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FINITE ELEMENT ANALYSIS OF A STATIC CONTACT PROBLEM WITH COULOMB FRICTION*

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Abstract. A unilateral contact problem with a variable coefficient of friction is solved by a simplest variant of the finite element technique. The coefficient of friction may depend on the magnitude of the tangential displacement. The existence of an approximate solution and some a priori estimates are proved.

Keywords: unilateral contact, Coulomb friction, finite elements, existence proofs

MSC 2000: 65N30, 73T05

Introduction

The problem of a unilateral contact with Coulomb friction attracted attention of many research workers both in engineering and mathematics. Among the numerous literature we have chosen the paper by Licht, Pratt and Raous [7], who proposed an efficient approximate method of solution on the basis of a simplest variant of the finite element method. They justified the method by numerical experiments and presented some theoretical numerical analysis, namely the proof of existence of a solution and some conditions guaranteeing its uniqueness. They restricted themselves, however, to a constant coefficient $\mathscr{F}$ of the Coulomb friction. See also the papers by Haslinger [5], [6] for similar results.

The aim of the present paper is to extend the above-mentioned results to the cases when the coefficient $\mathscr{F}$ is not constant, but depends on (i) the place ($\mathscr{F} =$

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$\mathscr{F}(x)$) or (ii) on the place and on the magnitude of the tangential displacement, i.e. $\mathscr{F} = \mathscr{F}(x, |u_T|)$.

The first section contains the definition of a continuous unilateral problem of contact with a variable coefficient of friction. In the second section an approximate problem is formulated by means of a simple finite element technique. We prove the existence of an approximate solution and some a priori estimates for the case $\mathscr{F} = \mathscr{F}(x)$. The proof is based on a fixed point theorem, like in [7] for $\mathscr{F} = \text{const.}$ The uniqueness is guaranteed if the ratio $\|\mathscr{F}\|_\infty^2/h_0$ is sufficiently small. (Here $\|\cdot\|_\infty$ is the standard norm in $C(\Gamma_C)$ and h_0 is the norm of the triangulation near the contact boundary Γ_C.)

The third section contains a proof of the existence theorem and some a priori estimates for the case $\mathscr{F} = \mathscr{F}(x, |u_T|)$. We employ the same method of proof as that used by Eck and Jarušek in [2], [3], i.e., a penalization and regularization, followed by a successive limiting process.

1. Setting of a continuous contact problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a polyhedral domain with Lipschitz boundary $\partial\Omega$. Assume that

$$\partial\Omega = \Gamma_U \cup \Gamma_F \cup \Gamma_C$$

is a mutually disjoint partition, Γ_U, Γ_F, Γ_C are of positive surface measure. Moreover, let Γ_C be an open subset of a straight line or of a plane

$$\{x\colon\ x = (x_1, \ldots, x_{d-1}, 0)\}.$$

Let the body occupying the domain Ω be elastic, so that the stress-strain relations are

$$\sigma_{ij} = a_{ijkl}\, e_{kl}, \tag{1.1}$$

where

$$e_{km} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k}\right)$$

and u is the displacement vector,

$$a_{ijkl} = a_{jikl} = a_{klij} \in L_\infty(\Omega),$$
$$a_{ijkm}\, \tau_{ij}\, \tau_{km} \geqslant \alpha_0\, \tau_{ij}\, \tau_{ij} \text{ for all symmetric } \tau_{ij} \text{ and a.a. } x \in \Omega,$$

with some positive α_0. Here we use the summation convention for repeated indices within the range $\{1, \ldots, d\}$.

The equations of equilibrium are

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \quad \text{in } \Omega, \quad 1 \leqslant i \leqslant d, \tag{1.2}$$

where $f \in [L_2(\Omega)]^d$ are given body forces. We consider the boundary conditions

$$\begin{aligned} u &= 0 \quad \text{on } \Gamma_U, \\ \sigma_{ij} n_j &= (T_0)_i, \quad 1 \leqslant i \leqslant d \quad \text{on } \Gamma_F, \end{aligned}$$

where $T_0 \in [L_2(\Gamma_F)]^d$ are given surface tractions and $\boldsymbol{n}$ denotes the unit outward normal vector.

On the part Γ_C a unilateral contact with friction is considered:

$$u_N \leqslant 0, \quad \sigma_N \leqslant 0, \quad u_N \sigma_N = 0 \tag{1.3}$$

$$\begin{aligned} &|\sigma_T| \leqslant \mathscr{F}(u_T)|\sigma_N|, \\ &u_T = 0 \Rightarrow |\sigma_T| < \mathscr{F}(0)|\sigma_N|, \\ &u_T \neq 0 \Rightarrow \sigma_T = -\mathscr{F}(u_T)|\sigma_N| u_T/|u_T|. \end{aligned} \tag{1.4}$$

Here

$$\begin{aligned} u_N &= u_i n_i, \quad u_{Ti} = u_i - u_N n_i, \\ \sigma_N &= \sigma_{ij} n_i n_j, \quad \sigma_{Ti} = \sigma_{ij} n_j - \sigma_N n_i, \quad 1 \leqslant i \leqslant d; \end{aligned}$$

$\mathscr{F}$ is the coefficient of the Coulomb friction, such that $\mathscr{F}(u_T) \equiv \mathscr{F}(x, |u_T|)$ is a bounded nonnegative function on $\Gamma_C \times [0, \infty)$ and $\mathscr{F}(x, \cdot)$ is Lipschitz continuous for almost all $x \in \Gamma_C$ with a constant C_L independent of x; $\mathscr{F}(\cdot, \xi)$ has a compact support in Γ_C.

We define the subspace

$$\boldsymbol{V} = \{v \in [H^1(\Omega)]^d \colon\ v = 0 \text{ on } \Gamma_U\},$$

the subset

$$\boldsymbol{K} = \{v \in \boldsymbol{V} \colon\ v_N \leqslant 0 \text{ on } \Gamma_C\},$$

the bilinear form

$$a(u, v) = \int_\Omega a_{ijkm}\, e_{ij}(u)\, e_{km}(v)\, \mathrm{d}x$$

and the linear functional

$$L(v) = \int_\Omega f_i v_i \, \mathrm{d}x + \int_{\Gamma_F} T_{0i}\, v_i \, \mathrm{d}s.$$

If $\omega \in \boldsymbol{V}$, $\sigma_{ij}(\omega) = a_{ijkm}\, e_{km}(\omega)$ and $\partial\sigma_{ij}(\omega)/\partial x_j + f_i = 0$ in Ω, the Green formula enables us to define a functional $t(\omega) = t(\sigma(\omega)) \in \boldsymbol{H}^{-1/2}(\Gamma_C)$ as follows:

$$\langle\langle t(\omega), v\rangle\rangle = a(\omega, \boldsymbol{P}v) - L(\boldsymbol{P}v) \quad \forall v \in [H_0^{1/2}(\Gamma_C)]^d, \tag{1.5}$$

where $\boldsymbol{P}v \in \boldsymbol{V}$ is any extension of v such that $\boldsymbol{P}v = 0$ on Γ_F, and $H_0^{1/2}(\Gamma_C)$ is the subspace of traces of functions from $H^1(\Omega)$ vanishing on $\Gamma_U \cup \Gamma_F$.

If $\sigma_{ij}(\omega) \in H^1(\Omega)$, the standard formula for surface stress vector holds:

$$t_i(\omega) = \sigma_{ij}(\omega)n_j \in L_2(\Gamma_C), \quad 1 \leqslant i \leqslant d,$$

and $\langle\langle\cdot,\cdot\rangle\rangle$ reduces to the inner product in $[L_2(\Gamma_C)]^d$.

Finally, we define the normal component of the surface stress vector

$$\langle t_N(\omega), w\rangle = \langle\langle t(\omega), \boldsymbol{n}w\rangle\rangle \quad \forall w \in H_0^{1/2}(\Gamma_C). \tag{1.6}$$

The *weak solution* of the contact problem is a function $u \in \boldsymbol{K}$ such that

$$a(u, v-u) - \big\langle t_N(u), \mathscr{F}(u_T)(|v_T| - |u_T|)\big\rangle \geqslant L(v-u) \quad \forall v \in \boldsymbol{K}. \tag{1.7}$$

For the existence and regularity of a weak solution we refer to Eck and Jarušek [2], [3], who considered even more general domains Ω and functions $\mathscr{F}(x, |u_T|)$.

2. Approximate contact problem

We shall approximate the problem (1.7) by a simplest finite element technique, i.e., by means of linear simplicial elements.

Assume that $\{\mathscr{T}_h\}, h \to 0+$, is a quasi-uniform (strongly regular) family of triangulations of the domain Ω (see [1], (17.13) for the definition). We introduce the following finite element spaces on simplexes $T \in \mathscr{T}_h$:

$$\begin{aligned}
X_h &= \{w \in C(\overline{\Omega})\colon\ w|_T \in P_1(T) \quad \forall\, T \in \mathscr{T}_h\},\\
\boldsymbol{V}_h &= \{w \in [X_h]^d\colon\ w = 0 \text{ on } \Gamma_U\},\\
\boldsymbol{K}_h &= \{v \in \boldsymbol{V}_h\colon\ v_N \leqslant 0 \text{ on } \Gamma_C\},\\
\tilde{X}_h &= \{w|_{\Gamma_C}\colon\ w \in X_h,\ w = 0 \text{ on } \partial\Gamma_C\} = X_h|_{\Gamma_C} \cap H_0^{1/2}(\Gamma_C).
\end{aligned}$$

The following discrete analog of the definitions (1.5), (1.6) will be used:

$$\big\langle\big\langle t^h(u), \tilde{v}\big\rangle\big\rangle = a(u, \boldsymbol{R}\tilde{v}) - L(\boldsymbol{R}\tilde{v}), \quad \tilde{v} \in [\tilde{X}_h]^d, \quad u \in \boldsymbol{V}_h, \tag{2.1}$$

$$\big\langle t_N^h(u), \tilde{w}\big\rangle = \big\langle\big\langle t^h(u), \tilde{w}\boldsymbol{n}\big\rangle\big\rangle, \quad \tilde{w} \in \tilde{X}_h, \quad u \in \boldsymbol{V}_h, \tag{2.2}$$

where $\boldsymbol{R}\colon [\tilde{X}_h]^d \to \boldsymbol{V}_h$ is a linear mapping such that $\boldsymbol{R}\tilde{v}(a_i) = \tilde{v}(a_i)$ at the nodes $a_i \in \Gamma_C$ and $\boldsymbol{R}\tilde{v} = 0$ at the other nodes of the triangulation $\mathscr{T}_h$.

Let Π^h denote the Lagrange interpolation operator of X_h restricted to the part Γ_C of the boundary, $\Pi^h\colon\ C^0(\overline{\Gamma_C}) \to \tilde{X}_h$, where C^0 denotes the space of continuous functions vanishing on $\partial\Gamma_C$.

The *approximate solution* is a function $u^h \in \boldsymbol{K}_h$ such that

$$a(u^h, v - u^h) - \big\langle t_N^h(u^h), \Pi^h\big(\mathscr{F}(u_T^h)(|v_T| - |u_T^h|)\big)\big\rangle \geqslant L(v - u^h) \quad \forall v \in \boldsymbol{K}_h. \tag{2.3}$$

The main result of the section is represented by the following

Theorem 2.1. *There exists at least one approximate solution u^h of* (2.3)*. Positive constants C_0 and M exist, independent of $\mathscr{F}$ and such that*

$$\|u^h\|_{1,\Omega} \leqslant \|L\|_{-1}/C_0,$$
$$\|t_N^h(u^h)\|_* \leqslant M\|L\|_{-1}h_0^{-1/2},$$

where

$$C_0 = \inf_{v\in V\setminus\{0\}} \frac{a(v,v)}{\|v\|_{1,\Omega}^2},$$

$\|L\|_{-1}$ is the norm of L in the dual space $([H^1(\Omega)]^d)'$; $\|\cdot\|_*$ is the norm in $(\tilde{X}_h)'$;

$$\|g\|_* = \sup_{\tilde{v}\in\tilde{X}_h} \frac{\langle g, \tilde{v}\rangle}{\|\tilde{v}\|_{0,\Gamma_C}},$$
$$h_0 = \max_{T\subset \operatorname{supp} \boldsymbol{R}\tilde{v}} (\operatorname{diam} T).$$

Let $\mathscr{R}\colon\ \tilde{X}_h \to X_h$ be the extension determined by the nodal values of $\tilde{z} \in \tilde{X}_h$ on Γ_C and by zero values at the other nodes of $\mathscr{T}_h$.

Lemma 2.1. *There exists a positive constant $\hat{C}$, independent of h_0 and such that*

$$\|\mathscr{R}\tilde{z}\|_{0,\Omega} \leqslant \hat{C}h_0^{1/2}\|\tilde{z}\|_{0,\Gamma_C} \quad \forall\tilde{z} \in \tilde{X}_h. \tag{2.4}$$

P r o o f. (i) Let $d = 2$. Consider a triangle $T_1(a_1a_2a_3)$, $a_1 = (0,0)$, $a_2 = (a_{12}, 0)$, $a_3 = (a_{13}, a_{23})$ and the barycentric coordinates

$$\lambda_1 = 1 - \lambda_2 - \lambda_3, \quad \lambda_2 = (x_1 - a_{13}x_2/a_{23})/a_{12}, \quad \lambda_3 = x_2/a_{23}.$$

We find that

$$\int_{T_1} \lambda_i^2 \,\mathrm{d}x = \frac{1}{6} \operatorname{meas} T_1 = \frac{1}{4}\, a_{23} \int_0^{a_{12}} \tilde{\lambda}_i^2 \,\mathrm{d}x_1, \quad i = 1, 2, \tag{2.5}$$

where $\tilde{\lambda}_i = \lambda_i|_{x_2=0}$. Furthermore, we have

$$\int_{T_1} \lambda_1 \lambda_2 \,\mathrm{d}x = \frac{1}{4}\, a_{23} \int_0^{a_{12}} \tilde{\lambda}_1 \tilde{\lambda}_2 \,\mathrm{d}x_1 = \frac{1}{12} \operatorname{meas} T_1. \tag{2.6}$$

Consequently, we obtain for $\mathscr{R}\tilde{z} = z_1\lambda_1 + z_2\lambda_2$, $\tilde{z} = z_1\tilde{\lambda}_1 + z_2\tilde{\lambda}_2$

$$\int_{T_1} (\mathscr{R}\tilde{z})^2 \,\mathrm{d}x = \frac{1}{4}\, a_{23} \int_0^{a_{12}} \tilde{z}^2 \,\mathrm{d}x_1. \tag{2.7}$$

For the adjacent triangle $T_2(a_1a_3a_4)$ (with $a_{24} > 0$) we derive

$$\int_{T_2} (\mathscr{R}\tilde{z})^2 \,\mathrm{d}x = \int_{T_2} z_1^2 \mu_1^2(x) \,\mathrm{d}x = \frac{1}{6}\, z_1^2 \operatorname{meas} T_2,$$

where μ_1 is a barycentric coordinate and (2.5) has been used. Since the family of triangulations is strongly regular,

$$\operatorname{meas} T_2 \leqslant C \operatorname{meas} T_1$$

holds with the constant C independent of h and therefore

$$\int_{T_2} (\mathscr{R}\tilde{z})^2 \,\mathrm{d}x \leqslant \frac{1}{6}\, z_1^2 C \operatorname{meas} T_1 \leqslant \tilde{C} z_1^2 a_{23} \int_0^{a_{12}} \tilde{\lambda}_1^2 \,\mathrm{d}x_1. \tag{2.8}$$

Due to the regularity of the family of triangulations, there exist at most M triangles with the vertex a_1, M being independent of h. Since $a_{23} \leqslant h_0$, adding the estimates of the type (2.7) and (2.8) we arrive at

$$\sum_j \int_{T_j} (\mathscr{R}\tilde{z})^2 \,\mathrm{d}x \leqslant h_0 \left(\frac{1}{4} + M\tilde{C} \right) \int_{\Gamma_C} \tilde{z}^2 \,\mathrm{d}x_1,$$

so that (2.4) follows.

(ii) $d = 3$. Consider a tetrahedron $T_1(a_1, a_2, a_3, a_4)$, where $a_1 = (0,0,0)$, $a_2 = (a_{12}, 0, 0)$, $a_3 = (a_{13}, a_{23}, 0)$, $a_4 = (a_{14}, a_{24}, a_{34})$, $a_{34} > 0$, $a_{12} > 0$. Using the barycentric coordinates λ_i, we derive

$$\int_{T_1} \lambda_i^2 \,\mathrm{d}x = \frac{1}{5}\, a_{34} \int_{\tilde{T}_1} \tilde{\lambda}_i^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2, \quad 1 \leqslant i \leqslant 3, \tag{2.9}$$

$$\int_{T_1} \lambda_i \lambda_j \,\mathrm{d}x = \frac{1}{5}\, a_{34} \int_{T_1} \tilde{\lambda}_i \tilde{\lambda}_j \,\mathrm{d}x_1 \,\mathrm{d}x_2, \quad i \neq j,\ 1 \leqslant i,\ j \leqslant 3, \tag{2.10}$$

where $\tilde{T}_1 = \tilde{T}_1(a_1, a_2, a_3)$. Then for $\mathscr{R}\tilde{z} = \sum\limits_{i=1}^{3} z_i\lambda_i$, $\tilde{z} = \sum\limits_{i=1}^{3} z_i\tilde{\lambda}_i$, $\tilde{\lambda}_i = \lambda_i|_{x_3=0}$ we obtain

$$\int_{T_1} (\mathscr{R}\tilde{z})^2 \,\mathrm{d}x = \frac{1}{5}\, a_{34} \int_{\tilde{T}_1} \tilde{z}^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2. \tag{2.11}$$

Next, let us consider the tetrahedron $T_2(a_2, a_3, a_4, b)$, where $b = (b_1, b_2, b_3)$, $b_3 > 0$. We may write

$$\int_{T_2} (\mathscr{R}\tilde{z})^2 \,\mathrm{d}x = z_2^2 \int_{T_2} \mu_2^2 \,\mathrm{d}x + z_3^2 \int_{T_2} \mu_3^2 \,\mathrm{d}x + 2z_2z_3 \int_{T_2} \mu_2\mu_3 \,\mathrm{d}x. \tag{2.12}$$

Using (2.9), we obtain

$$\int_{T_2} \mu_2^2 \,\mathrm{d}x \leqslant \frac{1}{5}\, h_0 \int_{\Delta} \tilde{\mu}_2^2 \,\mathrm{d}S, \quad \Delta = \Delta(a_2, a_3, a_4). \tag{2.13}$$

The results of part (i) and the definition of a strongly regular family of triangulations imply that

$$\int_{\Delta} \tilde{\mu}_2^2 \,\mathrm{d}S = \frac{1}{6} \operatorname{meas} \Delta \leqslant \frac{1}{12}\, h_0^2 = \tilde{C} \operatorname{meas} \tilde{T}_1 = C \int_{\tilde{T}_1} \tilde{\lambda}_2^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2.$$

Substituting this estimate into (2.13), we arrive at

$$\int_{T_2} \mu_2^2 \,\mathrm{d}x \leqslant \frac{1}{5}\, C h_0 \int_{\tilde{T}_1} \tilde{\lambda}_2^2 \,\mathrm{d}x_1 \,\mathrm{d}x_2. \tag{2.14}$$

In the same way we derive that

$$\int_{T_2} \mu_2\mu_3 \,\mathrm{d}x \leqslant \frac{1}{5}\, h_0 \int_{\Delta} \tilde{\mu}_2\tilde{\mu}_3 \,\mathrm{d}S = \frac{1}{60}\, h_0 \operatorname{meas} \Delta \leqslant \frac{1}{5}\, C h_0 \int_{\tilde{T}_1} \tilde{\lambda}_2\tilde{\lambda}_3 \,\mathrm{d}x_1 \,\mathrm{d}x_2. \tag{2.15}$$

There exist at most M tetrahedrons with the vertex a_i, $i = 1, 2, 3$, where M is independent of h. Combining the estimates (2.11), (2.12), (2.14) and (2.15), we are led to the estimate (2.4). □

The case $\mathscr{F} = \mathscr{F}(x)$.

First we introduce an auxiliary problem of unilateral contact with a given slip stress.

Let G be the set of *positive* linear functionals g on $\tilde{X}_h$. For any $g \in G$ let us define the problem $\mathbf{P}^{\mathrm{h}}_{\mathrm{g}}$ to find $u_g \in \boldsymbol{K}_h$ such that

$$a(u_g, v-u_g) + \big\langle g, \Pi^h\big(\mathscr{F}(|v_T| - |u_{gT}|)\big)\big\rangle \geqslant L(v-u_g) \quad \forall v \in \boldsymbol{K}_h. \tag{2.16}$$

Proposition 2.1. *The problem* $(\mathbf{P}^{\mathrm{h}}_{\mathrm{g}})$ *has a unique solution for any* $g \in G$.

Proof. Let us denote

$$J_1(u) = \big\langle g, \Pi^h(\mathscr{F}|u_T|)\big\rangle, \quad J_2(u) = \frac{1}{2}a(u,u) - L(u).$$

Since J_1 is convex, J_2 strictly convex and differentiable on $\boldsymbol{V}_h$, the inequality in $(\mathbf{P}^{\mathrm{h}}_{\mathrm{g}})$ is equivalent to the minimization of the sum $J = J_1 + J_2$ over the set $\boldsymbol{K}_h$.

We can show that the functional J_1 is Lipschitz continuous on $\boldsymbol{V}_h$, i.e.,

$$|J_1(u) - J_1(v)| \leqslant C_g \|\mathscr{F}\|_\infty \|u-v\|_{1,\Omega} \quad \forall u, v \in \boldsymbol{V}_h, \tag{2.17}$$

where $\|\cdot\|_\infty$ denotes the standard norm in $C(\overline{\Gamma_C})$.

Indeed, let $d = 2$. For any $v \in H^1(\Gamma_C)$ we have

$$\|\Pi^h v - v\|_{0,\Gamma_C} \leqslant C_\pi h_0 |v|_{1,\Gamma_C}$$

so that

$$\|\Pi^h v\|_{0,\Gamma_C} \leqslant C_\pi h_0 |v|_{1,\Gamma_C} + \|v\|_{0,\Gamma_C}. \tag{2.18}$$

We may write

$$\begin{aligned} |J_1(u) - J_1(v)| &\leqslant \|g\|_* \big\|\Pi^h\big(\mathscr{F}(|u_T| - |v_T|)\big)\big\|_{0,\Gamma_C} \\ &\leqslant \|g\|_* \|\mathscr{F}\|_\infty \big\|\Pi^h(|\,|u_T| - |v_T|\,|)\big\|_{0,\Gamma_C} \\ &\leqslant \|g\|_* \|\mathscr{F}\|_\infty \|\Pi^h(|w_T|)\|_{0,\Gamma_C} \end{aligned} \tag{2.19}$$

since

$$\big|\Pi^h\big(\mathscr{F}(|u_T| - |v_T|)\big)\big| \leqslant \|\mathscr{F}\|_\infty \Pi^h\big(\big||u_T| - |v_T|\big|\big) \leqslant \|\mathscr{F}\|_\infty \Pi^h(|w_T|),$$

where $w := u - v$. For $w_j \in X_h|_{\Gamma_C}$ the "inverse inequality"

$$\|w_j\|_{1,\Gamma_C} \leqslant C h_0^{-1} \|w_j\|_{0,\Gamma_C} \tag{2.20}$$

holds [1]. Using (2.18), (2.20) and the Trace Theorem, we obtain

$$(2.21)\qquad \begin{aligned}\|\Pi^h(|w_T|)\|_{0,\Gamma_C} &\leqslant C_\pi h_0 \big||w_1|\big|_{1,\Gamma_C} + \big\||w_1|\big\|_{0,\Gamma_C}\\ &\leqslant Ch_0\|w_1\|_{1,\Gamma_C} + \|w_1\|_{0,\Gamma_C}\\ &\leqslant \tilde{C}\|w_1\|_{0,\Gamma_C} \leqslant \tilde{C}C\|w\|_{1,\Omega}.\end{aligned}$$

Inserting (2.21) into (2.19), we arrive at (2.17).

Next, let $d = 3$. Let us consider

$$v := |w_j|, \quad w_j \in X_h|_{\Gamma_C}, \quad (j = 1, 2),$$

and realize that for any triangle $K \in \Gamma_C$ we may write (cf. [1], Theorem 3.16)

$$\text{(i)}\qquad \|\Pi_K v - v\|^2_{0,2,K} \leqslant C(\operatorname{meas} K)^{1-2/(2+\varepsilon)} h_K^2 |v|^2_{1,2+\varepsilon,K}, \quad \varepsilon > 0.$$

Since we have

$$\left|\frac{\partial w_j}{\partial x_i}\right| = \left|\frac{\partial |w_j|}{\partial x_i}\right| \quad \text{a.e. in } K \quad (i, j = 1, 2),$$

$$\text{(ii)}\qquad |v|^2_{1,2+\varepsilon,K} = |w_j|^2_{1,2+\varepsilon,K}$$

holds. By means of the "inverse assumption" (cf. [1], (3.2.33)), we may write

$$\text{(iii)}\qquad |w_j|^2_{1,2+\varepsilon,K} \leqslant C(h_0^2)^{2/(2+\varepsilon)-1} |w_j|^2_{1,2,K}.$$

Inserting (ii) and (iii) into (i), we obtain

$$\|\Pi_K v - v\|^2_{0,2,K} \leqslant Ch_K^2 |w_j|^2_{1,2,K}.$$

Summing over all $K \in \Gamma_C$, we arrive at the estimate

$$\|\Pi^h |w_j| - |w_j|\,\|_{0,\Gamma_C} \leqslant Ch_0 |w_j|_{1,\Gamma_C}, \quad j = 1, 2.$$

As a consequence, we have

$$\|\Pi^h |w_j|\,\|_{0,\Gamma_C} \leqslant \|w_j\|_{0,\Gamma_C} + Ch_0|w_j|_{1,\Gamma_C} \leqslant \tilde{C}\|w_j\|_{0,\Gamma_C}.$$

Since

$$\Pi^h(|w_T|) \leqslant \sum_{j=1}^{2} \Pi^h(|w_j|),$$

we obtain

$$\|\Pi^h(|w_T|)_{0,\Gamma_C} \leqslant \sum_{j=1}^{2} \|\Pi^h(|w_j|)\|_{0,\Gamma_C} \leqslant \tilde{C} \sum_{j=1}^{2} \|w_j\|_{0,\Gamma_C} \leqslant \tilde{C}C\|w\|_{1,\Omega}. \tag{2.21a}$$

Combining (2.21a) with (2.19), (2.17) follows.

As a consequence, the functional J is continuous and coercive on V_h by virtue of Korn's inequality and the non-negativeness of $J_1(u)$. Since the set $\boldsymbol{K}_h$ is convex and closed, a minimizer exists. The uniqueness follows from the fact that J_2 is strictly convex and J_1 is convex. □

Next let us define a mapping $\boldsymbol{T}\colon G \to (X_h)'$ by the formula

$$\boldsymbol{T}(g) = -t_N^h(u_g). \tag{2.22}$$

Lemma 2.2.

$$\boldsymbol{T}(G) \subset G.$$

P r o o f. Let $\tilde{w} \in \tilde{X}_h$, $\tilde{w} \geqslant 0$. We may write

$$\langle \boldsymbol{T}(g), \tilde{w}\rangle = \bigl\langle -t_N^h(u_g), \tilde{w}\bigr\rangle = a\bigl(u_g, \boldsymbol{R}(-\tilde{w}\boldsymbol{n})\bigr) - L\bigl(\boldsymbol{R}(-\tilde{w}\boldsymbol{n})\bigr). \tag{2.23}$$

If $v = u_g + \boldsymbol{R}(-\tilde{w}\boldsymbol{n})$, then $v \in \boldsymbol{K}_h$, since $(\boldsymbol{R}(-\boldsymbol{n}\tilde{w}))_N \leqslant 0$ on Γ_C. From the inequality $(\mathbf{P}_{\mathrm{g}}^{\mathrm{h}})$ we deduce

$$a(u_g, \boldsymbol{R}(-\tilde{w}\boldsymbol{n})) - L(\boldsymbol{R}(-\tilde{w}\boldsymbol{n})) \geqslant -\bigl\langle g, \Pi^h\bigl(\mathscr{F}(|u_{gT} + \boldsymbol{R}_T(-\tilde{w}\boldsymbol{n})| - |u_{gT}|)\bigr)\bigr\rangle = 0,$$

since $\boldsymbol{R}_T(-\tilde{w}\boldsymbol{n}) = 0$. Inserting this into (2.23), we obtain

$$\langle \boldsymbol{T}(g), \tilde{w}\rangle \geqslant 0.$$

□

Lemma 2.3. *The mapping $\boldsymbol{T}$ is Lipschitz continuous, i.e.,*

$$\|\boldsymbol{T}(g_2) - \boldsymbol{T}(g_1)\|_* \leqslant C h_0^{-1/2} \|\mathscr{F}\|_\infty \|g_2 - g_1\|_*,$$

where C is independent of h_0, $\mathscr{F}$, g_1, g_2.

P r o o f. Denote $u^1 := u_{g_1}$, $u^2 := u_{g_2}$ and choose an arbitrary $\tilde{w} \in \tilde{X}_h$. It is readily seen that

$$\bigl|\bigl\langle t_N^h(u^1) - t_N^h(u^2), \tilde{w}\bigr\rangle\bigr| = |a(u^1 - u^2, \boldsymbol{R}(\tilde{w}n))| \leqslant C_1 |u^1 - u^2|_{1,\Omega}\, |\mathscr{R}\tilde{w}|_{1,\Omega}, \tag{2.24}$$

since $n_j = 0$ and $\boldsymbol{R}_j(\tilde{w}n) = 0$ for $1 \leqslant j \leqslant d-1$, $n_d = -1$, $\boldsymbol{R}_d(\tilde{w}n) = -\mathscr{R}\tilde{w}$. Lemma 2.1 and the inverse inequality for elements of X_h yield

$$|\mathscr{R}\tilde{w}|_{1,\Omega} \leqslant C_2 h_0^{-1} \|\mathscr{R}\tilde{w}\|_{0,\Omega} \leqslant C_2 \hat{C} h_0^{-1/2} \|\tilde{w}\|_{0,\Gamma_C}. \tag{2.25}$$

Thus we have the following estimate from (2.24) and (2.25):

$$\|\boldsymbol{T}(g_1) - \boldsymbol{T}(g_2)\|_* \leqslant C_3 h_0^{-1/2} |u^1 - u^2|_{1,\Omega}. \tag{2.26}$$

On the other hand, the definition (2.16) and Korn's inequality imply

$$\begin{aligned} C_0 \|u^1 - u^2\|^2_{1,\Omega} &\leqslant a(u^1 - u^2, u^1 - u^2) \\ &\leqslant \big\langle g_1 - g_2, \Pi^h\big(\mathscr{F}(|u^2_T| - |u^1_T|)\big)\big\rangle \\ &\leqslant \|g_1 - g_2\|_* \big\|\Pi^h\big((|u^2_T| - |u^1_T|)\mathscr{F}\big)\big\|_{0,\Gamma_C}. \end{aligned} \tag{2.27}$$

Using (2.20) and (2.21) or (2.21a), we obtain

$$\big\|\Pi^h\big(\mathscr{F}(|u^2_T| - |u^1_T|)\big)\big\|_{0,\Gamma_C} \leqslant \|\mathscr{F}\|_\infty \|\Pi^h(|w_T|)\big\|_{0,\Gamma_C} \leqslant C\|\mathscr{F}\|_\infty \|u^2 - u^1\|_{1,\Omega}$$

so that (2.27) yields

$$C_0 \|u^2 - u^1\|_{1,\Omega} \leqslant C\|\mathscr{F}\|_\infty \|g_1 - g_2\|_*. \tag{2.28}$$

Combining (2.26) and (2.28), we arrive at

$$\|\boldsymbol{T}(g_1) - \boldsymbol{T}(g_2)\|_* \leqslant C_0^{-1} C \|\mathscr{F}\|_\infty h_0^{-1/2} \|g_1 - g_2\|_*.$$

□

Lemma 2.4. *There exists a constant $M > 0$, independent of h_0 and $\mathscr{F}$, such that*

$$\|\boldsymbol{T}(g)\|_* \leqslant M\|L\|_{-1} h_0^{-1/2} \quad \forall g \in G.$$

P r o o f. Setting $v := 0$ in the definition (2.16) and using Korn's inequality, we obtain

$$C_0 \|u_g\|^2_{1,\Omega} \leqslant a(u_g, u_g) \leqslant L(u_g) - \big\langle g, \Pi^h(\mathscr{F}|u_{gT}|)\big\rangle \leqslant L(u_g) \leqslant \|L\|_{-1} \|u_g\|_{1,\Omega}$$

so that

$$\|u_g\|_{1,\Omega} \leqslant C_0^{-1} \|L\|_{-1} \tag{2.29}$$

holds for all $g \in G$. We may write

$$(2.30) \qquad \begin{aligned} |\langle \boldsymbol{T}(g), \tilde{w}\rangle| &= |a(u_g, \boldsymbol{R}(\boldsymbol{n}\tilde{w})) - L(\boldsymbol{R}(\boldsymbol{n}\tilde{w}))| \\ &\leqslant C_1\|u_g\|_{1,\Omega}\,\|\mathscr{R}\tilde{w}\|_{1,\Omega} + \|L\|_{-1}\|\mathscr{R}\tilde{w}\|_{1,\Omega} \\ &\leqslant (C_0^{-1}C_1 + 1)\|L\|_{-1}\|\mathscr{R}\tilde{w}\|_{1,\Omega}. \end{aligned}$$

On the other hand,

$$\|\mathscr{R}\tilde{w}\|_{1,\Omega} \leqslant C_2 h_0^{-1}\|\mathscr{R}\tilde{w}\|_{0,\Omega} \leqslant C_2\hat{C}h_0^{-1/2}\|\tilde{w}\|_{0,\Gamma_C}$$

follows from the inverse inequality on the domain $\operatorname{supp}(\mathscr{R}\tilde{w})$ and from Lemma 2.1. Inserting this into (2.30), we arrive at

$$\|\boldsymbol{T}(g)\|_* \leqslant (1 + C_1/C_0)C_3 h_0^{-1/2}\|L\|^{-1}.$$

□

P r o o f o f T h e o r e m 2.1 i n c a s e $\mathscr{F} = \mathscr{F}(x)$. Let us denote

$$B(h_0) = \{g \in G\colon\ \|g\|_* \leqslant M\|L\|_{-1}h_0^{-1/2}\},$$

where the constant M is that of Lemma 2.4. Since the set $B(h_0)$ is bounded and closed in the dual space $(\tilde{X}_h)'$, $B(h_0)$ is compact and convex. By virtue of Lemma 2.3 the mapping $\boldsymbol{T}$ is continuous and $\boldsymbol{T}(B(h_0)) \subset B(h_0)$ holds by virtue of Lemma 2.4. As a consequence, the Brouwer Theorem yields the existence of a fixed point of $\boldsymbol{T}$.

It is easy to see that a solution of the problem (2.3) exists if and only if there exists a fixed point of T.

The a priori estimates of Theorem 2.1 follow from (2.29) and Lemma 2.4. □

Theorem 2.2. *There exists a positive constant C, independent of h_0, $\mathscr{F}$, and L such that the problem* (2.3) *has at most one solution provided*

$$h_0 > C\|\mathscr{F}\|_\infty^2.$$

P r o o f. If u and $\overline{u}$ are two solutions of (2.3), then

$$\begin{aligned} a(u, \overline{u} - u) - \left\langle t_N^h(u), \Pi^h\left(\mathscr{F}(|\overline{u}_T| - |u_T|)\right)\right\rangle &\geqslant L(\overline{u} - u), \\ a(\overline{u}, u - \overline{u}) - \left\langle t_N^h(\overline{u}), \Pi^h\left(\mathscr{F}(|u_T| - |\overline{u}_T|)\right)\right\rangle &\geqslant L(u - \overline{u}). \end{aligned}$$

By addition, we derive that

$$a(u - \overline{u}, \overline{u} - u) + \left\langle t_N^h(\overline{u}) - t_N^h(u), \Pi^h\left(\mathscr{F}(|\overline{u}_T| - |u_T|)\right)\right\rangle \geqslant 0.$$

By definitions (1.5), (1.6) we may therefore write

$$a(u-\overline{u},u-\overline{u}) \leqslant a\big(\overline{u}-u, \boldsymbol{R}\big(\boldsymbol{n}\Pi^h(\mathscr{F}(|\overline{u}_T|-|u_T|))\big)\big).$$

Denoting $w := \overline{u}-u$, we obtain

$$C_0\|w\|_{1,\Omega}^2 \leqslant C_1\|w\|_{1,\Omega}|\mathscr{U}_d|_{1,\Omega}, \tag{2.31}$$

where

$$\mathscr{U}_d = \mathscr{R}\big(\Pi^h(\mathscr{F}(|\overline{u}_T|-|u_T|))\big).$$

Since $\mathscr{U}_d \in X_h$, the inverse inequality and Lemma 2.1 imply

$$|\mathscr{U}_d|_{1,\Omega} \leqslant C_2 h_0^{-1}\|\mathscr{U}_d\|_{0,\Omega} \leqslant C_2\hat{C}h_0^{-1/2}\big\|\Pi^h\big(\mathscr{F}(|\overline{u}_T|-|u_T|)\big)\big\|_{0,\Gamma_C}. \tag{2.32}$$

Arguing as in the derivation of the estimates (2.20), (2.21), we obtain

$$\big\|\Pi^h\big(\mathscr{F}(|\overline{u}_T|-|u_T|)\big)\big\|_{0,\Gamma_C} \leqslant C_3\|\mathscr{F}\|_\infty\|w\|_{1,\Omega}. \tag{2.33}$$

Combining (2.31), (2.32) and (2.33), we arrive at

$$\|w\|_{1,\Omega} \leqslant C_0^{-1}C_1C_2\hat{C}C_3h_0^{-1/2}\|\mathscr{F}\|_\infty\|w\|_{1,\Omega}. \tag{2.34}$$

Let us denote $C_4 := C_0^{-1}C_1C_2\hat{C}C_3$ and assume that

$$C_4h_0^{-1/2}\|\mathscr{F}\|_\infty < 1. \tag{2.35}$$

Then $w=0$ follows from (2.34). □

Remark 2.1. It is easy to see that the mapping $\boldsymbol{T}$ defined by (2.22) is contractive if (2.35) holds. □

3. The case $\mathscr{F}=\mathscr{F}(x,|u_T|)$

Following the line of thoughts used by Eck and Jarušek in [2] and [3] for the continuous problem (1.7), we shall prove Theorem 2.1. Thus we will apply a penalization with respect to $t_N^h(u)$ and a regularization of the absolute values in the definition (2.3). After that, we will pass to the limit with the parameters of regularization and penalization.

Remark 3.1. The approach of the previous section, based on the fixed point, fails in the present case since we are not able to prove the continuity of the mapping $\boldsymbol{T}$ outside a small ball in $(\tilde{X}_h)'$, where the uniqueness for $(\mathbf{P}_{\mathrm{g}}^{\mathrm{h}})$ is guaranteed. □

Let us introduce the functionals

$$\Phi_\delta(u,v) = \int_{\Gamma_C} \Pi^h(\delta^{-1}[u_N]_+ v_N)\,\mathrm{d}s,$$
$$j_\delta(u,v) = \int_{\Gamma_C} \Pi^h\big(\delta^{-1}[u_N]_+ \mathscr{F}(u_T)|v_T|\big)\,\mathrm{d}s,$$

where δ is a positive parameter, and the problem $(\boldsymbol{P}_\delta)$: find $u \in \boldsymbol{V}_h$ such that

$$a(u,v-u) + \Phi_\delta(u,v-u) + j_\delta(u,v) - j_\delta(u,u) \geqslant L(v-u) \quad \forall v \in \boldsymbol{V}_h. \tag{3.1}$$

Let $\varepsilon > 0$ and let

$$\varphi_\varepsilon(t) = \begin{cases} |t| & \text{for } |t| \geqslant \varepsilon, \\ -\dfrac{|t|^4}{8\varepsilon^3} + \dfrac{3|t|^2}{4\varepsilon} + \dfrac{3}{8}\varepsilon & \text{for } |t| \leqslant \varepsilon \end{cases}$$

be a regularization of the absolute value $|t|$.

We define also

$$j_{\delta,\varepsilon}(u,v) = \int_{\Gamma_C} \Pi^h\big(\delta^{-1}[u_N]_+ \mathscr{F}(u_T)\varphi_\varepsilon(v_T)\big)\,\mathrm{d}s$$

and

$$\begin{aligned}\psi_{\delta,\varepsilon} &= \lim_{\lambda\to 0+}\big(j_{\delta,\varepsilon}(u,u+\lambda v) - j_{\delta,\varepsilon}(u,u)\big) \\ &= \int_{\Gamma_C} \Pi^h\big(\delta^{-1}[u_N]_+ \mathscr{F}(u_T)\,\mathrm{grad}\,\varphi_\varepsilon(u_T)\cdot v_T\big)\,\mathrm{d}s.\end{aligned}$$

The regularized problem (3.1), where j_δ is replaced by $j_{\delta,\varepsilon}$, is equivalent to the following variational equation $(\boldsymbol{P}_{\delta,\varepsilon})$: find $u \in \boldsymbol{V}_h$, such that

$$a(u,v) + \Phi_\delta(u,v) + \psi_{\delta,\varepsilon}(u,v) = L(v) \quad \forall v \in \boldsymbol{V}_h. \tag{3.2}$$

In what follows, we prove the existence of a solution of (3.2). Then passing to the limit successively with $\varepsilon \to 0+$ and $\delta \to 0+$, we obtain the existence of a solution of the problem (2.3).

Let us introduce the operators

$$A\colon \boldsymbol{V}_h \to \boldsymbol{V}_h', \quad Q\colon \boldsymbol{V}_h \to \boldsymbol{V}_h', \quad F\colon \boldsymbol{V}_h \to \boldsymbol{V}_h'$$

by the formulae

$$\langle Au, v\rangle = a(u,v), \quad \langle Qu, v\rangle = \Phi_\delta(u,v), \quad \langle Fu, v\rangle = \psi_{\delta,\varepsilon}(u,v)$$

and the operator $T\colon \boldsymbol{V}_h \to \boldsymbol{V}_h{}'$, $\quad T = A + Q + F$.

We can show that the operator T is continuous and coercive. To this end we need an auxiliary

Lemma 3.1. *For any $u, v, w \in [X_h]^d$, we have*

$$[u_N]_+ \leqslant |u_N| = |u_d|,$$

$$|v_T| = |v_1| \quad \text{for} \quad d = 2 \qquad \text{and} \qquad |v_T| \leqslant |v_1| + |v_2| \quad \text{for } d = 3,$$

$$|\Pi^h(|u_N|_+ v_N)| \leqslant \Pi^h([u_N]_+|v_N|) \leqslant \Pi^h(|u_d|\,|v_d|) \leqslant \|u_d\|_\infty \|v_d\|_\infty,$$

$$\Pi^h(|u_j|\,|w_T|) \leqslant \|u_j\|_\infty (\sum_{j=1}^{d-1} \|w_j\|_\infty).$$

P r o o f is obvious. □

Lemma 3.2. *The following assertions hold:*

(i) *A is continuous, linear and elliptic,*

(ii) *Q is continuous and $\langle Qv, v\rangle \geqslant 0$ for all $v \in \boldsymbol{V}_h$,*

(iii) *F is continuous and $\langle Fv, v\rangle \geqslant 0$ for all $v \in \boldsymbol{V}_h$.*

P r o o f. (i) is obvious.

(ii) Since $|[a]_+ - [b]_+| \leqslant |a - b|$ holds for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned} |\langle Qu - Qw, v\rangle| &\leqslant \delta^{-1} \int_{\Gamma_C} \big|\Pi^h\big(([u_N]_+ - [w_N]_+)v_n\big)\big|\,\mathrm{d}s \\ &\leqslant \delta^{-1} \int_{\Gamma_C} \Pi^h(|u_N - w_N|\,|v_N|)\,\mathrm{d}s \leqslant C\delta^{-1}\|u_d - w_d\|_\infty \|v_d\|_\infty. \end{aligned}$$

Hence Q is Lipschitz continuous. Since

$$[a]_+ a = ([a]_+)^2 \geqslant 0,$$

we have

$$\langle Qv, v\rangle = \delta^{-1} \int_{\Gamma_C} \Pi^h(|v_N|_+ v_N)\,\mathrm{d}s \geqslant 0.$$

(iii) We may write

$$\begin{aligned} |\langle Fu - Fw, v\rangle| = \delta^{-1} \bigg| \int_{\Gamma_C} \Big\{&\Pi^h([u_N]_+ \mathscr{F}(u_T)\nabla\varphi_\varepsilon(u_T) \cdot v_T \\ &- \Pi^h([w_N]_+ \mathscr{F}(w_T)\nabla\varphi_\varepsilon(w_T) \cdot v_T)\Big\}\,\mathrm{d}s\bigg| \\ \leqslant \delta^{-1} &\int_{\Gamma_C} |\Pi^h(J_1 + J_2 + J_3)|\,\mathrm{d}s \\ \leqslant \delta^{-1} &\int_{\Gamma_C} (|\Pi^h J_1| + |\Pi^h J_2| + |\Pi^h J_3|)\,\mathrm{d}s, \end{aligned} \tag{3.3}$$

where

$$J_1 = ([u_N]_+ - [w_N]_+)\mathscr{F}(u_T)\nabla\varphi_\varepsilon(u_T)\cdot v_T,$$
$$J_2 = [w_N]_+\mathscr{F}(u_T)\big(\nabla\varphi_\varepsilon(u_T) - \nabla\varphi_\varepsilon(w_T)\big)\cdot v_T,$$
$$J_3 = [w_N]_+\big(\mathscr{F}(u_T) - \mathscr{F}(w_T)\big)\nabla\varphi_\varepsilon(w_T)\cdot v_T.$$

We have

$$\int_{\Gamma_C} |\Pi^h J_1|\,\mathrm{d}s \leqslant C\|\mathscr{F}\|_\infty\|u_d - w_d\|_\infty\|\,|v_T|\,\|_\infty,$$

since $|\nabla\varphi_\varepsilon| \leqslant 1$ everywhere;

$$\int_{\Gamma_C} |\Pi^h J_2|\,\mathrm{d}s \leqslant C\|\mathscr{F}\|_\infty\|w_d\|_\infty\sum_{j=1}^{d-1}\|u_j - w_j\|_\infty\|\,|v_T|\,\|_\infty,$$

since

$$|\nabla\varphi_\varepsilon(u_T) - \nabla\varphi_\varepsilon(w_T)| \leqslant \frac{3}{2\varepsilon}|u_T - w_T|;$$

$$\int_{\Gamma_C} |\Pi^h J_3|\,\mathrm{d}s \leqslant CC_L\|w_d\|_\infty\sum_{j=1}^{d-1}\|u_j - w_j\|_\infty\|\,|v_T|\,\|_\infty$$

since

$$|\mathscr{F}(s) - \mathscr{F}(t)| \leqslant C_L|s-t| \quad \forall s,t\in[0,\infty) \text{ and a.a. } x\in\Gamma_C.$$

Inserting these estimates into (3.3), we obtain

$$\text{(3.4)}\quad |\langle Fu - Fw, v\rangle| \leqslant C\delta^{-1} \times\left\{\|\mathscr{F}\|_\infty\|u_d - w_d\|_\infty + (\|\mathscr{F}\|_\infty + C_L)\|w_d\|_\infty\sum_{j=1}^{d-1}\|u_j - w_j\|_\infty\right\}\|\,|v_T|\,\|_\infty$$

where $C \equiv C(\varepsilon)$, so that F is continuous. Finally, we have

$$\langle Fv, v\rangle = \delta^{-1}\int_{\Gamma_C}\Pi^h\big([v_N]_+\mathscr{F}(v_T)\nabla\varphi_\varepsilon(v_T)\cdot v_T\big)\,\mathrm{d}s \geqslant 0,$$

since

$$\nabla\varphi_\varepsilon(v_T)\cdot v_T \geqslant 0.$$

In fact, the latter inequality follows from the convexity of φ_ε and the fact that φ_ε attains its minimum at the origin. $\square$

Proposition 3.1. *The problem* ($\boldsymbol{P}_{\delta,\varepsilon}$) (3.2) *has at least one solution for any positive* δ *and* ε.

P r o o f follows from a general theorem—see [4], Theorem 2.5, since the operator $T = A + Q + F$ is continuous and coercive by Lemma 3.2. □

Proposition 3.2. *The problem* (3.1) ($\boldsymbol{P}_\delta$) *has at least one solution for any positive* δ.

P r o o f. Let us denote the solution of the problem (3.2) with parameters δ, ε by u_ε and let us substitute $v := u_\varepsilon$ in (3.2). We have

$$C_0\|u_\varepsilon\|_{1,\Omega}^2 \leqslant \langle Tu_\varepsilon, u_\varepsilon\rangle = L(u_\varepsilon) \leqslant \|L\|_{-1}\|u_\varepsilon\|_{1,\Omega}$$

so that

$$\|u_\varepsilon\|_{1,\Omega} \leqslant \|L\|_{-1}/C_0 \quad \forall\varepsilon > 0.$$

There exists an element $\omega \in \boldsymbol{V}_h$ and a sequence $\{\varepsilon_k\}$, $k \to \infty$, such that $\varepsilon_k \to 0$ and $u_k \to \omega$ hold for $u_k := u_{\varepsilon_k}$.

The equation (3.2) is equivalent to the variational inequality

$$a(u_k, v - u_k) + \Phi_\delta(u_k, v - u_k) + j_{\delta,\varepsilon_k}(u_k, v) - j_{\delta,\varepsilon_k}(u_k, u_k) \geqslant L(v - u_k) \quad \forall v \in \boldsymbol{V}_h.$$

Let us pass to the limit with $k \to \infty$ and use Lemma 3.2. Thus we obtain

$$a(u_k, v - u_k) \to a(\omega, v - \omega), \quad L(v - u_k) \to L(v - \omega), \tag{3.5}$$

$$\Phi_\delta(u_k, v - u_k) = \langle Qu_k, v - u_k\rangle \to \langle Q\omega, v - \omega\rangle = \Phi_\delta(\omega, v - \omega).$$

Next, we may write

$$\begin{aligned}
&|j_{\delta,\varepsilon_k}(u_k, v) - j_\delta(\omega, v)| \\
&\quad\leqslant |j_{\delta,\varepsilon_k}(u_k, v) - j_\delta(u_k, v)| + |j_\delta(u_k, v) - j_\delta(\omega, v)| \\
&\quad= J_1 + J_2, \\
J_1 &= \delta^{-1}\left|\int_{\Gamma_C} \Pi^h\big([u_{kN}]_+\mathscr{F}(u_{kT})\big(\varphi_{\varepsilon_k}(v_T) - |v_T|\big)\big)\,\mathrm{d}s\right| \\
&\leqslant \delta^{-1}\|\mathscr{F}\|_\infty \int_{\Gamma_C} \Pi^h\big(|u_{kN}|\,\big|\varphi_{\varepsilon_k}(v_T) - |v_T|\,\big|\big)\,\mathrm{d}s \\
&\leqslant C\delta^{-1}\|\mathscr{F}\|_\infty\varepsilon_k\|u_{kd}\|_\infty \to 0,
\end{aligned}$$

since

$$\big|\varphi_{\varepsilon_k}(v_T) - |v_T|\big| \leqslant \varepsilon_k;$$

$$
\begin{aligned}
J_2 &\leqslant \delta^{-1} \int_{\Gamma_C} \left|\Pi^h\big(([u_{kN}]_+ - |\omega_N|)\mathscr{F}(u_{kT})|v_T|\big)\right| \,\mathrm{d}s \\
&\quad + \delta^{-1} \int_{\Gamma_C} \left|\Pi^h\big([\omega_N]_+\big(\mathscr{F}(u_{kT}) - \mathscr{F}(\omega_T)\big)|v_T|\big)\right| \,\mathrm{d}s \\
&\leqslant C\delta^{-1}\Big\{\|\mathscr{F}\|_\infty \|u_{kd} - \omega_d\|_\infty + C_L\|\omega_d\|_\infty \sum_{j=1}^{d-1} \|u_{kj} - \omega_j\|_\infty\Big\} \|\,|v_T|\,\|_\infty \to 0.
\end{aligned}
$$

As a consequence, we get

$$
j_{\delta,\varepsilon_k}(u_k, v) \to j_\delta(\omega, v). \tag{3.6}
$$

In a similar way, we can write

$$
\begin{aligned}
|j_{\delta,\varepsilon_k}(u_k, u_k) - j_\delta(\omega,\omega)| &\leqslant |j_{\delta,\varepsilon_k}(u_k,u_k) - j_{\delta,\varepsilon_k}(u_k,\omega)| + |j_{\delta,\varepsilon_k}(u_k,\omega) - j_\delta(\omega,\omega)| \\
&= J_3 + J_4.
\end{aligned}
$$

From (3.6), $J_4 \to 0$ follows immediately. Finally, we have

$$
\begin{aligned}
J_3 &= \delta^{-1}\left|\int_{\Gamma_C} \Pi^h\big([u_{kN}]_+\mathscr{F}(u_{kT})\big(\varphi_{\varepsilon_k}(u_{kT}) - \varphi_{\varepsilon_k}(\omega_T)\big)\big)\,\mathrm{d}s\right| \\
&\leqslant C\delta^{-1}\|\mathscr{F}\|_\infty \|u_{kd}\|_\infty \sum_{j=1}^{d-1} \|u_{kj} - \omega_j\|_\infty \to 0
\end{aligned} \tag{3.7}
$$

using Lemma 3.1 and the estimate

$$
|\varphi_{\varepsilon_k}(u_{kT}) - \varphi_{\varepsilon_k}(\omega_T)| \leqslant |\,|u_{kT}| - |\omega_T|\,| \leqslant |u_{kT} - \omega_T|.
$$

Combining (3.5)–(3.7), we arrive at the inequality

$$
a(\omega, v-\omega) + \Phi_\delta(\omega, v-\omega) + j_\delta(\omega, v) - j_\delta(\omega,\omega) \geqslant L(v-\omega).
$$

As a consequence, ω is a solution of the problem ($\boldsymbol{P}_\delta$) (3.1). □

Next let us consider a solution $u := u_\delta$ of the problem (3.1) with a parameter δ and substitute $v := 0$ into (3.1). Then

$$
\begin{aligned}
a(u,u) + \Phi_\delta(u,u) &\leqslant j_\delta(u,0) - j_\delta(u,u) + L(u), \\
\Phi_\delta(u,u) &= \delta^{-1}\int_{\Gamma_C} \Pi^h([u_N]_+^2)\,\mathrm{d}s, \\
j_\delta(u,0) - j_\delta(u,u) &= -j_\delta(u,u) = -\delta^{-1}\int_{\Gamma_C} \Pi^h\big([u_N]_+\mathscr{F}(u_T)|u_T|\big)\,\mathrm{d}s \leqslant 0.
\end{aligned}
$$

We arrive at the estimate

$$C_0\|u\|^2_{1,\Omega} + \delta^{-1}\int_{\Gamma_C} \Pi^h([u_N]^2_+)\,\mathrm{d}s \leqslant \|L\|_{-1}\|u\|_{1,\Omega} \tag{3.8}$$

and at

Lemma 3.3. *There exists a positive constant C independent of δ and such that*

$$\|u_\delta\|_{1,\Omega} + \delta^{-1}\int_{\Gamma_C} \Pi^h([u_{\delta N}]^2_+)\,\mathrm{d}s \leqslant C$$

holds for all solutions u_δ of the problem (3.1).

P r o o f. The estimate (3.8) yields that

$$\|u_\delta\|_{1,\Omega} \leqslant \|L\|_{-1}/C_0, \tag{3.9}$$

and inserting this into the right-hand side of (3.8) we get

$$\delta^{-1}\int_{\Gamma_C} \Pi^h([u_{\delta N}]^2_+)\,\mathrm{d}s \leqslant \|L\|^2_{-1}/C_0.$$

□

As a consequence of Lemma 3.3, there exist $u \in \boldsymbol{V}_h$ and a sequence $\{\delta_k\}$, $k \to \infty$, such that $\delta_k \to 0$ and

$$u_k := u_{\delta_k} \to u. \tag{3.10}$$

Let us denote

$$G_k = \delta_k^{-1}[u_{kN}]_+$$

and define functionals $\mathscr{G}_k \in (\tilde{X}_h)'$ as follows:

$$\langle \mathscr{G}_k, \psi\rangle = \int_{\Gamma_C} \Pi^h(G_k\psi)\,\mathrm{d}s, \quad \psi \in \tilde{X}_h.$$

Each $\mathscr{G}_k$ is linear and bounded, since

$$|\langle \mathscr{G}_k, \psi\rangle| \leqslant C\|G_k\|_\infty\|\psi\|_\infty.$$

Let

$$\|\mathscr{G}_k\|' = \sup \langle \mathscr{G}_k, \psi\rangle / \|\psi\|_\infty \text{ for } \psi \in \tilde{X}_h \setminus \{0\}.$$

Lemma 3.4. *There exists a positive constant C such that*

$$\|\mathscr{G}_k\|' \leqslant C \quad \forall k \geqslant 1.$$

P r o o f. Let us insert

$$v = u_k \pm \boldsymbol{R}(\psi \boldsymbol{n}), \quad \psi \in \tilde{X}_h$$

into (3.1), where $\boldsymbol{R}$ is the mapping from (2.1). We obtain

$$a(u_k, \boldsymbol{R}(\psi\boldsymbol{n})) + \Phi_{\delta_k}(u_k, \boldsymbol{R}(\psi\boldsymbol{n})) = L(\boldsymbol{R}(\psi\boldsymbol{n})), \tag{3.11}$$

since $(\boldsymbol{R}(\psi\boldsymbol{n}))_T = 0$ and therefore

$$j_{\delta_k}(u_k, u_k \pm \boldsymbol{R}(\psi\boldsymbol{n})) = j_{\delta_k}(u_k, u_k).$$

The equation (3.11) implies that

$$\begin{aligned} |\Phi_{\delta_k}(u_k, \boldsymbol{R}(\psi\boldsymbol{n}))| &= |L(\boldsymbol{R}(\psi\boldsymbol{n})) - a(u_k, \boldsymbol{R}(\psi\boldsymbol{n}))| \\ &\leqslant (\|L\|_{-1} + C_1\|u_k\|_{1,\Omega})\|\boldsymbol{R}(\psi\boldsymbol{n})\|_{1,\Omega} \\ &\leqslant C_4\|\mathscr{R}\psi\|_{1,\Omega} \leqslant C_5\|\psi\|_{0,\Gamma_C}, \end{aligned} \tag{3.12}$$

where Lemma 3.3, the definition of $\boldsymbol{R}$, the inverse inequality and Lemma 2.1 have been used. Since $(\boldsymbol{R}(\psi\boldsymbol{n}))_N = \psi$, (3.12) and the definition of Φ_δ imply that

$$|\langle \mathscr{G}_k, \psi\rangle| = |\Phi_{\delta_k}(u_k, \boldsymbol{R}(\psi\boldsymbol{n}))| \leqslant C_6\|\psi\|_\infty,$$

where C_6 does not depend on δ. □

P r o o f o f T h e o r e m 2.1. By Lemma 3.4, there exist a functional $\mathscr{G} \in (\tilde{X}_h)'$ and a subsequence $\{\mathscr{G}_m\} \subset \{\mathscr{G}_k\}$ such that

$$\mathscr{G}_m \to \mathscr{G} \quad \text{in } (\tilde{X}_h)'. \tag{3.13}$$

Choose an arbitrary $v \in \boldsymbol{K}_h$. Since $v_N \leqslant 0$ on Γ_C, we have

$$\begin{aligned} \Phi_{\delta_m}(u_m, v - u_m) &= \delta_m^{-1}\int_{\Gamma_C} \Pi^h\big([u_{mN}]_+(v_N - u_{mN})\big)\,\mathrm{d}s \\ &\leqslant -\delta_m^{-1}\int_{\Gamma_C} \Pi^h\big([u_{mN}]_+ u_{mN}\big)\,\mathrm{d}s \leqslant 0. \end{aligned}$$

As a consequence, we may write

$$a(u_m, v - u_m) + j_{\delta_m}(u_m, v) - j_{\delta_m}(u_m, u_m) \geqslant L(v - u_m). \tag{3.14}$$

Passing to the limit with $m \to \infty$ and using (3.10), we obtain

$$a(u_m, v - u_m) \to a(u, v - u), \quad L(v - u_m) \to L(v - u).$$

Next, we have

$$\begin{aligned} j_{\delta_m}(u_m, v) - j_{\delta_m}(u_m, u_m) &= \int_{\Gamma_C} \Pi^h\big(G_m \mathscr{F}(u_{mT})(|v_T| - |u_{mT}|)\big)\,\mathrm{d}s \\ &= \int_{\Gamma_C} \Pi^h\big(G_m\big(\mathscr{F}(u_{mT}) - \mathscr{F}(u_T)\big)(|v_T| - |u_{mT}|)\big)\,\mathrm{d}s \\ &\quad + \int_{\Gamma_C} \Pi^h\big(G_m \mathscr{F}(u_T)(|v_T| - |u_T|)\big)\,\mathrm{d}s \\ &\quad + \int_{\Gamma_C} \Pi^h\big(G_m \mathscr{F}(u_T)(|u_T| - |u_{mT}|)\big)\,\mathrm{d}s \\ &= J_1 + J_2 + J_3. \end{aligned}$$

For any $\varphi \in C(\overline{\Gamma_C})$ we may write

$$\int_{\Gamma_C} \Pi^h(G_m\varphi)\,\mathrm{d}s = \int_{\Gamma_C} \Pi^h(G_m\Pi^h\varphi)\,\mathrm{d}s = \langle \mathscr{G}_m, \Pi^h\varphi \rangle.$$

Therefore, J_1 can be estimated as

$$\begin{aligned} |J_1| &= \big|\big\langle \mathscr{G}_m, \Pi^h\big(\big(\mathscr{F}(u_{mT}) - \mathscr{F}(u_T)\big)(|v_T| - |u_{mT}|)\big)\big\rangle\big| \\ &\leqslant C\|\Pi^h\big((\mathscr{F}(u_{mT}) - \mathscr{F}(u_T))(|v_T| - |u_{mT}|)\big)\|_\infty \\ &\leqslant CC_L\|\,|u_{mT}| - |u_T|\,\|_\infty\|\,|v_T| - |u_{mT}|\,\|_\infty \to 0, \end{aligned}$$

using also Lemma 3.4.

On the basis of (3.13) we obtain

$$J_2 = \big\langle \mathscr{G}_m, \Pi^h\big(\mathscr{F}(u_T)(|v_T| - |u_T|)\big)\big\rangle \to \big\langle \mathscr{G}, \Pi^h\big(\mathscr{F}(u_T)(|v_T| - |u_T|)\big)\big\rangle.$$

Finally,

$$|J_3| = \big\langle \mathscr{G}_m, \Pi^h\big(\mathscr{F}(u_T)(|u_T| - |u_{mT}|)\big)\big\rangle \leqslant C\|\mathscr{F}\|_\infty\|\,|u_T| - |u_{mT}|\,\|_\infty \to 0.$$

Employing these results in the limiting process of (3.14), we arrive at

$$a(u, v - u) + \big\langle \mathscr{G}, \Pi^h\big(\mathscr{F}(u_T)(|v_T| - |u_T|)\big)\big\rangle \geqslant L(v - u). \tag{3.15}$$

Lemma 3.3 yields the estimate

$$\int_{\Gamma_C} \Pi^h([u_{mN}]_+^2)\,\mathrm{d}s \leqslant C\delta_m.$$

Passing to the limit, we obtain

$$\int_{\Gamma_C} \Pi^h([u_N]_+^2)\,\mathrm{d}s = 0,$$

so that $[u_N]_+ = 0$ at all nodes of the triangulation of $\overline{\Gamma_C}$. Since $u_N \in X_h|_{\Gamma_C}$, we have $u_N \leqslant 0$ everywhere on Γ_C and $u \in \boldsymbol{K}_h$ follows.

Let us set

$$v = u_m \pm \boldsymbol{R}(\psi \boldsymbol{n}),$$

where $\psi = \Pi^h \varphi$ and $\varphi \in C_0(\overline{\Gamma}_C)$ as in the proof of Lemma 3.4. The definition of Φ_δ and (3.11) imply that

$$\Phi_{\delta_m}(u_m, \boldsymbol{R}(\psi\boldsymbol{n})) = \langle \mathscr{G}_m, \psi\rangle = L(\boldsymbol{R}(\psi\boldsymbol{n})) - a(u_m, \boldsymbol{R}(\psi\boldsymbol{n})).$$

Passing to the limit and using the definition (2.1), (2.2), we obtain

$$\langle \mathscr{G}, \psi\rangle = L(\boldsymbol{R}(\psi\boldsymbol{n})) - a(u, \boldsymbol{R}(\psi\boldsymbol{n})) = -\left\langle t_N^h(u), \psi\right\rangle. \tag{3.16}$$

If we set

$$\psi = \Pi^h\big(\mathscr{F}(u_T)(|v_T| - |u_T|)\big),$$

the inequality (3.15) can be rewriten as

$$a(u, v-u) - \left\langle t_N^h(u), \Pi^h\big(\mathscr{F}(u_T)(|v_T| - |u_T|)\big)\right\rangle \geqslant L(v-u).$$

Thus u is a solution of the problem (2.3). The estimate

$$\|u\|_{1,\Omega} \leqslant \|L\|_{-1}/C_0$$

is an immediate consequence of (3.10) and (3.9).

From (3.16) we deduce

$$\begin{aligned}\left|\left\langle t_N^h(u), \psi\right\rangle\right| &\leqslant (\|L\|_{-1} + C_1\|u\|_{1,\Omega})\|\boldsymbol{R}(\psi\boldsymbol{n})\|_{1,\Omega}\\ &\leqslant (1 + C_1 C_0^{-1})\|L\|_{-1} C h_0^{-1/2}\|\psi\|_{0,\Gamma_C}\end{aligned}$$

as in the proof of Lemma (2.4). Consequently,

$$\|t_N^h(u)\|_* \leqslant M h_0^{-1/2}\|L\|_{-1}$$

follows. □

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