Applications of Mathematics

Liping Liu; Pekka Neittaanmäki; Michal Křížek

Second-order optimality conditions for nondominated solutions of multiobjective programming with $C^{1,1}$ data

Applications of Mathematics, Vol. 45 (2000), No. 5, 381–397

Persistent URL: `http://dml.cz/dmlcz/134446`

SECOND-ORDER OPTIMALITY CONDITIONS FOR NONDOMINATED SOLUTIONS OF MULTIOBJECTIVE PROGRAMMING WITH $C^{1,1}$ DATA*

Liping Liu, Toronto, Pekka Neittaanmäki, Jyväskylä
and Michal Křížek, Praha

(Received April 9, 1999)

Abstract. We examine new second-order necessary conditions and sufficient conditions which characterize nondominated solutions of a generalized constrained multiobjective programming problem. The vector-valued criterion function as well as constraint functions are supposed to be from the class $C^{1,1}$. Second-order optimality conditions for local Pareto solutions are derived as a special case.

Keywords: multiobjective programming, nonsmooth constrained optimization, second-order optimality conditions, nondominated solutions, local Pareto optimal solutions

MSC 2000: 90C31

1. Introduction

In various real-life problems we need to minimize simultaneously several criteria over some admissible set of points. One way of treating this problem is to minimize their weighted average, but then we usually lose information. Another way is to apply methods and algorithms of multiobjective programming for minimization of vector-valued functions (see, e.g., [12], [14], [21], [23]), where usually the first or second-order optimality conditions are employed.

The first-order optimality conditions for multiobjective programming have been considered by many authors. However, the second-order optimality conditions for

* The work was supported by the COMAS graduate school at Jyväskylä and by grant No. 201/1998/1452 of the Grant Agency of the Czech Republic.

multiobjective programming were derived by few authors ([3], [5], [8], [12], [16], [21]) only. In [8], second-order necessary conditions and sufficient conditions are introduced in the case the vector objective function is from the class C^2 and the constraints are from $C^{1,1}$. But we sometimes need to minimize criteria with less smoothness. For instance, if f is from C^2 then the penalty function $(f^+)^2$, where f^+ stands for the positive part of f, is generally only from the class $C^{1,1}$ but not C^2 (compare [7]). The aim of this paper is to generalize the above-mentioned results from [5], [8], [16] to nonsmooth data and also to extend some results concerning nonlinear programming from [10] to multiobjective programming. Moreover, we obtain second-order conditions for local Pareto optimal solutions with $C^{1,1}$ data (C^2 data) as a by-product of our approach.

For both the unconstrained and constrained nonlinear minimization problems with $C^{1,1}$ data, the second-order necessary optimality conditions were derived in [2], [8], [9], [10], [13], [17], [18], [20]. Second-order sufficient conditions were also obtained in [2], [6], [8], [9], [10], [13], [18], [19]. However, these conditions cannot be naturally generalized for multiobjective programming.

Throughout the paper, E_n stands for an n-dimensional Euclidean space equipped with the Euclidean norm $\|\cdot\|$. By $\mathscr{B}_\varepsilon(x)$ we denote an open ball in E_n with radius ε and centre x. Let $S \subset E_n$ be a nonempty open set and let $f(x) = \big(f_1(x), \ldots, f_m(x)\big)^T$, $x \in S$. Recall that f is said to be a $C^{1,1}$ vector function on S (we will write $f \in C^{1,1}(S)$) if f is continuously differentiable and its gradient

$$\nabla f(x) = \big(\nabla f_1(x)^T, \ldots, \nabla f_m(x)^T\big)^T = \begin{bmatrix} \partial f_1(x)/\partial x_1 & \ldots & \partial f_1(x)/\partial x_n \\ \vdots & & \vdots \\ \partial f_m(x)/\partial x_1 & \ldots & \partial f_m(x)/\partial x_n \end{bmatrix}_{m\times n}$$

is locally Lipschitz continuous on S, i.e.,

$$\forall \overline{x} \in S \quad \exists C > 0 \quad \exists \varepsilon > 0 \quad \forall x, y \in \mathscr{B}_\varepsilon(\overline{x})\colon\ \|\nabla f(x) - \nabla f(y)\| \leqslant C\|x - y\|.$$

Note that ∇f is differentiable almost everywhere by Rademacher's theorem (see [15]).

If a vector function $f \in C^2(S)$, $\nabla^2 f_i(x)$ is the $n \times n$ Hessian matrix of f_i ($i = 1, \ldots, m$) at x, then

$$\nabla^2 f(x) = \begin{bmatrix} \nabla^2 f_1(x) & & 0 \\ & \ddots & \\ 0 & & \nabla^2 f_m(x) \end{bmatrix}_{mn\times mn}$$

is called the Hessian matrix of a vector function f at x.

The generalized Hessian matrix of a vector function $f \in C^{1,1}(S)$ at $x \in S$ is denoted by

$$\partial^2 f(x) = \mathrm{Co}\{M \mid \exists\, \{x^i\}_{i=1}^{\infty}\colon\ x^i \to x \text{ with } f \text{ twice differentiable at } x^i \text{ such that } \nabla^2 f(x^i) \to M\},$$

where “Co” stands for the convex hull. This definition was introduced for $m = 1$ by Hiriart-Urruty et al. in 1984 (see [4]). They proved that if $\overline{x} \in S$ is a local minimum for the problem

$$\min_{x \in S} f(x) \tag{1.1}$$

then there exists a matrix $A \in \partial^2 f(\overline{x})$ such that $(Ad, d) \geqslant 0$ for all $d \in E_n$. But there are also examples (see [4], [10]) showing that

$$\exists A \in \partial^2 f(\overline{x}) \quad \forall d \in E_n\colon\ (Ad, d) < 0. \tag{1.2}$$

Recall that the generalized second-order directional derivative of $f \in C^{1,1}(S)$ at $x \in S$ in a direction $d \in E_n$ was defined by Liu in [8] as

$$\begin{aligned}\partial^2_* f(x)(d, d) = \{\varphi(x; d) \in E_m \mid \exists\, \{t_i\}_{i=1}^{\infty}\colon\ t_i \to 0^+ \Longrightarrow \\ 2t_i^{-2}\bigl(f(x + t_i d) - f(x) - t_i \nabla f(x) d\bigr) \to \varphi(x; d)\}.\end{aligned} \tag{1.3}$$

This approach differs from that presented in [11]. Now we show that the set (1.3) is always nonempty. Since $f \in C^{1,1}(S)$, there exists $C > 0$ such that for any $i \in \{1, 2, \ldots\}$ and any $j \in \{1, \ldots, m\}$ there exists $\tilde{x}_i^j \in [x, x + t_i d] \subset E_n$ such that

$$\begin{aligned}\|t_i^{-2}\bigl(f_j(x + t_i d) - f_j(x) - t_i \nabla f_j(x) d\bigr)\| &= t_i^{-2}\|t_i \nabla f_j(\tilde{x}_i^j) d - t_i \nabla f_j(x) d\| \\ &\leqslant t_i^{-1}\|\nabla f_j(\tilde{x}_i^j) - \nabla f_j(x)\|\, \|d\| \leqslant C t_i^{-1}\|\tilde{x}_i^j - x\|\, \|d\| \leqslant C\|d\|^2.\end{aligned} \tag{1.4}$$

Hence, the sequence

$$\{t_i^{-2}\bigl(f(x + t_i d) - f(x) - t_i \nabla f(x) d\bigr)\}_{i=1}^{\infty}$$

is bounded for any $x \in S$ and any $d \in E_n$, and thus it has at least one accumulation point.

Moreover, Liu proved in [8] that if $f \in C^{1,1}(S)$ and $\overline{x} \in S$ is a local minimum for (1.1) then for all $\varphi(\overline{x}; d) \in \partial^2_* f(\overline{x})(d, d)$, $d \in E_n$, we have $\varphi(\overline{x}; d) \geqslant 0$ (compare (1.2)).

Obviously, if f is twice differentiable at x then

$$\partial_*^2 f(x)(d,d) = \{\mathbb{D}^T \nabla^2 f(x) D\},$$

where

$$\mathbb{D} = \begin{bmatrix} d & & 0 \\ & \ddots & \\ 0 & & d \end{bmatrix}_{mn\times m} \quad \text{and} \quad D = \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix}_{mn\times 1}.$$

Setting

$$\partial^2 f(x)(d,d) = \{\mathbb{D}^T M D \mid M \in \partial^2 f(x)\}$$

for $f \in C^{1,1}(S)$, then by [8] we get

$$\partial_*^2 f(x)(d,d) \subset \partial^2 f(x)(d,d),$$

but it may also happen (see [10]) that $\partial_*^2 f(x)(d,d) \mathbb{D} \partial^2 f(x)(d,d)$.

If $f \in C^{1,1}(S)$ is convex then by a contradiction argument we can easily derive from (1.3) that

$$\partial_*^2 f(x)(d,d) \subset E_m^+ \quad \forall x \in S \quad \forall d \in E_n,$$

where E_m^+ denotes the set of vectors from E_m with nonnegative components.

2. Auxiliary lemmas

Throughout the paper we assume that the vertex of any cone is at the origin. Consider a constrained multiobjective minimization problem

$$\text{(2.1)} \qquad \begin{cases} \min f(x) \\ \mathscr{R} = \{x \in S \mid g(x) \in K\}, \end{cases}$$

where $f = (f_1, \dots, f_m)^T \in C^{1,1}(S)$, $g = (g_1, \dots, g_l)^T \in C^{1,1}(S)$, K is a closed cone in E_l and $\mathscr{R}$ is supposed to be nonempty.

Let W be a convex cone in E_m. Recall that a vector $\overline{x} \in \mathscr{R}$ is said to be a *local nondominated solution* associated with W for problem (2.1), if

$$\exists \varepsilon > 0 \quad \nexists\, x \in \mathscr{B}_\varepsilon(\overline{x}) \cap \mathscr{R}\colon\ f(\overline{x}) \in f(x) + W, \quad f(\overline{x}) \neq f(x).$$

If $W = E_m^+$ then the local nondominated solution $\overline{x}$ is also known as a local Pareto optimal (efficient) solution, i.e., there exists no other $\overline{x} \in \mathscr{R}$ such that $f(x) \leqslant f(\overline{x})$ and $f(x) \neq f(\overline{x})$.

From now on we assume that

(A) $\operatorname{cl} W \subset E_m$ and $K \subset E_l$ are polyhedral cones with nonempty interiors.

That is, $W^0 \neq \emptyset$ and $K^0 \neq \emptyset$. Note that a polyhedral cone is always closed and convex (see [1], p. 65).

Denote by W^* and K^* the polar cones of W and K, respectively, i.e., $K^* = \{q \in E_l \mid q^T u \leqslant 0 \quad \forall u \in K\}$. Since W^* and K^* are also polyhedral cones, there exist sets of vectors $P = \{p_1, \ldots, p_k\} \subset E_m$ and $Q = \{q_1, \ldots, q_s\} \subset E_l$ determining the edges of polyhedral cones such that

$$-W^* = \Big\{p \in E_m \mid p = \sum_{j=1}^{k} \alpha_j p_j,\ \alpha_j \in E_1^+,\ p_j \in P,\ j = 1, \ldots, k\Big\}$$

and

$$K^* = \Big\{q \in E_l \mid q = \sum_{j=1}^{s} \beta_j q_j,\ \beta_j \in E_1^+,\ q_j \in Q,\ j = 1, \ldots, s\Big\}.$$

Recall that a cone $K \subset E_l$ is said to be *acute* if there exists an open half-space

$$H = \{x \in E_l \mid a^T x > 0,\ a \neq 0\}$$

such that

$$\operatorname{cl} K \subset H \cup \{0\}.$$

Let $\overline{x} \in \operatorname{cl} S$. Then the *cone of interior directions* to S at $\overline{x}$ is given by

$$I(S, \overline{x}) = \{x \in E_n \mid \exists\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall y \in x + \mathscr{B}_\varepsilon(0) \quad \forall \lambda \in (0, \delta) \Longrightarrow \overline{x} + \lambda y \in S\}.$$

For simplicity we write $I^*(S, \overline{x})$ instead of $\big(I(S, \overline{x})\big)^*$.

To prove Lemma 2.3 and Theorem 3.3 we need the following two lemmas.

Lemma 2.1. *Let $\mathscr{K}$ be a cone in E_m. Then*

(i) *$\mathscr{K}$ is acute if and only if $(\mathscr{K}^*)^0 \neq \emptyset$.*

(ii) *If $\mathscr{K}$ is acute then*

$$(\mathscr{K}^*)^0 = \{q \in E_m \mid q^T u < 0 \quad \forall u \in \operatorname{cl} \mathscr{K},\ u \neq 0\}.$$

Proof. See [22]. $\square$

Lemma 2.2. *Let $\mathscr{K}$ be a convex cone in E_l, $\overline{x} \in \operatorname{cl}\mathscr{K}$, and let $I(\mathscr{K},\overline{x})$ be the cone of interior directions to $\mathscr{K}$ at $\overline{x}$. If $\mathscr{K}$ is open then*

$$I(\mathscr{K},\overline{x}) = \{x - \alpha\overline{x} \mid x \in \mathscr{K},\ \alpha \geqslant 0\}.$$

Proof. See [1], p. 129. □

Next, we introduce some further notation.

If $f \in C^{1,1}(S)$, $g \in C^{1,1}(S)$, $g(x) \in K$, $\nabla f(x)d \in -\operatorname{cl} W$ and $\nabla g(x)d \in \operatorname{cl} I(K^0, g(x))$ for $d \in E_n$ and $x \in \mathscr{R}$, then we put

$$\begin{aligned} P_1(x,d) &= \{p_j \in P \mid p_j \in I^*\bigl(-W^0, \nabla f(x)d\bigr)\} \\ &= \{p_j \in P \mid p_j \in -W^*,\ p_j^T \nabla f(x)d = 0\}, \\ P_2(x,d) &= \{p_j \in P \mid p_j \in -W^*,\ p_j^T \nabla f(x)d < 0\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} Q_1(x,d) &= \{q_j \in Q \mid q_j \in I^*(I(K^0, g(x)), \nabla g(x)d)\} \\ &= \{q_j \in Q \mid q_j \in K^*,\ q_j^T g(x) = 0,\ q_j^T \nabla g(x)d = 0\}, \\ Q_2(x,d) &= \{q_j \in Q \mid q_j \in K^*,\ q_j^T g(x) = 0,\ q_j^T \nabla g(x)d < 0\}, \\ Q_3(x,d) &= \{q_j \in Q \mid q_j \in K^*,\ q_j^T g(x) < 0\}, \end{aligned} \tag{2.3}$$

where equalities in (2.2) and in (2.3) follow from Lemma 2.2. Obviously,

$$P_1(x,d) \cap P_2(x,d) = \emptyset, \tag{2.4}$$

$$P = P_1(x,d) \cup P_2(x,d) \tag{2.5}$$

and

$$Q_r(x,d) \cap Q_v(x,d) = \emptyset, \quad r \neq v, \quad r,v = 1,2,3, \tag{2.6}$$

$$Q = Q_1(x,d) \cup Q_2(x,d) \cup Q_3(x,d). \tag{2.7}$$

Throughout the paper we suppose that

$$D_L(x) = \{d \in E_n \mid \nabla f(x)d \in -\operatorname{cl} W, \quad \nabla g(x)d \in \operatorname{cl}\, I(K^0, g(x))\}$$

is nonempty. Moreover, we set

$$L(x) = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} \in E_{m+l}, \qquad x \in S. \tag{2.8}$$

Then we have $L \in C^{1,1}(S)$ and thus for

$$\gamma(x;d) = \begin{bmatrix} \varphi(x;d) \\ \psi(x;d) \end{bmatrix} \in \partial_*^2 L(x)(d,d) \subset E_{m+l}$$

we can define

$$Z_L(x;\varphi,\psi,d) = \left\{ \begin{bmatrix} z \\ z \end{bmatrix} \in E_{2n} \,\middle|\, \begin{bmatrix} \nabla f(x) & 0 \\ 0 & \nabla g(x) \end{bmatrix} \begin{bmatrix} z \\ z \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \varphi(x;d) \\ \psi(x;d) \end{bmatrix} \right.$$
$$\left. \in \begin{bmatrix} I\big(-W^0, \nabla f(x)d\big) \\ I\big(I(K^0, g(x)), \nabla g(x)d\big) \end{bmatrix} \right\}.$$

It is easy to see that $Z_L(x;\varphi,\psi,d)$ is a convex set.

Lemma 2.3. *Let* (A) *hold, let* $f \in C^{1,1}(S)$ *and* $g \in C^{1,1}(S)$*. If* $\overline{x} \in \mathscr{R}$ *is a local nondominated solution associated with* W *for problem* (2.1)*, then for any* $d \in D_L(\overline{x})$ *and any* $\gamma(\overline{x};d) = \begin{bmatrix} \varphi(\overline{x};d) \\ \psi(\overline{x};d) \end{bmatrix} \in \partial_*^2 L(\overline{x})(d,d)$ *we have*

$$Z_L(\overline{x};\varphi,\psi,d) = \emptyset.$$

P r o o f. Suppose that this is not true. Then there exist $\overline{d} \in D_L(\overline{x})$, $\overline{z} \in E_n$ and

$$\gamma(\overline{x};\overline{d}) = \begin{bmatrix} \varphi(\overline{x};\overline{d}) \\ \psi(\overline{x};\overline{d}) \end{bmatrix} \in \partial_*^2 L(\overline{x})(\overline{d},\overline{d})$$

such that

$$\begin{bmatrix} \nabla f(\overline{x}) & 0 \\ 0 & \nabla g(\overline{x}) \end{bmatrix} \begin{bmatrix} \overline{z} \\ \overline{z} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \varphi(\overline{x};\overline{d}) \\ \psi(\overline{x};\overline{d}) \end{bmatrix} \in \begin{bmatrix} I(-W^0, \nabla f(\overline{x})\overline{d}) \\ I\big(I(K^0, g(\overline{x})), \nabla g(\overline{x})\overline{d}\big) \end{bmatrix}. \tag{2.9}$$

Moreover, there exists a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \to 0^+$ as $i \to \infty$, such that

$$\begin{bmatrix} 2t_i^{-2}\big(f(\overline{x}+t_i\overline{d}) - f(\overline{x}) - t_i \nabla f(\overline{x})\overline{d}\big) \\ 2t_i^{-2}\big(g(\overline{x}+t_i\overline{d}) - g(\overline{x}) - t_i \nabla g(\overline{x})\overline{d}\big) \end{bmatrix} \to \begin{bmatrix} \varphi(\overline{x};\overline{d}) \\ \psi(\overline{x};\overline{d}) \end{bmatrix}. \tag{2.10}$$

Hence,

$$\varphi(\overline{x};\overline{d}) \in \partial_*^2 f(\overline{x})(\overline{d},\overline{d}),$$
$$\psi(\overline{x};\overline{d}) \in \partial_*^2 g(\overline{x})(\overline{d},\overline{d}).$$

We shall continue in two steps:

1) First we prove that for the sequence $\{t_i\}_{i=1}^{\infty}$ there exists a sufficiently large i_1 such that

$$g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})\in K \quad \forall i>i_1. \tag{2.11}$$

By (2.10) and (2.9), there exists a sufficiently large i_2 such that

$$\begin{aligned} & g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-g(\overline{x})-t_i\nabla g(\overline{x})\overline{d} \\ &= t_i^2\big(\nabla g(\overline{x})\overline{z}+\frac{1}{2}\psi(\overline{x};\overline{d})+\varepsilon_1(t_i)\big) \\ &\in I\big(I(K^0,g(\overline{x})),\nabla g(\overline{x})\overline{d}\big) \quad \forall i>i_2, \end{aligned} \tag{2.12}$$

where $\varepsilon_1(t_i)\to 0$ as $i\to\infty$.

(i) Let $q_j\in Q_1(\overline{x},\overline{d})$. Then from (2.12) we see that

$$q_j^T\big(g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-g(\overline{x})-t_i\nabla g(\overline{x})\overline{d}\big)\leqslant 0$$

and

$$q_j^T g(\overline{x})=0, \qquad q_j^T\nabla g(\overline{x})\overline{d}=0.$$

Hence,

$$q_j^T g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})\leqslant 0.$$

(ii) Let $q_j\in Q_2(\overline{x},\overline{d})$. Then

$$q_j^T\nabla g(\overline{x})\overline{d}<0, \qquad q_j^T g(\overline{x})=0$$

and thus there exists a sufficiently large i_3 such that

$$q_j^T g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})=q_j^T\big(g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-g(\overline{x})\big)\leqslant 0 \quad \forall i>i_3.$$

(iii) Let $q_j\in Q_3(\overline{x},\overline{d})$. Then $q_j^T g(\overline{x})<0$. Moreover, since $g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})$ is continuous in t_i, there exists a sufficiently large i_4 such that

$$q_j^T g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})\leqslant 0 \quad \forall i>i_4.$$

Set $i_1=\max\{i_2,i_3,i_4\}$. Then for any $q_j\in Q(\overline{x},\overline{d})$, i.e., for any $q\in K^*$, by (2.6) and (2.7) we have

$$q_j^T g(\overline{x}+t_i\overline{d}+t_i^2\overline{z})\leqslant 0 \quad \forall i>i_1.$$

So finally we get (2.11).

2) Secondly we prove that there exists a sufficiently large i_5 such that

$$f(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-f(\overline{x})\in -W^0 \quad \forall i>i_5. \tag{2.13}$$

By (2.10) and (2.9), there exists a sufficiently large i_6 such that

$$\begin{aligned} &f(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-f(\overline{x})-t_i\nabla f(\overline{x})\overline{d}\\ &\quad = t_i^2\big(\nabla f(\overline{x})\overline{z}+\frac{1}{2}\varphi(\overline{x};\overline{d})+\varepsilon_2(t_i)\big)\\ &\quad \in I(-W^0,\nabla f(\overline{x})\overline{d}) \quad \forall i>i_6, \end{aligned} \tag{2.14}$$

where $\varepsilon_2(t_i)\to 0$ as $i\to\infty$.

(i) Let $p_j\in P_1(\overline{x};\overline{d})$. Since $W^0\neq\emptyset$, then we know from Lemma 2.1 (i) that the cone $I^*(-W^0,\nabla f(\overline{x})\overline{d})$ is acute. By Lemma 2.1 (ii) and (2.14) we have

$$\begin{aligned} &p_j^T\big(f(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-f(\overline{x})\big)\\ &\quad = p_j^T\big(f(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-f(\overline{x})-t_i\nabla f(\overline{x})\overline{d}\big)<0 \quad \forall i>i_6. \end{aligned}$$

(ii) Let $p_j\in P_2(\overline{x};\overline{d})$. Then we see that $p_j^T\nabla f(\overline{x})\overline{d}<0$ and thus there exists a sufficiently large i_7 such that

$$p_j^T\big(f(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-f(\overline{x})\big)<0 \quad \forall i>i_7.$$

Put $i_5=\max\{i_6,i_7\}$. Then for any $p_j\in P(\overline{x};\overline{d})$, by (2.4) and (2.5) we get

$$p_j^T\big(f(\overline{x}+t_i\overline{d}+t_i^2\overline{z})-f(\overline{x})\big)<0 \quad \forall i>i_5.$$

Consequently, (2.13) is valid.

Now, by setting $i_0=\max\{i_1,i_5\}$ and combining 1) with 2), we obtain

$$\overline{x}+t_i\overline{d}+t_i^2\overline{z}\in\mathscr{R} \quad \forall i>i_0$$

and

$$f(\overline{x})\in f(\overline{x}+t_i\overline{d}+t_i^2\overline{z})+W^0 \quad \forall i>i_0,$$

which contradicts the fact that $\overline{x}\in\mathscr{R}$ is a local nondominated solution associated with W for problem (2.1). $\square$

Lemma 2.4. *Let A be an $m \times n$ matrix, B an $l \times n$ matrix, U_1, U_2 nonempty convex cones in E_m, E_l, respectively, $b_1 \in E_m$ and $b_2 \in E_l$. If the system*

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} z \\ z \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \tag{2.15}$$

has no solution then there exist $p \in U_1^*$ and $q \in U_2^*$ such that $\begin{bmatrix} p \\ q \end{bmatrix} \neq 0$,

$$p^T A + q^T B = 0, \qquad p^T b_1 + q^T b_2 \geqslant 0.$$

Proof. Similarly to [8] we set

$$V = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \,\middle|\, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} z \\ z \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \ z \in E_n \right\} \quad \text{and} \quad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}.$$

Obviously, V is a convex set in E_{m+l} and U is a convex cone in E_{m+l}.

Suppose that system (2.15) has no solution. Then $V \cap U = \emptyset$. By the separation theorem for disjoint convex sets, there exist $p \in E_m$ and $q \in E_l$ such that $\begin{bmatrix} p \\ q \end{bmatrix} \neq 0$ and

$$(p^T, q^T) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \geqslant (p^T, q^T) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \forall \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in V \quad \forall \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in U,$$

i.e.,

$$p^T(Az + b_1) + q^T(Bz + b_2) \geqslant p^T u_1 + q^T u_2 \quad \forall z \in E_n \quad \forall u_1 \in U_1 \quad \forall u_2 \in U_2. \tag{2.16}$$

Letting $u_1 \to 0$ and $u_2 \to 0$, we get $(p^T A + q^T B)z + p^T b_1 + q^T b_2 \geqslant 0$.

Since z is arbitrary, we have

$$p^T A + q^T B = 0, \quad p^T b_1 + q^T b_2 \geqslant 0.$$

Moreover, $u_1 \in U_1$ and $u_2 \in U_2$ are also arbitrary and thus from (2.16) we find by a contradiction argument that

$$p^T u_1 \leqslant 0 \quad \forall u_1 \in U_1, \quad q^T u_2 \leqslant 0 \quad \forall u_2 \in U_2.$$

Hence, $p \in U_1^*$ and $q \in U_2^*$. $\square$

In the next theorem we establish necessary conditions for the existence of nondominated solutions.

Theorem 3.1. *Let* (A) *hold, let the cone* $W \subset E_m$ *be acute,* $f \in C^{1,1}(S)$ *and* $g \in C^{1,1}(S)$. *If* $\overline{x} \in \mathscr{R}$ *is a local nondominated solution associated with* W *for problem* (2.1), *then for any direction* $d \in D_L(\overline{x})$ *and any* $\gamma(\overline{x}; d) = \begin{bmatrix} \varphi(\overline{x}; d) \\ \psi(\overline{x}; d) \end{bmatrix} \in \partial_*^2 L(\overline{x})(d, d)$ *there exists a vector* $\begin{bmatrix} \overline{p} \\ \overline{q} \end{bmatrix} \neq 0$ *satisfying*

$$\overline{p} \in -W^*, \quad \overline{q} \in K^*, \tag{3.1}$$

$$\overline{p}^T \nabla f(\overline{x}) + \overline{q}^T \nabla g(\overline{x}) = 0, \tag{3.2}$$

$$\overline{q}^T g(\overline{x}) = 0, \tag{3.3}$$

$$\overline{p}^T \nabla f(\overline{x}) d = 0, \quad \overline{q}^T \nabla g(\overline{x}) d = 0, \tag{3.4}$$

$$\overline{p}^T \varphi(\overline{x}; d) + \overline{q}^T \psi(\overline{x}; d) \geqslant 0. \tag{3.5}$$

Proof. Let $d \in D_L(\overline{x})$ and $\gamma(\overline{x}; d) = \begin{bmatrix} \varphi(\overline{x}; d) \\ \psi(\overline{x}; d) \end{bmatrix} \in \partial_*^2 L(\overline{x})(d, d)$. Then by Lemma 2.3 we have $Z_L(\overline{x}; \varphi, \psi, d) = \emptyset$, i.e., the system

$$\begin{bmatrix} \nabla f(\overline{x}) & 0 \\ 0 & \nabla g(\overline{x}) \end{bmatrix} \begin{bmatrix} z \\ z \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \varphi(\overline{x}; d) \\ \psi(\overline{x}; d) \end{bmatrix} \in \begin{bmatrix} I(-W^0, \nabla f(\overline{x}) d) \\ I\bigl(I(K^0, g(\overline{x})), \nabla g(\overline{x}) d\bigr) \end{bmatrix}$$

has no solution. From Lemma 2.4 we see that there exist $\overline{p} \in I^*\bigl(-W^0, \nabla f(\overline{x}) d\bigr)$, $\overline{q} \in I^*\bigl(I(K^0, g(\overline{x})), \nabla g(\overline{x}) d\bigr)$ and $\begin{bmatrix} \overline{p} \\ \overline{q} \end{bmatrix} \neq 0$ such that

$$\overline{p}^T \nabla f(\overline{x}) + \overline{q}^T \nabla g(\overline{x}) = 0,$$
$$\overline{p}^T \varphi(\overline{x}; d) + \overline{q}^T \psi(\overline{x}; d) \geqslant 0.$$

Moreover, since

$$I^*\bigl(-W^0, \nabla f(\overline{x}) d\bigr) = \{p \in -W^* \mid p^T \nabla f(\overline{x}) d = 0\},$$
$$I^*\bigl(I(K^0, g(\overline{x})), \nabla g(\overline{x}) d\bigr) = \{q \in K^* \mid q^T g(\overline{x}) = 0,\ q^T \nabla g(\overline{x}) d = 0\}$$

hold due to (2.2) and (2.3), we get (3.1)–(3.5). □

If f and g are twice continuously differentiable, we immediately obtain Theorem 1 in [5] from Theorem 3.1 above:

Corollary 3.2. *Under the assumptions of Theorem* 3.1, *if* $f \in C^2(S)$ *and* $g \in C^2(S)$ *then for any* $d \in D_L(\overline{x})$ *there exists a vector* $\begin{bmatrix} \overline{p} \\ \overline{q} \end{bmatrix} \neq 0$ *satisfying* (3.1)–(3.4) *and*

$$\overline{p}^T \mathbb{D}^T \nabla^2 f(\overline{x}) D + \overline{q}^T \widetilde{\mathbb{D}}^T \nabla^2 g(\overline{x}) \widetilde{D} \geqslant 0,$$

where

$$\widetilde{\mathbb{D}} = \begin{bmatrix} d & & 0 \\ & \ddots & \\ 0 & & d \end{bmatrix}_{ln\times l}, \quad \widetilde{D} = \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix}_{ln\times 1}.$$

Further, we introduce sufficient conditions for nondominated solutions.

Theorem 3.3. *Let* (A) *hold,* $f \in C^{1,1}(S)$, $g \in C^{1,1}(S)$ *and* $\overline{x} \in \mathscr{R}$. *If for any* $d \in D_L(\overline{x})$, $d \neq 0$, *and for any* $\gamma(\overline{x}; d) = \begin{bmatrix} \varphi(\overline{x}; d) \\ \psi(\overline{x}; d) \end{bmatrix} \in \partial^2_* L(\overline{x})(d, d)$ *there exists a vector* $\begin{bmatrix} \overline{p} \\ \overline{q} \end{bmatrix} \neq 0$ *satisfying* (3.1)–(3.4) *and*

$$\overline{p}^T \varphi(\overline{x}; d) + \overline{q}^T \psi(\overline{x}; d) > 0, \tag{3.6}$$

then $\overline{x}$ *is a local nondominated solution associated with* W *for problem* (2.1).

P r o o f. Suppose $\overline{x}$ is not a local nondominated solution associated with W for problem (2.1). Then there exists a sequence $\{x^i\} \subset \mathscr{R}$, $x^i \to \overline{x}$ as $i \to \infty$, such that for all i,

$$f(\overline{x}) \in f(x^i) + W, \qquad f(\overline{x}) \neq f(x^i). \tag{3.7}$$

Without loss of generality, we may suppose

$$x^i = \overline{x} + t_i d_i,$$

where $t_i \in E_1$, $t_i \to 0^+$, $\|d_i\| = 1$, $d_i \to d$ as $i \to \infty$.

Since $L \in C^{1,1}(S)$, it is easy to prove like in (1.4) that the sequence

$$\{2t_i^{-2}\big(L(\overline{x} + t_i d_i) - L(\overline{x}) - t_i \nabla L(\overline{x}) d_i\big)\}_{i=1}^{\infty} \tag{3.8}$$

is bounded. So, there exists a convergent subsequence and thus we will assume for simplicity that the whole sequence (3.8) is convergent. Denote its limit by $\overline{\gamma}(\overline{x}; d)$.

Now we prove that

$$\overline{\gamma}(\overline{x}; d) \in \partial^2_* L(\overline{x})(d, d). \tag{3.9}$$

By (3.8), the definition of $\overline{\gamma}(\overline{x}; d)$ and the mean-value theorem, we see that there exists a sequence $\varepsilon_i \to 0$ as $i \to \infty$ such that

$$\begin{aligned}\frac{1}{2}\overline{\gamma}(\overline{x}; d) &= t_i^{-2}\big(L(\overline{x} + t_i d_i) - L(\overline{x}) - t_i \nabla L(\overline{x}) d_i\big) + \varepsilon_i \\ &= t_i^{-2}\int_0^1 \big(t_i \nabla L(\overline{x} + s t_i d_i) d_i - t_i \nabla L(\overline{x}) d_i\big)\,\mathrm{d}s + \varepsilon_i \\ &= t_i^{-1}\int_0^1 \big(\nabla L(\overline{x} + s t_i d_i) - \nabla L(\overline{x})\big)(d_i - d)\,\mathrm{d}s \\ &\quad + t_i^{-1}\int_0^1 \big(\nabla L(\overline{x} + s t_i d_i) - \nabla L(\overline{x} + s t_i d)\big) d\,\mathrm{d}s \\ &\quad + t_i^{-1}\int_0^1 \big(\nabla L(\overline{x} + s t_i d) - \nabla L(\overline{x})\big) d\,\mathrm{d}s + \varepsilon_i.\end{aligned} \tag{3.10}$$

Recall that ∇L is locally Lipschitz continuous and $\|d_i\| = \|d\| = 1$. Consequently, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}&\left|t_i^{-1}\int_0^1 \big(\nabla L(\overline{x} + s t_i d_i) - \nabla L(\overline{x})\big)(d_i - d)\,\mathrm{d}s\right| \\ &\quad\leqslant t_i^{-1}\int_0^1 \|\nabla L(\overline{x} + s t_i d_i) - \nabla L(\overline{x})\|\,\|d_i - d\|\,\mathrm{d}s \\ &\quad\leqslant t_i^{-1}\int_0^1 C_1 s t_i \|d_i\|\,\|d_i - d\|\,\mathrm{d}s = C_1\|d_i - d\| \to 0 \quad \text{as} \quad i \to \infty\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}&\left|t_i^{-1}\int_0^1 \big(\nabla L(\overline{x} + s t_i d_i) - \nabla L(\overline{x} + s t_i d)\big) d\,\mathrm{d}s\right| \\ &\quad\leqslant t_i^{-1}\int_0^1 \|\nabla L(\overline{x} + s t_i d_i) - \nabla L(\overline{x} + s t_i d)\|\,\|d\|\,\mathrm{d}s \\ &\quad\leqslant C_2\|d_i - d\| \to 0 \qquad \text{as} \quad i \to \infty.\end{aligned}$$

From this, (3.10), (3.11) and the mean-value theorem we have

$$\frac{1}{2}\overline{\gamma}(\overline{x}; d) = t_i^{-2}\big(L(\overline{x} + t_i d) - L(\overline{x}) - t_i \nabla L(\overline{x}) d\big) + \varepsilon_i', \tag{3.12}$$

where $\varepsilon_i' \to 0$ as $i \to \infty$. So we see from (3.12) and (1.3) that (3.9) holds. Hence, for

$$\overline{\gamma}(\overline{x}; d) = \begin{bmatrix} \varphi(\overline{x}; d) \\ \psi(\overline{x}; d) \end{bmatrix},$$

we get

$$\varphi(\overline{x};d) \in \partial_*^2 f(\overline{x})(d,d), \tag{3.13}$$

$$\psi(\overline{x};d) \in \partial_*^2 g(\overline{x})(d,d). \tag{3.14}$$

Now we check whether

$$d \in D_L(\overline{x}). \tag{3.15}$$

From (3.14) we find that for any i there exists an ε_g such that

$$\frac{1}{2}\psi(\overline{x};d) = t_i^{-2}\big(g(\overline{x}+t_i d) - g(\overline{x}) - t_i \nabla g(\overline{x})d\big) + \varepsilon_g, \tag{3.16}$$

where $\varepsilon_g \to 0$ as $i \to \infty$. Moreover, since $K^0 \neq \emptyset$ and $\overline{x} \in \mathscr{R}$, we get by (3.16) and Lemma 2.2 that

$$t_i\big(\nabla g(\overline{x})d + \frac{1}{2}t_i\psi(\overline{x};d) - t_i\varepsilon_g\big) = g(\overline{x}+t_i d) - g(\overline{x}) \in I(K^0, g(\overline{x})).$$

The positivity of t_i implies that

$$\nabla g(\overline{x})d + \frac{1}{2}t_i\psi(\overline{x};d) - t_i\varepsilon_g \in I(K^0, g(\overline{x})), \tag{3.17}$$

and since $t_i \to 0^+$ for $i \to \infty$, we obtain

$$\nabla g(\overline{x})d \in \operatorname{cl} I(K^0, g(\overline{x})). \tag{3.18}$$

Similarly to (3.17) and using also (3.7), we find that

$$\nabla f(\overline{x})d + \frac{1}{2}t_i\varphi(\overline{x};d) - t_i\varepsilon_f \in -W, \tag{3.19}$$

where $\varepsilon_f \to 0$ as $i \to \infty$. Therefore,

$$\nabla f(\overline{x})d \in -\operatorname{cl} W. \tag{3.20}$$

In view of (3.18) and (3.20) we get (3.15).

By the assumptions of the theorem, there exists a vector $\begin{bmatrix}\overline{p}\\ \overline{q}\end{bmatrix} \neq 0$ satisfying (3.1)–(3.4). As $\overline{p} \in -W^*$ and $\overline{q} \in K^*$ then like in (2.2) we get $\overline{q} \in I^*(K^0, g(\overline{x}))$ by (3.3) and Lemma 2.2. Using (3.17), (3.19) and (3.2), we have

$$\overline{p}^T\varphi(\overline{x};d) + \overline{q}^T\psi(\overline{x};d) - 2(\varepsilon_f + \varepsilon_g) \leqslant 0.$$

Letting $i \to \infty$, we come to

$$\overline{p}^T\varphi(\overline{x};d) + \overline{q}^T\psi(\overline{x};d) \leqslant 0,$$

which contradicts (3.6). $\square$

In the case that both functions f and g are twice continuously differentiable, we have Theorem 3 in [5] from Theorem 3.3 as:

Corollary 3.4. *Under the assumptions of Theorem* 3.3, *if* $f \in C^2(S)$, $g \in C^2(S)$, $\overline{x} \in \mathscr{R}$, *and if for any* $d \in D_L(\overline{x})$, $d \neq 0$, *there exists* $\begin{bmatrix} \overline{p} \\ \overline{q} \end{bmatrix} \neq 0$ *satisfying* (3.1)–(3.4) *and*

$$\overline{p}^T \mathbb{D}^T \nabla^2 f(\overline{x}) D + \overline{q}^T \widetilde{\mathbb{D}}^T \nabla^2 g(\overline{x}) \widetilde{D} > 0,$$

then $\overline{x}$ *is a local nondominated solution associated with* W *for problem* (2.1).

4. Second-order optimality conditions for Pareto optimal solutions

In this section we set $W = E_m^+$ and $K = -E_l^+$. Then we may write (2.1) as

$$\begin{cases} \min\big(f_1(x), \ldots, f_m(x)\big) \\ \mathscr{R} = \{x \in S \mid g_j(x) \leqslant 0,\ j = 1, \ldots, l\}, \end{cases} \tag{4.1}$$

and

$$D_L(\overline{x}) = \{d \in E_n \mid \nabla f_i(\overline{x})^T d \leqslant 0\ \forall i = 1, \ldots, m;\ \nabla g_j(\overline{x})^T d \leqslant 0\ \forall j \in J(\overline{x})\},$$

where $J(\overline{x}) = \{j \in \{1, \ldots, l\} \mid g_j(\overline{x}) = 0\}$. From Theorems 3.1 and 3.3 we easily get the second-order optimality conditions for local Pareto optimal solutions.

Theorem 4.1. *Let* $f_i \in C^{1,1}(S)$, $i = 1, \ldots, m$, *and* $g_j \in C^{1,1}(S)$, $j = 1, \ldots, l$. *If* $\overline{x} \in \mathscr{R}$ *is a local Pareto optimal solution for problem* (4.1) *then for any direction* $d \in D_L(\overline{x})$ *and any* $\gamma(\overline{x}; d) = \big(\varphi_1(\overline{x}; d), \ldots, \varphi_m(\overline{x}; d), \psi_1(\overline{x}; d), \ldots, \psi_l(\overline{x}; d)\big)^T \in \partial_*^2 L(\overline{x})(d, d)$ *there exist* $0 \leqslant \lambda \in E_m$ *and* $0 \leqslant \mu \in E_l$, $\begin{bmatrix} \lambda \\ \mu \end{bmatrix} \neq 0$, *such that*

$$\sum_{i=1}^{m} \lambda_i \nabla f_i(\overline{x}) + \sum_{j=1}^{l} \mu_j \nabla g_j(\overline{x}) = 0, \tag{4.2}$$

$$\mu_j g_j(\overline{x}) = 0 \quad \forall j = 1, \ldots, l, \tag{4.3}$$

$$\sum_{i=1}^{m} \lambda_i \nabla f_i(\overline{x})^T d = 0, \quad \sum_{j=1}^{l} \mu_j \nabla g_j(\overline{x})^T d = 0, \tag{4.4}$$

$$\sum_{i=1}^{m} \lambda_i \varphi_i(\overline{x}; d) + \sum_{j=1}^{l} \mu_j \psi_j(\overline{x}; d) \geqslant 0. \tag{4.5}$$

Theorem 4.2. *Let $f_i \in C^{1,1}(S)$, $i = 1, \ldots, m$, and $g_j \in C^{1,1}(S)$, $j = 1, \ldots, l$. If $\overline{x} \in \mathscr{R}$, and if for any $\gamma(\overline{x}; d) = \big(\varphi_1(\overline{x}; d), \ldots, \varphi_m(\overline{x}; d), \psi_1(\overline{x}; d), \ldots, \psi_l(\overline{x}; d)\big)^T \in \partial^2_* L(\overline{x})(d, d)$ and for any $d \in D_L(\overline{x})$, $d \neq 0$, there exist $0 \leqslant \lambda \in E_m$ and $0 \leqslant \mu \in E_l$, $\begin{bmatrix} \lambda \\ \mu \end{bmatrix} \neq 0$, satisfying* (4.2)–(4.4) *and*

$$\sum_{i=1}^{m} \lambda_i \varphi_i(\overline{x}; d) + \sum_{j=1}^{l} \mu_j \psi_j(\overline{x}; d) > 0, \tag{4.6}$$

then $\overline{x}$ is a local Pareto optimal solution for problem (4.1).

R e m a r k 4.3. If f and g are twice continuously differentiable, then formula (4.5) takes the form

$$d^T \left(\sum_{i=1}^{m} \lambda_i \nabla^2 f_i(\overline{x}) + \sum_{j=1}^{l} \mu_j \nabla^2 g_j(\overline{x}) \right) d \geqslant 0 \tag{4.7}$$

and formula (4.6) also reduces to (4.7) with strict inequality. Similar statements for C^2 data are given in [16].

Acknowledgement. The authors wish to thank Kaisa Miettinen for fruitful discussions.

References

[1] *M. S. Bazaraa, C. M. Shetty*: Foundations of Optimization. LN in Econom. and Math. Systems, vol. 122, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[2] *A. Ben-Tal*: Second-order and related extremality conditons in nonlinear programming. J. Optim. Theory Appl. *31* (1980), 143–165.
[3] *A. Cambini, L. Martein and R. Cambini*: A New Approach to Second Order Optimality Conditions in Vector Optimization. Advances in Multiple Objective and Goal Programming, LN in Econom. and Math. Systems, vol. 455, Springer, Berlin, 1997.
[4] *J. B. Hiriart-Urruty, J. J. Strodiot and V. H. Nguyen*: Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data. Appl. Math. Optim. *11* (1984), 43–56.
[5] *S. Huang*: Second-order conditions for nondominated solution in generalized multiobjective mathematical programming. J. Systems Sci. Math. Sci. *5* (1985), 172–184. (In Chinese.)
[6] *D. Klatte, K. Tammer*: On second-order sufficient optimality conditions for $C^{1,1}$ optimization problems. Optimization *19* (1988), 169–179.
[7] *M. Křížek, P. Neittaanmäki*: Finite Element Approximation of Variational Problems and Applications. Longman, Harlow, 1990.
[8] *L. Liu*: The second-order conditions of nondominated solutions for $C^{1,1}$ generalized multiobjective mathematical programming. J. Systems Sci. Math. Sci. *4* (1991), 128–138.

[9] *L. Liu*: The second order conditions for $C^{1,1}$ nonlinear mathematical programming In: Proc. Prague Math. Conf. 96, Math. Inst. Acad. Sci., Prague, 1996. pp. 153–158.

[10] *L. Liu, M. Křížek*: The second order optimality conditions for nonlinear mathematical programming with $C^{1,1}$ data. Appl. Math. *42* (1997), 311–320.

[11] *D. T. Luc*: Taylor's formula for $C^{k,1}$ functions. SIAM J. Optim. *5* (1995), 659–669.

[12] *Ch. Malivert*: First and second order optimality conditions in vector optimization. Ann. Sci. Math. Québec *14* (1990), 65–79.

[13] *G. P. McCormick*: Second order conditions for constrained minima. SIAM. J. Appl. Math. *15* (1967), 641–652.

[14] *K. Miettinen*: Nonlinear Multiobjective Optimization. Kluwer, Dordrecht, 1998.

[15] *S. Sáks*: Theory of the Integral. Hafner Publishing Co., New York, 1937.

[16] *S. Wang*: Second-order necessary and sufficient conditions in multiobjective programming. Numer. Funct. Anal. Optim. *12* (1991), 237–252.

[17] *D. E. Ward*: Characterizations of strict local minima and necessary conditions for weak sharp minima. J. Optim. Theory Appl. *80* (1994), 551–571.

[18] *D. E. Ward*: A comparison of second-order epiderivatives: calculus and optimality conditions. J. Math. Anal. Appl. *193* (1995), 465–482.

[19] *X. Q. Yang*: Generalized second-order derivatives and optimality conditions. Nonlinear Anal. *23* (1994), 767–784.

[20] *X. Q. Yang, V. Jeyakumar*: Generalized second-order directional derivatives and optimization with $C^{1,1}$ functions. Optimization *26* (1992), 165–185.

[21] *X. Q. Yang, V. Jeyakumar*: First and second-order optimality conditions for convex composite multiobjective optimization. J. Optim. Theory Appl. *95* (1997), 209–224.

[22] *P. L. Yu*: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjective. J. Optim. Theory Appl. *17* (1974), 320–377.

[23] *P. L. Yu*: Multiple-Criteria Decision Making: Concepts, Techniques and Extensions. Plenum Press, New York, 1985.

Authors' addresses: *Liping Liu*, Fields Institute, 222 College Street, Toronto, Ontario, M5T 3J1 Canada, e-mail: `lipingl@unbsj.ca`, *Pekka Neittaanmäki*, Laboratory of Scientific Computing, Department of Mathematics, University of Jyväskylä, P. O. Box 35, FIN–40351, Jyväskylä, Finland, e-mail: `pn@math.jyu.fi`, *Michal Křížek*, Mathematical Institute, Academy of Sciences, Žitná 25, CZ-115 67 Prague 1, Czech Republic, e-mail: `krizek@math.cas.cz`.