Applications of Mathematics

Vratislava Mošová

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Applications of Mathematics, Vol. 45 (2000), No. 6, 401–410

Persistent URL: http://dml.cz/dmlcz/134448

SOME ESTIMATES FOR THE OSCILLATION OF THE DEFORMATION GRADIENT

Vratislava Mošová, Olomouc

(Received August 10, 1998)

Abstract. As a measure of deformation we can take the difference $D\vec{\varphi}-R$, where $D\vec{\varphi}$ is the deformation gradient of the mapping $\vec{\varphi}$ and R is the deformation gradient of the mapping $\vec{\gamma}$, which represents some proper rigid motion. In this article, the norm $\|D\vec{\varphi}-R\|_{L^p(\Omega)}$ is estimated by means of the scalar measure $e(\vec{\varphi})$ of nonlinear strain. First, the estimates are given for a deformation $\vec{\varphi}\in W^{1,p}(\Omega)$ satisfying the condition $\vec{\varphi}\big|_{\partial\Omega}=\vec{\mathrm{id}}$. Then we deduce the estimate in the case that $\vec{\varphi}(x)$ is a bi-Lipschitzian deformation and $\vec{\varphi}\big|_{\partial\Omega}\neq\vec{\mathrm{id}}$.

Keywords: hyperelastic material, deformation gradient, strain tensor, matrix and spectral norms, bi-Lipschitzian map

MSC 2000: 73G05

1. Introduction

An important aspect of the theory of elasticity is Korn's inequality. It enables us to show that the functional of potential energy associated with a linear problem is coercive. What is the analogue in the case of finite elasticity? R. V. Kohn introduced a new measure of nonlinear strain $e(\vec{\varphi})$ and found estimates for deformation in the case of hyperelastic material, such as rubber.

We start from the description of the mathematical model given by R. V. Kohn. (See [3] for details.) Let $\Omega\subset\mathbb{R}^n$ be a domain and let the mapping $\vec{\varphi}\colon\ \Omega\to\mathbb{R}^n$ be differentiable a.e. on Ω. The deformation of Ω is usually described by means of the gradient $D\vec{\varphi}$. In what follows, we use the terms "deformation" and "deformation gradient" for $\vec{\varphi}$ and $D\vec{\varphi}$, respectively. Hyperelastic material is governed by a stored energy function $W(\vec{x},D\vec{\varphi})$. If, in addition, this material is homogeneous and isotropic, then W can be expressed as a symmetric function of eigenvalues

$0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_n$ of the nonnegative-definite and symmetric matrix

$$U = \left(D\vec{\varphi}^T D\vec{\varphi}\right)^{1/2}. \tag{1}$$

This fact can be used for the definiton of the scalar measure $e(\vec{\varphi})$ of the deformation:

$$e(\vec{\varphi}) = e_1(\vec{\varphi}) + e_2(\vec{\varphi}) + e_3(\vec{\varphi}), \tag{2}$$

where

$$e_1(\vec{\varphi}) = (\lambda_n - 1)_+ \tag{3}$$

$$e_2(\vec{\varphi}) = (\lambda_2, \ldots, \lambda_n - 1)_+ \tag{4}$$

$$e_3(\vec{\varphi}) = |\det U - 1| \tag{5}$$

and a_+ is the positive part of a.

R. V. Kohn found an estimate for the oscillation $D\vec{\varphi}$ in the form

$$\|D\vec{\varphi} - R\|^2_{L^2(\Omega)} \leqslant C\|e(\vec{\varphi}) + |U^2 - I|\,\|_{L^1(\Omega)},$$

where R is some orthogonal matrix and I is the unit matrix. In this paper, the generalization of Kohn's estimate in the case $p > 2$ is obtained.

First, we find estimates for the deformation $\vec{\varphi}\big|_{\partial\Omega} = \vec{\mathrm{id}}$. The idea how to measure the deformation is the following: Imagine that in Ω there are stretched elastic bands pinned on the boundary of this domain. Every deformation of the domain must evoke stretching of at least one of these elastic bands. So the deformation can be controlled by stretching of elastic bands.

We begin with some auxiliary estimates for directional derivative in the direction $\vec{\theta}$ which represents the deformation of the band parallel with this direction. After that we estimate the oscillation of the gradient $D\vec{\varphi}$.

Secondly, we consider the case $\vec{\varphi}\big|_{\partial\Omega} \neq \vec{\mathrm{id}}$. The results obtained can be used for the study of coercivity in the theory of elasticity for hyperelastic materials.

2. The estimate in the case $\vec{\varphi}|_{\partial\Omega} = \vec{\mathrm{id}}$

In what follows, we denote by $\langle\vec{x},\vec{y}\rangle$ the inner product of two vectors $\vec{x}$, $\vec{y}$ in $\mathbb{R}^n$ and by $|\vec{x}| = \langle\vec{x},\vec{x}\rangle^{1/2}$ its norm.

For a square matrix A we define its matrix norm as

$$|A| = \left[\mathrm{tr}(A^T A)\right]^{1/2} \tag{6}$$

and its spectral norm as

$$\|A\| = \sup_{\vec{x}\neq 0} \frac{|A\vec{x}|}{|\vec{x}|}. \tag{7}$$

Lemma 1. *Let Ω be a domain in $\mathbb{R}^n$ and $\vec{\varphi} \in C^1(\overline{\Omega}, \mathbb{R}^n)$, then*

$$|\mathrm{tr}(D\vec{\varphi}^T D\vec{\varphi} - I)| \leqslant n\|D\vec{\varphi}^T D\vec{\varphi} - I\| \tag{8}$$

pointwise in Ω.

P r o o f. From the definition of the spectral norm (7) we have

$$\|D\vec{\varphi}^T D\vec{\varphi} - I\| = \sup_{\vec{x}\neq 0} \frac{|(U^2 - I)\vec{x}|}{|\vec{x}|} = \max_{i=1,2,\dots,n} \{|\lambda_i^2 - 1|\}.$$

Consequently,

$$\begin{aligned} |\mathrm{tr}(D\vec{\varphi}^T D\vec{\varphi} - I)| &= \left|\sum_{i=1}^n (\lambda_i^2 - 1)\right| \leqslant \sum_{i=1}^n |\lambda_i^2 - 1| \leqslant n \max_{i=1,2,\dots,n} \{|\lambda_i^2 - 1|\} \\ &= n\|D\vec{\varphi}^T D\vec{\varphi} - I\|. \end{aligned}$$

□

In what follows, we suppose that the deformation $\vec{\varphi}(\vec{x})$ of the reference configuration Ω is given by the formula

$$\vec{\varphi}(\vec{x}) = \vec{x} + \vec{u}(\vec{x}), \tag{9}$$

where $\vec{u}\colon \overline{\Omega} \to \mathbb{R}^n$ is the displacement.

Lemma 2. *Let Ω be a domain in $\mathbb{R}^n$, $\vec{\varphi}\colon \overline{\Omega} \to \mathbb{R}^n$ a deformation which is differentiable on Ω and $\vec{\theta}$ a unit vector in $\mathbb{R}^n$. Let us denote the derivative in the direction $\vec{\theta}$ by D_θ. Then for every $k \geqslant 1$, the following estimate holds:*

$$|D_\theta\langle\vec{u},\vec{\theta}\rangle|^k \leqslant c_1[e_1(\vec{\varphi})^k + 2e_1(\vec{\varphi}) - D_\theta\langle\vec{u},\vec{\theta}\rangle]. \tag{10}$$

P r o o f. According to (9) the derivative in the direction $\vec{\theta}$ satisfies

$$D_\theta\langle\vec{u},\vec{\theta}\rangle = D_\theta\langle\vec{\varphi},\vec{\theta}\rangle - \langle\vec{\theta},\vec{\theta}\rangle \leqslant |\langle D\vec{\varphi},\vec{\theta}\rangle| - 1 \leqslant (\lambda_n - 1)_+ = e_1(\vec{\varphi}). \tag{11}$$

Because $D_\theta\langle\vec{u},\vec{\theta}\rangle - e_1(\vec{\varphi}) \leqslant 0$, we have

$$|D_\theta\langle\vec{u},\vec{\theta}\rangle| \leqslant |D_\theta\langle\vec{u},\vec{\theta}\rangle - e_1(\vec{\varphi})| + e_1(\vec{\varphi}) = 2e_1(\vec{\varphi}) - D_\theta\langle\vec{u},\vec{\theta}\rangle. \tag{12}$$

By the polar decomposition theorem we can consider the deformation gradient $D\vec{\varphi}$ in the form

$$D\vec{\varphi} = \widetilde{R}U$$

where $\widetilde{R}$ is some proper orthogonal matrix and U is the matrix (1).

Let c_0 be a constant satisfying

$$|\widetilde{R} - I| \leqslant c_0. \tag{13}$$

We distinguish two cases:

a) If $|D_\theta\langle\vec{u},\vec{\theta}\rangle| \leqslant 2c_0$, then

$$|D_\theta\langle\vec{u},\vec{\theta}\rangle|^k \leqslant (2c_0)^{k-1}|D_\theta\langle\vec{u},\vec{\theta}\rangle|.$$

From (12) we have

$$|D_\theta\langle\vec{u},\vec{\theta}\rangle|^k \leqslant (2c_0)^{k-1}[2e_1(\vec{\varphi}) - D_\theta\langle\vec{u},\vec{\theta}\rangle]. \tag{14}$$

b) If $|D_\theta\langle\vec{u},\vec{\theta}\rangle| \geqslant 2c_0$, then we can use the fact that

$$\begin{aligned}
|D_\theta\langle\vec{u},\vec{\theta}\rangle| &\leqslant |\langle D\vec{u}\,\vec{\theta},\vec{\theta}\rangle| \leqslant |\langle D\vec{u}^T D\vec{u}\vec{\theta},\vec{\theta}\rangle|^{1/2} \leqslant \big[\mathrm{tr}\big((D\vec{u})^T D\vec{u}\big)\big]^{1/2} = |D\vec{u}|,\\
|D\vec{u}| &= |D\vec{\varphi} - I| = |\widetilde{R}U - I| \leqslant |\widetilde{R} - I|\,|U - I| + |\widetilde{R} - I| + |U - I|\\
&= (|\widetilde{R} - I| + 1)|U - I| + |\widetilde{R} - I|
\end{aligned}$$

and

$$|U - I| = \{\mathrm{tr}[(U - I)^T(U - I)]\}^{1/2} = \left[\sum_{i=1}^n (\lambda_i - 1)^2\right]^{1/2} \leqslant \sqrt{n}\,e_1(\vec{\varphi}).$$

Using the relation (13) we now obtain

$$|D_\theta\langle\vec{u},\vec{\theta}\rangle| \leqslant (c_0 + 1)\sqrt{n}\,e_1(\vec{\varphi}) + c_0. \tag{15}$$

This inequality and the assumption b) imply

$$(c_0+1)\sqrt{n}\,e_1(\vec\varphi) \geqslant |D_\theta\langle\vec u,\vec\theta\rangle| - c_0 \geqslant \frac{1}{2}|D_\theta\langle\vec u,\vec\theta\rangle|$$

and so

$$|D_\theta\langle\vec u,\vec\theta\rangle|^k \leqslant [2\sqrt{n}\,(c_0+1)]^k e_1(\vec\varphi)^k.$$

Putting

$$c_1 = \max\{[2\sqrt{n}\,(c_0+1)]^k, (2c_0)^{k-1}\},$$

we obtain from (14) and (15) the estimate (10). □

Lemma 3. *If $\Omega\subset\mathbb{R}^n$ is a bounded domain with a Lipschitz boundary, $k\geqslant 1$, $\vec\varphi\in W^{1,k}(\Omega,\mathbb{R}^n)$, $\vec\varphi = \vec{\mathrm{id}}$ on $\partial\Omega$, then*

$$\int_\Omega |\mathrm{tr}(D\vec\varphi - I)|^k\,\mathrm{d}\vec x \leqslant c_2\int_\Omega [e_1(\vec\varphi)^k + 2e_1(\vec\varphi)]\,\mathrm{d}\vec x. \tag{16}$$

P r o o f. Denote by $l_\theta(x)$ the line which goes through the point $\vec x$ and is parallel with the directon $\vec\theta$. From (10) we have

$$\begin{aligned}\int_{l_\theta(x)\cap\Omega} |D_\theta\langle\vec u,\vec\theta\rangle|^k\,\mathrm{d}s &\leqslant c_1\int_{l_\theta(x)\cap\Omega}\big(e_1(\vec\varphi)^k + 2e_1(\vec\varphi) - D_\theta\langle\vec u,\vec\theta\rangle\big)\,\mathrm{d}s\\ &= c_1\bigg(\int_{l_\theta(x)\cap\Omega}[e_1(\vec\varphi)^k + 2e_1(\vec\varphi)]\,\mathrm{d}s - |\vec u|\big|_{l_\theta(x)\cap\partial\Omega}\bigg).\end{aligned}$$

If $\vec\varphi\big|_{\partial\Omega} = \vec{\mathrm{id}}$, then $\vec u\big|_{\partial\Omega} = 0$ and

$$\int_{l_\theta(x)\cap\Omega} |D_\theta\langle\vec u,\vec\theta\rangle|^k\,\mathrm{d}s \leqslant c_1\int_{l_\theta(x)\cap\Omega}\big(e_1(\vec\varphi)^k + 2e_1(\vec\varphi)\big)\,\mathrm{d}s.$$

The integration over $\vec\theta^\perp$ gives

$$\int_\Omega |D_\theta\langle\vec u,\vec\theta\rangle|^k\,\mathrm{d}\vec x \leqslant c_1\int_\Omega\big(e_1(\vec\varphi)^k + 2e_1(\vec\varphi)\big)\,\mathrm{d}\vec x. \tag{17}$$

Let now $\vec\theta_1,\dots,\vec\theta_n$ be an orthonormal base. From the inequality

$$\Big(\sum_{i=1}^n a_i\Big)^k \leqslant n^{k-1}\sum_{k=1}^n a_i^k \tag{18}$$

(which holds for all $a_i \geqslant 0$, $k \geqslant 1$) and from (17) we conclude

$$\int_\Omega |\operatorname{tr}(D\vec{\varphi} - I)|^k \,\mathrm{d}\vec{x} = \int_\Omega |\operatorname{tr} D\vec{u}|^k \,\mathrm{d}\vec{x} = \int_\Omega \Big|\sum_{i=1}^n D_{\vec{\theta}_i}\langle \vec{u}, \vec{\theta}_i\rangle\Big|^k \,\mathrm{d}\vec{x}$$
$$\leqslant n^{k-1} \sum_{i=1}^n \int_\Omega |D_{\vec{\theta}_i}\langle \vec{u}, \vec{\theta}_i\rangle|^k \,\mathrm{d}\vec{x} \leqslant c_2 \int_\Omega \big(e_1(\vec{\varphi})^k + 2e_1(\vec{\varphi})\big) \,\mathrm{d}\vec{x},$$

where $c_2 = n^k c_1$. $\square$

In the proof of Lemma 3 we have formalized the heuristic idea from the introduction. The line segments $l_{\theta_i}(x) \cap \Omega$ represent the above mentioned elastic bands pinned at $l_{\theta_i}(x) \cap \partial\Omega$.

Theorem 1 (Case $\vec{\varphi}\big|_{\partial\Omega} = \vec{\mathrm{id}}$)**.**
Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded domain with a Lipschitz boundary, $p \geqslant 2$, $\vec{\varphi} \in W^{1,p}(\Omega, \mathbb{R}^n)$, $\vec{\varphi}\big|_{\partial\Omega} = \vec{\mathrm{id}}$. Then

$$\text{(19)}\quad \|D\vec{\varphi} - I\|_{L^p(\Omega)} \leqslant K_1 \bigg[\Big(\int_\Omega e_1(\vec{\varphi})^{p/2} \,\mathrm{d}\vec{x}\Big)^{1/p} + \Big(\int_\Omega e_1(\vec{\varphi}) \,\mathrm{d}\vec{x}\Big)^{1/p} + \Big(\int_\Omega \|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2} \,\mathrm{d}\vec{x}\Big)^{1/p}\bigg].$$

P r o o f. From (6), (18), (8) we have

$$\begin{aligned}|D\vec{\varphi} - I|^p &= \{\operatorname{tr}[(D\vec{\varphi} - I)^T (D\vec{\varphi} - I)]\}^{p/2} \\ &= \operatorname{tr}(D\vec{\varphi}^T D\vec{\varphi} - I) - 2\operatorname{tr}(D\vec{\varphi} - I)]^{p/2} \\ &\leqslant 2^{\frac{p}{2}-1} |\operatorname{tr}(D\vec{\varphi}^T D\vec{\varphi} - I)|^{p/2} + 2^{p-1} |\operatorname{tr}(D\vec{\varphi} - I)|^{p/2} \\ &\leqslant 2^{\frac{p-2}{2}} n^{\frac{p}{2}} \|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2} + 2^{p-1} |\operatorname{tr}(D\vec{\varphi} - I)|^{p/2}.\end{aligned}$$

Hence using (16), (18) we obtain

$$\begin{aligned}\|D\vec{\varphi} - I\|_{L_p(\Omega)} &= \Big(\int_\Omega |D\vec{\varphi}(x) - I|^p \,\mathrm{d}\vec{x}\Big)^{1/p} \\ &\leqslant [2^{\frac{p}{2}-1} n^{p/2} \Big(\int_\Omega \|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2} \,\mathrm{d}\vec{x}\Big) + c_2 2^{p-1} \int_\Omega \big(e_1(\vec{\varphi})^{p/2} + 2e_1(\vec{\varphi})\big) \,\mathrm{d}\vec{x}]^{1/p} \\ &\leqslant 3^{\frac{1-p}{p}} \bigg[2^{-1/p}(2n)^{1/2} \Big(\int_\Omega \|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2} \,\mathrm{d}\vec{x}\Big)^{1/p} + c_2^{1/p} 2^{\frac{p-1}{p}} \Big(\int_\Omega e_1(\vec{\varphi})^{p/2} \,\mathrm{d}\vec{x}\Big)^{1/p} \\ &\quad + c_2^{1/p} 2 \Big(\int_\Omega e_1(\vec{\varphi}) \,\mathrm{d}\vec{x}\Big)^{1/p}\bigg].\end{aligned}$$

If we denote $K_1 = 3^{\frac{1-p}{p}} \max\Big\{2^{-1/p}(2n)^{1/2},\ c_2^{1/p} 2^{\frac{p-1}{p}},\ 2c_2^{1/p}\Big\}$, we obtain the assertion of the theorem. $\square$

3. The estimate in the case $\vec{\varphi}\big|_{\partial\Omega} \neq \vec{\mathrm{id}}$

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a domain. The map $\vec{\varphi}\colon \Omega \to \mathbb{R}^n$ is said to be *bi-Lipschitzian* if there is an inverse map $\vec{\varphi}^{-1}\colon \vec{\varphi}(\Omega) \to \Omega$ such that both $\vec{\varphi}$ and $\vec{\varphi}^{-1}$ are Lipschitzian.

Theorem 2 (R. V. Kohn)**.** *Let Ω be a bounded, Lipschitzian domain in $\mathbb{R}^n$ $(n \geqslant 2)$ and let $1 \leqslant p$ $(p \neq n)$. There is a constant $c(\Omega, p)$ such that, for any bi-Lipschitzian map $\vec{\varphi}\colon \Omega \to \mathbb{R}^n$, there exists a rigid motion $\vec{\gamma}$ satisfying*
(i) *if $1 \leqslant p < n$, then*

$$\|\vec{\varphi} - \vec{\gamma}\|_{L^q(\Omega)} + \|\vec{\varphi} - \vec{\gamma}\|_{L^p(\partial\Omega)} \leqslant c(\Omega, p)\|e(\vec{\varphi})\|_{L^p(\Omega)} \text{ with } q = \frac{np}{n-p}; \tag{20}$$

(ii) *if $p > n$, then*

$$\sup_{x\in\Omega} |\vec{\varphi}(x) - \vec{\gamma}(x)| \leqslant c(\Omega, p)\|e(\vec{\varphi})\|_{L^p(\Omega)}.$$

P r o o f. See [3]. $\square$

It is evident from the theorem that there is an approximating rigid motion $\vec{\gamma}$ for the deformation $\vec{\varphi}$. Let us denote $D\vec{\gamma} = R$ and compare it with the deformation gradient $D\vec{\varphi}$.

Theorem 3 (Case $\vec{\varphi}\big|_{\partial\Omega} \neq \vec{\mathrm{id}}$)**.**
If $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded domain with a Lipschitzian boundary, $p \geqslant 2$, then there is a constant K_2 such that for any bi-Lipschitzian map $\vec{\varphi}\colon \Omega \to \mathbb{R}^n$ there exists an orthogonal matrix R satisfying

$$\begin{aligned}\|D\vec{\varphi} - R\|_{L^p(\Omega)} \leqslant K_2\biggl[\Bigl(\int_\Omega e_1(\vec{\varphi})^{p/2}\,\mathrm{d}\vec{x}\Bigr)^{1/p} + \Bigl(\int_\Omega e(\vec{\varphi})\,\mathrm{d}\vec{x}\Bigr)^{1/p} \\ + \Bigl(\int_\Omega \|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2}\,\mathrm{d}\vec{x}\Bigr)^{1/p}\biggr].\end{aligned} \tag{21}$$

P r o o f. Let $\vec{\gamma}$ be a rigid motion from Theorem 2. We can put

$$\vec{u}(\vec{x}) = \vec{\gamma}^{-1} \cdot \vec{\varphi}(\vec{x}) - \vec{x}. \tag{22}$$

Then

$$D\vec{u}(\vec{x}) = R^{-1}D\vec{\varphi}(\vec{x}) - I,$$

where $R^{-1} = D\vec{\gamma}^{-1}$ is an orthogonal matrix independent of x, and

$$|D\vec{\varphi} - R| \leqslant \sqrt{n}\,\|D\vec{\varphi} - R\| = \sqrt{n}\,\|R^{-1}D\vec{\varphi} - I\| \leqslant \sqrt{n}\,|R^{-1}D\vec{\varphi} - I|.$$

In the same way as in Theorem 1 we show that

$$\begin{aligned}|R^{-1}D\vec{\varphi} - I|^p &= \{\mathrm{tr}[(R^{-1}D\vec{\varphi} - I)^T(R^{-1}D\vec{\varphi} - I)]\}^{p/2}\\ &= [\mathrm{tr}\,(D\vec{\varphi}^T D\vec{\varphi} - I) - 2\,\mathrm{tr}(R^{-1}D\vec{\varphi} - I)]^{p/2}\\ &\leqslant 2^{\frac{p-2}{2}} n^{p/2}\|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2} + 2^{p-1}|\,\mathrm{tr}(R^{-1}D\vec{\varphi} - I)|^{p/2}.\end{aligned}$$

It means that

$$\begin{aligned}\int_\Omega |D\vec{\varphi} - R|^p \,\mathrm{d}\vec{x} &\leqslant n^{p/2}\int_\Omega |R^{-1}D\vec{\varphi} - I|^p \,\mathrm{d}\vec{x}\\ &\leqslant c_3\Big[\int_\Omega \|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2}\,\mathrm{d}\vec{x} + \int_\Omega |\mathrm{tr}(R^{-1}D\vec{\varphi} - I)|^{p/2}\,\mathrm{d}\vec{x}\Big],\end{aligned} \tag{23}$$

where

$$c_3 = \max\big\{2^{\frac{p}{2}-1}n^p, 2^{p-1}n^{p/2}\big\}.$$

We need to estimate $\int_\Omega |\mathrm{tr}(R^{-1}D\vec{\varphi} - I)|^{p/2}\,\mathrm{d}\vec{x}$. According to (10) we have

$$\begin{aligned}|D_\theta\langle\vec{u},\vec{\theta}\rangle|^{p/2} &\leqslant c_1[e_1(\vec{\gamma}^{-1}\centerdot\vec{\varphi})^{p/2} + 2e_1(\vec{\gamma}^{-1}\centerdot\vec{\varphi}) - D_\theta\langle\vec{u},\vec{\theta}\rangle]\\ &\leqslant c_1[e_1(\vec{\varphi})^{p/2} + 2e_1(\vec{\varphi}) - D_\theta\langle\vec{u},\vec{\theta}\rangle].\end{aligned}$$

If we integrate along the line $l_\theta(x)$, we obtain

$$\int_{l_\theta(x)\cap\Omega} |D_\theta\langle\vec{u},\vec{\theta}\rangle|^{p/2}\,\mathrm{d}s \leqslant c_1\Big\{\int_{l_\theta(x)\cap\Omega}\big(e_1(\vec{\varphi})^{p/2} + 2e_1(\vec{\varphi})\big)\,\mathrm{d}s + |\vec{u}|\big|_{l_\theta(x)\cap\partial\Omega}\Big\}.$$

Integration over $\vec{\theta}^\perp$ gives

$$\int_\Omega |D_\theta\langle\vec{u},\vec{\theta}\rangle|^{p/2}\,\mathrm{d}\vec{x} \leqslant c_1\Big[\int_\Omega \big(e_1(\vec{\varphi})^{p/2} + 2e_1(\vec{\varphi})\big)\,\mathrm{d}\vec{x} + \int_{\partial\Omega}|\vec{u}|\,\mathrm{d}\vec{a}\Big]. \tag{24}$$

Let $\theta_1,\dots,\theta_n$ be an orthonormal basis in $\mathbb{R}^n$. From the definition of the trace, from (18) and (24) we have

$$\begin{aligned}\int_\Omega |\mathrm{tr}(R^{-1}D\vec{\varphi} - I)|^{p/2}\,\mathrm{d}\vec{x} &= \int_\Omega |\,\mathrm{tr}\,D\vec{u}|^{p/2}\,\mathrm{d}\vec{x} \leqslant n^{\frac{p-2}{2}}\sum_{i=1}^n \int_\Omega |D_{\theta_i}\langle\vec{u},\vec{\theta}_i\rangle|^{p/2}\,\mathrm{d}\vec{x}\\ &\leqslant c_1 n^{p/2}\Big[\int_\Omega \big(e_1(\vec{\varphi})^{p/2} + 2e_1(\vec{\varphi})\big)\,\mathrm{d}\vec{x} + \int_{\partial\Omega}|\vec{u}|\,\mathrm{d}\vec{a}\Big].\end{aligned} \tag{25}$$

Because

$$\|\vec{\varphi} - \vec{\gamma}\|_{L^q(\Omega)} \geqslant 0,$$

we have from (20) setting $p = 1$ that

$$\|\vec{\varphi} - \vec{\gamma}\|_{L^1(\partial\Omega)} \leqslant c\|e(\vec{\varphi})\|_{L^1(\Omega)}.$$

It means that

$$\int_{\partial\Omega} |\vec{u}| \,\mathrm{d}\vec{a} = \int_{\partial\Omega} |\vec{\gamma}^{-1} \cdot \vec{\varphi}(\vec{x}) - \vec{x}| \,\mathrm{d}\vec{a} \leqslant c_4 \int_\Omega e(\vec{\varphi}) \,\mathrm{d}\vec{x}.$$

From here and from (25) we see that

$$\begin{aligned}\int_\Omega |\mathrm{tr}(R^{-1}D\vec{\varphi} - I)|^{p/2} \,\mathrm{d}\vec{x} &\leqslant c_1 n^{p/2}\Big[\int_\Omega \big(e_1(\vec{\varphi})^{p/2} + 2e_1(\vec{\varphi})\big) \,\mathrm{d}\vec{x} + c_4 \int_\Omega e(\vec{\varphi}) \,\mathrm{d}\vec{x}\Big] \\ &\leqslant c_5 \int_\Omega \big(e_1(\vec{\varphi})^{p/2} + e(\vec{\varphi})\big) \,\mathrm{d}\vec{x},\end{aligned}$$

where $c_5 = c_1 n^{p/2} \max\{2, c_4 + 2\}$.

We substitute this result into (23). We have

$$\int_\Omega |D\vec{\varphi} - R|^p \,\mathrm{d}\vec{x} \leqslant c_3\Big[\int_\Omega \|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2} \,\mathrm{d}\vec{x} + c_5 \int_\Omega \big(e_1(\vec{\varphi})^{p/2} + e(\vec{\varphi})\big) \,\mathrm{d}\vec{x}\Big].$$

If we use the inequality (18) and denote $K_2 = 3^{\frac{1-p}{p}} c_3^{1/p} \max\{1, c_5^{1/p}\}$, we obtain

$$\|D\vec{\varphi} - R\|_{L^p(\Omega)} = \Big(\int_\Omega |D\vec{\varphi} - R|^p \,\mathrm{d}\vec{x}\Big)^{1/p}$$

$$\leqslant K_2\Big[\Big(\int_\Omega \|D\vec{\varphi}^T D\vec{\varphi} - I\|^{p/2} \,\mathrm{d}\vec{x}\Big)^{1/p} + \Big(\int_\Omega e_1(\vec{\varphi})^{p/2} \,\mathrm{d}\vec{x}\Big)^{1/p} + \Big(\int_\Omega e(\vec{\varphi}) \,\mathrm{d}\vec{x}\Big)^{1/p}\Big].$$

$\square$

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Author's address: *Vratislava Mošová*, Department of Mathematical Information Science, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, email: `mosova@upol.cz`.