrm{ad}}^h,\ \alpha \in U_{\mathrm{ad}}, \\ \hat{\mathbf{u}}_h(\alpha_h) \to \hat{\mathbf{u}}(\alpha) \text{ in } \hat{V}, \\ \mathbf{u}_h(\alpha_h) \in V_h(\alpha_h) \text{ and } \mathbf{u}(\alpha) \in V(\alpha) \\ \quad \text{being the solutions of } (\mathrm{P}_h(\alpha_h)) \\ \quad \text{and } (\mathrm{P}(\alpha)),\ \text{respectively}, \end{array} \right\} \Rightarrow \lim_{h \to 0^+} S_h(\alpha_h) = S(\alpha).$$

Lemma 4.3. *Let $\alpha_h \rightrightarrows \alpha (h \to 0^+)$ in $[0,1],\ \alpha_h \in U_{\mathrm{ad}}^h,\ \alpha \in U_{\mathrm{ad}}$. Let $\mathbf{u}_h(\alpha_h)$ be the solutions of $(\mathrm{P}_h(\alpha_h))$. Then*

$$\hat{\mathbf{u}}_h(\alpha_h) \to \tilde{\mathbf{u}} \quad \text{in } \hat{V},$$

where $\tilde{\mathbf{u}}\big|_{\Omega(\alpha)} \equiv \mathbf{u}(\alpha)$ is the solution of $(\mathrm{P}(\alpha))$ and $\tilde{\mathbf{u}} = \mathbf{0}$ a.e. in $\hat{\Omega} \setminus \overline{\Omega(\alpha)}$ (thus $\tilde{\mathbf{u}} = \hat{\mathbf{u}}(\alpha)$).

P r o o f. As in Lemma 3.2 one can construct a subsequence $\{\mathbf{u}_{h_j}(\alpha_{hj})\}$ such that

$$\hat{\mathbf{u}}_{h_j}(\alpha_{hj}) \rightharpoonup \tilde{\mathbf{u}} \quad \text{in } \hat{V}, \tag{4.2}$$

where moreover

$$\tilde{\mathbf{u}}\big|_{\Omega(\alpha)} \in V(\alpha) \quad \text{and} \quad \tilde{\mathbf{u}} = \mathbf{0} \quad \text{a.e. in } \hat{\Omega} \setminus \overline{\Omega(\alpha)}.$$

Apart from the proof of the strong convergence (of the full sequence) in (4.2), it remains to show that $\tilde{\mathbf{u}}\big|_{\Omega(\alpha)}$ is the solution of $(\mathrm{P}(\alpha))$. Denote $\mathbf{u}^\star \equiv \tilde{\mathbf{u}}\big|_{\Omega(\alpha)}$. As in

Lemma 3.2 it can be proved that $\mathbf{u}^\star = \mathbf{0}$ on Γ_α and on $\Gamma_u(\alpha)$, thus $\mathbf{u}^\star \in V(\alpha)$. Let $\mathbf{u}(\alpha) \in V(\alpha)$ be the solution of $(\mathrm{P}(\alpha))$. Then

$$\mathbf{u}^\star = \mathbf{u}(\alpha) \quad \text{a.e. in } \Omega(\alpha) \tag{4.3}$$

should be proved. Obviously

$$\Phi_{\Omega(\alpha)}(\mathbf{u}^\star) \geqslant \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha))$$

holds. Consequently, the opposite inequality is to be proved. According to Lemma 3.1 there exists a sequence

$$\{\mathbf{w}_i\} \subset \left[\mathcal{E}\big(\overline{\Omega(\alpha)}\big)\right]^2 \cap V(\alpha)$$

such that

$$\mathbf{w}_i \to \mathbf{u}(\alpha) \quad \text{in } V(\alpha) \tag{4.4}$$

and for i fixed there exists $h_{j_0}(i)$ such that (3.1) and (3.2) are met $\forall h_j \leqslant h_{j_0}(i)$. Denote by

$$\mathbf{w}_{ih_j} = r_{h_j}\big(\hat{\mathbf{w}}_i\big|_{\Omega(\alpha_{h_j})}\big) \in V_{h_j}(\alpha_{h_j})$$

the Lagrangian interpolation of $\hat{\mathbf{w}}_i\big|_{\Omega(\alpha_{h_j})}$. Then the definition of $\big(P_{h_j}(\alpha_{h_j})\big)$ yields

$$\Phi_{\Omega(\alpha_{h_j})}(\mathbf{w}_{ihj}) \geqslant \Phi_{\Omega(\alpha_{h_j})}(\mathbf{u}_{h_j}(\alpha_{h_j}))$$

and also

$$\Phi_{\hat{\Omega}}(\hat{\mathbf{w}}_{ih_j}) \geqslant \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}_{h_j}(\alpha_{h_j})) \ \ \forall h_j \leqslant h_{j_0}(i). \tag{4.5}$$

From the well-known approximation properties of the Lagrange interpolation one obtains

$$\hat{\mathbf{w}}_{ih_j} \to \hat{\mathbf{w}}_i \text{ in } \hat{V} \quad \text{ as } h_j \to 0^+.$$

Furthermore, this convergence and the continuity of $\Phi_{\hat{\Omega}}$ on $\hat{V}$ yield

$$\lim_{h_j \to 0^+} \Phi_{\hat{\Omega}}(\hat{\mathbf{w}}_{ih_j}) = \Phi_{\hat{\Omega}}(\hat{\mathbf{w}}_i) \tag{4.6}$$

and the weak lower semi-continuity of $\Phi_{\hat{\Omega}}$ on $\hat{V}$ together with (4.2) leads to

$$\liminf_{h_j \to 0^+} \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}_{h_j}(\alpha_{hj})) \geqslant \Phi_{\hat{\Omega}}(\tilde{\mathbf{u}}) = \Phi_{\Omega(\alpha)}(\mathbf{u}^\star). \tag{4.7}$$

Combining (4.5)–(4.7) one obtains

$$\begin{aligned}(4.8)\qquad \Phi_{\Omega(\alpha)}(\mathbf{w}_i) = \Phi_{\hat{\Omega}}(\hat{\mathbf{w}}_i) &= \lim_{h_j\to 0^+} \Phi_{\hat{\Omega}}(\hat{\mathbf{w}}_{ih_j}) \\ &\geqslant \liminf_{h_j\to 0^+} \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}_{h_j}(\alpha_{hj})) \geqslant \Phi_{\Omega(\alpha)}(\mathbf{u}^\star).\end{aligned}$$

Passing to the limit ($i\to\infty$) in (4.8) it can finally be concluded that

$$\Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)) \geqslant \Phi_{\Omega(\alpha)}(\mathbf{u}^\star),$$

making use of (4.4). Thus

$$\Phi_{\Omega(\alpha)}(\mathbf{u}^\star) = \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha))$$

and, as a consequence, (4.3) holds. The fact that the subsequence $\{\mathbf{u}_{h_j}(\alpha_{h_j})\}$ in (4.2) strongly converges, can be proved in the same way as in the continuous case (see Lemma 3.2). In addition, again, since $\mathbf{u}(\alpha)$ is unique, the convergence holds for the whole sequence, i.e.

$$\hat{\mathbf{u}}_h(\alpha_h) \to \hat{\mathbf{u}}(\alpha) \quad \text{in } \hat{V} \text{ as } h\to 0^+,$$

which completes the proof. □

Finally, a theorem dealing with the relation between the solutions of the optimization problems (R_h) and (R) will be presented.

Theorem 4.4. *Let* (PZ) *be satisfied. Let* $\{\alpha_h^\star\}$, $h\to 0^+$, *be a sequence of solutions of* (R_h) *and let* $\mathbf{u}_h(\alpha_h^\star)\in V_h(\alpha_h^\star)$ *be the unique solutions of* ($\mathrm{P}_h(\alpha_h^\star)$). *Then for* $\alpha_h^\star \rightrightarrows \alpha^\star$ *in* $[0,1]$

$$(4.9)\qquad \hat{\mathbf{u}}_h(\alpha_h^\star) \to \hat{\mathbf{u}}(\alpha^\star) \quad \textit{in } \hat{V} \textit{ as } h\to 0^+$$

holds and, moreover, $\alpha^\star$ *is a solution of* (R) *and* $\hat{\mathbf{u}}(\alpha^\star)\big|_{\Omega(\alpha^\star)}$ *solves* ($\mathrm{P}(\alpha^\star)$).

Proof. Due to the compactness of U_{ad}, it may be assumed that

$$\alpha_h^\star \rightrightarrows \alpha^\star \quad \text{in } [0,1] \text{ as } h\to 0^+$$

and $\alpha^\star\in U_{\mathrm{ad}}$. Let $\mathbf{u}_h(\alpha_h^\star)$ be the unique solutions of ($\mathrm{P}_h(\alpha_h^\star)$). By virtue of Lemma 4.1

$$\hat{\mathbf{u}}_h(\alpha_h^\star) \to \hat{\mathbf{u}}(\alpha^\star) \quad \text{in } \hat{V} \text{ as } h\to 0^+$$

holds and $\hat{\mathbf{u}}(\alpha^\star)\big|_{\Omega(\alpha^\star)} = \mathbf{u}(\alpha^\star)$ is the solution of $(\mathrm{P}(\alpha^\star))$. It remains to prove that $\alpha^\star$ is a solution of (R), i.e. that

$$S(\alpha^\star) \leqslant S(\alpha) \ \ \forall \alpha \in U_{\mathrm{ad}}. \tag{4.10}$$

The assumption (PZ) implies

$$\lim_{h\to 0^+} S_h(\alpha_h^\star) = S(\alpha^\star). \tag{4.11}$$

Take an arbitrary $\alpha \in U_{\mathrm{ad}}$. There exists a sequence $\{\alpha_k\}, \alpha_k \in U_{\mathrm{ad}}^k$ such that

$$\alpha_k \rightrightarrows \alpha \text{ in } [0,1] \text{ as } k \to 0^+$$

(for the proof see [8]). Consequently, it is possible to use again Lemma 4.1 obtaining

$$\hat{\mathbf{u}}_k(\alpha_k) \to \hat{\mathbf{u}}(\alpha) \quad \text{in } \hat{V}$$

and according to the assumption (PZ)

$$\lim_{k\to 0^+} S_k(\alpha_k) = S(\alpha) \tag{4.12}$$

holds. The formulation of (R_h) results in

$$S_h(\alpha_h^\star) \leqslant S_h(\alpha_h) \ \ \forall h. \tag{4.13}$$

Finally, (4.11)–(4.13) on a suitable filter of indices $\{h_f\}$ implies

$$S(\alpha^\star) \leqslant S(\alpha).$$

Since $\alpha \in U_{\mathrm{ad}}$ was arbitrary, (4.10) holds. $\square$

In the last part of this paper it will be proved that the cost functionals introduced in Section 3 satisfy the stronger assumption (PZ). Thus, it has to be proved that if

$$\alpha_h \rightrightarrows \alpha \text{ in } [0,1] \ \&\ \hat{\mathbf{u}}_h(\alpha_h) \to \hat{\mathbf{u}}(\alpha) \quad \text{in } \hat{V}$$

with $\mathbf{u}_h(\alpha_h)$ and $\mathbf{u}(\alpha)$ being solutions of $(\mathrm{P}_h(\alpha_h))$ and $(\mathrm{P}(\alpha))$, respectively, then

$$\lim_{h\to 0^+} \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)) = \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)) \tag{4.14}$$

and

$$\lim_{h\to 0+} \int\limits_{\Omega(\alpha_h)} \left(\mathbf{u}_h(\alpha_h) - \mathbf{u}_d\right)^2 \mathrm{d}\mathbf{x} = \int\limits_{\Omega(\alpha)} \left(\mathbf{u}(\alpha) - \mathbf{u}_d\right)^2 \mathrm{d}\mathbf{x} \tag{4.15}$$

are satisfied.

Let (4.14) be proved first. From the weak lower semi-continuity of $\Phi_{\hat{\Omega}}$ on $\hat{V}$ it follows that

$$\begin{aligned}\Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)) = \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}(\alpha)) &\leqslant \liminf_{h\to 0+} \Phi_{\hat{\Omega}}(\hat{\mathbf{u}}_h(\alpha_h)) \\ &= \liminf_{h\to 0+} \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)).\end{aligned} \tag{4.16}$$

According to Lemma 3.1 there exists a sequence $\{\mathbf{w}_i\} \subset \left[\mathcal{E}\left(\overline{\Omega(\alpha)}\right)\right]^2 \cap V(\alpha)$ such that

$$\hat{\mathbf{w}}_i \to \hat{\mathbf{u}}(\alpha) \quad \text{in } \hat{V}, \quad i \to \infty,$$

and for i fixed there exists $h_0(i)$ such that (3.1) and (3.2) hold $\forall h \leqslant h_0(i)$. Denote by $\mathbf{w}_{ih} = r_h\left(\hat{\mathbf{w}}_i\big|_{\Omega(\alpha_h)}\right) \in V_h(\alpha_h)$ the Lagrangian interpolation of $\hat{\mathbf{w}}_i\big|_{\Omega(\alpha_h)}$. Then

$$\Phi_{\Omega(\alpha_h)}(\mathbf{w}_{ih}) \geqslant \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)) \ \ \forall h \leqslant h_0(i). \tag{4.17}$$

Since

$$\hat{\mathbf{w}}_{ih} \to \hat{\mathbf{w}}_i \quad \text{in } \hat{V} \text{ as } h \to 0^+,$$

we have, using also (4.17),

$$\begin{aligned}\liminf_{h\to 0+} \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)) &\leqslant \liminf_{h\to 0+} \Phi_{\Omega(\alpha_h)}(\mathbf{w}_{ih}) \\ &= \lim_{h\to 0+} \Phi_{\hat{\Omega}}(\hat{\mathbf{w}}_{ih}) = \Phi_{\hat{\Omega}}(\hat{\mathbf{w}}_i) = \Phi_{\Omega(\alpha)}(\mathbf{w}_i).\end{aligned} \tag{4.18}$$

Passing to the limit ($i \to \infty$) in (4.18) and using the continuity of $\Phi_{\hat{\Omega}}$ on $\hat{V}$ we obtain

$$\liminf_{h\to 0+} \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)) \leqslant \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)),$$

which together with (4.16) enables us to get

$$\liminf_{h\to 0+} \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)) = \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)). \tag{4.19}$$

From (4.17) it also follows that

$$\limsup_{h\to 0+} \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)) \leqslant \limsup_{h\to 0+} \Phi_{\Omega(\alpha_h)}(\mathbf{w}_{ih}) = \Phi_{\Omega(\alpha)}(\mathbf{w}_i)$$

for any i. Hence,

$$\limsup_{h\to 0^+} \Phi_{\Omega(\alpha_h)}(\mathbf{u}_h(\alpha_h)) = \Phi_{\Omega(\alpha)}(\mathbf{u}(\alpha)).$$

From this and (4.19), (4.14) follows.

The verification of (4.15) is straightforward.

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Author's address: *Zuzana Dimitrovová*, IDMEC, Instituto Superior Técnico, Av. Rovisco Pais 1, 1096 Lisbon, Portugal, e-mail: `zdimitro@dem.ist.utl.pt`.