Reiner Vanselow

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ABOUT DELAUNAY TRIANGULATIONS AND DISCRETE MAXIMUM PRINCIPLES FOR THE LINEAR CONFORMING FEM APPLIED TO THE POISSON EQUATION

Reiner Vanselow, Dresden

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Abstract. The starting point of the analysis in this paper is the following situation: "In a bounded domain in $\mathbb{R}^2$, let a finite set of points be given. A triangulation of that domain has to be found, whose vertices are the given points and which is 'suitable' for the linear conforming Finite Element Method (FEM)." The result of this paper is that for the discrete Poisson equation and under some weak additional assumptions, only the use of Delaunay triangulations preserves the maximum principle.

Keywords: linear conforming finite element method, Delaunay triangulation, discrete maximum principle

MSC 2000: 65N30, 65N50

1. Introduction

Finite Element Methods (FEM) are often used in numerical solution algorithms for partial differential equations.

In this paper we consider the Poisson equation and only such conforming methods which use triangulations of the domain and where the degrees of freedom are function values. For the major part of the article, the linear conforming FEM is discussed.

The starting point of the analysis in this paper is the following situation:

"In a bounded domain in $\mathbb{R}^2$, which is not necessarily convex, let a finite set of points be given. A triangulation of that domain has to be found, whose vertices are the given points and which is 'suitable' for the linear conforming Finite Element Method (FEM)."

Obviously, in general there exist different triangulations with the property that the set of vertices is equal to that of the given points.

It is known (see Section 3) that a Delaunay triangulation implies advantageous properties for the linear conforming FEM. If an additional condition denoted by (V2) is satisfied near the boundary, then the off-diagonal elements of the coefficient matrix of the FEM are not positive, which is sufficient for a discrete maximum principle.

In this paper it is proved conversely that, if the matrix has this property, then the triangulation has to be a Delaunay one and the condition (V2) has to be satisfied.

Additionally, a new discrete maximum principle is presented which transfers the property of the continuous problem to the discrete one in a natural way. Starting from this, it is easy to prove that the above property of the matrix is also necessary for this discrete maximum principle.

The last two results are true not only for the linear conforming FEM, but for the FEM with the Mini element, too.

2. Finite element method and Delaunay triangulation

2.1. The boundary value problem.

Let us assume that $\Omega \subset \mathbb{R}^2$ is an open, simply connected and bounded polygonal domain, not necessarily convex, with the boundary $\Gamma = \partial\Omega$. Further, let $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \cup \Gamma_2 = \Gamma$, Γ_1 being closed and the measure of Γ_1 being greater than zero.

Consider the boundary value problem

$$\text{(BVP)} \qquad \begin{aligned} -\operatorname{div}\operatorname{grad} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ n^T \operatorname{grad} u &= 0 \quad \text{on } \Gamma_2. \end{aligned}$$

The vector n denotes the unit outer normal on Γ.

2.2. Triangulation and notation.

Let a triangulation $T = \{t\}$ of Ω be given with the usual properties (cf. [2], [5]). Especially, the intersection of different triangles $t_1, t_2 \in T$ may only contain one common edge or vertex. Boundaries like that in Fig. 1 are allowed, too. In that case, the location of points and the triangulation represented in Fig. 1 are permitted, because the points and triangles on both sides of the boundary are independent of each other.

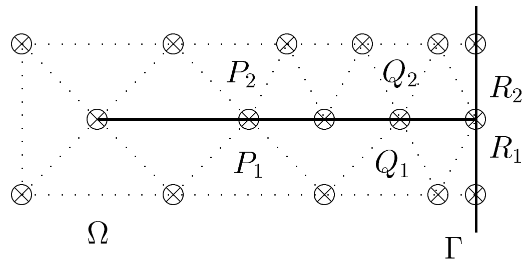


Figure 1. Triangulation of a domain Ω with a crack (detail).

We introduce the following notation:

$$\begin{aligned}
C(T) &= \{P \in \overline{\Omega}\colon\ P \text{ is a vertex of a triangle } t \in T\},\\
C_i(T) &= C(T) \cap \Omega,\\
C_b(T) &= C(T) \cap \Gamma,\\
CI(T) &= C_i(T) \cup \{P \in C_b(T)\colon\ P \in \Gamma_2\},\\
CB(T) &= \{P \in C_b(T)\colon\ P \in \Gamma_1\},\\
NE(P) &= \{Q \in C_b(T)\colon\ P \text{ and } Q \text{ are neighbours on } \Gamma\} \quad \text{for } P \in C_b(T).
\end{aligned}$$

As in Fig. 1, two points in $C(T)$ may have an equal geometric location (as e.g. R_1 and R_2).

2.3. FEM solution with linear conforming finite elements.

A weak formulation of the boundary value problem (BVP) reads:
Find $u \in V := \{v \in H^1(\Omega)\colon\ v\big|_{\Gamma_1} = 0\}$ such that

$$a(u,v) := \iint\limits_{\Omega} (\operatorname{grad} u)^T \operatorname{grad} v \,\mathrm{d}\Omega = \iint\limits_{\Omega} f v \,\mathrm{d}\Omega =: d(f,v) \ \ \forall v \in V.$$

Using linear conforming finite elements the corresponding FEM is determined as follows:
Find $u_h^G \in V_h(T) := \{v \in C^0(\overline{\Omega})\colon\ v\big|_{t\in T} \in P_1(\mathbb{R}^2) \text{ and } v(Q) = 0 \text{ for } Q \in CB(T)\}$ such that

$$a(u_h^G, v_h) = d(f, v_h) \ \ \forall v_h \in V_h(T) \tag{2.1}$$

with $P_1(\mathbb{R}^2) := \operatorname{span}\{1, x, y\}$.

In the definition of $V_h(T)$ and in some later notation, the index h hints at a change from a continuous to a discrete problem.

Now, using a nodal basis in $V_h(T)$ with functions Φ_P for all $P \in CI(T)$ and denoting the vector of the function values of the solution u_h^G of the FEM (2.1) at the points $P \in CI(T)$ by u^{FE} a linear system of equations arises, which has the form

$$L^{FE} u^{FE} = b^{FE}. \tag{2.2}$$

Here, the stiffness matrix L^{FE}, the vector u^{FE} and the right-hand side b^{FE} are defined by

$$L_{kl}^{FE} = a(\Phi_P, \Phi_Q), \quad u_k^{FE} = u_h^G(P) \quad \text{and} \quad b_k^{FE} = d(f, \Phi_P) \quad k, l = 1, \ldots, m \tag{2.3}$$

with $m = \dim V_h(T) = |CI(T)|$, whereby we assume $0 < m < \infty$. Further, the point P belongs to the index k and the point Q to the index l.

Remark 1. With the last sentence a duality in notation is introduced. So, in the sequel we will use e.g. for the same matrix element the notation L_{kl}^{FE} as well as L_{PQ}^{FE} .

2.4. Delaunay triangulation and condition (V2).

Delaunay triangulations can be defined in different ways. Here we will use, as in [7]:

Definition 1′. A triangulation T is called a Delaunay one iff for any triangle $t \in T$ there exists no point $P \in C(T)$ in the interior of its circumcircle.

For nonconvex domains it is necessary to modify this definition (cf. [4]). Therefore, for a triangle $t \in T$ we introduce the following notation (see Fig. 2):

$$\begin{aligned} K_t &= \text{interior of the circumcircle of the triangle } t, \\ K_t(\Omega) &= K_t \cap \Omega, \\ K_t(\Omega, t) &= \text{the largest simply connected set in } K_t(\Omega) \\ &\quad\ \text{which includes the interior of } t. \end{aligned}$$

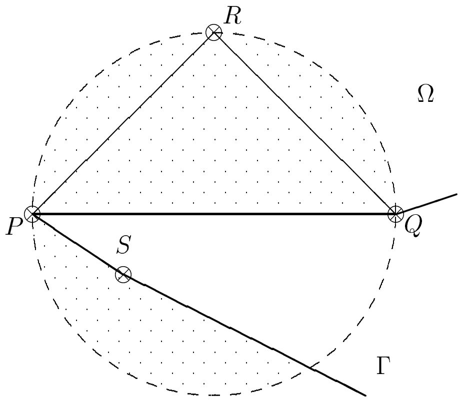
Definition 1. A triangulation T is called a Delaunay one iff for any triangle $t \in T$ the set $C(T) \cap K_t(\Omega, t)$ is empty.

Remark 2. For convex domains Definitions 1 and 1′ are equivalent.

From [9], we recall the following condition for convex domains:

For any two points $P_1, P_2 \in C_b(T)$ with $P_2 \in NE(P_1)$ we have for $\widetilde{P} = \frac{1}{2}(P_1 + P_2)$:

$$\|P_i - \widetilde{P}\|_2 \leqslant \|Q - \widetilde{P}\|_2 \text{ for all } Q \in C(T) \text{ with } Q \neq P_i, \ \ i = 1, 2. \tag{V2'}$$



(a) $K_t(\Omega)$ (dotted area) (b) $K_t(\Omega, t)$ (dotted area)

Figure 2. Triangle $t = \triangle PQR$ and the corresponding sets $K_t(\Omega)$ and $K_t(\Omega, t)$ (detail).

For nonconvex domains it is necessary to modify this condition. Therefore, for any two points $P, Q \in C_b(T)$ with $Q \in NE(P)$ we define the half-plane $H(P, Q; \Omega)$. It is bounded by $g(P, Q)$, which denotes the line through the points P and Q, and lies along the straight line $\overline{PQ}$ in the direction to the domain Ω (i.e. in Fig. 1: $H(Q_1, R_1; \Omega)$ below $\overline{Q_1R_1}$ and $H(Q_2, R_2; \Omega)$ above $\overline{Q_2R_2}$).

The modified condition (V2') reads:

For any two points $P_1, P_2 \in C_b(T)$ with $P_2 \in NE(P_1)$ we have for $\widetilde{P} = \frac{1}{2}(P_1 + P_2)$:

(V2) $\|P_i - \widetilde{P}\|_2 \leqslant \|Q - \widetilde{P}\|_2$ for all $Q \in C(T) \cap H(P_1, P_2; \Omega)$ with $Q \neq P_i$, $i = 1, 2$.

R e m a r k 3. For convex domains the conditions (V2) and (V2') are equivalent.

3. Some known results

3.1. About the FEM (2.1) and Delaunay triangulations.

For arbitrary triangulations T of Ω and the FEM (2.1) the following is true (cf. [2], [8]):

Proposition 1. *Let $t = \triangle PRQ$ be an arbitrary triangle of T with $P \notin \Gamma$. Then L^{FE}_{PQ} is a sum of two addends, where the one which originated from integration over the triangle $\triangle PRQ$, is given by*

$$ll_{PQ}(R) := -\frac{1}{2}\cot[\angle(PRQ)].$$

Conclusion 1. *Let $P \notin \Gamma_1$ and let $\triangle PR_1Q$ and $\triangle PR_2Q$ with $R_1 \neq R_2$ be two triangles of the triangulation T which have the common edge $\overline{PQ}$. Then we have*

$$L^{FE}_{PQ} = ll_{PQ}(R_1) + ll_{PQ}(R_2) = -\frac{1}{2}\frac{\sin[\angle(PR_1Q) + \angle(PR_2Q)]}{\sin[\angle(PR_1Q)]\sin[\angle(PR_2Q)]}.$$

Proposition 2. *We have*

a) $L^{FE}_{PP} > 0$ *for all* $P \in CI(T)$, *and*

b) $L^{FE}_{PP} = -\sum_{Q \neq P} L^{FE}_{PQ}$ *if* $Q \in CI(T)$ *holds for all* $Q \in C(T)$ *with* $L^{FE}_{PQ} \neq 0$.

For Delaunay triangulations and the FEM (2.1) the following is true (cf. [9]).

Proposition 3. *If the triangulation is a Delaunay one and* (V2) *is satisfied, then we have*

a) $L^{FE}_{PQ} \leqslant 0$ *for* $Q \neq P$, *and*

b) $L^{FE}_{PP} \begin{cases} = -\sum_{Q \neq P} L^{FE}_{PQ} > 0 & \textit{if } Q \in CI(T) \textit{ holds for all } Q \in C(T) \textit{ with } L^{FE}_{PQ} < 0, \\ > -\sum_{Q \neq P} L^{FE}_{PQ} \geqslant 0 & \textit{otherwise.} \end{cases}$

3.2. About maximum principles.

For the Poisson equation the following continuous maximum principle is true (cf. e.g. [2]):

Proposition 4 [continuous maximum principle]**.** *For an arbitrary function* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *let*

$$-\operatorname{div}\operatorname{grad} u \leqslant 0 \quad \textit{in } \Omega.$$

Then the maximum of u lies on the boundary of Ω. Moreover, if u has a (local) maximum in Ω, then u is constant on $\overline{\Omega}$.

This continuous maximum principle is not only a mathematical property of the solution for the Poisson equation with nonpositive right-hand side. It reflects also a property of problems modeled by the Poisson equation. That's why it is of interest whether or not a numerical method preserves this maximum principle (cf. also [2], [6], [8], [10]).

To get a discrete maximum principle and to deduce it only from properties of matrices, we have to enlarge the $m \times m$ matrix L^{FE} defined by (2.3) in such a way that we include the influence of the whole boundary in the matrix. That's why we introduce the $m \times \hat{m}$ matrix $L = (L^{FE} \hat{L}^{FE})$ with $\hat{m} = |C(T)|$ and

$$L_{kl} = \begin{cases} L_{kl}^{FE} & \text{for } k, l = 1, 2, \dots, m \\ \hat{L}_{kl}^{FE} & \text{for } k = 1, 2, \dots, m \ \text{ and } \ l = m+1, m+2, \dots, \hat{m}, \end{cases}$$

where the elements of the $m \times (\hat{m} - m)$ matrix $\hat{L}^{FE}$ are defined by

$$\hat{L}_{kl}^{FE} = a(\Phi_P, \Phi_Q) \quad \text{for } k = 1, 2, \dots, m \ \text{ and } \ l = m+1, m+2, \dots, \hat{m}.$$

Here Φ_Q is the nodal basis function in $\hat{V}_h(T) = \{v_h \in C^0(\overline{\Omega})\colon \ v_h\big|_{t \in T} \in P_1(\mathbb{R}^2)\}$ for the point Q.

Further, the m-dimensional vector u^{FE} is enlarged to an $\hat{m}$-dimensional vector U.

Proposition 5 [a discrete maximum principle]**.** *Let the function f in* (BVP) *satisfy $f \leqslant 0$ and let $u^{FE} \in R^m$ be the corresponding solution of* (2.2)*. If the above matrix $L = (L^{FE} \hat{L}^{FE})$ has the property $L_{PQ} \leqslant 0$ for $Q \neq P$, then*

(MAX) $\quad u_P^{FE} = \max\limits_{Q \in II_P} U_Q$ *with $P \in CI(T)$ results in $U_Q = const$ for all $Q \in II_P$.*

Here $II_P := \{Q \in C(T)\colon \ L_{PQ} \neq 0\}$.

P r o o f (cf. also Section I.3 in [2]). At the beginning, let us point out the fact that the statement is also true in the case of nonhomogeneous boundary conditions on Γ_1, therefore we give the proof in that case.

The following systems of linear equations are equivalent:

$$L^{FE} u^{FE} = B \quad \Longleftrightarrow \quad LU = b^{FE},$$

where b^{FE} is defined by (2.3) and B by

$$B_P = b_P^{FE} - \sum_{Q \notin CI(T)} \hat{L}_{PQ}^{FE} U_Q \ \ \forall P \in CI(T).$$

First, because of Proposition 2 we have $L_{PP} = L_{PP}^{FE} > 0$.
Further, it can be shown that $L_{PP} = -\sum\limits_{Q \neq P} L_{PQ}$.

Now, let us assume that there exists a point $P \in CI(T)$ and a point $R \in II_P$ with $U_R < u_P^{FE} = \max\limits_{Q \in II_P} U_Q$.

Therefore, because $f \leqslant 0$ results in $b_P^{FE} \leqslant 0$, we obtain

$$0 \geqslant b_P^{FE} = B_P + \sum_{Q \notin CI(T)} \hat{L}_{PQ}^{FE} U_Q = \sum_{Q \in CI(T)} L_{PQ}^{FE} u_Q^{FE} + \sum_{Q \notin CI(T)} \hat{L}_{PQ}^{FE} U_Q$$
$$= \sum_{Q \in II_P} L_{PQ} U_Q \geqslant L_{PR} U_R + u_P^{FE} \sum_{Q \in II_P, Q \neq R} L_{PQ} > u_P^{FE} \sum_{Q \in II_P} L_{PQ} = 0,$$

which is a contradiction, so that the statement follows. □

By Proposition 3 we get

Conclusion 2. *Let the function f in* (BVP) *satisfy $f \leqslant 0$. If the triangulation is a Delaunay one and* (V2) *is satisfied, then for the corresponding solution u^{FE} of* (2.2) *the discrete maximum principle* (MAX) *is true.*

4. Necessary conditions for the nonpositivity of the off-diagonal entries of the matrix L

Whereas Proposition 3 gives sufficient conditions for the property $L_{PQ} \leqslant 0$ for $Q \neq P$, we prove now that these conditions are also necessary under some weak additional assumptions (to get an idea of these additional assumptions (A) and (B) see also Fig. 4).

Theorem 1. *Let a triangulation T of Ω be given such that for any $t = \triangle P_1 P_2 P_3 \in T$ we have*
(A) *$P_i \in CI(T)$ for $i = 1$, $i = 2$ or $i = 3$ and*
(B) *$P_i, P_j \in CB(T)$ with $i \neq j$ and $i, j \in \{1, 2, 3\}$ implies $P_i \in NE(P_j)$.*
If the matrix L defined in Section 3.2 *has the property $L_{PQ} \leqslant 0$ for $Q \neq P$, then the triangulation is a Delaunay one.*

Proof. First, for $P_1 \neq P_2$ and $P_3 \notin g(P_1, P_2)$ we define the half-plane $\hat{H}(P_1, P_2; P_3)$. It is bounded by the line $g(P_1, P_2)$ and contains the point P_3.

Suppose that the assumptions of the theorem are satisfied, but the triangulation is not a Delaunay one, i.e. by Definition 1 there exist a triangle $t = \triangle PQR \in T$ and a corresponding point S with $S \in C(T) \cap K_t(\Omega, t)$.

Further, let us assume, without loss of generality, that $S \notin \hat{H}(P, Q; R)$, which yields in $\overline{PQ} \not\subset \Gamma$ (this is possible because of assumption (A)).

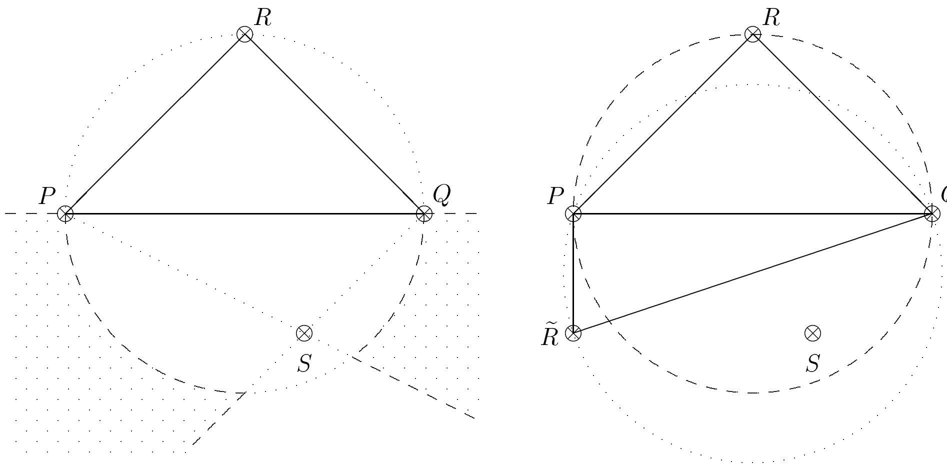
Because of assumption (B) we get $P \in CI(T)$ or $Q \in CI(T)$. Let us assume $P \in CI(T)$, which implies, that L_{PQ} exists, i.e. in the linear system of equations there is one row which corresponds to the point P, and one column which corresponds to the point Q.

Case 1: $\triangle PQS \in T$.

By $S \in K_t(\Omega, t)$ with $t = \triangle PQR$ it follows that

$$\angle(PRQ) + \angle(PSQ) > \pi.$$

By Conclusion 1 this results in $L_{PQ} > 0$, which is a contradiction.



(a) possible location of $\widetilde{R} \in C(T)$ (dotted area)

(b) concrete $\widetilde{R}$, triangles $\triangle PRQ$ and $\triangle P\tilde{R}Q$ and their circumcircles

Figure 3. Possible location of $\tilde{R} \in C(T)$ in the proof of Theorem 1 and one special location (detail).

Case 2: $\triangle PQS \notin T$.

Because of $\overline{PQ} \not\subset \Gamma$ there exists a point $\widetilde{R} \in C(T)$ with $\triangle PQ\widetilde{R} \in T$ and with the following properties for its location (see also Fig. 3a):

$$\widetilde{R} \notin K_t(\Omega, t) \quad \text{and} \quad \widetilde{R} \in \hat{H}(P, Q; S) \cap [\hat{H}(P, S; Q) \cup \hat{H}(Q, S; P)].$$

Namely, if $\widetilde{R} \in K_t(\Omega, t)$, then Case 1 can be applied, whereas S has to be substituted by $\widetilde{R}$. And, if $\widetilde{R} \notin \hat{H}(P, S; Q)$ and $\widetilde{R} \notin \hat{H}(Q, S; P)$, then we get $S \in \triangle PQ\widetilde{R}$, which is a contradiction for a triangulation.

Obviously, $K_t(\Omega, t) \cap \hat{H}(P, Q; S) \subset K_{\widetilde{t}}(\Omega, \widetilde{t})$ holds for $t = \triangle PQR$ and $\widetilde{t} = \triangle PQ\widetilde{R}$, which results in $S \in K_{\widetilde{t}}(\Omega, \widetilde{t})$.

Now, let us assume $\widetilde{R} \in \hat{H}(Q, S; P)$ (see also Fig. 3b). Then the same situation is true for the triangle $\triangle PQ\widetilde{R}$, the edge $\overline{Q\widetilde{R}}$ and the point S as for the triangle $\triangle PQR$, the edge $\overline{PQ}$ and the point S, which means either the Case 1 or Case 2 is possible again.

Because of the finite number of points in $C(T)$, Case 1 must occur at some stage, which, however, leads to a contradiction.

Thus, the theorem is proved. $\square$

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied. If the matrix* L *defined in Section* 3.2 *has the property* $L_{PQ} \leqslant 0$ *for* $Q \neq P$, *then the following is true:*

If for two points $P, Q \in C_b(T)$ *with* $Q \in NE(P)$ *the condition* (V2) *is not satisfied, then this results in* $P \notin CI(T)$ *and* $Q \notin CI(T)$.

Proof. Suppose that the assumptions of the theorem are satisfied and that the condition (V2) is not satisfied for two points $P, Q \in C_b(T)$ with $Q \in NE(P)$, but $P \in CI(T)$ holds.

First, because the condition (V2) is not satisfied for the two points P and Q, there exists a point $S \in C(T) \cap H(P, Q; \Omega)$ for which we get

$$\|S - \widetilde{S}\|_2 < \|P - \widetilde{S}\|_2 = \|Q - \widetilde{S}\|_2 =: r \quad \text{with } \widetilde{S} = \tfrac{1}{2}(P + Q).$$

Further, because of $\overline{PQ} \subset \Gamma$, there exists a point $R \in C(T) \cap H(P, Q; \Omega)$ such that the triangle $t = \triangle PQR$ is the unique one in T with the edge $\overline{PQ}$ ($R = S$ is possible).

Now, because of $P \in CI(T)$, it follows that L_{PQ} exists (in the same sense as above).

If R lies outside the circle around $\widetilde{S}$ with the radius r, then S lies inside the circumcircle of $\triangle PQR$, which contradicts the fact that T is a Delaunay triangulation.

If R lies inside the circle around $\widetilde{S}$ with the radius r, then it follows that $\angle(PRQ) > \frac{\pi}{2}$ and, by Proposition 1, $L_{PQ} > 0$, which is a contradiction, too.

Consequently, the theorem is proved. $\square$

5. A necessary and sufficient condition for a new discrete maximum principle

For the boundary value problem (BVP), in Proposition 5 the discrete maximum principle (MAX) for the corresponding solution u^{FE} of (2.2) is stated under the assumption that the matrix L defined in Section 3.2 has the property $L_{PQ} \leqslant 0$ for $Q \neq P$.

In what follows, for another discrete maximum principle we prove that this condition is sufficient as well as necessary. This new discrete maximum principle transfers the continuous maximum principle of the Poisson equation, which uses only local properties of the function f, to the discrete problem in a natural way.

Therefore, for $P \in C(T)$ we introduce the notation

$$TR_P = \{t \in T\colon\ P \text{ is a vertex of } t\}$$

with $TR_P \subset \Omega$ and define the "discrete maximum principle at $P \in CI(T)$".

Definition 2 (discrete maximum principle at $P \in CI(T)$)**.** For the boundary value problem (BVP), let u^{FE} be the corresponding solution of 2.2. Then the discrete maximum principle is satisfied at $P \in CI(T)$ iff $f \leqslant 0$ in TR_P implies

$$u_P^{FE} = \max_{Q \in II_P} U_Q \quad \text{results in} \quad U_Q = \text{const} \quad \text{for all } Q \in II_P.$$

Theorem 3. *For the boundary value problem* (BVP), *let u^{FE} be the corresponding solution of* 2.2. *Then the discrete maximum principle is satisfied at $P \in CI(T)$ iff the matrix L defined in Section* 3.2 *fulfils $L_{PQ} \leqslant 0$ for all $Q \neq P$.*

Proof. $\Rightarrow$: Let us assume $L_{PR} > 0$ for some $R \in C(T)$ with $R \neq P$.

First we choose $f \equiv 0$ in TR_P, which results in $b_P^{FE} = 0$ as well as in $U_R = -\dfrac{L_{PP}}{L_{PR}}$ and $U_Q = 0$ for all $Q \in II_P$ with $Q \neq R$ and $Q \neq P$ (because f may be arbitrary outside of TR_P, such a choice is possible).

Because of $L_{PP} > 0$ (see Proposition 2) we have $U_R < 0$. Together with $u_P^{FE} = 1$, which follows from $b_P^{FE} = 0$, the above choice of U_Q for $Q \in II_P$ with $Q \neq R$ and the k-th equation of the linear system $LU = b^{FE}$, this contradicts the fact that the discrete maximum principle is satisfied at P.

$\Leftarrow$: This direction may be proved analogously to Proposition 5. $\square$

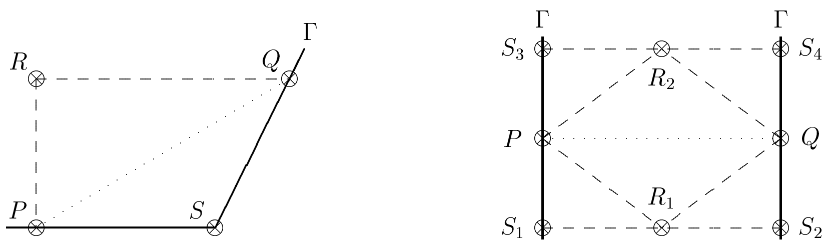
Remark 4. There are some discrete maximum principles for which it is known that the condition $L_{PQ}^{FE} \leqslant 0$ for $Q \neq P$ is not necessary (cf. [8]). For others (cf. [2], [6], [10]) the author does not know any results concerning the fact whether or not this condition is also necessary.

6. Discussion

6.1 About the assumed properties of the triangulation in Theorem 1.
It is possible to formulate Theorems 1 and 2 independently of the boundary conditions.

Theorem 1′. *If for all boundary value problems* (BVP) *with any possible choice of* Γ_1 *the matrices* L *defined in Section* 3.2 *have the property* $L_{PQ} \leqslant 0$ *for* $Q \neq P$, *then the triangulation is a Delaunay one and the condition* (V2) *is satisfied.*

This is true, because Γ_1 can be chosen in such a way that $|CI(T)| = |C(T)| - 2$ holds, so that the assumptions (A) and (B) in Theorem 1 are obviously satisfied.



(a) $\triangle PQS$ does not satisfy (A) (b) $\triangle PQR_1$ and $\triangle PQR_2$ do not satisfy (B)

Figure 4. Examples with $\Gamma_1 = \Gamma$, where the matrix L has the property $L_{kl} \leqslant 0$ for $l \neq k$, but the triangulation is not a Delaunay because of the dotted edges (detail).

Fig. 4 shows that, if the assumptions (A) and (B) are not satisfied, then the triangulation is not necessarily a Delaunay one.

6.2. About discrete maximum principles.
In [2], [6], [8] and [10], different formulations of a discrete maximum principle can be found. For instance, using Proposition 5 we can show

Conclusion 3. *Let the function* f *in* (BVP) *satisfy* $f \leqslant 0$ *and let* $u^{FE} \in R^m$ *be the corresponding solution of* (2.2). *If the matrix* $L = (L^{FE}\hat{L}^{FE})$ *defined in Section* 3.2 *has the property* $L_{PQ} \leqslant 0$ *for* $Q \neq P$, *then the maximum of* U *lies on* Γ_1, *i.e.*

$$\max_{Q \in CI(T)} u_Q^{FE} \leqslant \max_{Q \in CB(T)} U_Q.$$

Remark 5. If the assumptions (A) and (B) in Theorem 1 are satisfied for the given triangulation, then it is even possible to deduce

$$u_P^{FE} = \max_{Q \in II_P} U_Q \text{ for all } P \in CI(T) \text{ results in } U_Q = \text{const for all } Q \in C(T).$$

If these assumptions are not satisfied, then this is true only in a subset of Ω, namely in one which is associated with the point P in a certain sense. E.g. in Fig. 4a, the value U_S may be arbitrary.

R e m a r k 6. Generalizing Definition 1.11 in [10] about the associated directed graph $G(C)$ of an $n \times n$ matrix C to the case of non-quadratic matrices, we find that the assumptions (A) and (B) in Theorem 1 are satisfied iff the associated directed graph $G(C)$ of the $m \times \hat{m}$ matrix L is strongly connected.

By virtue of Proposition 1.12 in [10] it can also be deduced that the assumption (B) in Theorem 1 is satisfied iff the $m \times \hat{m}$ matrix L is irreducible.

6.3. Extension of the discrete maximum principle to a greater class of methods and an application.

In a natural way, the extension of the maximum principle given in Definition 2 is possible to the class of all methods using only the function values to determine the degrees of freedom.

Therefore, we introduce the notation

$$\widetilde{TR}_P = \{t \in T\colon\ \Phi_P \neq 0 \text{ in } t\}$$

with $\widetilde{TR}_P \subset \Omega$, where Φ_P is the nodal basis function for the point P.

Definition 3 (discrete maximum principle at $P \in \widetilde{CI}(T)$**).** For the boundary value problem (BVP), let an FEM be given which uses only the function values to determine the degrees of freedom. Let u^{FEM} be the corresponding solution of the linear system of equations

$$L^{FEM} u^{FEM} = b^{FEM}$$

obtained by a nodal basis. Then the discrete maximum principle is satisfied at $P \in \widetilde{CI}(T)$ iff $f \leqslant 0$ in $\widetilde{TR}_P$ implies

$$u_P^{FEM} = \max_{Q \in \widetilde{II}_P} \widetilde{U}_Q \text{ results in } \widetilde{U}_Q = \text{const for all } Q \in \widetilde{II}_P.$$

Here $\widetilde{CI}(T) := \widetilde{C}(T) \cap [\Omega \cup \Gamma_2]$, $\widetilde{C}(T)$ is the set of all points where the degrees of freedom are given, $\widetilde{II}_P := \{Q \in \widetilde{C}(T)\colon\ \widetilde{L}_{PQ} \neq 0\}$, while $\widetilde{L}$ and $\widetilde{U}$ denote the extensions of the matrix L^{FEM} and the vector u^{FEM}, respectively (in the same sense as L and U in Section 3.2 for the FEM (2.1)).

R e m a r k 7. Because of $\widetilde{C}(T) = C(T)$, $\widetilde{CI}(T) = CI(T)$ and $\widetilde{TR}_P = TR_P$ for all $P \in \widetilde{CI}(T) = CI(T)$, for the FEM from Section 2.3 Definitions 2 and 3 are equivalent.

In the sequel, necessary and sufficient conditions are given such that the above discrete maximum principle is satisfied for the FEM with the Mini element at all points $P \in \widetilde{CI}(T)$. Thereby, the Mini element is a finite element which is commonly used for flow problems as the Stokes equation (cf. [3]). Here, it is applied to the Poisson equation.

For the Stokes equation, the Mini element is used as one possibility to overcome stability problems with other finite elements like the P_1-P_1-element (cf. [3]).

For the Poisson equation, because of (6.2) it is not much more than an academic example.

If $CS(T)$ denotes the set of all centers of gravity belonging to the triangles, then the method can be characterized by the space

$$\widetilde{V}_h(T) = \left\{ v \in C^0(\overline{\Omega}) \colon \; v\big|_{t \in T} \in \widetilde{P}_1(t) \;\text{ and }\; v(Q) = 0 \;\text{ for }\; Q \in CB(T) \right\}$$

with

$$\widetilde{P}_1(t) = P_1(\mathbb{R}^2) \cup \operatorname{span}\{\Phi_P \Phi_Q \Phi_R\} \qquad \text{for } t = \triangle PQR,$$

where the function $\Phi_P \Phi_Q \Phi_R$ is named the triangle-bubble function, too.

The degree of freedom, which is in comparison with the FEM defined in Section 2.3 still vacant in each triangle, is determined by the function value at the centre of gravity of this triangle.

Obviously, we have

$$\widetilde{C}(T) = C(T) \cup CS(T) \qquad \text{and} \qquad \widetilde{CI}(T) = CI(T) \cup CS(T),$$

so that the generalization in Definition 3 is necessary.

Theorem 4. *Let the function f in* (BVP) *satisfy $f \leqslant 0$ and let a triangulation T of Ω be given. Then, for a FEM solution with the Mini element, the discrete maximum principle is satisfied at all $P \in \widetilde{CI}(T)$ for any possible choice of Γ_1 iff the triangulation is a Delaunay one and the condition* (V2) *is satisfied.*

P r o o f. If for a triangle $t \in T$ the functions Φ_i, $i = 1, 2, 3$, denote the nodal basis functions in $V_h(T)$ (for the FEM (2.1)), then, obviously, those in $\widetilde{V}_h(T)$ (for the FEM with the Mini element) are given by

$$\widetilde{\Phi}_4 = 27 \Phi_1 \Phi_2 \Phi_3 \qquad \text{and} \qquad \widetilde{\Phi}_i = \Phi_i - \tfrac{1}{3} \widetilde{\Phi}_4, \;\; i = 1, 2, 3.$$

Here $\widetilde{\Phi}_4$ denotes the nodal basis function, which corresponds to the centre of gravity. Now, by virtue of the identity

$$\iint\limits_t (\operatorname{grad} \widetilde{\Phi}_4)^T \operatorname{grad} \Phi_i \, \mathrm{d}\Omega = \oint\limits_{\partial t} \widetilde{\Phi}_4 (\operatorname{grad} \Phi_i)^T n(t) \, \mathrm{d}\Gamma - \iint\limits_t \widetilde{\Phi}_4 \operatorname{div}(\operatorname{grad} \Phi_i) \, \mathrm{d}\Omega$$

with the outer normal direction $n(t)$ of the triangle t, which is true because of Green's formula, it is easy to verify for $i \in \{1,2,3\}$ that

$$a(\widetilde{\Phi}_4, \Phi_i) = \iint\limits_t (\operatorname{grad} \widetilde{\Phi}_4)^T \operatorname{grad} \Phi_i \, \mathrm{d}\Omega = 0.$$

This yields for $Q \in \widetilde{C}(T)$ with $Q \neq P$:

- in the case of $P \in CS(T)$

$$L_{PQ}^{FEM} = \begin{cases} -\frac{1}{3} L_{PP}^{FEM} & \text{for } Q \in t_P, \\ 0 & \text{for } Q \notin t_P, \end{cases}$$

  where the triangle $t_P \in T$ is uniquely defined by $t_P = \widetilde{TR}_P$,
- in the case of $P \in CI(T)$, using $TR_P = \widetilde{TR}_P$,

$$L_{PQ}^{FEM} = \begin{cases} -\frac{1}{3} L_{QQ}^{FEM} & \text{for } Q \in CS(T) \cap TR_P, \\ L_{PQ}^{FE} + \frac{1}{9} \sum\limits_{R \in \widetilde{II}_{PQ}} L_{RR}^{FEM} & \text{for } Q \in CI(T) \cap TR_P, \\ 0 & \text{for } Q \notin TR_P, \end{cases}$$

  where $\widetilde{II}_{PQ}$ is defined for $P, Q \in CI(T)$ and $P \neq Q$ by

$$\widetilde{II}_{PQ} := \{R \in CS(I)\colon\ R \in TR_P \text{ and } R \in TR_Q\}.$$

  For $P \in CI(T)$ we also obtain

$$b_P^{FEM} = b_P^{FE} - \frac{1}{3} \sum_{Q \in CS(T) \cap TR_P} b_Q^{FEM}.$$

  In these formulas, L_{PQ}^{FE} and b_P^{FE} denote the elements of the matrix L^{FE} and the vector b^{FE}, respectively, for the FEM (2.1).

This results in the equations:

- in the case of $P \in CS(T)$

$$L_{PP}^{FEM} u_P^{FEM} - \frac{1}{3} L_{PP}^{FEM} \sum_{Q \in CI(T) \cap t_P} \widetilde{U}_Q = b_P^{FEM}, \tag{6.1}$$

- in the case of $P \in CI(T)$

$$
\begin{aligned}
L_{PP}^{FEM} u_P^{FEM} - \frac{1}{3} \sum_{Q \in CS(T) \cap TR_P} & L_{QQ}^{FEM} u_Q^{FEM} \\
& + \sum_{Q \in [CI(T) \cap TR_P] \setminus \{P\}} \Bigl(L_{PQ}^{FE} + \frac{1}{9} \sum_{R \in \widetilde{II}_{PQ}} L_{RR}^{FEM} \Bigr) \widetilde{U}_Q \\
& = b_P^{FE} - \frac{1}{3} \sum_{Q \in CS(T) \cap TR_P} b_Q^{FEM}.
\end{aligned}
$$

In the case of $P \in CI(T)$, substituting all unknowns u_Q^{FEM} by $Q \in CS(T)$, easy calculation yields

$$
L_{PP}^{FE} u_P^{FEM} + \sum_{Q \neq P} L_{PQ}^{FE} \widetilde{U}_Q = b_P^{FE}. \tag{6.2}
$$

From equation (6.2) it follows that for the Poisson equation at all points $P \in CI(T)$ the solution for the FEM with the Mini element is the same as for the FEM (2.1).

Taking into account (6.1), the statement follows from Theorem 1′. □

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Author's address: *Reiner Vanselow*, Institut für Numerische Mathematik, Technische Universität Dresden, Mommsenstrasse 13, 01062 Dresden, Germany, e-mail: `vanselow@math.tu-dresden.de`.