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TRANSFER OF BOUNDARY CONDITIONS FOR DIFFERENCE EQUATIONS*

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Abstract. It is well-known that the idea of transferring boundary conditions offers a universal and, in addition, elementary means how to investigate almost all methods for solving boundary value problems for ordinary differential equations. The aim of this paper is to show that the same approach works also for discrete problems, i.e., for difference equations. Moreover, it will be found out that some results of this kind may be obtained also for some particular two-dimensional problems.

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1. Preliminaries

It is well-known (see, e.g., Taufer [1972]) that the method of transfer of boundary conditions yields an elementary frame into which most methods for solving boundary value problems for ordinary *differential* equations can be included. Namely, this concerns such of them which are based on transforming this problem to initial value problems.

The main idea of this approach starts, in the model case of a differential equation of the second order, with the following observation: Any solution of the differential equation

$$-[p(x)y'(x)]' + q(x)y(x) = f(x) \quad \text{in} \quad [a,b] \tag{1.1}$$

which satisfies, moreover, a linear boundary condition of the type

$$-\alpha_0 p(x_0)y'(x_0) + \beta_0 y(x_0) = \gamma_0 \tag{1.2}$$

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for some fixed $x_0 \in [a,b]$ has to satisfy in $[a,b]$ a *first order* linear differential equation. In other words, there have to exist functions $\alpha, \beta, \gamma\colon\ [a,b] \to \mathbb{R}$ such that

$$-\alpha(x)p(x)y'(x) + \beta(x)y(x) = \gamma(x) \tag{1.3}$$

is satisfied for any $x \in [a,b]$. Moreover, the functions α, β and γ are solutions of some initial value problems.

Since (1.3) has exactly the same form as (1.2) it may be viewed as the result of transferring the condition (1.2) to a general point of the given interval.

The idea of transferring boundary conditions may be very easily utilized for solving a two point boundary value problem for the equation (1.1): by transferring the left- and right-hand boundary condition into a common point we obtain a system of two linear algebraic equations for the value of the solution and its derivative at this point.

The aim of this paper is to show that the same approach works also for discrete problems, i.e., for difference equations. Further, it will be found out that some results of this kind may be obtained also for some two-dimensional problems.

2. Transfer of boundary conditions for one dimensional difference equations

As a model example we will consider a linear difference equation of the second order

$$c_{n-1}x_{n-1} + a_n x_n + b_n x_{n+1} = g_n, \quad n = 1, \dots, N-1, \tag{2.1}$$

with boundary conditions

$$a_0 x_0 + b_0 x_1 = g_0 \tag{2.2}$$

and

$$c_{N-1}x_{N-1} + a_N x_N = g_N. \tag{2.3}$$

Hence, we deal in fact with the system of linear algebraic equations

$$\mathsf{A}_N \mathsf{x} = \mathsf{g}, \tag{2.4}$$

where $\mathsf{x} = (x_0, \dots, x_N)^{\mathrm{T}}$ and $\mathsf{g} = (g_0, \dots, g_N)^{\mathrm{T}}$ are $(N+1)$-dimensional vectors and A_N is an $(N+1) \times (N+1)$ tridiagonal matrix

$$\mathsf{A}_N = \begin{bmatrix} a_0 & b_0 & 0 & \dots & 0 \\ c_0 & & & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & & & b_{N-1} \\ 0 & \dots & 0 & c_{N-1} & a_N \end{bmatrix}. \tag{2.5}$$

It is seen almost at the first glance that any solution of the difference equation (2.1) which satisfies, moreover, the boundary condition (2.2) must fulfil a *first order* difference equation. Thus, the boundary condition of the type (2.2) can be transferred to any point similarly as in the continuous case. Since the coefficients in the transferred boundary condition are not determined uniquely there are many forms of the transfer of boundary conditions with different properties. Two of them are described in the following theorems.

Theorem 2.1. *Let $c_n \neq 0$ for $n = 0, \dots, N-2$ and let $x_0, \dots, x_N$ be the solution of* (2.1) *which satisfies, moreover, the boundary condition* (2.2). *Then we have*

$$z_n x_n - \frac{b_n}{c_{n-1}} z_{n-1} x_{n+1} = r_n \tag{2.6}$$

for $n = 0, \dots, N-1$, where the sequence $\{z_n\}$ is defined by the difference equation

$$z_{n+1} = -\frac{a_{n+1}}{c_n} z_n - \frac{b_n}{c_{n-1}} z_{n-1}, \quad n = 0, \dots, N-2, \tag{2.7}$$

with the initial conditions

$$z_0 = a_0, \quad z_{-1} = -c_{-1}, \tag{2.8}$$

where c_{-1} is an arbitrary real number different from zero and the sequence $\{r_n\}$ is defined by the difference equation

$$r_{n+1} = r_n - \frac{g_{n+1}}{c_n} z_n, \quad n = 0, \dots, N-2, \tag{2.9}$$

with the initial condition

$$r_0 = g_0. \tag{2.10}$$

Proof. Put

$$h_n = z_n x_n - \frac{b_n}{c_{n-1}} z_{n-1} x_{n+1} - r_n, \quad n = 0, \dots, N-1. \tag{2.11}$$

Using (2.9), we obtain, after simple manipulations, that

$$\begin{aligned} h_{n+1} - h_n = {} & \left(z_{n+1} + \frac{a_{n+1}}{c_n} z_n + \frac{b_n}{c_{n-1}} z_{n-1} \right) x_{n+1} \\ & - \frac{1}{c_n} (b_{n+1} x_{n+2} + a_{n+1} x_{n+1} + c_n x_n - g_{n+1}) z_n \end{aligned} \tag{2.12}$$

for $n = 0, \ldots, N-2$. Taking into account (2.3) and (2.7), we see that this expression is equal to zero. Consequently, the sequence $\{h_n\}$ is constant. But if we put $n = 0$ in (2.11) and use (2.8) and (2.2), we have

$$h_0 = z_0 x_0 - \frac{b_0}{c_{-1}} z_{-1} x_1 - g_0 = 0. \tag{2.13}$$

Hence, the entries of the sequence $\{h_n\}$ are equal to zero for $n = 0, \ldots, N-1$, and this fact proves the theorem. □

Theorem 2.2. *Let $a_n > 0$ for $n = 0, \ldots, N$, $b_0 \leqslant 0$, $b_n < 0$ for $n = 1, \ldots, N-1$, $c_n < 0$ for $n = 0, \ldots, N-2$, $c_{N-1} \leqslant 0$ and let A_N be diagonally dominant. Then*

$$d_n x_n + b_n x_{n+1} = u_n \tag{2.14}$$

for $n = 0, \ldots, N-1$, where the sequence $\{d_n\}$ is defined by

$$d_{n+1} = a_{n+1} - \frac{c_n b_n}{d_n}, \quad n = 0, \ldots, N-2, \tag{2.15}$$

with the initial condition

$$d_0 = a_0, \tag{2.16}$$

and the sequence $\{u_n\}$ is defined by

$$u_{n+1} = g_{n+1} - \frac{c_n u_n}{d_n}, \quad n = 0, \ldots, N-2, \tag{2.17}$$

with the initial condition

$$u_0 = g_0. \tag{2.18}$$

Proof. The proof of this theorem will be an easy consequence of the following lemma. □

Lemma 2.1. *Let $a_n > 0$ for $n = 0, \ldots, N$, $b_0 \leqslant 0$, $b_n < 0$ for $n = 1, \ldots, N-1$, $c_n < 0$ for $n = 0, \ldots, N-2$, $c_{N-1} \leqslant 0$ and let A_N be diagonally dominant. Then the sequence defined by* (2.7) *with the initial conditions* (2.8) *satisfies*

$$z_n > 0, \quad n = 0, \ldots, N-1. \tag{2.19}$$

Proof. First of all remember that the diagonal dominance of the matrix A_N means that its row sums are nonnegative. The proof of the assertion of the lemma will be performed by induction. According to (2.7) we have $z_1 = b_0 - a_0 a_1/c_0$. Consequently,

$$z_1 - \frac{b_1}{c_0} z_0 = b_0 - \frac{a_0}{c_0}(a_1 + b_1) \geqslant b_0 - \frac{a_0}{c_0}(-c_0) = a_0 + b_0 \geqslant 0, \tag{2.20}$$

since the diagonal dominancy implies $a_1 + b_1 \geqslant -c_0$, and, obviously, $-a_0/c_0 > 0$. But from (2.20) we obtain that

$$z_1 \geqslant \frac{b_1}{c_0} z_0 = \frac{b_1}{c_0} a_0 > 0. \tag{2.21}$$

Thus, for $n = 1$, the relations (2.19) and

$$z_n - \frac{b_n}{c_{n-1}} z_{n-1} \geqslant 0 \tag{2.22}$$

are satisfied. Suppose now that the inequalities (2.19) and (2.22) are satisfied for $n = m \leqslant N - 2$ and prove, first of all, that (2.22) holds also for $n = m + 1$. We have

$$\begin{aligned} z_{m+1} - \frac{b_{m+1}}{c_m} z_m &= -\frac{a_{m+1}}{c_m} z_m - \frac{b_m}{c_{m-1}} z_{m-1} - \frac{b_{m+1}}{c_m} z_m \\ &= -\frac{z_m}{c_m}(a_{m+1} + b_{m+1}) - \frac{b_m}{c_{m-1}} z_{m-1} \\ &\geqslant -\frac{z_m}{c_m}(-c_m) - \frac{b_m}{c_{m-1}} z_{m-1} \geqslant 0 \end{aligned} \tag{2.23}$$

since $a_{m+1} + b_{m+1} \geqslant -c_m$ for $m \leqslant N - 2$ as follows from the diagonal dominancy of A_N, and $(-z_m)/c_m > 0$ according to the induction hypothesis. But from (2.23) and the induction hypothesis the validity of (2.19) for $n = m+1$ follows immediately. The proof of the lemma is complete. □

P r o o f of Theorem 2.2. We have $z_n > 0$ for $n = 0, \dots, N - 1$ according to Lemma 2.1. Hence, the equations (2.6) may be multiplied for $n = 0, \dots, N - 1$ by $-c_{n-1}/z_{n-1}$. If we put

$$d_n = -c_{n-1} \frac{z_n}{z_{n-1}}, \quad n = 0, \dots, N - 1, \tag{2.24}$$

and

$$u_n = -\frac{c_{n-1} r_n}{z_{n-1}}, \quad n = 0, \dots, N - 1, \tag{2.25}$$

we obtain (2.14). The recurrence relations (2.15) and (2.17) now follow from (2.7) and (2.9) by easy computation. Theorem is proved. □

If we replace the boundary condition (2.2) by (2.3) the same investigations as above lead to the following parallels of Theorems 2.1 and 2.2:

Theorem 2.1a. *Let $b_n \neq 0$ for $n = 1, \ldots, N-1$ and let $x_0, \ldots, x_N$ be the solution of* (2.1) *which satisfies, moreover, the boundary condition* (2.3). *Then we have*

$$-\frac{c_{n-1}}{b_n}\hat{z}_{n+1}x_{n-1} + \hat{z}_n x_n = \hat{r}_n \tag{2.6a}$$

for $n = N, \ldots, 1$, where the sequence $\{\hat{z}_n\}$ is defined by the difference equation

$$\hat{z}_{n-1} = -\frac{a_{n-1}}{b_{n-1}}\hat{z}_n - \frac{c_{n-1}}{b_n}\hat{z}_{n+1}, \quad n = N, \ldots, 2, \tag{2.7a}$$

with the initial conditions

$$\hat{z}_N = a_N, \quad \hat{z}_{N+1} = -b_N, \tag{2.8a}$$

with b_N being an arbitrary real number different from zero and the sequence $\{\hat{r}_n\}$ is defined by the difference equation

$$\hat{r}_{n-1} = \hat{r}_n - \frac{g_{n-1}}{b_{n-1}}\hat{z}_n, \quad n = N, \ldots, 2, \tag{2.9a}$$

with the initial condition

$$\hat{r}_N = g_N. \tag{2.10a}$$

Theorem 2.2a. *Let $a_n > 0$ for $n = 0, \ldots, N$, $b_0 \leqslant 0$, $b_n < 0$ for $n = 1, \ldots, N-1$, $c_n < 0$ for $n = 0, \ldots, N-2$, $c_{N-1} \leqslant 0$ and let A_N be diagonally dominant. Then*

$$c_n x_n + \hat{d}_{n+1}x_{n+1} = \hat{u}_{n+1} \tag{2.14a}$$

for $n = N-1, \ldots, 0$, where the sequence $\{\hat{d}_n\}$ is defined by

$$\hat{d}_{n-1} = a_{n-1} - \frac{c_{n-1}b_{n-1}}{\hat{d}_n}, \quad n = N, \ldots, 2, \tag{2.15a}$$

with the initial condition

$$\hat{d}_N = a_N, \tag{2.16a}$$

and the sequence $\{\hat{u}_n\}$ *by*

$$\hat{u}_{n-1} = g_{n-1} - \frac{b_{n-1}\hat{u}_n}{\hat{d}_n}, \quad n = N, \dots, 2, \tag{2.17a}$$

with the initial condition

$$\hat{u}_N = g_N. \tag{2.18a}$$

The reader has certainly noted that we have obtained, in fact, variants of the Gaussian elimination method.

Let us conclude this section with a remark which may be of some interest. If the assumptions of Theorems 2.2 and 2.2a are satisfied then two consecutive components of the solution vector x of the system (2.4) fulfil

$$\begin{aligned} d_n x_n + b_n x_{n+1} &= u_n, \\ c_n x_n + \hat{d}_{n+1} x_{n+1} &= \hat{u}_{n+1}. \end{aligned} \tag{2.26}$$

The entries of the sequences $\{d_n\}$, $\{u_n\}$, $\{\hat{d}_{n+1}\}$ and $\{\hat{u}_{n+1}\}$ are computed from the recurrence relations (2.15), (2.17), (2.15a) and (2.17a) and it is necessary to store them *only* for those n's which are the indices of those components of the solution vector which we want to know. Thus, if we are interested only in few components of the solution vector (as compared with the total number of them) the equations (2.26) give us such a modification of the Gaussian elimination method which enables us to solve extremely large systems of linear equations with tridiagonal matrices (having millions of unknowns) on usual PC's without any problems with storage.

Naturally, the same is true if we start with Theorems 2.1 and 2.1a instead of Theorems 2.2 and 2.2a. The system analogous to (2.26) is now

$$\begin{aligned} z_n x_n - \frac{b_n}{c_{n-1}} z_{n-1} x_{n+1} &= r_n, \\ -\frac{c_n}{b_{n+1}} \hat{z}_{n+2} x_n + \hat{z}_{n+1} x_{n+1} &= \hat{r}_{n+1} \end{aligned} \tag{2.27}$$

and it holds under more general assumptions. On the other hand, the entries of the matrix of the system (2.27) may grow extremely rapidly even for very reasonably behaved matrices. This fact may bring some stability problems.

3. Discrete analogue of the Laplace equation

As an example of the application of the idea of transferring boundary conditions in the two-dimensional case, we will deal here with a discrete analogue of the Dirichlet problem for the Laplace equation on a rectangle. Hence, we will investigate the system

$$
\begin{gathered}
4u_{ij} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = 0, \\
i = 1, \dots, n-1, \quad j = 1, \dots, m-1,
\end{gathered} \tag{3.1}
$$

with boundary conditions

$$
u_{0j} = f_j, \quad u_{nj} = g_j, \quad u_{i0} = u_{im} = 0. \tag{3.2}
$$

(General Dirichlet boundary conditions are easily obtained by superposition.)

Let us begin with introducing some notation. Let

$$
\mathsf{v}^{(\nu)} = (v_1^{(\nu)}, \dots, v_{m-1}^{(\nu)})^{\mathrm{T}}, \quad \nu = 1, \dots, m-1, \tag{3.3}
$$

be the complete system of orthonormal eigenvectors of the $(m-1) \times (m-1)$ matrix

$$
\mathsf{P} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & & & \ddots & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & \ddots & & & -1 \\ 0 & \dots & 0 & -1 & 2 \end{bmatrix}. \tag{3.4}
$$

Further, let V be the ($(m-1) \times (m-1)$ orthogonal) matrix defined by

$$
\mathsf{V} = \left(\mathsf{v}^{(1)}, \dots, \mathsf{v}^{(m-1)}\right). \tag{3.5}
$$

Finally, let

$$
\mathsf{u}_i = \left(u_{i1}, \dots, u_{i,m-1}\right)^{\mathrm{T}}, \quad i = 0, \dots, n, \tag{3.6}
$$

and $c_\nu^{(i)}$ $(i = 0, \dots, n)$ be the Fourier coefficients of the vectors u_i with respect to the vectors (3.3), i.e.,

$$
c_\nu^{(i)} = \sum_{j=1}^{m-1} u_{ij} v_j^{(\nu)} = \left(\mathsf{v}^{(\nu)}\right)^{\mathrm{T}} \mathsf{u}_i. \tag{3.7}
$$

Note that u_0 and u_n are known vectors and, consequently, also the quantities $c_\nu^{(0)} = \tilde{c}_\nu^{(0)}$ and $c_\nu^{(n)} = \tilde{c}_\nu^{(n)}$ can be supposed to be known. In the actual situation, the direct computation gives that the components of the vectors $v^{(\nu)}$ from (3.3) are given by

$$v_j^{(\nu)} = \sqrt{2h}\sin\frac{\nu\pi j}{m}, \quad j = 1, \ldots, m-1, \quad h = \frac{1}{m}. \tag{3.8}$$

Hence, the sums in (3.7) may be computed very quickly by the fast Fourier transform.

Now we have all prepared to be able to formulate the main result of this section.

Theorem 3.1. *Let u_{ij}, $i = 1, \ldots, n-1$, $j = 1, \ldots, m-1$, be the solution of* (3.1) *satisfying the boundary conditions $u_{0j} = f_j$, $j = 1, \ldots, m-1$, $u_{i0} = u_{im} = 0$, $i = 1, \ldots, n-1$. Then we have*

$$D_i V^{\mathrm{T}} u_i - V^{\mathrm{T}} u_{i+1} = r_i \tag{3.9}$$

for $i = 1, \ldots, n-1$ where

$$D_i = \operatorname{diag}\big(d_i^{(1)}, \ldots, d_i^{(m-1)}\big), \tag{3.10}$$

$$r_i = \big(r_i^{(1)}, \ldots, r_i^{(m-1)}\big)^{\mathrm{T}}, \tag{3.11}$$

the sequences $\{d_i^{(\nu)}\}$ and $\{r_i^{(\nu)}\}$ are defined by the recurrences

$$d_{i+1}^{(\nu)} = 2 + \lambda_\nu - \frac{1}{d_i^{(\nu)}}, \quad i = 1, \ldots, n-2, \quad d_1^{(\nu)} = 2 + \lambda_\nu, \tag{3.12}$$

$$r_{i+1}^{(\nu)} = \frac{1}{d_i^{(\nu)}} r_i^{(\nu)}, \qquad i = 1, \ldots, n-2, \quad r_1^{(\nu)} = \tilde{c}_\nu^{(0)}, \tag{3.13}$$

for $\nu = 1, \ldots, m-1$, and λ_ν are the eigenvalues corresponding to the eigenvectors $v^{(\nu)}$ of P.

P r o o f. Taking into account the definition of the numbers $c_\nu^{(i)}$, we can write the solution of (3.1), (3.2) in the form

$$u_{ij} = \sum_{\nu=1}^{m-1} c_\nu^{(i)} v_j^{(\nu)}. \tag{3.14}$$

Then the $c_\nu^{(i)}$'s satisfy

$$-c_\nu^{(i-1)} + (2 + \lambda_\nu) c_\nu^{(i)} - c_\nu^{(i+1)} = 0, \quad i = 1, \ldots, n-1, \tag{3.15}$$

with the boundary conditions

$$c_\nu^{(0)} = \tilde{c}_\nu^{(0)}, \quad c_\nu^{(n)} = \tilde{c}_\nu^{(n)}. \tag{3.16}$$

If we now apply Theorem 2.2 to the difference equation (3.15) with the first condition of the boundary conditions (3.16) (note that $\lambda_\nu > 0$) we obtain

$$d_i^{(\nu)} c_\nu^{(i)} - c_\nu^{(i+1)} = r_i^{(\nu)} \tag{3.17}$$

for $i = 1, \ldots, n-1$ and $\nu = 1, \ldots, m-1$. If we put, moreover,

$$\mathsf{c}_i = \left(c_1^{(i)}, \ldots, c_{m-1}^{(i)}\right)^{\mathrm{T}}, \tag{3.18}$$

the equation (3.17) may be rewritten in the form

$$\mathsf{D}_i \mathsf{c}_i - \mathsf{c}_{i+1} = \mathsf{r}_i. \tag{3.19}$$

On the other hand, we have

$$\mathsf{V}^{\mathrm{T}} \mathsf{u}_i = \begin{bmatrix} \left(\mathsf{v}^{(1)}\right)^{\mathrm{T}} \\ \vdots \\ \left(\mathsf{v}^{(m-1)}\right)^{\mathrm{T}} \end{bmatrix} \mathsf{u}_i = \begin{bmatrix} \left(\mathsf{v}^{(1)}\right)^{\mathrm{T}} \mathsf{u}_i \\ \vdots \\ \left(\mathsf{v}^{(m-1)}\right)^{\mathrm{T}} \mathsf{u}_i \end{bmatrix} = \mathsf{c}_i \tag{3.20}$$

where the last equality follows from (3.7). Substituting (3.20) into (3.19), we obtain (3.9). The theorem is proved. □

Thus, the equation (3.8) represents the result of the transfer of the left boundary condition of the problem (3.1), (3.2).

Analogously, if we apply Theorem 2.2a to the difference equation (3.15) with the second condition of the boundary conditions (3.16) we obtain

Theorem 3.1a. *Let u_{ij}, $i = 1, \ldots, n-1$, $j = 1, \ldots, m-1$, be the solution of (3.1) satisfying the boundary conditions $u_{nj} = g_j$, $j = 1, \ldots, m-1$, $u_{i0} = u_{im} = 0$, $i = 1, \ldots, n-1$. Then we have*

$$-\mathsf{V}^{\mathrm{T}} \mathsf{u}_i + \hat{\mathsf{D}}_{i+1} \mathsf{V}^{\mathrm{T}} \mathsf{u}_{i+1} = \hat{\mathsf{r}}_{i+1} \tag{3.9a}$$

for $i = n-2, \ldots, 0$ where

$$\hat{\mathsf{D}}_i = \operatorname{diag}\left(\hat{d}_i^{(1)}, \ldots, \hat{d}_i^{(m-1)}\right), \tag{3.10a}$$

$$\hat{\mathsf{r}}_i = \left(\hat{r}_i^{(1)}, \ldots, \hat{r}_i^{(m-1)}\right)^{\mathrm{T}}, \tag{3.11a}$$

the sequences $\{\hat{d}_i^{(\nu)}\}$ *and* $\{\hat{r}_i^{(\nu)}\}$ *are defined by the recurrences*

$$\hat{d}_{i-1}^{(\nu)} = 2 + \lambda_\nu - \frac{1}{\hat{d}_i^{(\nu)}}, \quad i = n-1, \ldots, 2, \quad \hat{d}_{n-1}^{(\nu)} = 2 + \lambda_\nu, \tag{3.12a}$$

$$\hat{r}_{i-1}^{(\nu)} = \frac{1}{\hat{d}_i^{(\nu)}} \hat{r}_i^{(\nu)}, \quad i = n-1, \ldots, 2, \quad \hat{r}_n^{(\nu)} = \tilde{c}_\nu^{(n)} \tag{3.13a}$$

for $\nu = 1, \ldots, m-1$, *and* λ_ν *are the eigenvalues corresponding to the eigenvectors* $\mathsf{v}^{(\nu)}$ *of* P.

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