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A remark on the local Lipschitz continuity of vector hysteresis operators

A REMARK ON THE LOCAL LIPSCHITZ CONTINUITY OF
VECTOR HYSTERESIS OPERATORS

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Abstract. It is known that the vector stop operator with a convex closed characteristic Z of class C^1 is locally Lipschitz continuous in the space of absolutely continuous functions if the unit outward normal mapping n is Lipschitz continuous on the boundary ∂Z of Z. We prove that in the regular case, this condition is also necessary.

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1. Introduction

Mathematical models of multidimensional hysteresis phenomena in elastoplasticity or ferromagnetism are often based on the variational inequality (see e.g. [1], [2], [4], [5], [10], [12], [15], [16])

$$
(1.1) \quad
\begin{cases}
\langle \dot{u}(t) - \dot{x}(t), x(t) - w \rangle \geqslant 0 \ \ \forall w \in Z, \\
x(t) \in Z \ \ \forall t \in [0, T], \\
x(0) = x^0 \in Z,
\end{cases}
$$

where $u \in W^{1,1}(0, T; X)$ is a given function, X a Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle$, $Z \subset X$ is a convex closed set, $t \in [0, T]$ is the time variable and the dot denotes the derivative with respect to t.

The existence of a unique solution $x \in W^{1,1}(0, T; X)$ to problem (1.1) is a special case of classical results for evolution variational inequalities, cf. e.g. [3], [10].

In stochastics, inequality (1.1) is known as a special case of the *Skorokhod problem* ([8], [9]). In the theory of hysteresis operators, the solution mapping

$$(1.2) \qquad \mathscr{S} \colon Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X) \colon (x^0, u) \mapsto x$$

is called the *stop operator with characteristic Z* and its properties have been systematically studied (see [11], [16], [12], [13]) together with its extension to the space $C([0,T];X)$ of continuous functions. The dynamics described by the operator $\mathscr{S}$ is a special case of a *sweeping process*, see [14].

Analytical properties of the stop in the space $W^{1,1}(0,T;X)$ endowed with the norm

$$(1.3) \qquad |u|_{1,1} := |u(0)| + \int_0^T |\dot{u}(t)| \, \mathrm{d}t$$

depend substantially on the geometry of the characteristic Z. The operator $\mathscr{S} \colon Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ is always continuous, see Theorem I.3.12 of [12]. It was conjectured without proof in [11] that this mapping is Lipschitz continuous if Z is a polyhedron and locally Lipschitz continuous if the boundary ∂Z of Z is smooth. These statements have been rigorously proved only recently in [7] and [6], respectively. In [6], it was shown that the Lipschitz continuity of the mapping

$$(1.4) \qquad n \colon \partial Z \to \partial B_1(0)$$

(by $B_r(z)$ we denote the ball centered at $z \in X$ with radius $r > 0$), which with each $x \in \partial Z$ associates the unit outward normal $n(x)$ to Z at the point x, is sufficient for the local Lipschitz continuity of the stop. Another proof which also yields an explicit upper bound for the Lipschitz coefficient (optimal if Z is a ball) can be found in [13] as a generalization of the technique used in [5] for the ball.

Example 3.2 of [6] shows that the stop is not necessarily locally Lipschitz continuous if the mapping n is only 1/2-Hölder continuous. The aim of this paper is to fill the gap and to prove that the local Lipschitz continuity cannot be expected if ∂Z is of class C^1 and the ratio $|n(x) - n(y)| / |x - y|$, $x, y \in \partial Z$, is unbounded.

Let us note that this is not just an academic question. A precise upper bound for the Lipschitz coefficient of the stop has been substantially exploited in [5] for proving the well-posedness of constitutive laws of elastoplasticity with nonlinear kinematic hardening.

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2. Main result

We consider the simplest case $X = \mathbb{R}^2$ and fix a convex closed set $Z \subset X$ of class C^1 in such a way that there exists a point $x^* \in \partial Z$ for which we have

$$(2.1) \qquad \lim_{\substack{x \to x^* \\ x \in \partial Z}} |n(x) - n(x^*)| \, / \, |x - x^*| = +\infty.$$

By shifting and rotating the coordinate system we may assume that $x^* = 0$ and that there exists $\varepsilon > 0$ such that

$$(2.2) \qquad Z \cap \left([-\varepsilon, \varepsilon]^2\right) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in [-\varepsilon, \varepsilon]^2; \ b \geqslant G(a) \right\},$$

where $G \colon [-\varepsilon, \varepsilon] \to \mathbb{R}^+$ is a convex function, $G(0) = 0$, and its derivative $g = G'$ is continuous, increasing, $g(0) = 0$ and $\lim_{a \to 0+} g(a)/a = +\infty$ (see Fig. 1).

We make the following simplifying assumptions.

Hypothesis 2.1.
  (i) $G \colon [-\varepsilon, \varepsilon] \to \mathbb{R}^+$ *is convex and even, $G(0) = 0$,*
 (ii) $g = G'$ *is increasing and concave in $[0, \varepsilon[$, $g(0) = 0$, $g'(0+) = +\infty$.*

The rest of this paper is devoted to the proof of the following result.

Theorem 2.2. *Let $Z \subset \mathbb{R}^2$ be a convex closed set satisfying condition (2.2) and Hypothesis 2.1. Then for every $R > 0$ there exists a function $u \in W^{1,1}(0, 1; \mathbb{R}^2)$ such that $|u|_{1,1} \leqslant 1$, and initial conditions $x^0, y^0 \in Z$ such that the functions $x = \mathscr{S}(x^0, u), y = \mathscr{S}(y^0, u)$, where $\mathscr{S}$ is the stop operator (1.2), satisfy the inequality*

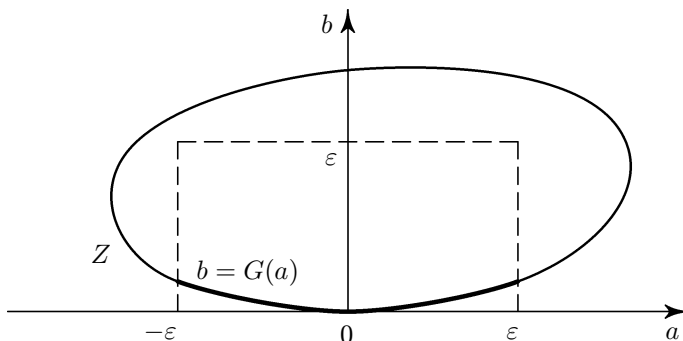$$(2.3) \qquad \int_0^1 |\dot{x}(t) - \dot{y}(t)| \, \mathrm{d}t \geqslant R \left| x^0 - y^0 \right|.$$

Figure 1. The convex characteristic Z.

3. Proof of Theorem 2.2

We follow the construction from Example 3.2 of [6]. Taking a smaller $\varepsilon > 0$ if necessary, we may assume that

$$(3.1) \qquad \varepsilon < \frac{1}{2\sqrt{2}}, \quad g(\varepsilon) < \frac{1}{\sqrt{2}}.$$

We fix some $a_0 \in \,]0, \varepsilon[$ (arbitrary, for the moment) and construct a sequence $\{a_k; k \in \mathbb{N} \cup \{0\}\}$ by induction in the following way. Let $a_0 > a_1 > \ldots > a_k > 0$ be already given and let us consider the differential equation

$$(3.2) \qquad \dot{r}_k = \frac{1 - g(a_k - t)\, g(r_k)}{1 + g^2(r_k)}, \quad r_k(0) = 0,$$

in the domain $(t, r_k) \in \mathscr{D}_k := [0, a_k] \times [0, a_k]$. The function

$$F \colon (t, r_k) \mapsto \frac{1 - g(a_k - t)g(r_k)}{1 + g^2(r_k)}$$

is continuous in $\mathscr{D}_k$ and $0 < F(t, r_k) < 1$ whenever $(t, r_k) \in \mathscr{D}_k$, $r_k > 0$. Moreover, the function $r_k \mapsto F(t, r_k)$ is decreasing in $[0, a_k]$ for every $t \in [0, a_k]$; problem (3.2) therefore admits in $\mathscr{D}_k$ a unique maximal solution $r_k \colon [0, a_k] \to [0, a_k]$, $0 < \dot{r}_k(t) < 1$ for all $t \in \,]0, a_k[$. Putting

$$(3.3) \qquad a_{k+1} := r_k(a_k)$$

we thus have $0 < a_{k+1} < a_k$ and the induction step is complete. By construction, we moreover have for every $k \in \mathbb{N} \cup \{0\}$

$$(3.4) \qquad a_{k+1} \geqslant a_k \frac{1 - g^2(a_k)}{1 + g^2(a_k)} \geqslant a_k \big(1 - 2g^2(a_k)\big).$$

For $k \in \mathbb{N} \cup \{0\}$ put

$$(3.5) \qquad t_0 := 0, \quad t_{k+1} := t_k + a_k, \quad T := \sum_{k=0}^{\infty} a_k \leqslant \infty.$$

We choose two points $x^0, y^0 \in Z$ in the form

$$(3.6) \qquad x^0 := \begin{pmatrix} -a_0 \\ G(a_0) \end{pmatrix}, \quad y^0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

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and define functions $\bar{u}, \bar{x}, \bar{y} \colon [0, T[\to \mathbb{R}^2$ by the formulas

$$(3.7) \qquad\qquad \bar{u}(0) := 0, \quad \bar{x}(0) := x^0, \quad \bar{y}(0) := y^0,$$

$$(3.8) \qquad \bar{u}(t) := \begin{cases} \bar{u}(t_j) + \begin{pmatrix} t - t_j \\ G(t_{j+1} - t) - G(a_j) \end{pmatrix} & \text{for } t \in \,]t_j, t_{j+1}], \ \ j \text{ even}, \\[2ex] \bar{u}(t_j) + \begin{pmatrix} t_j - t \\ G(t_{j+1} - t) - G(a_j) \end{pmatrix} & \text{for } t \in \,]t_j, t_{j+1}], \ \ j \text{ odd}, \end{cases}$$

$$(3.9) \qquad \bar{x}(t) := \begin{cases} \bar{x}(t_j) + \bar{u}(t) - \bar{u}(t_j) & \text{for } t \in \,]t_j, t_{j+1}], \ \ j \text{ even}, \\[2ex] \begin{pmatrix} -r_j(t - t_j) \\ G(r_j(t - t_j)) \end{pmatrix} & \text{for } t \in \,]t_j, t_{j+1}], \ \ j \text{ odd}, \end{cases}$$

$$(3.10) \qquad \bar{y}(t) := \begin{cases} \begin{pmatrix} r_j(t - t_j) \\ G(r_j(t - t_j)) \end{pmatrix} & \text{for } t \in \,]t_j, t_{j+1}], \ \ j \text{ even}, \\[2ex] \bar{y}(t_j) + \bar{u}(t) - \bar{u}(t_j) & \text{for } t \in \,]t_j, t_{j+1}], \ \ j \text{ odd}, \end{cases}$$

where $r_j \colon [0, a_j] \to [0, a_{j+1}]$ is the solution of equation (3.2) for $j \in \mathbb{N} \cup \{0\}$.

Let us check by induction that we have

$$(3.11) \qquad\qquad \bar{x} = \mathscr{S}(x^0, \bar{u}), \quad \bar{y} = \mathscr{S}(y^0, \bar{u}) \qquad \text{in } [0, T[.$$

Assume that identities (3.11) hold for $t \in [0, t_k]$, and let for instance k be even, $k \geqslant 0$ (the case 'k odd' is analogous). For $k \geqslant 2$ we have

$$(3.12) \qquad\qquad \bar{x}(t_k) = \begin{pmatrix} -r_{k-1}(t_k - t_{k-1}) \\ G(r_{k-1}(t_k - t_{k-1})) \end{pmatrix} = \begin{pmatrix} -a_k \\ G(a_k) \end{pmatrix},$$

$$(3.13) \qquad \begin{aligned} \bar{y}(t_k) &= \bar{y}(t_{k-1}) + \bar{u}(t_k) - \bar{u}(t_{k-1}) \\ &= \begin{pmatrix} r_{k-2}(t_{k-1} - t_{k-2}) \\ G(r_{k-2}(t_{k-1} - t_{k-2})) \end{pmatrix} - \begin{pmatrix} a_{k-1} \\ G(a_{k-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

for $k = 0$ the above identities hold by the choice (3.6), (3.7) of initial conditions. For $t \in \,]t_k, t_{k+1}]$ we have by definition

$$\bar{x}(t) := \bar{x}(t_k) + \bar{u}(t) - \bar{u}(t_k) = \begin{pmatrix} t - t_{k+1} \\ G(t_{k+1} - t) \end{pmatrix}, \qquad \bar{y}(t) := \begin{pmatrix} r_k(t - t_k) \\ G(r_k(t - t_k)) \end{pmatrix}.$$

In particular, both $\bar{x}$, $\bar{y}$ are absolutely continuous in $[0, t_{k+1}]$ and $\bar{x}(t)$, $\bar{y}(t)$ belong to Z for all $t \in [t_k, t_{k+1}]$. Since $\dot{\bar{x}}(t) = \dot{\bar{u}}(t)$ for all $t \in \,]t_k, t_{k+1}[$, the function $\bar{x}$ is

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automatically a solution of problem (1.1) in $[0, t_{k+1}]$. The same argument applies to $\bar{y}$ provided we check that the inequality

$$(3.14) \qquad \langle \dot{\bar{u}}(t) - \dot{\bar{y}}(t), \bar{y}(t) - w \rangle \geqslant 0 \ \ \forall w \in Z$$

holds in $]t_k, t_{k+1}[$.

Equation (3.2) yields

$$(3.15) \qquad \dot{r}_k(t - t_k) = \frac{1 - g(t_{k+1} - t) \, g(r_k(t - t_k))}{1 + g^2(r_k(t - t_k))} \qquad \text{for } t \in \,]t_k, t_{k+1}[,$$

hence

$$(3.16) \qquad \dot{\bar{u}}(t) - \dot{\bar{y}}(t) = \frac{g(t_{k+1} - t) + g(r_k(t - t_k))}{\sqrt{1 + g^2(r_k(t - t_k))}} \, n(\bar{y}(t)),$$

where

$$(3.17) \qquad n(\bar{y}(t)) := \frac{1}{\sqrt{1 + g^2(r_k(t - t_k))}} \begin{pmatrix} g(r_k(t - t_k)) \\ -1 \end{pmatrix}$$

is the unit outward normal to Z at the point $\bar{y}(t)$ and inequality (3.14) follows from the convexity of Z. We have thus proved that identities (3.11) are fulfilled.

An elementary computation yields for all $j \in \mathbb{N} \cup \{0\}$

$$(3.18) \qquad \int_{t_j}^{t_{j+1}} |\dot{\bar{u}}(t)| \, \mathrm{d}t = \int_{t_j}^{t_{j+1}} \sqrt{1 + g^2(t_{j+1} - t)} \, \mathrm{d}t$$

$$= \int_0^{a_j} \sqrt{1 + g^2(s)} \, \mathrm{d}s \leqslant \sqrt{2} \, a_j,$$

$$(3.19) \qquad \int_{t_j}^{t_{j+1}} |\dot{\bar{x}}(t) - \dot{\bar{y}}(t)| \, \mathrm{d}t = \int_{t_j}^{t_{j+1}} \frac{g(t_{j+1} - t) + g(r_j(t - t_j))}{\sqrt{1 + g^2(r_j(t - t_j))}} \, \mathrm{d}t$$

$$\geqslant \frac{1}{\sqrt{2}} \int_{t_j}^{t_{j+1}} g(t_{j+1} - t) \, \mathrm{d}t = \frac{1}{\sqrt{2}} \, G(a_j).$$

The proof of Theorem 2.2 consists in choosing an appropriate value of a_0 in the above construction and putting

$$(3.20) \qquad u(t) := \begin{cases} \bar{u}(t) & \text{for } t \in [0, t_n], \\ \bar{u}(t_n) & \text{for } t \in \,]t_n, 1], \end{cases}$$

with some n depending on a_0 such that $t_n < 1$. More precisely, we choose n to be the integer part of $1/(\sqrt{2} \, a_0)$,

$$(3.21) \qquad n := \left[\frac{1}{\sqrt{2} \, a_0} \right],$$

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and, according to assumption (3.1), we have

$$(3.22) \qquad \frac{1}{2\sqrt{2}} \leqslant na_0 \leqslant \frac{1}{\sqrt{2}}.$$

Definition (3.5) yields

$$t_n = \sum_{k=0}^{n-1} a_k \leqslant na_0 \leqslant \frac{1}{\sqrt{2}} < 1,$$

hence formula (3.20) is meaningful. Inequality (3.18) yields

$$(3.23) \qquad |u|_{1,1} = \int_0^1 |\dot{u}(t)| \, dt = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\dot{\bar{u}}(t)| \, dt \leqslant \sqrt{2} \sum_{k=0}^{n-1} a_k \leqslant 1.$$

Let now $R > 0$ be given. The proof will be complete if we check that inequality (2.3) holds for a suitable choice of a_0.

Let us first estimate the integral $\int_0^1 |\dot{x}(t) - \dot{y}(t)| \, dt$ from below. We obviously have $x = \bar{x}$, $y = \bar{y}$ in $[0, t_n]$, $\dot{x} = \dot{y} = 0$ in $]t_n, 1[$, consequently

$$(3.24) \qquad \int_0^1 |\dot{x}(t) - \dot{y}(t)| \, dt = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\dot{\bar{x}}(t) - \dot{\bar{y}}(t)| \, dt \geqslant \frac{1}{\sqrt{2}} \sum_{k=0}^{n-1} G(a_k)$$

according to inequality (3.19).

We define auxiliary functions

$$(3.25) \qquad \varphi(s) := 2sg^2(s) \quad \Phi(s) := \int_s^\varepsilon \frac{dr}{\varphi(r)} \qquad \text{for } s \in \,]0, \varepsilon].$$

Then $\Phi' = -1/\varphi$, $\Phi(\varepsilon) = 0$, $\Phi(0+) = +\infty$, $\varphi(0) = 0$ and Hypothesis 2.1 (ii) entails $\lim_{s \to 0+} \varphi'(s) = 0$. Inequality (3.4) can be written in the form

$$(3.26) \qquad a_{k+1} \geqslant a_k - \varphi(a_k),$$

which implies that

$$(3.27) \qquad \Phi(a_{k+1}) - \Phi(a_k) = \int_{a_{k+1}}^{a_k} \frac{dr}{\varphi(r)} \leqslant \frac{a_k - a_{k+1}}{\varphi(a_{k+1})} \leqslant \frac{\varphi(a_k)}{\varphi(a_k - \varphi(a_k))}$$

for $k \in \mathbb{N} \cup \{0\}$. Note that

$$(3.28) \qquad \lim_{s \to 0+} \frac{\varphi(s) - \varphi(s - \varphi(s))}{\varphi(s)} = \lim_{s \to 0+} \frac{1}{\varphi(s)} \int_{s-\varphi(s)}^s \varphi'(r) \, dr = 0,$$

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hence

$$\text{(3.29)} \qquad \lim_{s \to 0+} \frac{\varphi(s)}{\varphi(s - \varphi(s))} = 1.$$

Consequently, we can put

$$\text{(3.30)} \qquad \alpha := \sup_{s \in \,]0,\varepsilon]} \frac{\varphi(s)}{\varphi(s - \varphi(s))} < \infty$$

and from inequality (3.27) it follows that

$$\text{(3.31)} \qquad \Phi(a_{k+1}) - \Phi(a_k) \leqslant \alpha \ \ \forall k \in \mathbb{N} \cup \{0\}.$$

Let $\Phi^{-1} \colon \mathbb{R}^+ \to \,]0, \varepsilon]$ be the inverse function to Φ. Summing up the above inequalities over k, we obtain

$$\text{(3.32)} \qquad a_k \geqslant \Phi^{-1}(\Phi(a_0) + \alpha k) \ \ \forall k \in \mathbb{N} \cup \{0\}.$$

Combining relations (3.32) and (3.22), we have

$$\text{(3.33)} \quad \sum_{k=0}^{n-1} G(a_k) \geqslant \sum_{k=0}^{n-1} G\big(\Phi^{-1}(\Phi(a_0) + \alpha k)\big) \geqslant \int_0^n G\big(\Phi^{-1}(\Phi(a_0) + \alpha x)\big) \,\mathrm{d}x$$

$$\geqslant \int_0^{\frac{1}{2\sqrt{2}a_0}} G\big(\Phi^{-1}(\Phi(a_0) + \alpha x)\big) \,\mathrm{d}x.$$

The estimates (3.33) and (3.24) together with the elementary inequality $\big|x^0 - y^0\big| = \sqrt{a_0^2 + G^2(a_0)} \leqslant \sqrt{2}a_0$ show that Theorem 2.2 will be proved if

$$\text{(3.34)} \qquad \limsup_{s \to 0+} \frac{1}{s} \int_0^{\frac{1}{2\sqrt{2}s}} G\big(\Phi^{-1}(\Phi(s) + \alpha x)\big) \,\mathrm{d}x = \infty,$$

that is,

$$\text{(3.35)} \qquad \limsup_{s \to 0+} \frac{1}{s} \int_{\Phi(s)}^{\Phi(s) + \frac{\beta}{s}} G(\Phi^{-1}(y)) \,\mathrm{d}y = \infty \qquad \text{with} \quad \beta = \frac{\alpha}{2\sqrt{2}}.$$

By Hypothesis 2.1 (ii), we have $2G(z) \geqslant zg(z)$ and $g(z) \leqslant g(s)$ for $0 < z < s < \varepsilon$, hence

$$\text{(3.36)} \qquad \frac{1}{s} \int_{\Phi(s)}^{\Phi(s) + \frac{\beta}{s}} G(\Phi^{-1}(y)) \,\mathrm{d}y = \frac{1}{2s} \int_{\Phi^{-1}(\Phi(s) + \frac{\beta}{s})}^{s} \frac{G(z)}{zg^2(z)} \,\mathrm{d}z$$

$$\geqslant \frac{1}{4g(s)} \left(1 - \frac{1}{s} \Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right)\right).$$

Let us define an auxiliary function $\psi(v) := 1/\Phi^{-1}(v)$ for $v > 0$. Then $\psi(0) = 1/\varepsilon$, $\lim_{v \to +\infty} \psi(v) = +\infty$, ψ is increasing in $\mathbb{R}^+$ and satisfies the differential equation

$$(3.37) \qquad \psi'(v) = 2\psi(v) \, g^2\left(\frac{1}{\psi(v)}\right).$$

By the change of variables $s = 1/\psi(v)$ we obtain

$$(3.38) \qquad \frac{1}{s}\Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right) = \frac{\psi(v)}{\psi(v + \beta\psi(v))}.$$

According to the Mean Value Theorem, for all $v > 0$ we have

$$(3.39) \qquad \frac{\psi(v + \beta\psi(v))}{\psi(v)} = 1 + \beta\psi'(m(v))$$

for some $m(v) \in [v, v + \beta\psi(v)]$. Using Eq. (3.37) and the fact that the function $s \mapsto g(s)/s$ is decreasing, we obtain

$$(3.40) \qquad \begin{aligned} \frac{\psi(v + \beta\psi(v))}{\psi(v)} &= 1 + 2\beta\,\psi(m(v))g^2\left(\frac{1}{\psi(m(v))}\right) \\ &\geqslant 1 + 2\beta\,\frac{\psi^2(v)g^2\left(\frac{1}{\psi(v)}\right)}{\psi(m(v))} \\ &\geqslant 1 + 2\beta\,\frac{\psi^2(v)g^2\left(\frac{1}{\psi(v)}\right)}{\psi(v + \beta\psi(v))}, \end{aligned}$$

hence

$$(3.41) \qquad \frac{\psi(v + \beta\psi(v))}{\psi(v)} \geqslant \frac{1}{2} + \left(\frac{1}{4} + 2\beta\psi(v)g^2\left(\frac{1}{\psi(v)}\right)\right)^{1/2} \quad \forall v > 0.$$

In terms of $s = 1/\psi(v)$, the above inequality reads

$$(3.42) \qquad \frac{1}{s}\Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right) \leqslant \left(\frac{1}{2} + \left(\frac{1}{4} + 2\beta\,\frac{g^2(s)}{s}\right)^{1/2}\right)^{-1} \quad \forall s \in]0, \varepsilon],$$

and we conclude that for all $s \in]0, \varepsilon]$ we have

$$(3.43) \qquad \frac{1}{g(s)}\left(1 - \frac{1}{s}\Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right)\right) \geqslant 2\beta\,\frac{g(s)}{s}\left(\frac{1}{2} + \left(\frac{1}{4} + 2\beta\,\frac{g^2(s)}{s}\right)^{1/2}\right)^{-2}.$$

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Taking into account estimates (3.36) and (3.43), we see that relation (3.35) is fulfilled provided

$$(3.44) \qquad \limsup_{s \to 0+} \frac{g(s)}{s} \left(\frac{1}{2} + \left(\frac{1}{4} + 2\beta \frac{g^2(s)}{s} \right)^{1/2} \right)^{-2} = +\infty.$$

We distinguish two cases.

A. $\exists \gamma > 0 \colon \limsup\limits_{s \to 0+} g^2(s)/s \geqslant \gamma$.

The function $x \mapsto x \left(1/2 + (1/4 + x)^{1/2} \right)^{-2}$ is increasing for $x > 0$, hence

$$\limsup_{s \to 0+} \frac{g^2(s)}{s} \left(\frac{1}{2} + \left(\frac{1}{4} + 2\beta \frac{g^2(s)}{s} \right)^{1/2} \right)^{-2} \geqslant \gamma \left(\frac{1}{2} + \left(\frac{1}{4} + 2\beta\gamma \right)^{1/2} \right)^{-2} > 0$$

and $\lim\limits_{s \to 0+} 1/g(s) = +\infty$, which yields the assertion.

B. $\lim\limits_{s \to 0+} g^2(s)/s = 0$.

Then

$$\lim_{s \to 0+} \left(\frac{1}{2} + \left(\frac{1}{4} + 2\beta \frac{g^2(s)}{s} \right)^{1/2} \right)^{-2} = 1$$

and $\lim\limits_{s \to 0+} g(s)/s = +\infty$, with the same conclusion as above.

Theorem 2.2 is proved. □

References

[1] *H.-D. Alber*: Materials with Memory. Lecture Notes in Mathematics, Vol. 1682. Springer-Verlag, Berlin-Heidelberg, 1998.

[2] *A. Bergqvist*: Magnetic vector hysteresis model with dry friction-like pinning. Physica B *233* (1997), 342–347.

[3] *H. Brézis*: Opérateurs Maximaux Monotones. North-Holland Math. Studies, Amsterdam, 1973.

[4] *M. Brokate*: Elastoplastic constitutive laws of nonlinear kinematic hardening type. In: Functional Analysis with Current Applications in Science, Technology and Industry (M. Brokate, A. H. Siddiqi, eds.). Pitman Res. Notes Math. Ser., 377, Longman, Harlow, 1998, pp. 238–272.

[5] *M. Brokate, P. Krejčí*: Wellposedness of kinematic hardening models in elastoplasticity. Math. Modelling Numer. Anal. *32* (1998), 177–209.

[6] *W. Desch*: Local Lipschitz continuity of the stop operator. Appl. Math. *43* (1998), 461–477.

[7] *W. Desch, J. Turi*: The stop operator related to a convex polyhedron. J. Differential Equations *157* (1999), 329–347.

[8] *P. Dupuis, H. Ishii*: On Lipschitz continuity of the solution mapping to the Skorokhod problem. Stochastics Stochastics Rep. *35* (1991), 31–62.

[9] *P. Dupuis, A. Nagurney*: Dynamical systems and variational inequalities. Ann. Oper. Res. *44* (1993), 9–42.

[10] *G. Duvaut, J.-L. Lions*: Inequalities in Mechanics and Physics. Springer-Verlag, Berlin, 1976, French edition: Dunod, Paris, 1972.

[11] *M. A. Krasnoseľskiĭ, A. V. Pokrovskiĭ*: Systems with Hysteresis. Springer-Verlag, Berlin, 1989, Russian edition: Nauka, Moscow, 1983.

[12] *P. Krejčí*: Hysteresis, Convexity and Dissipation in Hyperbolic Equations. Gakuto Int. Ser. Math. Sci. Appl., Vol. 8. Gakkōtosho, Tokyo, 1996.

[13] *P. Krejčí*: Evolution variational inequalities and multidimensional hysteresis operators. In: Nonlinear Differential Equations (P. Drábek, P. Krejčí and P. Takáč, eds.). Research Notes in Mathematics, Vol. 404. Chapman & Hall/CRC, London, 1999, pp. 47–110.

[14] *J.-J. Moreau*: Evolution problem associated with a moving convex set in a Hilbert space. J. Differential Eqations *26* (1977), 347–374.

[15] *J. Nečas, I. Hlaváček*: Mathematical Theory of Elastic and Elastico-Plastic Bodies: An Introduction. Elsevier, Amsterdam, 1981.

[16] *A. Visintin*: Differential Models of Hysteresis. Springer-Verlag, Berlin-Heidelberg, 1994.

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