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A FINITE ELEMENT CONVERGENCE ANALYSIS FOR 3D STOKES EQUATIONS IN CASE OF VARIATIONAL CRIMES*

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Abstract. We investigate a finite element discretization of the Stokes equations with nonstandard boundary conditions, defined in a bounded three-dimensional domain with a curved, piecewise smooth boundary. For tetrahedral triangulations of this domain we prove, under general assumptions on the discrete problem and without any additional regularity assumptions on the weak solution, that the discrete solutions converge to the weak solution. Examples of appropriate finite element spaces are given.

Keywords: Stokes equations, nonstandard boundary conditions, finite element method, approximation of boundary

MSC 2000: 65N30, 35Q30

1. Formulation of the problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain the boundary of which consists of the closures of sets Γ^D and Γ^N, and let us consider the following Stokes equations:

$$-\nu\,\Delta\,\boldsymbol{u} + \nabla\,p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{u}_b \qquad \text{on } \Gamma^D, \tag{1.3}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Gamma^N, \tag{1.4}$$

$$(\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n})\,(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})\,\boldsymbol{n} = \boldsymbol{\varphi} \qquad \text{on } \Gamma^N. \tag{1.5}$$

The equations describe a slow motion of a viscous incompressible fluid (e.g. of molten glass) and the symbols used have the following meanings: $\boldsymbol{u}$ is the velocity,

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p is the pressure, ν is the kinematic viscosity, which is assumed to be constant and positive, $\boldsymbol{f}$ is an outer volume force, $\boldsymbol{n}$ is the unit outward normal vector to the boundary $\partial\Omega$ of the domain Ω, $\boldsymbol{I}$ is the identity tensor and $\boldsymbol{\varphi}$ is a surface force acting in the tangential direction to Γ^N. The tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ is a tensor defined by

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \qquad \forall\, \boldsymbol{c} \in \mathbb{R}^3.$$

Thus, $(\boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n})$ is a projection operator onto the plane tangent to Γ^N and (1.5) can be componentwise written as

$$\sum_{j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) n_j - \sum_{j,k=1}^{3} \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) n_i\, n_j\, n_k = \varphi_i, \qquad i = 1, 2, 3.$$

We assume that

$$\text{(1.6)} \qquad \boldsymbol{f} \in L^{\frac{6}{5}}(\Omega)^3,$$

$$\text{(1.7)} \qquad \boldsymbol{u}_b \in H^{\frac{1}{2}}(\partial\Omega)^3, \qquad \boldsymbol{u}_b \cdot \boldsymbol{n} = 0 \text{ on } \Gamma^N, \qquad \int_{\partial\Omega} \boldsymbol{u}_b \cdot \boldsymbol{n} \,\mathrm{d}\sigma = 0,$$

$$\text{(1.8)} \qquad \boldsymbol{\varphi} \in H^{\frac{1}{2}}(\partial\Omega)^3, \qquad \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma^N.$$

The third condition in (1.7) and the second condition in (1.8) are necessary conditions for the solvability of (1.1)–(1.5).

For investigating effects caused by an approximation of the domain Ω by a polyhedral domain, we need certain assumptions on the regularity of the boundary of Ω. First, we introduce a property of subsets of $\partial\Omega$, analogous to the Lipschitz property of domains in $\mathbb{R}^2$. To this end, we employ local Cartesian coordinate systems from the definition of the Lipschitz continuity of $\partial\Omega$ (see Section 4, below Lemma 4.3). In each of these local coordinate systems, a part of $\partial\Omega$ is represented by the graph of a function $f \in C^{0,1}([-a, a]^2)$, where $a > 0$ is a constant common to all the coordinate systems.

Definition 1.1. A relatively open set $\Gamma \subset \partial\Omega$ is of class $C^{0,1*}$ if there exists a finite number of the above-mentioned local coordinate systems such that the graphs of the functions f on $(-a, a)^2$ cover the whole boundary $\partial\Omega$ and, in each of these coordinate systems, the projection of the respective part of Γ into the square $(-a, a)^2$ is a set with a Lipschitz-continuous boundary.

The boundary of Ω is assumed to be Lipschitz-continuous and to consist of closures of disjoint relatively open C^2 surfaces $\Gamma_i \subset \partial\Omega$, $i = 1, \ldots, K$, which are of class $C^{0,1*}$, have piecewise C^2 boundaries and satisfy, for some $K^D \in \{1, \ldots, K\}$,

$$\Gamma^D = \operatorname{int} \bigcup_{i=1}^{K^D} \overline{\Gamma}_i, \qquad \Gamma^N = \operatorname{int} \bigcup_{i=K^D+1}^{K} \overline{\Gamma}_i.$$

Thus, we have

$$\partial\Omega = \overline{\Gamma}^D \cup \overline{\Gamma}^N, \qquad \Gamma^D \cap \Gamma^N = \emptyset, \qquad \operatorname{meas}_2(\Gamma^D) > 0, \qquad \operatorname{meas}_2(\Gamma^N) \geqslant 0.$$

Further, we assume that each surface Γ_i can be extended to a C^2 surface $\widetilde{\Gamma}_i$ satisfying $\operatorname{dist}(\partial\widetilde{\Gamma}_i, \partial\Gamma_i) > 0$. Finally, we suppose that there exists an extension $\boldsymbol{m} \in W^{2,\infty}(\mathbb{R}^3)^3$ of the unit outward normal vector to Γ^N, i.e., $\boldsymbol{m}|_{\Gamma^N} = \boldsymbol{n}|_{\Gamma^N}$.

The $C^{0,1*}$ regularity of the sets Γ_i guarantees that no zero angle between two parts of $\partial\Gamma_i$ can arise at a point where $\partial\Gamma_i$ is not C^2. This is necessary for constructing regular triangulations of Γ_i.

Remark 1.1. Let Γ^N be a $C^{k,1}$ surface of class $C^{0,1*}$, where $k \geqslant 1$. Then there exists a function $\boldsymbol{m} \in W^{k,\infty}(\mathbb{R}^3)^3$ satisfying $\boldsymbol{m}|_{\Gamma^N} = \boldsymbol{n}|_{\Gamma^N}$ (see Lemma 4.7 in Section 4). If Γ^N is such that there exists a domain Ω^* with a $C^{k,1}$ boundary satisfying $\Gamma^N \subset \partial\Omega^*$, then there exists an extension $\boldsymbol{m} \in W^{k,\infty}(\mathbb{R}^3)^3$ of the normal vector to $\partial\Omega^*$ and the $C^{0,1*}$ property of Γ^N is not necessary.

Properties of a finite element discretization of (1.1)–(1.5) defined by using a tetrahedral triangulation of Ω were already investigated in [7], where it was shown that, under the assumption $\boldsymbol{u} \in H^k(\Omega)^3$, $p \in H^{k-1}(\Omega)$ with $k \in \{2, 3\}$, the discrete solutions converge to the weak solution $\boldsymbol{u}$, p with the convergence order $\frac{k}{2}$. The aim of the present paper is to find conditions which guarantee the convergence of the discrete solutions to the weak solution $\boldsymbol{u} \in H^1(\Omega)^3$, $p \in L^2_0(\Omega)$ without any additional regularity assumptions on $\boldsymbol{u}$ and p. The basic difficulty is that, due to the approximation of the boundary of Ω, the discrete test functions for the velocity cannot be used as test functions in the weak formulation. In view of the low regularity of the weak solution, this difficulty cannot be circumvented using the classical formulation (1.1)–(1.5) as in the proof of Theorem 5.1 in [7] and we deal with the question how to approximate the discrete test functions by suitable functions belonging to the spaces from the weak formulation.

For the two-dimensional case, such approximations were already constructed in [11] by using natural extensions of discrete test functions near non-Dirichlet boundaries and transforming discrete test functions onto curved (ideal) triangles near Dirichlet boundaries. However, these techniques are very difficult to extend to three dimensions. Therefore, we develop a new and rather simple approach which can be applied in two as well as three dimensions. The basic idea is to replace any function v_h from a discrete test function space V_h by a function $v_h^0 \in V_h$ vanishing on elements intersecting the closure of a Dirichlet boundary, and to extend v_h^0 using the Nikolskij method outside the approximating domain Ω_h.

The approximation of the computational domain is one of the so-called variational crimes in the finite element method, which have been investigated since the early 1970s. We mention at least the fundamental paper [3] and the book [10] published at that time. In these and many further papers, convergence results were derived under the assumption that the weak solution possesses higher regularity. The first paper, where the convergence of finite element solutions to a weak solution were proved without additional regularity assumptions, was the above-mentioned paper [11] treating general isoparametric approximations of a two-dimensional computational domain. The present paper shows how to prove the convergence of finite element solutions to a weak solution without higher regularity in the three-dimensional case and suggests an alternate way for proving the results of [11].

The plan of the paper is as follows. In Section 2, we introduce a weak formulation of (1.1)–(1.5) and mention some related results. In Section 3, we define triangulations of Ω and of the sets Γ^D and Γ^N. Further, we introduce finite element spaces approximating the spaces from the weak formulation and define a finite element discretization of (1.1)–(1.5). In Section 4, we summarize auxiliary results needed for convergence investigations in the subsequent sections. In Section 5 we explain, on the example of Poisson's equation, the idea of the convergence proof and establish a fundamental result on the approximation of discrete test functions by test functions from a weak formulation. Section 6 is devoted to convergence investigations for the discretization of (1.1)–(1.5) and, finally, in Section 7, we give examples of finite element spaces satisfying the assumptions introduced in Section 3.

Throughout the paper we use standard notation which can be found e.g. in [5]. We only mention a few of the symbols. We denote by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$ the usual norm and seminorm, respectively, in the Sobolev space $W^{k,p}(\Omega)$ and, for $p = 2$, we drop the second index and use the notation $\|\cdot\|_{k,\Omega}$, $|\cdot|_{k,\Omega}$ and $H^k(\Omega) \equiv W^{k,2}(\Omega)$. The space of functions $v \in L^2(\Omega)$ satisfying $\int_\Omega v \,\mathrm{d}x = 0$ is denoted by $L_0^2(\Omega)$. For an integer $k \geqslant 0$, we denote by $P_k(\Omega)$ the space of all polynomials defined on Ω, of degrees less than or equal to k. The notation C, $\widetilde{C}$ and $\overline{C}$ is used to denote generic constants independent of h.

2. Weak formulation

Denoting

$$\mathbf{V} = \{\boldsymbol{v} \in H^1(\Omega)^3;\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma^D,\ \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma^N\}$$

and

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= \frac{\nu}{2} \int_\Omega (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}) \cdot (\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}) \,\mathrm{d}x, \\ b(\boldsymbol{v}, p) &= -\int_\Omega p \operatorname{div} \boldsymbol{v} \,\mathrm{d}x, \\ \langle \boldsymbol{g}, \boldsymbol{v} \rangle &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{d}x + \nu \int_{\Gamma^N} \boldsymbol{\varphi} \cdot \boldsymbol{v} \,\mathrm{d}\sigma, \end{aligned}$$

we introduce the following weak formulation.

Definition 2.1. Let $\widetilde{\boldsymbol{u}}_b \in H^1(\Omega)^3$ be any function satisfying

$$\widetilde{\boldsymbol{u}}_b|_{\partial\Omega} = \boldsymbol{u}_b. \tag{2.1}$$

Then functions $\boldsymbol{u}$, p are a *weak solution* of problem (1.1)–(1.5) if

$$\boldsymbol{u} - \widetilde{\boldsymbol{u}}_b \in \mathbf{V}, \qquad p \in L^2_0(\Omega) \tag{2.2}$$

and

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \langle \boldsymbol{g}, \boldsymbol{v} \rangle \qquad \forall\, \boldsymbol{v} \in \mathbf{V}, \tag{2.3}$$

$$b(\boldsymbol{u}, q) = 0 \qquad \forall\, q \in L^2_0(\Omega). \tag{2.4}$$

R e m a r k 2.1. In view of (2.4) and (1.7), we have $b(\boldsymbol{u}, q) = 0\ \ \forall\, q \in L^2(\Omega)$ and hence any weak solution satisfies the condition $\operatorname{div} \boldsymbol{u} = 0$.

The derivation of the weak formulation can be found in [7], Section 2, where the following results were proved: problem (2.2)–(2.4) is uniquely solvable, any classical solution $\boldsymbol{u} \in C^2(\overline{\Omega})^3$, $p \in C^1(\overline{\Omega}) \cap L^2_0(\Omega)$ of (1.1)–(1.5) is a weak solution of (1.1)–(1.5) and any weak solution of (1.1)–(1.5) satisfying $\boldsymbol{u} \in C^2(\overline{\Omega})^3$, $p \in C^1(\overline{\Omega})$ is a classical solution of (1.1)–(1.5).

We assume that we are given a family $\boldsymbol{\mathcal{T}} = \{\mathcal{T}_h\}$ of triangulations of the domain Ω having the following properties. Given $h > 0$, the set $\mathcal{T}_h$ consists of a finite number of closed tetrahedra called elements and denoted by T, $\operatorname{diam}(T) \leqslant h$ for any $T \in \mathcal{T}_h$, all vertices of any $T \in \mathcal{T}_h$ belong to $\overline{\Omega}$, and any two different elements of the triangulation $\mathcal{T}_h$ are either disjoint or possess either a common vertex or a common edge or a common face. The elements of a triangulation make up a polyhedral domain

$$\Omega_h = \operatorname{int} \bigcup_{T \in \mathcal{T}_h} T$$

representing an approximation of Ω. We assume that the boundary of Ω_h is Lipschitz-continuous. The faces of the elements of $\mathcal{T}_h$ (being open two-dimensional sets) will be denoted by T' and the faces belonging to the boundary of Ω_h will be called boundary faces. We assume that, for any boundary face T', there exists $i \in \{1, \ldots, K\}$ such that all three vertices of T' belong to $\overline{\Gamma}_i$. Further, we assume that the set of the vertices of any triangulation $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$ contains all points at which the boundary of some of the sets Γ_i is not C^2. The family of the triangulations is assumed to be regular, i.e., there exists a number $\sigma > 0$ such that, for any $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$ and any $T \in \mathcal{T}_h$,

$$\frac{h_T}{\varrho_T} \leqslant \sigma, \tag{3.1}$$

where

$$h_T = \operatorname{diam}(T), \qquad \varrho_T = \sup_{B \subset T \text{ is a ball}} \operatorname{diam}(B).$$

As usual, the parameters h in $\{\mathcal{T}_h\}$ are supposed to constitute a set whose only accumulation point is zero. Saying "any $h > 0$", we shall still mean "any $\mathcal{T}_h \in \boldsymbol{\mathcal{T}}$". Since the set of the parameters h is bounded, we can introduce a bounded domain $\widetilde{\Omega} \subset \mathbb{R}^3$ with a Lipschitz-continuous boundary such that $\overline{\Omega} \subset \widetilde{\Omega}$ and $\overline{\Omega}_h \subset \widetilde{\Omega}$ for any $h > 0$.

For any $i \in \{1, \ldots, K\}$, the set Γ_i will be approximated by a relatively open set $\Gamma_{ih} \subset \partial\Omega_h$ consisting of boundary faces whose all vertices belong to $\overline{\Gamma}_i$. We assume that $\Gamma_{ih} \cap \Gamma_{jh} = \emptyset$ for any $i \neq j$ and that $\bigcup_{i=1}^{K} \overline{\Gamma}_{ih} = \partial\Omega_h$. For any boundary face T' we denote by $x^c_{T'}$ the barycentre of T' and by $x^b_{T'}$ the nearest point to $x^c_{T'}$ lying on $\partial\Omega$ and satisfying $(x^c_{T'} - x^b_{T'}) \perp T'$. For small h, any boundary face $T' \subset \Gamma_{ih}$ is supposed to satisfy $x^b_{T'} \in \Gamma_i$. This additional assumption is needed since generally it can happen that all vertices of a boundary face belong to $\overline{\Gamma}_i \cap \overline{\Gamma}_j$ for some $i \neq j$. The sets Γ^D and Γ^N will be approximated by relatively open sets Γ^D_h and Γ^N_h, respectively, defined by

$$\Gamma_h^D = \operatorname{int} \bigcup_{i=1}^{K^D} \overline{\Gamma}_{ih}, \qquad \Gamma_h^N = \operatorname{int} \bigcup_{i=K^D+1}^{K} \overline{\Gamma}_{ih}.$$

To define a discretization of (1.1)–(1.5), we approximate the spaces $\mathbf{V}$ and $L_0^2(\Omega)$ from the weak formulation by finite element spaces $\mathbf{V}_h \subset H^1(\Omega_h)^3$ and $\mathrm{Q}_h \subset L_0^2(\Omega_h)$, respectively. Before formulating their properties, let us give an example of $\mathbf{V}_h$ and Q_h.

E x a m p l e 3.1. The simplest finite element subspace of $H^1(\Omega_h)$ approximating the space $H^1(\Omega)$ is the space

$$\mathrm{V}_{1,h} = \{v \in C(\overline{\Omega}_h);\ v|_T \in P_1(T)\ \ \forall\, T \in \mathcal{T}_h\}.$$

If we want to approximate the space $\mathbf{V}$, we also have to take care of the boundary conditions. For example, we can set

$$\boldsymbol{V}_{1,h} = \{\boldsymbol{v} \in [\mathrm{V}_{1,h}]^3;\ \boldsymbol{v} = \boldsymbol{0}\ \text{ on } \Gamma_h^D,\ \ (\boldsymbol{v} \cdot \boldsymbol{n})(x) = 0\ \text{ at any vertex } x \in \Gamma_h^N\}.$$

If $\widetilde{r}_h \in \mathcal{L}(H^2(\Omega)^3, [\mathrm{V}_{1,h}]^3)$ is the Lagrange interpolation operator (i.e., $(\widetilde{r}_h\, \boldsymbol{v})(x) = \boldsymbol{v}(x)$ for any $\boldsymbol{v} \in H^2(\Omega)^3$ and any vertex x of $\mathcal{T}_h$), then we have (cf. [2], p. 124, Theorem 15.3)

$$\|\boldsymbol{v} - \widetilde{r}_h\, \boldsymbol{v}\|_{1,\Omega_h} \leqslant C\, h\, \|\boldsymbol{v}\|_{2,\widetilde{\Omega}} \qquad \forall\, \boldsymbol{v} \in H^2(\widetilde{\Omega})^3$$

and, moreover, $\widetilde{r}_h \in \mathcal{L}(\mathbf{V} \cap H^2(\Omega)^3, \boldsymbol{V}_{1,h})$. Sometimes it can be convenient to set

$$\boldsymbol{V}_{1,h} = \{\boldsymbol{v} \in [\mathrm{V}_{1,h}]^3;\ \boldsymbol{v} = \boldsymbol{0}\ \text{ on } \Gamma_h^D,\ \ (\boldsymbol{v} \cdot \boldsymbol{n}_h^*)(x) = 0\ \text{ at any vertex } x \in \Gamma_h^N\},$$

where $\boldsymbol{n}_h^*$ approximates $\boldsymbol{n}$. In a straightforward way, we can introduce an operator $r_h \in \mathcal{L}(\mathbf{V} \cap H^2(\Omega)^3, \boldsymbol{V}_{1,h})$ satisfying

$$\|\boldsymbol{v} - r_h\, \boldsymbol{v}\|_{1,\Omega_h} \leqslant C\, h^\gamma\, \|\boldsymbol{v}\|_{2,\widetilde{\Omega}} \qquad \forall\, \boldsymbol{v} \in H^2(\widetilde{\Omega})^3,\ \boldsymbol{v}|_\Omega \in \mathbf{V}$$

with $\gamma \in [0,1]$ depending on the quality of the approximation of $\boldsymbol{n}$.

The simplest finite element subspace of $L_0^2(\Omega_h)$ approximating $L_0^2(\Omega)$ is the space

$$\mathrm{Q}_h = \{q \in L_0^2(\Omega_h);\ q|_T \in P_0(T)\ \ \forall\, T \in \mathcal{T}_h\}.$$

An operator $s_h \in \mathcal{L}(L_0^2(\Omega) \cap H^1(\Omega), \mathrm{Q}_h)$ defined using L^2 projections onto elements of the triangulation (see e.g. [5], pp. 102 and 126) satisfies

$$\|q - s_h\, q\|_{0,\Omega_h} \leqslant C\, h\, \|q\|_{1,\widetilde{\Omega}} \qquad \forall\, q \in H^1(\widetilde{\Omega}),\ q|_\Omega \in L_0^2(\Omega).$$

Unfortunately, due to the saddle-point character of (2.2)–(2.4), the spaces $\boldsymbol{V}_{1,h}$, Q_h cannot be used in a finite element discretization of (1.1)–(1.5) since they do not satisfy the so-called Babuška-Brezzi condition (cf. assumption A3 below). Therefore, we have to enlarge the space $\boldsymbol{V}_{1,h}$. For example, we can set

$$\mathbf{V}_h = \boldsymbol{V}_{1,h} \oplus \operatorname{span}\{p_{T'}\, \boldsymbol{n}_{T'}\}_{T' \not\subset \partial\Omega_h},$$

where $\boldsymbol{n}_{T'}$ are normal vectors to the faces T' and $p_{T'} \in H^1_0(\Omega_h)$ are piecewise cubic functions assigned to inner faces T' of the triangulation which have their supports in the two elements adjacent to T' and satisfy $\int_{T'} p_{T'}\, \mathrm{d}\sigma = |T'|$ (see e.g. [5], pp. 144–146).

In the sequel, we shall formulate general abstract assumptions on the spaces $\mathbf{V}_h$ and Q_h approximating the spaces $\mathbf{V}$ and $L^2_0(\Omega)$, respectively, which are fulfilled in particular for the spaces from Example 3.1.

We choose an integer $k \geqslant 1$ and assume that we are given spaces

$$\mathbf{V}_h \subset \{\boldsymbol{v} \in C(\overline{\Omega}_h)^3;\ \boldsymbol{v}|_T \in P_k(T)^3 \quad \forall\, T \in \mathcal{T}_h,\ \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma^D_h\}$$

and $\mathrm{Q}_h \subset L^2_0(\Omega_h)$ possessing the following properties:

A1: There exist an integer $l_1 \geqslant 2$, a real number $\gamma_1 > 0$ and an operator $r_h \in \mathcal{L}(\mathbf{V} \cap H^{l_1}(\Omega)^3, \mathbf{V}_h)$ such that

$$\|\boldsymbol{v} - r_h\, \boldsymbol{v}\|_{1,\Omega_h} \leqslant C\, h^{\gamma_1}\, \|\boldsymbol{v}\|_{l_1,\widetilde{\Omega}} \quad \forall\, \boldsymbol{v} \in H^{l_1}(\widetilde{\Omega})^3,\ \boldsymbol{v}|_\Omega \in \mathbf{V}.$$

Any $\boldsymbol{v}_h \in \mathbf{V}_h$ satisfies $\boldsymbol{v}_h \cdot \boldsymbol{n}^*_h = 0$ at any vertex of $\mathcal{T}_h$ lying in Γ^N_h, where $\boldsymbol{n}^*_h$ is a function defined at the vertices of $\mathcal{T}_h$ and satisfying for some fixed $\alpha \in (0,1]$

$$|\boldsymbol{n}^*_h(x) - \boldsymbol{n}(x)| \leqslant C\, h_T^{1+\alpha} \quad \text{at any vertex } x \in \Gamma^N_h, \tag{3.2}$$

with $T \in \mathcal{T}_h$ being any element containing the vertex x.

A2: There exist an integer $l_2 \geqslant 1$, a real number $\gamma_2 > 0$ and an operator $s_h \in \mathcal{L}(L^2_0(\Omega) \cap H^{l_2}(\Omega), \mathrm{Q}_h)$ such that

$$\|q - s_h\, q\|_{0,\Omega_h} \leqslant C\, h^{\gamma_2}\, \|q\|_{l_2,\widetilde{\Omega}} \qquad \forall\, q \in H^{l_2}(\widetilde{\Omega}),\ q|_\Omega \in L^2_0(\Omega).$$

A3: There exists a constant $\beta > 0$ independent of h such that

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h, \boldsymbol{v}_h \neq \mathbf{0}} \frac{\displaystyle\int_{\Omega_h} q_h \operatorname{div} \boldsymbol{v}_h\, \mathrm{d}x}{\|\boldsymbol{v}_h\|_{1,\Omega_h}} \geqslant \beta\, \|q_h\|_{0,\Omega_h} \qquad \forall\, q_h \in \mathrm{Q}_h. \tag{3.3}$$

R e m a r k 3.1. Due to the assumption on the extension $\boldsymbol{m}$, the normal vector $\boldsymbol{n}$ is continuous on Γ^N so that (3.2) makes sense.

R e m a r k 3.2. The piecewise polynomial character of the functions from $\mathbf{V}_h$ simplifies the proof of Theorem 5.1. However, it would be also possible to consider more general finite element functions.

R e m a r k 3.3. Note that the only assumptions made on the space Q_h are the inclusion $\mathrm{Q}_h \subset L^2_0(\Omega_h)$ and assumptions A2 and A3.

R e m a r k 3.4. Further examples of spaces $\mathbf{V}_h$ and Q_h satisfying the above assumptions will be given in Section 7.

Since the finite element functions are defined on Ω_h instead of Ω, we replace the forms a, b and $\boldsymbol{g}$ by

$$\begin{aligned} a_h(\boldsymbol{u},\boldsymbol{v}) &= \frac{\nu}{2}\int_{\Omega_h} (\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^{\mathrm{T}})\cdot(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^{\mathrm{T}})\,\mathrm{d}x,\\ b_h(\boldsymbol{v},p) &= -\int_{\Omega_h} p\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x,\\ \langle \boldsymbol{g}_h^*,\boldsymbol{v}\rangle &= \int_{\Omega_h} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \nu\int_{\Gamma_h^N} \boldsymbol{\varphi}\cdot\boldsymbol{v}\,\mathrm{d}\sigma, \end{aligned}$$

respectively, where $\boldsymbol{f}$ and $\boldsymbol{\varphi}$ are now considered as extended onto $\widetilde{\Omega}$, i.e.,

$$\boldsymbol{f}\in L^{\frac{6}{5}}(\widetilde{\Omega})^3, \qquad \boldsymbol{\varphi}\in H^1(\widetilde{\Omega})^3.$$

Such extensions exist and can be defined in various ways. Further, we introduce a function $\widetilde{\boldsymbol{u}}_{bh}\in H^1(\Omega_h)^3$ approximating the Dirichlet boundary condition $\boldsymbol{u}_b$ in the sense that

$$\lim_{h\to 0}\|E\,\widetilde{\boldsymbol{u}}_b-\widetilde{\boldsymbol{u}}_{bh}\|_{1,\Omega_h}=0, \tag{3.4}$$

where $E\colon H^1(\Omega)^3\to H^1(\widetilde{\Omega})^3$ is an extension operator and $\widetilde{\boldsymbol{u}}_b\in H^1(\Omega)^3$ is an arbitrary but fixed function satisfying (2.1). In view of inequality (4.1) in the next section, the validity of (3.4) is not influenced by the choice of E. (In fact, here we use the notation $E\,\widetilde{\boldsymbol{u}}_b$ instead of "any function $\bar{\boldsymbol{u}}_b\in H^1(\widetilde{\Omega})^3$ satisfying $\bar{\boldsymbol{u}}_b|_\Omega=\widetilde{\boldsymbol{u}}_b$".) If $\widetilde{\boldsymbol{u}}_b\in H^{l_1}(\Omega)^3$, where l_1 is the integer from assumption A1, then the function $\widetilde{\boldsymbol{u}}_{bh}$ is usually defined as $\widetilde{\boldsymbol{u}}_{bh}=\widetilde{r}_h\,\widetilde{\boldsymbol{u}}_b$, where $\widetilde{r}_h\in\mathcal{L}(H^{l_1}(\Omega)^3,H^1(\Omega_h)^3)$ is an operator which maps functions from $H^{l_1}(\Omega)^3$ into a finite element space and satisfies

$$\|\boldsymbol{v}-\widetilde{r}_h\,\boldsymbol{v}\|_{1,\Omega_h}\leqslant C\,h^{\gamma_1}\,\|\boldsymbol{v}\|_{l_1,\widetilde{\Omega}} \qquad \forall\,\boldsymbol{v}\in H^{l_1}(\widetilde{\Omega})^3.$$

Since typical finite element functions are piecewise polynomial, we can assume that an exact evaluation of the bilinear forms a_h and b_h requires only a small number of computational operations. This, however, cannot be assumed in the case of the functional $\boldsymbol{g}_h^*$ and therefore we replace it by some easily computable functional $\boldsymbol{g}_h \in [\mathbf{V}_h]'$ satisfying

$$\lim_{h\to 0} \|\boldsymbol{g}_h - \boldsymbol{g}_h^*\|_{[\mathbf{V}_h]'} = 0. \tag{3.5}$$

For example, if $\boldsymbol{f}$ and $\boldsymbol{\varphi}$ are continuous, we can obtain $\boldsymbol{g}_h$ by evalutaing $\boldsymbol{g}_h^*$ using numerical integration (cf. [2], Chapter IV) or by replacing $\boldsymbol{f}$ and $\boldsymbol{\varphi}$ by piecewise polynomial functions. In the latter case, defining $\boldsymbol{g}_h$ using piecewise linear interpolates of $\boldsymbol{f}$ and $\boldsymbol{\varphi}$, we have $\|\boldsymbol{g}_h - \boldsymbol{g}_h^*\|_{[\mathbf{V}_h]'} \leqslant C\, h^{\frac{3}{2}}\, (|\boldsymbol{f}|_{2,\widetilde{\Omega}} + |\boldsymbol{\varphi}|_{2,\widetilde{\Omega}})$ for any $\boldsymbol{f}, \boldsymbol{\varphi} \in H^2(\widetilde{\Omega})^3$ and, moreover, $\boldsymbol{g}_h$ depends only on the values of $\boldsymbol{f}|_{\overline{\Omega}}$ and $\boldsymbol{\varphi}|_{\overline{\Gamma}^N}$.

Now we can introduce a family of discrete problems corresponding to the weak formulation (2.2)–(2.4).

Definition 3.1. The functions $\boldsymbol{u}_h$, p_h are a *discrete solution* of problem (1.1)–(1.5) if

$$\boldsymbol{u}_h - \widetilde{\boldsymbol{u}}_{bh} \in \mathbf{V}_h, \qquad p_h \in \mathrm{Q}_h \tag{3.6}$$

and

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = \langle \boldsymbol{g}_h, \boldsymbol{v}_h \rangle \qquad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h, \tag{3.7}$$

$$b_h(\boldsymbol{u}_h, q_h) = 0 \qquad \forall\, q_h \in \mathrm{Q}_h. \tag{3.8}$$

It is easy to show that problem (3.6)–(3.8) has a unique solution (cf. [7]).

Remark 3.5. It is of advantage to construct the function $\widetilde{\boldsymbol{u}}_{bh}$ and the space $\mathbf{V}_h$ in such a way that $\int_{\partial\Omega_h} \widetilde{\boldsymbol{u}}_{bh} \cdot \boldsymbol{n}_h \,\mathrm{d}\sigma = 0$ (cf. Remark 3.6) and

$$\int_{\partial\Omega_h} \boldsymbol{v}_h \cdot \boldsymbol{n}_h \,\mathrm{d}\sigma = 0 \qquad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h \tag{3.9}$$

(cf. Remark 3.7), where $\boldsymbol{n}_h$ is the unit outward normal vector to $\partial\Omega_h$. Then the equation (3.8) is satisfied for any $q_h \in \mathrm{Q}_h \oplus \mathbb{R}$ and $b_h(\boldsymbol{v}_h, p_h) = b_h(\boldsymbol{v}_h, p_h + c)$ holds for any $\boldsymbol{v}_h \in \mathbf{V}_h$, $c \in \mathbb{R}$. Therefore, the stiffness matrix corresponding to the bilinear form b_h can be constructed by using basis functions from $\mathrm{Q}_h \oplus \mathbb{R}$ which do not belong to $L^2_0(\Omega_h)$. This increases the sparsity of this stiffness matrix.

Remark 3.6. Let us describe a way how to construct a function $\widetilde{\boldsymbol{u}}_{bh} \in H^1(\Omega_h)^3$ satisfying $\int_{\partial\Omega_h} \widetilde{\boldsymbol{u}}_{bh} \cdot \boldsymbol{n}_h \,\mathrm{d}\sigma = 0$ and approximating a function $\widetilde{\boldsymbol{u}}_b \in H^2(\widetilde{\Omega})^3$ with $\int_{\partial\Omega} \widetilde{\boldsymbol{u}}_b \cdot \boldsymbol{n} \,\mathrm{d}\sigma = 0$. First we introduce a function $\boldsymbol{q} \in H^2(\widetilde{\Omega})^3$ satisfying $\int_{\partial\Omega} \boldsymbol{q} \cdot \boldsymbol{n} \,\mathrm{d}\sigma \neq$

0. For small h we have $\int_{\partial\Omega_h} \boldsymbol{q}\cdot\boldsymbol{n}_h \,\mathrm{d}\sigma \neq 0$ and we can set $\alpha_h = \int_{\partial\Omega_h} \widetilde{\boldsymbol{u}}_b\cdot\boldsymbol{n}_h \,\mathrm{d}\sigma / \int_{\partial\Omega_h} \boldsymbol{q}\cdot \boldsymbol{n}_h \,\mathrm{d}\sigma$. Then $|\alpha_h| \leqslant C\,h$ and, denoting $\widetilde{\boldsymbol{u}}_{bh}^* = \widetilde{\boldsymbol{u}}_b - \alpha_h\,\boldsymbol{q}$, we have $\int_{\partial\Omega_h} \widetilde{\boldsymbol{u}}_{bh}^*\cdot\boldsymbol{n}_h \,\mathrm{d}\sigma = 0$. For any face $T' \subset \partial\Omega_h$ of an element $T \in \mathcal{T}_h$ we introduce a function $p_{T'} \in H^1(\Omega_h)$ with $\operatorname{supp} p_{T'} = T$, $p_{T'}|_T \in P_3(T)$, $p_{T'}|_{\partial T\setminus T'} = 0$ and $\int_{T'} p_{T'} \,\mathrm{d}\sigma = |T'|$. Further, we denote by $\boldsymbol{n}_{T'}$ the unit outward (with respect to $\partial\Omega_h$) normal vector to T'. Let $\widetilde{r}_h \in \mathcal{L}(H^2(\Omega)^3, [\mathrm{V}_{1,h}]^3)$ be the Lagrange interpolation operator from Example 3.1. Setting

$$\widetilde{\boldsymbol{u}}_{bh} = \widetilde{r}_h\widetilde{\boldsymbol{u}}_{bh}^* + \sum_{T'\subset\partial\Omega_h} \alpha_{T'}\,p_{T'}\,\boldsymbol{n}_{T'} \qquad \text{with} \qquad \alpha_{T'} = \frac{1}{|T'|}\int_{T'} (\widetilde{\boldsymbol{u}}_{bh}^* - \widetilde{r}_h\widetilde{\boldsymbol{u}}_{bh}^*)\cdot\boldsymbol{n}_{T'} \,\mathrm{d}\sigma,$$

we have $\widetilde{\boldsymbol{u}}_{bh} \in H^1(\Omega_h)^3$ and $\int_{\partial\Omega_h} \widetilde{\boldsymbol{u}}_{bh}\cdot\boldsymbol{n}_h \,\mathrm{d}\sigma = 0$. Since

$$\frac{1}{|T'|}\int_{T'} \boldsymbol{v}\cdot\boldsymbol{n}_{T'} \,\mathrm{d}\sigma \leqslant C\,(h_T^{-3/2}\,\|\boldsymbol{v}\|_{0,T} + h_T^{-1/2}\,|\boldsymbol{v}|_{1,T}) \qquad \forall\,\boldsymbol{v} \in H^1(\Omega_h)^3$$

and $\|p_{T'}\|_{1,T} \leqslant C\,h_T^{1/2}$, where T is the element adjacent to T', we also obtain $\|\widetilde{\boldsymbol{u}}_b - \widetilde{\boldsymbol{u}}_{bh}\|_{1,\Omega_h} \leqslant C\,h$.

R e m a r k 3.7. If all traces of functions from $\mathbf{V}_h$ coincide with traces of piecewise linear functions from $\mathbf{V}_h$ (which is the case for $\mathbf{V}_h$ from Example 3.1 and for some of the examples of $\mathbf{V}_h$ in Section 7), then condition (3.9) is satisfied for the following choice of the function $\boldsymbol{n}_h^*$: at any vertex $x \in \Gamma_h^N$ we set

$$\boldsymbol{n}_h^*(x) = \frac{\boldsymbol{n}_h'(x)}{|\boldsymbol{n}_h'(x)|} \qquad \text{with} \qquad \boldsymbol{n}_h'(x) = \sum_{T'\subset\partial\Omega_h,\, x\in\overline{T'}} \operatorname{meas}_2(T')\,\boldsymbol{n}_{T'}, \tag{3.10}$$

where $\boldsymbol{n}_{T'}$ denotes the unit outward normal vector to the boundary face T'. Let us show that (3.9) really holds. Consider an arbitrary vertex $x^* \in \Gamma^N$. It suffices to show that (3.9) is satisfied for a piecewise linear function $\boldsymbol{v}_h^* \in \mathbf{V}_h$ satisfying $\boldsymbol{v}_h^*(x) = \mathbf{0}$ at any vertex $x \neq x^*$. For such a $\boldsymbol{v}_h^*$, we have

$$\begin{aligned}\int_{\partial\Omega_h} \boldsymbol{v}_h^*\cdot\boldsymbol{n}_h \,\mathrm{d}\sigma &= \sum_{T'\subset\partial\Omega_h,\, x^*\in\overline{T'}} \int_{T'} \boldsymbol{v}_h^*\cdot\boldsymbol{n}_{T'} \,\mathrm{d}\sigma \\ &= \frac{1}{3}\,\boldsymbol{v}_h^*(x^*)\cdot \sum_{T'\subset\partial\Omega_h,\, x^*\in\overline{T'}} \operatorname{meas}_2(T')\,\boldsymbol{n}_{T'} = \frac{1}{3}\,(\boldsymbol{v}_h^*\cdot\boldsymbol{n}_h')(x^*) = 0.\end{aligned}$$

If Γ^N is a C^3 surface and some uniformity assumptions on the triangulations of Γ^N are satisfied, then it can be shown that $\boldsymbol{n}_h^*$ defined by (3.10) satisfies (3.2) (cf. [6], pp. 132–135).

4. Auxiliary results

In this section we summarize a lot of auxiliary results which will be needed in the subsequent two sections. First, in Lemmas 4.1–4.3, we present several results, the proofs of which can be found in [7].

Lemma 4.1. *There exists a constant $C > 0$ independent of h such that*

$$\text{meas}_3(\Omega \setminus \Omega_h \cup \Omega_h \setminus \Omega) \leqslant C\,h^2, \tag{4.1}$$

$$\|v\|_{0,\partial\Omega_h} \leqslant C\,\|v\|_{1,\Omega_h} \qquad \forall\, v \in H^1(\Omega_h), \tag{4.2}$$

$$\left| \int_{\Gamma^N} \theta\, v \,\mathrm{d}\sigma - \int_{\Gamma^N_h} \theta\, v \,\mathrm{d}\sigma \right| \leqslant C\,h\,\|\theta\|_{1,\widetilde{\Omega}}\,\|v\|_{1,\widetilde{\Omega}} \qquad \forall\, \theta, v \in H^1(\widetilde{\Omega}), \tag{4.3}$$

$$C\,\|\boldsymbol{v}_h\|^2_{1,\Omega_h} \leqslant a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \qquad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h. \tag{4.4}$$

Lemma 4.2. *Any face $T' \subset \partial\Omega_h$ can be associated with a set $\Gamma_{T'} \subset \partial\Omega$ in such a way that*

$$\overline{\Gamma}^D = \bigcup_{T' \subset \Gamma^D_h} \overline{\Gamma}_{T'}, \qquad \overline{\Gamma}^N = \bigcup_{T' \subset \Gamma^N_h} \overline{\Gamma}_{T'}$$

and, for any $T' \subset \partial\Omega_h$,

$$\Gamma_{T'} \subset \{\, x \in \mathbb{R}^3\,;\ \text{dist}(x, T') < C\,h^2_{T'}\},\ T' \subset \{\, x \in \mathbb{R}^3\,;\ \text{dist}(x, \Gamma_{T'}) < C\,h^2_{T'}\}, \tag{4.5}$$

where $h_{T'} = \text{diam}(T')$ and C is independent of T' and h. In particular,

$$\Gamma^D_h \subset \{x \in \mathbb{R}^3\,;\ \text{dist}(x, \Gamma^D) < C\,h^2\}$$

and hence

$$\text{dist}(x, \Gamma^D) \leqslant \text{dist}(x, \Gamma^D_h) + C\,h^2 \qquad \forall\, x \in \mathbb{R}^3. \tag{4.6}$$

Lemma 4.3. *There exists an extension operator $E_h \in \mathcal{L}(H^1(\Omega_h), H^1_0(\widetilde{\Omega}))$ satisfying*

$$(E_h\, v)|_{\Omega_h} = v \qquad \text{and} \qquad \|E_h\, v\|_{1,\widetilde{\Omega}} \leqslant C\,\|v\|_{1,\Omega_h} \qquad \forall\, v \in H^1(\Omega_h),$$

where the constant C is independent of h.

The operator E_h was constructed in [7] using the Nikolskij method (cf. [8], p. 75, Theorem 3.9). Since the properties of E_h play an important role in this paper, we repeat the precise definition of E_h here.

In view of the Lipschitz continuity of $\partial\Omega$, there exist positive real numbers a and b, M local Cartesian coordinate systems and functions $f_1, \ldots, f_M \in C^{0,1}([-a,a]^2)$ such that, in the r-th local coordinate system ($r = 1, \ldots, M$), we have for any $\overline{x} \in (-a,a)^2$

$$\begin{aligned}
&(\overline{x}, f_r(\overline{x})) \in \partial\Omega, \\
&f_r(\overline{x}) < x_3 < f_r(\overline{x}) + b \quad \Rightarrow \quad (\overline{x}, x_3) \in \Omega, \\
&f_r(\overline{x}) - b < x_3 < f_r(\overline{x}) \quad \Rightarrow \quad (\overline{x}, x_3) \notin \overline{\Omega}
\end{aligned}$$

and $\partial\Omega$ is covered by the graphs $\{(\overline{x}, f_r(\overline{x}));\ \overline{x} \in (-a/4, a/4)^2\}$, $r = 1, \ldots, M$. We will suppose that $h < \min\{a/8, b/8\}$ and $\operatorname{dist}(\partial\widetilde{\Omega}, \partial\Omega) > \max\{a, b\}$. In each local coordinate system we introduce a continuous piecewise linear function f_{rh} defined on $[-3a/4, 3a/4]^2$ and describing the corresponding part of $\partial\Omega_h$. All M graphs of these functions cover the whole boundary of Ω_h and there exists a constant $\widetilde{C}$ independent of h such that, for $r = 1, \ldots, M$, we have

$$\|f_r - f_{rh}\|_{0,\infty,(-3a/4,3a/4)^2} \leqslant \widetilde{C}\, h^2, \qquad |f_{rh}|_{1,\infty,(-3a/4,3a/4)^2} \leqslant \widetilde{C}. \tag{4.7}$$

For $r = 1, \ldots, M$ we define, in the local coordinate systems, sets

$$\begin{aligned}
\mathrm{U}_r &= \{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in (-a/2, a/2)^2,\ f_r(\overline{x}) - b/2 < x_3 < f_r(\overline{x}) + b/2\}, \\
\widetilde{\mathrm{U}}_r &= \{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in (-3a/8, 3a/8)^2,\ f_r(\overline{x}) - b/4 < x_3 < f_r(\overline{x}) + b/4\}.
\end{aligned}$$

Then, for h sufficiently small, we have

$$\mathrm{U}_r \cap \Omega_h = \{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in (-a/2, a/2)^2,\ f_{rh}(\overline{x}) < x_3 < f_r(\overline{x}) + b/2\}.$$

Further, we introduce an open set $\mathrm{U}_{M+1} \subset \Omega$ such that

$$\Omega \subset \bigcup_{r=1}^{M+1} \mathrm{U}_r \qquad \text{and} \qquad \widetilde{\mathrm{U}}_r \cap \mathrm{U}_{M+1} = \emptyset, \quad r = 1, \ldots, M.$$

Then, again for h sufficiently small,

$$\mathrm{U}_{M+1} \subset \Omega_h \qquad \text{and} \qquad \Omega_h \subset \Omega \cup \bigcup_{r=1}^{M} \widetilde{\mathrm{U}}_r.$$

We define an operator $E_{M+1}\colon\ H_0^1(\mathrm{U}_{M+1}) \to H_0^1(\widetilde{\Omega})$ by

$$E_{M+1}\,v = \begin{cases} v & \text{in } \mathrm{U}_{M+1}, \\ 0 & \text{elsewhere,} \end{cases}$$

where $v \in H_0^1(\mathrm{U}_{M+1})$. For $r = 1, \dots, M$, we denote

$$V_{rh} = \{v \in H^1(\mathrm{U}_r \cap \Omega_h);\ v = 0 \text{ on } \partial(\mathrm{U}_r \cap \Omega_h) \setminus \partial\Omega_h\}$$

and define operators $E_{rh}\colon\ V_{rh} \to H_0^1(\widetilde{\Omega})$ in the following way. We extend any $v \in V_{rh}$ by zero in

$$\{(\overline{x}, x_3) \in \mathbb{R}^3\,;\ \overline{x} \in (-a/2, a/2)^2,\ x_3 \geqslant f_r(\overline{x}) + b/2\}$$

and set

$$(E_{rh}\,v)(\overline{x}, x_3) = \begin{cases} v(\overline{x}, x_3) & \text{for } \overline{x} \in (-a/2, a/2)^2,\ x_3 \geqslant f_{rh}(\overline{x}), \\ v(\overline{x}, 2\,f_{rh}(\overline{x}) - x_3) & \text{for } \overline{x} \in (-a/2, a/2)^2,\ x_3 \leqslant f_{rh}(\overline{x}), \\ 0 & \text{elsewhere.} \end{cases}$$

For small h we have $\Omega_h \cap \operatorname{supp}(E_{rh}\,v) \subset \mathrm{U}_r$. According to [8], p. 27, Proposition 2.3, there exist functions $\psi_r \in C_0^\infty(\mathrm{U}_r)$, $r = 1, \dots, M+1$, such that

$$\sum_{r=1}^{M+1} \psi_r(x) = 1 \qquad \forall\ x \in \Omega \cup \bigcup_{r=1}^{M} \widetilde{\mathrm{U}}_r$$

and we set

$$E_h\,v = E_{M+1}(v\,\psi_{M+1}) + \sum_{r=1}^{M} E_{rh}(v\,\psi_r), \qquad v \in H^1(\Omega_h),$$

which completes the definition of E_h. We remark that here and in the following statement, we regard a function $v \in H^1(\Omega_h)$ as a fixed representative of the corresponding equivalence class so that a uniquely determined value of v is known at any point of Ω.

Now, we can prove the following localization property of E_h.

Lemma 4.4. *For h sufficiently small, any $v \in H^1(\Omega_h)$ and any $x \in \widetilde{\Omega} \setminus \overline{\Omega}_h$ for which $E_h\,v(x) \neq 0$, there exists $x^* \in \Omega_h$ with $v(x^*) \neq 0$ satisfying*

$$(1/\overline{C})\ \operatorname{dist}(x, \Sigma) \leqslant \operatorname{dist}(x^*, \Sigma) \leqslant \overline{C}\ \operatorname{dist}(x, \Sigma) \qquad \textit{for any set } \Sigma \subset \partial\Omega_h, \tag{4.8}$$

where $\overline{C}$ depends only on Ω and the constant $\widetilde{C}$ from (4.7).

P r o o f. For $r = 1, \dots, M$ we introduce, in the local coordinate systems, sets

$$\begin{aligned}\overline{\mathrm{U}}_r &= \{(\overline{x}, x_3) \in \mathbb{R}^3 \, ; \; \overline{x} \in (-a/2, a/2)^2, \; f_r(\overline{x}) - 5b/8 < x_3 < f_r(\overline{x}) + 5b/8\},\\ \widehat{\mathrm{U}}_r &= \{(\overline{x}, x_3) \in \mathbb{R}^3 \, ; \; \overline{x} \in (-3a/4, 3a/4)^2, \; f_r(\overline{x}) - 3b/4 < x_3 < f_r(\overline{x}) + 3b/4\}\end{aligned}$$

and denote

$$\varepsilon = \min_{r=1,\dots,M} \operatorname{dist}(\partial \overline{\mathrm{U}}_r, \partial \widehat{\mathrm{U}}_r).$$

Let us consider any $v \in H^1(\Omega_h)$ and any $x \in \widetilde{\Omega} \setminus \overline{\Omega}_h$ satisfying $E_h\, v(x) \neq 0$. Then there exists $r \in \{1, \dots, M\}$ such that $E_{rh}(v\, \psi_r)(x) \neq 0$ and therefore, in the r-th local coordinate system, we have $v(\overline{x}, 2\, f_{rh}(\overline{x}) - x_3) \neq 0$, where $(\overline{x}, x_3) = x$. We denote $x^* = (\overline{x}, 2\, f_{rh}(\overline{x}) - x_3)$. Then $x^* \in \mathrm{U}_r \cap \Omega_h$ and hence it follows from (4.7) that $x \in \overline{\mathrm{U}}_r$ for $h^2 < b/(16\, \widetilde{C})$. Consider any $\Sigma \subset \partial\Omega_h$. If $\operatorname{dist}(x^*, \Sigma) \geqslant \varepsilon$, then $\operatorname{dist}(x, \Sigma) \leqslant \operatorname{diam}(\widetilde{\Omega}) \leqslant [\operatorname{diam}(\widetilde{\Omega})/\varepsilon] \operatorname{dist}(x^*, \Sigma)$. Thus, it suffices to consider $\operatorname{dist}(x^*, \Sigma) < \varepsilon$. Let $z^* \in \overline{\Sigma}$ be such that $|x^* - z^*| = \operatorname{dist}(x^*, \Sigma)$. Then $z^* \in \widehat{\mathrm{U}}_r$ and since, for h sufficiently small,

$$\widehat{\mathrm{U}}_r \cap \partial\Omega_h = \{(\overline{x}, x_3) \in \widehat{\mathrm{U}}_r \, ; \; x_3 = f_{rh}(\overline{x})\},$$

we have $z^* = (\overline{z}^*, f_{rh}(\overline{z}^*))$. Setting $y = (\overline{x}, f_{rh}(\overline{x})) = (\overline{x}^*, f_{rh}(\overline{x}^*))$ and applying (4.7), we obtain

$$|y - z^*|^2 = |\overline{x}^* - \overline{z}^*|^2 + |f_{rh}(\overline{x}^*) - f_{rh}(\overline{z}^*)|^2 \leqslant (1 + \widetilde{C}^2)\, |\overline{x}^* - \overline{z}^*|^2 \leqslant (1 + \widetilde{C}^2)\, |x^* - z^*|^2.$$

Therefore,

$$\begin{aligned}\operatorname{dist}(x, \Sigma) &\leqslant |x - z^*| \leqslant |x - y| + |y - z^*| = |x^* - y| + |y - z^*|\\ &\leqslant |x^* - z^*| + 2\, |y - z^*| \leqslant \Big(1 + 2\sqrt{1 + \widetilde{C}^2}\Big) \operatorname{dist}(x^*, \Sigma),\end{aligned}$$

which proves the first part of (4.8). The second part follows analogously. □

Corollary 4.1. *For h sufficiently small and any $v \in H^1(\Omega_h) \cap C(\overline{\Omega}_h)$ we have*

$$\sup_{x \in \operatorname{supp}(E_h\, v)} \operatorname{dist}(x, \Gamma_h^D) \leqslant \overline{C} \sup_{x \in \operatorname{supp} v} \operatorname{dist}(x, \Gamma_h^D), \tag{4.9}$$

$$\inf_{x \in \operatorname{supp}\, v} \operatorname{dist}(x, T') \leqslant \overline{C} \inf_{x \in \operatorname{supp}(E_h\, v)} \operatorname{dist}(x, T'), \tag{4.10}$$

where $T' \subset \partial\Omega_h$ is any boundary face of $\mathcal{T}_h$ and $\overline{C}$ is the constant from Lemma 4.4.

P r o o f. Let $E_h\, v(x) \neq 0$ for some $x \in \widetilde{\Omega} \setminus \overline{\Omega}_h$. Then, according to Lemma 4.4, there exists $x^* \in \Omega_h$ with $v(x^*) \neq 0$ satisfying

$$\operatorname{dist}(x, \Gamma_h^D) \leqslant \overline{C}\ \operatorname{dist}(x^*, \Gamma_h^D) \leqslant \overline{C} \sup_{x^* \in \operatorname{supp}\, v} \operatorname{dist}(x^*, \Gamma_h^D),$$
$$\overline{C}\ \operatorname{dist}(x, T') \geqslant \operatorname{dist}(x^*, T') \geqslant \inf_{x^* \in \operatorname{supp}\, v} \operatorname{dist}(x^*, T') \qquad \forall\, T' \subset \partial\Omega_h.$$

Since $\operatorname{dist}(\cdot, \Sigma)$ is continuous for any $\Sigma \subset \partial\Omega_h$, we obtain the corollary. □

Using the above notation, we can also prove the following result.

Lemma 4.5. *There exists a constant C such that, for any set $\Sigma \subset \Gamma^N$, we have*

$$\operatorname{dist}(x, \partial\Sigma) \leqslant C\ \operatorname{dist}(x, \Gamma^D) \qquad \forall\, x \in \Sigma. \tag{4.11}$$

P r o o f. Let $\overline{\mathrm{U}}_r$, $\widehat{\mathrm{U}}_r$ and ε be like in the proof of Lemma 4.4 and let us consider any $x \in \Sigma$. If $\operatorname{dist}(x, \Gamma^D) \geqslant \varepsilon$, then (4.11) holds with $C = \operatorname{diam}(\Omega)/\varepsilon$. Thus, let $\operatorname{dist}(x, \Gamma^D) < \varepsilon$. Let $r \in \{1, \ldots, M\}$ be such that $x \in \overline{\mathrm{U}}_r$ and let $y \in \overline{\Gamma}^D$ satisfy $|x - y| = \operatorname{dist}(x, \Gamma^D)$. Then $y \in \widehat{\mathrm{U}}_r$ and we have $x = (\overline{x}, f_r(\overline{x}))$, $y = (\overline{y}, f_r(\overline{y}))$. Let $z = (\overline{z}, f_r(\overline{z}))$ be such that $z \in \partial\Sigma$ and $\overline{z} = \alpha\, \overline{x} + (1 - \alpha)\, \overline{y}$ with $\alpha \in [0, 1]$. Then $|\overline{x} - \overline{z}| \leqslant |\overline{x} - \overline{y}| \leqslant |x - y|$ and since

$$|x - z|^2 = |\overline{x} - \overline{z}|^2 + |f_r(\overline{x}) - f_r(\overline{z})|^2 \leqslant (1 + |f_r|^2_{1,\infty,(-a,a)^2})\, |\overline{x} - \overline{z}|^2,$$

we obtain the lemma. □

Lemma 4.6. *Denoting*

$$U_h(A) = \{x \in \widetilde{\Omega};\ \operatorname{dist}(x, A) \leqslant C\, h\} \tag{4.12}$$

for any set $A \subset \widetilde{\Omega}$, we have

$$\int_{\Gamma^N \cap U_h(\Gamma^D)} 1\, \mathrm{d}\sigma \leqslant \widetilde{C}\, h. \tag{4.13}$$

P r o o f. Consider any $i \in \{K^D{+}1, \ldots, K\}$ and let the M local coordinate systems introduced above be the ones from the definition of the $C^{0,1*}$ property of Γ_i. The only difference from the situation above is that $\partial\Omega$ is covered by the graphs $\{(\overline{x}, f_r(\overline{x}))$; $\overline{x} \in (-a, a)^2\}$, $r = 1, \ldots, M$, but generally not by the graphs $\{(\overline{x}, f_r(\overline{x}))$; $\overline{x} \in (-a/4, a/4)^2\}$, $r = 1, \ldots, M$. Let

$$S_{ri} = \{\overline{x} \in (-a, a)^2;\ (\overline{x}, f_r(\overline{x})) \in \Gamma_i\}$$

be the projection of the respective part of Γ_i into the (x_1, x_2)-plane of the r-th local coordinate system and let

$$\Sigma_{ri} = \{(\overline{x}, f_r(\overline{x}));\ \overline{x} \in S_{ri}\}.$$

Lemma 4.5 implies that $\Sigma_{ri} \cap U_h(\Gamma^D) \subset \Sigma_{ri} \cap U_h(\partial\Sigma_{ri})$ and hence, for proving (4.13), it suffices to show that

$$\int_{\Sigma_{ri}\cap U_h(\partial\Sigma_{ri})} 1 \,\mathrm{d}\sigma \leqslant \int_{S_{ri}\cap U_h(\partial S_{ri})} \sqrt{1 + |\nabla f_r(\overline{x})|^2}\,\mathrm{d}\sigma \leqslant C\,h, \qquad r = 1, \ldots, M,$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$. This immediately follows from the Lipschitz continuity of ∂S_{ri}. $\square$

Further, let us prove the assertion from Remark 1.1.

Lemma 4.7. *Let Γ^N be a $C^{k,1}$ surface of class $C^{0,1*}$, where $k \geqslant 1$. Then there exists a function $\boldsymbol{m} \in W^{k,\infty}(\mathbb{R}^3)^3$ satisfying $\boldsymbol{m}|_{\Gamma^N} = \boldsymbol{n}|_{\Gamma^N}$.*

P r o o f. Like in the proof of Lemma 4.6, we assume that the M local coordinate systems introduced above are the ones from the definition of the $C^{0,1*}$ property of Γ^N and we denote, for $r = 1, \ldots, M$,

$$S_r = \{\overline{x} \in (-a, a)^2;\ (\overline{x}, f_r(\overline{x})) \in \Gamma^N\}, \qquad \Sigma_r = \{(\overline{x}, f_r(\overline{x}));\ \overline{x} \in S_r\}.$$

Note that $f_r|_{S_r} \in C^{k,1}(\overline{S_r})$ and hence, setting

$$\boldsymbol{m}_r(\overline{x}, x_3) = \frac{(\nabla f_r(\overline{x}), -1)}{\sqrt{1 + |\nabla f_r(\overline{x})|^2}} \qquad \text{for} \quad \overline{x} \in S_r,\ x_3 \in \mathbb{R},$$

we obtain a function $\boldsymbol{m}_r \in C^{k-1,1}(\overline{S_r} \times \mathbb{R})^3$ satisfying $\boldsymbol{m}_r|_{\Sigma_r} = \boldsymbol{n}|_{\Sigma_r}$. Since ∂S_r is Lipschitz-continuous, the function $\boldsymbol{m}_r$ can be extended to a function $\boldsymbol{m}_r \in W^{k,\infty}(\mathbb{R}^3)^3$ (cf. [9], p. 181, Theorem 5). We denote, for $r = 1, \ldots, M$,

$$\mathrm{U}_r^* = \{(\overline{x}, x_3) \in \mathbb{R}^3;\ \overline{x} \in (-a, a)^2,\ f_r(\overline{x}) - b < x_3 < f_r(\overline{x}) + b\}.$$

According to [8], p. 27, Proposition 2.3, there exist functions $\psi_r \in C_0^\infty(\mathrm{U}_r^*)$, $r = 1, \ldots, M$, satisfying $\sum\limits_{r=1}^{M} \psi_r(x) = 1$ for any $x \in \partial\Omega$. Setting $\boldsymbol{m} = \sum\limits_{r=1}^{M} \boldsymbol{m}_r\,\psi_r$, we obtain $\boldsymbol{m} \in W^{k,\infty}(\mathbb{R}^3)^3$ with $\boldsymbol{m}|_{\Gamma^N} = \boldsymbol{n}|_{\Gamma^N}$. $\square$

Finally, we derive a few consequences of the regularity assumption (3.1).

Lemma 4.8. *There exist constants K_1, $K_2 > 0$ depending only on σ such that*

$$\operatorname{card}\{T^* \in \mathcal{T}_h\,;\ T^* \cap T \neq \emptyset\} \leqslant K_1 \qquad \forall\, T \in \mathcal{T}_h \tag{4.14}$$

and

$$h_T \leqslant K_2\, h_{T^*} \qquad \forall\, T, T^* \in \mathcal{T}_h,\ T \cap T^* \neq \emptyset. \tag{4.15}$$

P r o o f. Let x^* be any vertex of $\mathcal{T}_h$ and let

$$\mathcal{M} = \{T \in \mathcal{T}_h\,;\ x^* \in T\}, \qquad \mathrm{M} = \operatorname{card} \mathcal{M}.$$

Consider any $T \in \mathcal{M}$ and let x^*, x^1, x^2, x^3 be the vertices of T. We construct a tetrahedron $\widetilde{T}$ having the vertices x^*, $x^* + \alpha\,(x^1 - x^*)$, $x^* + \alpha\,(x^2 - x^*)$, $x^* + \alpha\,(x^3 - x^*)$, where $\alpha > 0$ is such that $\widetilde{T}$ is contained in the unit ball $B = \{x \in \mathbb{R}^3\,;\ |x - x^*| \leqslant 1\}$ and at least one of the vertices of $\widetilde{T}$ lies on ∂B. In this way, we modify all elements from $\mathcal{M}$. Then the interiors of the tetrahedra $\widetilde{T}$ are again disjoint and we have $h_{\widetilde{T}}/\varrho_{\widetilde{T}} = h_T/\varrho_T$ and $h_{\widetilde{T}} \geqslant 1$ for any $\widetilde{T}$. Therefore, using (3.1), we obtain

$$\frac{4}{3}\,\pi = \operatorname{meas}_3(B) \geqslant \sum_{T \in \mathcal{M}} \operatorname{meas}_3(\widetilde{T}) \geqslant \frac{\pi}{6} \sum_{T \in \mathcal{M}} \varrho_{\widetilde{T}}^3 \geqslant \frac{\pi}{6\,\sigma^3} \sum_{T \in \mathcal{M}} h_{\widetilde{T}}^3 \geqslant \frac{\pi\,\mathrm{M}}{6\,\sigma^3},$$

which means that $\mathrm{M} \leqslant 8\,\sigma^3$. Thus, (4.14) holds with $K_1 = 32\,\sigma^3$.

Let $T, T^* \in \mathcal{T}_h$ have a common edge of length l. Then, by (3.1), $h_T \leqslant \sigma\,\varrho_T \leqslant \sigma\,l \leqslant \sigma\,h_{T^*}$. Thus, it follows from the proof of (4.14) that (4.15) holds with $K_2 = \sigma^{8\sigma^3}$. $\square$

Lemma 4.9. *For any $\delta \in (0, 1/(4\,\sigma^3 K_2))$ and for any face T' of the triangulation $\mathcal{T}_h$ we have*

$$O_\delta(T') \equiv \{\, x \in \mathbb{R}^3\,;\ \operatorname{dist}(x, T') < \delta\, h_{T'}\} \subset (\mathbb{R}^3 \setminus \overline{\Omega}_h) \cup \bigcup_{T \in \mathcal{T}_h,\, T \cap \overline{T'} \neq \emptyset} T. \tag{4.16}$$

P r o o f. Let T' be a given face of $\mathcal{T}_h$ and let $Q_{T'}$ be the set on the right-hand side of the inclusion (4.16). Let x be a vertex of T'. A ball around x with the radius $R^1_{h,x} = \min\{\varrho_T\,;\ T \in \mathcal{T}_h,\ x \in T\}$ lies in $Q_{T'}$ and, according to (3.1) and (4.15), we have $R^1_{h,x} \geqslant \min\{h_T/\sigma\,;\ T \in \mathcal{T}_h,\ x \in T\} \geqslant h_{T'}/(\sigma\, K_2)$. Let l be an edge of T' and let $x \in l$ be a point different from the end points x^1, x^2 of l. A ball around x with the radius $R^2_{h,x} = \min\{\varrho_T\,|x - x^1|/l,\ \varrho_T\,|x - x^2|/l\,;\ T \in \mathcal{T}_h,\ l \subset T\}$ lies in $Q_{T'}$ and, in view of (3.1), we have $R^2_{h,x} \geqslant \min\{|x - x^1|, |x - x^2|\}/\sigma$. Therefore, any ball with the radius $h_{T'}/(2\,\sigma^2 K_2)$ around an $x \in \partial T'$ lies in $Q_{T'}$. Consider any $x \in T'$. A ball around x with the radius $R^3_{h,x} = \min\{\varrho_T \operatorname{dist}(x, \partial T')/h_{T'}\,;\ T \in \mathcal{T}_h,\ x \in T\}$ lies in $Q_{T'}$ and since $R^3_{h,x} \geqslant \operatorname{dist}(x, \partial T')/\sigma$, we infer that any ball with the radius $h_{T'}/(4\,\sigma^3 K_2)$ around an $x \in \overline{T'}$ lies in $Q_{T'}$. $\square$

It is appropriate to measure the error of the discrete solution introduced in Definition 3.1 in norms defined on the approximating domain Ω_h. However, the weak solution of (1.1)–(1.5) is defined in Ω and it is generally not known in $\Omega_h \setminus \overline{\Omega}$. Therefore, it is desirable to extend the functions $\boldsymbol{u}$, p to functions defined in $\widetilde{\Omega}$ and belonging to Sobolev spaces of the same type as $\boldsymbol{u}$, p belong to, respectively. We will denote the extension of $\boldsymbol{u}$ (or p) as $E\,\boldsymbol{u}$ (or $E\,p$), where E is an extension operator. The operator E can be constructed, for instance, by the Nikolskij method (cf. [8], p. 75, Theorem 3.9) or by the Calderon method (cf. [8], p. 80, Theorem 3.10). In both cases, E depends on the order of differentiability of $\boldsymbol{u}$ and the Nikolskij operator generally also requires higher regularity of $\partial\Omega$. It is also possible to construct an operator E which simultaneously extends all orders of differentiability with very small requirements on the regularity of $\partial\Omega$ (cf. [9], p. 181, Theorem 5). In what follows, we will not specify which extension operator we are using and the reader can think of some of the above-mentioned operators. Since we will need neither continuity nor linearity of E, we can also imagine that the extension $E\,\boldsymbol{u}$ is defined for each function $\boldsymbol{u}$ in a particular way.

5. Idea of the convergence proof

We shall explain the idea of the convergence proof for (3.6)–(3.8) on the example of the Poisson equation

$$-\Delta\,\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega.$$

Denoting

$$\begin{aligned} \widetilde{a}(\boldsymbol{u},\boldsymbol{v}) &= \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x, &\qquad \langle\boldsymbol{f},\boldsymbol{v}\rangle &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x,\\ \widetilde{a}_h(\boldsymbol{u},\boldsymbol{v}) &= \int_{\Omega_h} \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,\mathrm{d}x, &\qquad \langle\boldsymbol{f}_h,\boldsymbol{v}\rangle &= \int_{\Omega_h} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x, \end{aligned}$$

we can introduce the weak formulation

$$\text{Find } \boldsymbol{u}\in H^1_0(\Omega)^3:\ \widetilde{a}(\boldsymbol{u},\boldsymbol{v}) = \langle\boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v}\in H^1_0(\Omega)^3 \tag{5.1}$$

and the discrete problem

$$\text{Find } \boldsymbol{u}_h\in \mathbf{V}_h:\ \widetilde{a}_h(\boldsymbol{u}_h,\boldsymbol{v}_h) = \langle\boldsymbol{f}_h,\boldsymbol{v}_h\rangle \quad \forall\,\boldsymbol{v}_h\in \mathbf{V}_h, \tag{5.2}$$

where $\mathbf{V}_h$ is the general finite element space from Section 3 with $\Gamma_h^D = \partial\Omega_h$. For any $\boldsymbol{z}_h\in\mathbf{V}_h$, we have

$$|\boldsymbol{u}_h-\boldsymbol{z}_h|^2_{1,\Omega_h} = \widetilde{a}_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{z}_h) + \widetilde{a}_h(\boldsymbol{u}-\boldsymbol{z}_h,\boldsymbol{u}_h-\boldsymbol{z}_h),$$

where $\boldsymbol{u}$ is considered to be extended by zero outside Ω, and hence we obtain using the triangular inequality

$$(5.3)\qquad |\boldsymbol{u}-\boldsymbol{u}_h|_{1,\Omega_h}\leqslant \sup_{\boldsymbol{v}_h\in\mathbf{V}_h,\boldsymbol{v}_h\neq\mathbf{0}}\frac{\widetilde{a}_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}_h)}{|\boldsymbol{v}_h|_{1,\Omega_h}}+2\inf_{\boldsymbol{z}_h\in\mathbf{V}_h}|\boldsymbol{u}-\boldsymbol{z}_h|_{1,\Omega_h}.$$

This is a particular case of the well-known general abstract error estimate already published in [3], p. 414, Theorem 1. We also refer to [11], p. 178, Theorem 4, where a modification of the abstract error estimate is given for the case when nonhomogeneous Dirichlet boundary conditions are used.

According to Lemma 6.1 from the next section, the second term on the right-hand side of (5.3) converges to zero. Thus, for proving that $|\boldsymbol{u}-\boldsymbol{u}_h|_{1,\Omega_h}\to 0$ for $h\to 0$, it suffices to show that

$$(5.4)\qquad \lim_{h\to 0}\sup_{\boldsymbol{v}_h\in\mathbf{V}_h,\boldsymbol{v}_h\neq\mathbf{0}}\frac{\widetilde{a}_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}_h)}{|\boldsymbol{v}_h|_{1,\Omega_h}}=0.$$

This is easy if Ω is convex (which is assumed by many authors) since then $\Omega_h\subset\Omega$ and any $\boldsymbol{v}_h\in\mathbf{V}_h$ can be extended by zero outside Ω_h to a function $\boldsymbol{v}_h\in H^1_0(\Omega)^3$. Setting $\boldsymbol{v}=\boldsymbol{v}_h$ in (5.1) and subtracting (5.1) from (5.2), we obtain $\widetilde{a}_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}_h)=0$, which is the Galerkin orthogonality, well known from the conforming case. If Ω is nonconvex, then the functions from $\mathbf{V}_h$ generally have nonzero values on some parts of $\partial\Omega$ and hence any extension of $\boldsymbol{v}_h\in\mathbf{V}_h$ generally cannot be used as a test function in (5.1). Therefore, any function $\boldsymbol{v}_h\in\mathbf{V}_h$ has to be *approximated* by a function $\boldsymbol{v}_h^0\in H^1(\widetilde{\Omega})^3$ satisfying $\boldsymbol{v}_h^0|_\Omega\in H^1_0(\Omega)^3$. Then, using (5.2) and (5.1) with $\boldsymbol{v}=\boldsymbol{v}_h^0|_\Omega$, we obtain

$$\widetilde{a}_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}_h)=\langle\boldsymbol{f}_h,\boldsymbol{v}_h\rangle-\widetilde{a}_h(\boldsymbol{u},\boldsymbol{v}_h)=\langle\boldsymbol{f}_h,\boldsymbol{v}_h\rangle-\langle\boldsymbol{f},\boldsymbol{v}_h^0\rangle+\widetilde{a}(\boldsymbol{u},\boldsymbol{v}_h^0)-\widetilde{a}_h(\boldsymbol{u},\boldsymbol{v}_h).$$

Since

$$\langle\boldsymbol{f},\boldsymbol{v}_h^0\rangle-\langle\boldsymbol{f}_h,\boldsymbol{v}_h\rangle=\langle\boldsymbol{f},\boldsymbol{v}_h^0-E_h\,\boldsymbol{v}_h\rangle+\int_{\Omega\setminus\Omega_h}\boldsymbol{f}\cdot E_h\,\boldsymbol{v}_h\,\mathrm{d}x-\int_{\Omega_h\setminus\Omega}\boldsymbol{f}\cdot\boldsymbol{v}_h\,\mathrm{d}x,$$

$$\widetilde{a}(\boldsymbol{u},\boldsymbol{v}_h^0)-\widetilde{a}_h(\boldsymbol{u},\boldsymbol{v}_h)=\widetilde{a}(\boldsymbol{u},\boldsymbol{v}_h^0-E_h\,\boldsymbol{v}_h)+\int_{\Omega\setminus\Omega_h}\nabla\boldsymbol{u}\cdot\nabla(E_h\,\boldsymbol{v}_h)\,\mathrm{d}x,$$

we get

$$\begin{aligned}\widetilde{a}_h(\boldsymbol{u}_h-\boldsymbol{u},\boldsymbol{v}_h)\leqslant C\,(&\|\boldsymbol{f}\|_{0,\frac{6}{5},\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega\cup\mathrm{supp}(\boldsymbol{v}_h^0-E_h\,\boldsymbol{v}_h)}\\&+|\boldsymbol{u}|_{1,\Omega\setminus\Omega_h\cup\mathrm{supp}(\boldsymbol{v}_h^0-E_h\,\boldsymbol{v}_h)})(\|\boldsymbol{v}_h\|_{1,\Omega_h}+\|\boldsymbol{v}_h^0\|_{1,\widetilde{\Omega}}).\end{aligned}$$

Thus, for proving (5.4), it suffices to construct the function $\boldsymbol{v}_h^0$ in such a way that

$$\operatorname{meas}_3(\operatorname{supp}(\boldsymbol{v}_h^0 - E_h\,\boldsymbol{v}_h)) \leqslant A_h, \qquad \lim_{h\to 0} A_h = 0, \qquad \|\boldsymbol{v}_h^0\|_{1,\widetilde{\Omega}} \leqslant C\,\|\boldsymbol{v}_h\|_{1,\Omega_h},$$

where A_h is independent of $\boldsymbol{v}_h$ and C is independent of both $\boldsymbol{v}_h$ and h. (Note that, extending $\boldsymbol{v}_h$ by zero outside Ω_h, we have $\|\boldsymbol{v}_h\|_{1,\Omega_h} = \|\boldsymbol{v}_h\|_{1,\widetilde{\Omega}} \leqslant C\,|\boldsymbol{v}_h|_{1,\Omega_h}$.)

In the following theorem, we will construct a function $\boldsymbol{v}_h^0$ having the above properties by modifying $\boldsymbol{v}_h$ on elements near the Dirichlet boundary and by extending the modified function using the operator E_h. The basic feature of this construction is that $\boldsymbol{v}_h^0|_{\Omega_h} \in \mathbf{V}_h$ and that $\boldsymbol{v}_h^0$ vanishes on elements intersecting the closure of the Dirichlet boundary. The theorem also represents a basis for the construction of $\boldsymbol{v}_h^0$ in the case when the set Γ^N in the definition of $\mathbf{V}$ is nonempty. Therefore, now we again consider $\Gamma_h^D \subset \partial\Omega_h$ (i.e., not necessarily $\Gamma_h^D = \partial\Omega_h$).

Theorem 5.1. *Let $k \geqslant 1$ be a given integer and let us denote*

$$\widetilde{W}_h = \{v \in C(\overline{\Omega}_h);\ v|_T \in P_k(T)\ \ \forall\, T \in \mathcal{T}_h,\ \ v(x) = 0 \text{ at any vertex } x \in \overline{\Gamma}_h^D\}.$$

Then, for h sufficiently small, any $v \in \widetilde{W}_h$ can be written as $v = v^0 + v^b$ with v^0, $v^b \in \widetilde{W}_h$, $E_h\,v^0|_{\Gamma^D} = 0$, $v^b = v$ or $v^b = 0$ at any vertex of $\mathcal{T}_h$, and

$$\|v^b\|_{1,\Omega_h} \leqslant C\,\|v\|_{1,\Omega_h}, \tag{5.5}$$

$$\operatorname{supp}(E_h\,v^b) \subset U_h(\Gamma^D) \equiv \{x \in \widetilde{\Omega};\ \operatorname{dist}(x,\Gamma^D) \leqslant C\,h\}, \tag{5.6}$$

where the constant C is independent of v and h.

P r o o f. We denote by $\widehat{T}$ the standard reference element having the vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. For any tetrahedron T there exists a regular affine mapping $F_T\colon \widehat{T} \to T$ which maps $\widehat{T}$ onto T (cf. e.g. [2]). For any $T \in \mathcal{T}_h$ and any $v \in L^1(T)$ we introduce the notation

$$\widehat{v}_T = v \circ F_T.$$

According to [2], Section 15, there exist positive constants C and $\widetilde{C}$ depending only on σ such that

$$C\,h_T^{\frac{3}{2}}\,\|\widehat{v}_T\|_{0,\widehat{T}} \leqslant \|v\|_{0,T} \leqslant h_T^{\frac{3}{2}}\,\|\widehat{v}_T\|_{0,\widehat{T}} \qquad \forall\, v \in L^2(T),\ T \in \mathcal{T}_h, \tag{5.7}$$

$$C\,h_T^{\frac{1}{2}}\,|\widehat{v}_T|_{1,\widehat{T}} \leqslant |v|_{1,T} \leqslant \widetilde{C}h_T^{\frac{1}{2}}\,|\widehat{v}_T|_{1,\widehat{T}} \qquad \forall\, v \in H^1(T),\ T \in \mathcal{T}_h. \tag{5.8}$$

Let $\dim P_k(\widehat{T}) = d$ and let the points $\widehat{x}^1,\dots,\widehat{x}^d \in \widehat{T}$ form the principal lattice of order k of the tetrahedron $\widehat{T}$ (cf. [2], p. 70, or [5], p. 99). Let $\hat{b}_1,\dots,\hat{b}_d \in P_k(\widehat{T})$

satisfy $\hat{b}_i(\widehat{x}^j) = \delta_{ij}$, $i,j = 1,\dots,d$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Then $\{\hat{b}_i\}_{i=1}^d$ is a basis of $P_k(\widehat{T})$. For any $T \in \mathcal{T}_h$, we set

$$b_{T,i} = \hat{b}_i \circ F_T^{-1}, \qquad x_T^i = F_T(\widehat{x}^i), \qquad i = 1,\dots,d. \tag{5.9}$$

Then $\{b_{T,i}\}_{i=1}^d$ is a basis of $P_k(T)$ satisfying $b_{T,i}(x_T^j) = \delta_{ij}$ and hence, for any $v \in \widetilde{W}_h$ and any $T \in \mathcal{T}_h$, we have

$$v|_T = \sum_{i=1}^d \alpha_{T,i}\, b_{T,i} \qquad \text{with} \qquad \alpha_{T,i} = v(x_T^i). \tag{5.10}$$

For convenience, we define sets

$$\begin{aligned}
\mathcal{G}_h &= \{T \in \mathcal{T}_h;\ T \cap \overline{\Gamma}_h^D \neq \emptyset\}, & G_h &= \bigcup_{T \in \mathcal{G}_h} T,\\
\mathcal{I}_h &= \{T \in \mathcal{T}_h;\ T \cap G_h = \emptyset\}, & I_h &= \bigcup_{T \in \mathcal{I}_h} T,\\
\mathcal{K}_h &= \mathcal{T}_h \setminus \{\mathcal{G}_h \cup \mathcal{I}_h\}, & K_h &= \overline{\Omega}_h \setminus \{G_h \cup I_h\}.
\end{aligned}$$

Consider any $v \in \widetilde{W}_h$ and define a function $v^b \in \widetilde{W}_h$ by

$$\begin{aligned}
&v^b|_{G_h} = v|_{G_h},\\
&v^b|_{I_h} = 0,\\
&v^b(x_T^i) = 0 \qquad \forall\ x_T^i \in K_h,\ i \in \{1,\dots,d\},\ T \in \mathcal{K}_h.
\end{aligned}$$

It follows from (5.9) and (5.10) that

$$\widehat{v}_T = \sum_{i=1}^d \alpha_{T,i}\, \widehat{b}_i.$$

Choose $T \in \mathcal{G}_h$ and let the mapping F_T map the point 0 onto the vertex $x \in T$ with $v(x) = 0$. Clearly, the seminorm $|\cdot|_{1,\widehat{T}}$ is a norm on the space $\{\widehat{v} \in P_k(\widehat{T});\ \widehat{v}(0) = 0\}$ and hence, in view of the equivalence of norms on finite-dimensional spaces, there exists a constant C depending only on k such that

$$\sum_{i=1}^d (\alpha_{T,i})^2 \leqslant C\, |\widehat{v}_T|_{1,\widehat{T}}^2.$$

Thus, using (5.8) we obtain

$$h_T \sum_{i=1}^d (\alpha_{T,i})^2 \leqslant C\, |v|_{1,T}^2 \qquad \forall\ T \in \mathcal{G}_h. \tag{5.11}$$

Consider any $T \in \mathcal{K}_h$. Then

$$v^b|_T = \sum_{i=1}^{d} \alpha^b_{T,i}\, b_{T,i} \qquad \text{with} \qquad \alpha^b_{T,i} = v^b(x^i_T).$$

By the definiton of v^b, we have $\alpha^b_{T,i} = 0$ if $x^i_T \in I_h \cup K_h$. If $x^i_T \in G_h$, then $x^i_T = x^j_{T^*}$ for some $j \in \{1, \dots, d\}$ and $T^* \in \mathcal{G}_h$ with $T^* \cap T \neq \emptyset$ and hence $\alpha^b_{T,i} = v^b(x^j_{T^*}) = v(x^j_{T^*}) = \alpha_{T^*,j}$. Therefore, we obtain using (4.15) and (5.11)

$$h_T \sum_{i=1}^{d} (\alpha^b_{T,i})^2 \leqslant h_T \sum_{\substack{T^* \in \mathcal{G}_h,\\ T^* \cap T \neq \emptyset}} \sum_{i=1}^{d} (\alpha_{T^*,i})^2 \leqslant \sum_{\substack{T^* \in \mathcal{G}_h,\\ T^* \cap T \neq \emptyset}} K_2\, h_{T^*} \sum_{i=1}^{d} (\alpha_{T^*,i})^2 \leqslant C \sum_{\substack{T^* \in \mathcal{G}_h,\\ T^* \cap T \neq \emptyset}} |v|^2_{1,T^*},$$

which in view of (5.8) implies

$$|v^b|^2_{1,K_h} = \sum_{T \in \mathcal{K}_h} |v^b|^2_{1,T} \leqslant C \sum_{T \in \mathcal{K}_h} h_T \sum_{i=1}^{d} (\alpha^b_{T,i})^2 \leqslant \widetilde{C} \sum_{T \in \mathcal{K}_h} \sum_{\substack{T^* \in \mathcal{G}_h,\\ T^* \cap T \neq \emptyset}} |v|^2_{1,T^*}. \tag{5.12}$$

By virtue of (4.14), each T^* appears at most K_1 times on the right-hand side of (5.12) and hence we obtain $|v^b|_{1,K_h} \leqslant C\,|v|_{1,G_h}$. Therefore, $|v^b|_{1,\Omega_h} \leqslant C\,|v|_{1,G_h} \leqslant C\,|v|_{1,\Omega_h}$. Using $\|\cdot\|_{0,\widehat{T}}$ instead of $|\cdot|_{1,\widehat{T}}$ and (5.7) instead of (5.8) in the above derivation, we also obtain $\|v^b\|_{0,\Omega_h} \leqslant C\,\|v\|_{0,\Omega_h}$, which gives (5.5). Since $\operatorname{dist}(x, \Gamma^D_h) \leqslant 2\,h$ for any $x \in \operatorname{supp} v^b$, relation (5.6) follows by using (4.9) and (4.6).

It remains to prove the important property $E_h\, v^0|_{\Gamma^D} = 0$. Consider any $\delta \in (0, 1/(4\,\sigma^3 K_2))$ and any face $T' \subset \Gamma^D_h$. According to (4.16), $O_\delta(T') \cap \overline{\Omega}_h \subset G_h$ and hence the function $v^0 \equiv v - v^b$ vanishes in $O_\delta(T') \cap \overline{\Omega}_h$. Thus, $\operatorname{dist}(x, T') \geqslant \delta\, h_{T'}$ for any $x \in \operatorname{supp} v^0$ and using (4.10), we infer that $E_h\, v^0$ vanishes in $O_{\delta/\overline{C}}(T')$. For $h < \delta/(C\,\overline{C})$, where C is the constant from (4.5), the first inclusion in (4.5) implies that $\overline{\Gamma}_{T'} \subset O_{\delta/\overline{C}}(T')$ and hence $E_h\, v^0|_{\overline{\Gamma}_{T'}} = 0$. Consequently, $E_h\, v^0|_{\Gamma^D} = 0$. □

From Theorem 5.1 we obtain the following corollary fitted to the above discretization of the Poisson equation.

Corollary 5.1. *Let $k \geqslant 1$ be a given integer and let us denote*

$$W_h = \{v \in C(\overline{\Omega}_h);\ v|_T \in P_k(T) \ \ \forall\, T \in \mathcal{T}_h,\ \ v = 0 \ \text{ on } \Gamma^D_h\}.$$

Then, for any $v \in W_h$, there exist functions v^0, $v^b \in H^1_0(\widetilde{\Omega})$ satisfying

$$E_h\, v = v^0 + v^b, \qquad v^0|_{\Gamma^D} = 0, \qquad \|v^0\|_{1,\widetilde{\Omega}} \leqslant C\,\|v\|_{1,\Omega_h}, \qquad \operatorname{supp} v^b \subset U_h(\Gamma^D).$$

P r o o f. For $h < h_0$ with h_0 sufficiently small, the assertion immediately follows from Theorem 5.1. For $h \geqslant h_0$, it suffices to set $v^0 = 0$. □

6. Convergence result for the Stokes equations

The approximation properties A1 and A2 have been formulated only for sufficiently regular fuctions from the spaces $\mathbf{V}$ and $L^2_0(\Omega)$. The following two lemmas show that any function from these spaces can be approximated by functions from $\mathbf{V}_h$ or Q_h, respectively, with an arbitrarily high precision if $h \to 0$.

Lemma 6.1. *Let* $\overline{\mathbf{V} \cap H^{l_1}(\Omega)^3} = \mathbf{V}$, *where* l_1 *is the integer from assumption* A1. *Then*

$$\lim_{h\to 0} \inf_{\boldsymbol{v}_h \in \mathbf{V}_h} \|E\,\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\Omega_h} = 0 \qquad \forall\, \boldsymbol{v} \in \mathbf{V}, \tag{6.1}$$

where $E\colon\ H^1(\Omega)^3 \to H^1(\widetilde{\Omega})^3$ *is an arbitrary extension operator.*

P r o o f. Let $\overline{E}\colon\ H^{l_1}(\Omega)^3 \to H^{l_1}(\widetilde{\Omega})^3$ be an extension operator (cf. the end of Section 4) and consider any $\boldsymbol{v} \in \mathbf{V}$. Then for any $\bar{\boldsymbol{v}} \in \mathbf{V} \cap H^{l_1}(\Omega)^3$ we have

$$\begin{aligned}\|E\,\boldsymbol{v} - r_h\,\bar{\boldsymbol{v}}\|_{1,\Omega_h} &\leqslant \|E\,\boldsymbol{v} - \overline{E}\,\bar{\boldsymbol{v}}\|_{1,\Omega_h} + \|\overline{E}\,\bar{\boldsymbol{v}} - r_h\,\bar{\boldsymbol{v}}\|_{1,\Omega_h}\\ &\leqslant \|\boldsymbol{v} - \bar{\boldsymbol{v}}\|_{1,\Omega_h\cap\Omega} + \|E\,\boldsymbol{v} - \overline{E}\,\bar{\boldsymbol{v}}\|_{1,\Omega_h\setminus\Omega} + C\,h^{\gamma_1}\,\|\overline{E}\,\bar{\boldsymbol{v}}\|_{l_1,\widetilde{\Omega}}.\end{aligned}$$

Consider any $\varepsilon > 0$. Since $\overline{\mathbf{V} \cap H^{l_1}(\Omega)^3} = \mathbf{V}$, there exists $\bar{\boldsymbol{v}} \in \mathbf{V} \cap H^{l_1}(\Omega)^3$ such that $\|\boldsymbol{v} - \bar{\boldsymbol{v}}\|_{1,\Omega_h\cap\Omega} < \varepsilon/3$. In view of (4.1), we find $h_1 > 0$ such that $\|E\,\boldsymbol{v} - \overline{E}\,\bar{\boldsymbol{v}}\|_{1,\Omega_h\setminus\Omega} < \varepsilon/3$ for $h \in (0, h_1)$ and hence there exists $h_2 \in (0, h_1)$ such that

$$\inf_{\boldsymbol{v}_h \in \mathbf{V}_h} \|E\,\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\Omega_h} \leqslant \|E\,\boldsymbol{v} - r_h\,\bar{\boldsymbol{v}}\|_{1,\Omega_h} < \varepsilon \qquad \forall\, h \in (0, h_2).$$

□

R e m a r k 6.1. If $\boldsymbol{m} \in H^{l_1}(\mathbb{R}^3)^3$ (cf. Remark 1.1) and the set Γ^D is of class $C^{0,1*}$, then it can be shown that the density assumption $\overline{\mathbf{V} \cap H^{l_1}(\Omega)^3} = \mathbf{V}$ is satisfied (see [6], p. 110, Lemma 3.13).

Lemma 6.2. *We have*

$$\lim_{h\to 0} \inf_{q_h \in \mathrm{Q}_h} \|E\,q - q_h\|_{0,\Omega_h} = 0 \qquad \forall\, q \in L^2_0(\Omega), \tag{6.2}$$

where $E\colon\ L^2(\Omega) \to L^2(\widetilde{\Omega})$ *is an arbitrary extension operator.*

P r o o f. Since $\overline{L^2_0(\Omega) \cap C^\infty(\overline{\Omega})} = L^2_0(\Omega)$, the lemma can be proved analogously as Lemma 6.1. □

If $\Gamma^N \neq \emptyset$, approximations of functions from $\mathbf{V}_h$ by functions from $\mathbf{V}$ are more difficult to construct than in Corollary 5.1. A suitable decomposition of extensions of the functions from $\mathbf{V}_h$ is established, using the results of Theorem 5.1, in the following theorem.

Theorem 6.1. *Let $k \geqslant 1$ be a given integer and let us denote*

$$\boldsymbol{W}_h = \{\boldsymbol{v} \in C(\overline{\Omega}_h)^3;\ \boldsymbol{v}|_T \in P_k(T)^3 \quad \forall\, T \in \mathcal{T}_h, \quad \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \Gamma_h^D, \\ (\boldsymbol{v} \cdot \boldsymbol{n}_h^*)(x) = 0 \text{ at any vertex } x \in \Gamma_h^N\},$$

where $\boldsymbol{n}_h^*$ is the function from assumption A1. *Then, for any $\boldsymbol{v} \in \boldsymbol{W}_h$, there exist functions $\bar{\boldsymbol{v}}^0$, $\bar{\boldsymbol{v}}^b$, $\bar{\boldsymbol{v}} \in H_0^1(\widetilde{\Omega})^3$ and $u^b \in H_0^1(\widetilde{\Omega})$ satisfying*

$$E_h\, \boldsymbol{v} = \bar{\boldsymbol{v}}^0 + \bar{\boldsymbol{v}} + \bar{\boldsymbol{v}}^b + u^b\, \boldsymbol{m}, \qquad \bar{\boldsymbol{v}}^0|_\Omega \in \mathbf{V}, \tag{6.3}$$

$$\|\bar{\boldsymbol{v}}^0\|_{1,\widetilde{\Omega}} + \|\bar{\boldsymbol{v}}\|_{1,\widetilde{\Omega}} + \|\bar{\boldsymbol{v}}^b\|_{1,\widetilde{\Omega}} + \|u^b\|_{1,\widetilde{\Omega}} \leqslant C\, \|\boldsymbol{v}\|_{1,\Omega_h}, \tag{6.4}$$

$$\|\bar{\boldsymbol{v}}\|_{1,\Omega_h} \leqslant C\, h^\alpha\, \|\boldsymbol{v}\|_{1,\Omega_h}, \qquad \|\bar{\boldsymbol{v}}\|_{0,6,\Omega_h} \leqslant C\, h^{1+\alpha}\, \|\boldsymbol{v}\|_{1,\Omega_h}, \tag{6.5}$$

$$\operatorname{supp}\, \bar{\boldsymbol{v}}^b \subset U_h(\Gamma^D), \qquad \operatorname{supp}\, u^b \subset U_h(\partial\Omega), \tag{6.6}$$

where $\boldsymbol{m}$ is the extension of $\boldsymbol{n}|_{\Gamma^N}$ from Section 1, *α is the constant from* (3.2), *U_h was defined in* (4.12) *and C is independent of $\boldsymbol{v}$ and h.*

P r o o f. Let h be sufficiently small and consider any $\boldsymbol{v} \in \boldsymbol{W}_h$. Then, by Theorem 5.1, there exist functions $\boldsymbol{v}^0$, $\bar{\boldsymbol{v}}^b \in H_0^1(\widetilde{\Omega})^3$ such that

$$\boldsymbol{v}^0|_{\Omega_h},\, \bar{\boldsymbol{v}}^b|_{\Omega_h} \in \{\boldsymbol{v} \in C(\overline{\Omega}_h)^3;\ \boldsymbol{v}|_T \in P_k(T)^3 \quad \forall\, T \in \mathcal{T}_h, \\ \boldsymbol{v}(x) = \boldsymbol{0} \text{ at any vertex } x \in \overline{\Gamma}_h^D,\ (\boldsymbol{v} \cdot \boldsymbol{n}_h^*)(x) = 0 \text{ at any vertex } x \in \Gamma_h^N\}$$

and

$$E_h\, \boldsymbol{v} = \boldsymbol{v}^0 + \bar{\boldsymbol{v}}^b, \qquad \boldsymbol{v}^0|_{\Gamma^D} = \boldsymbol{0}, \qquad \|\bar{\boldsymbol{v}}^b\|_{1,\widetilde{\Omega}} \leqslant C\, \|\boldsymbol{v}\|_{1,\Omega_h}, \qquad \operatorname{supp} \bar{\boldsymbol{v}}^b \subset U_h(\Gamma^D).$$

Let $\boldsymbol{m}_h^* \in C(\overline{\Omega}_h)^3$ be a piecewise linear function satisfying $\boldsymbol{m}_h^*(x) = \boldsymbol{m}(x)$ at any vertex $x \in \overline{\Omega}_h \setminus \Gamma_h^N$ and $\boldsymbol{m}_h^*(x) = \boldsymbol{n}_h^*(x)$ at any vertex $x \in \Gamma_h^N$, where $\boldsymbol{m}$ is the extension of $\boldsymbol{n}|_{\Gamma^N}$ from Section 1. Setting $u = \boldsymbol{v}^0|_{\Omega_h} \cdot \boldsymbol{m}_h^*$, we have

$$u \in \{v \in C(\overline{\Omega}_h);\ v|_T \in P_{k+1}(T) \quad \forall\, T \in \mathcal{T}_h, \quad v(x) = 0 \ \text{ at any vertex } x \in \partial\Omega_h\}$$

and hence, according to Theorem 5.1, there exist functions u^0, $u^b \in H_0^1(\widetilde{\Omega})$ such that

$$E_h\, u = u^0 + u^b, \quad u^0|_\Omega \in H_0^1(\Omega), \quad \|u^b\|_{1,\widetilde{\Omega}} \leqslant C\, \|u\|_{1,\Omega_h}, \quad \operatorname{supp}\, u^b \subset U_h(\partial\Omega).$$

Let $\boldsymbol{m}_h \in C(\overline{\Omega}_h)^3$ be the piecewise linear interpolate of $\boldsymbol{m}|_{\Omega_h}$. Then, according to [2], p. 124, Theorem 15.3, we have $|\boldsymbol{m}-\boldsymbol{m}_h|_{j,\infty,\Omega_h} \leqslant C\,h^{2-j}\,|\boldsymbol{m}|_{2,\infty,\widetilde{\Omega}}$, $j=0,1$, with a constant C independent of h. Further, using (3.2), we obtain $|(\boldsymbol{m}_h - \boldsymbol{m}_h^*)(x)| \leqslant C\,h_T^{1+\alpha}$ for any vertex x of $\mathcal{T}_h$ and any element $T \in \mathcal{T}_h$ containing the vertex x. Therefore, $|\boldsymbol{m}_h - \boldsymbol{m}_h^*|_{j,\infty,\Omega_h} \leqslant C\,h^{1+\alpha-j}$, $j=0,1$, and hence

$$|\boldsymbol{m} - \boldsymbol{m}_h^*|_{j,\infty,\Omega_h} \leqslant C\,h^{1+\alpha-j}, \quad j=0,1. \tag{6.7}$$

Thus, $\|\boldsymbol{m}_h^*\|_{1,\infty,\Omega_h} \leqslant C$, which implies

$$\|u^0\|_{1,\widetilde{\Omega}} + \|u^b\|_{1,\widetilde{\Omega}} \leqslant C\,\|u\|_{1,\Omega_h} \leqslant \widetilde{C}\,\|\boldsymbol{v}^0\|_{1,\Omega_h} \leqslant \overline{C}\,\|\boldsymbol{v}\|_{1,\Omega_h}.$$

Set

$$\begin{aligned}\bar{\boldsymbol{v}}^0 &= \boldsymbol{v}^0 - (\boldsymbol{v}^0\cdot\boldsymbol{m})\,\boldsymbol{m} + u^0\,\boldsymbol{m},\\ \bar{\boldsymbol{v}} &= [\boldsymbol{v}^0\cdot\boldsymbol{m} - E_h\,u]\,\boldsymbol{m}.\end{aligned}$$

Then $\bar{\boldsymbol{v}}^0$, $\bar{\boldsymbol{v}} \in H_0^1(\widetilde{\Omega})^3$ and (6.3) and (6.4) hold. Since $\bar{\boldsymbol{v}}|_{\Omega_h} = [\boldsymbol{v}^0\cdot(\boldsymbol{m}-\boldsymbol{m}_h^*)]\,\boldsymbol{m}$, we obtain (6.5) by (6.7). If h is not sufficiently small, we can set $\bar{\boldsymbol{v}}^0 = \boldsymbol{0}$, $\bar{\boldsymbol{v}} = \boldsymbol{0}$, $u^b = 0$ and $\bar{\boldsymbol{v}}^b = E_h\,\boldsymbol{v}$. □

Now we are in position to prove a convergence result for the discretization of the Stokes equations. The basic techniques are the same as for the Poisson equation in the preceding section, as for conforming discretizations of the Poisson equation with nonhomogeneous Dirichlet boundary conditions in [4] and as for conforming discretizations of the Stokes equations in [5], Chapter II.

Theorem 6.2. *Let* $\overline{\mathbf{V}\cap H^{l_1}(\Omega)^3} = \mathbf{V}$, *where* l_1 *is the integer from assumption* A1, *and let* $\boldsymbol{u}$, p *be the weak solution of* (1.1)–(1.5). *Then the discrete solutions* $\boldsymbol{u}_h$, p_h *of problem* (1.1)–(1.5) *satisfy*

$$\lim_{h\to 0}\{\|E_1\,\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} + \|E_0\,p - p_h\|_{0,\Omega_h}\} = 0,$$

where $E_1\colon\ H^1(\Omega)^3 \to H^1(\widetilde{\Omega})^3$, $E_0\colon\ L^2(\Omega)\to L^2(\widetilde{\Omega})$ *are arbitrary extension operators.*

P r o o f. For simplicity, we denote by $\boldsymbol{u}$ and p the extensions $E_1\,\boldsymbol{u}$ and $E_0\,p$, respectively. Using the decomposition from Theorem 6.1, we infer that, for any

$\boldsymbol{v}_h \in \mathbf{V}_h$,

$$\begin{aligned}
|a(\boldsymbol{u}, \bar{\boldsymbol{v}}_h^0) - a_h(\boldsymbol{u}, \boldsymbol{v}_h)| &\leqslant C\,(|\boldsymbol{u}|_{1,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega\cup U_h(\partial\Omega)} + h^\alpha\,|\boldsymbol{u}|_{1,\Omega})\,\|\boldsymbol{v}_h\|_{1,\Omega_h},\\
|b(\bar{\boldsymbol{v}}_h^0, p) - b_h(\boldsymbol{v}_h, p)| &\leqslant C\,(\|p\|_{0,\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega\cup U_h(\partial\Omega)} + h^\alpha\,\|p\|_{0,\Omega})\,\|\boldsymbol{v}_h\|_{1,\Omega_h},\\
\left|\int_\Omega \boldsymbol{f}\cdot\bar{\boldsymbol{v}}_h^0\,\mathrm{d}x - \int_{\Omega_h} \boldsymbol{f}\cdot\boldsymbol{v}_h\,\mathrm{d}x\right| &\leqslant C\,(\|\boldsymbol{f}\|_{0,\frac{6}{5},\Omega\setminus\Omega_h\cup\Omega_h\setminus\Omega\cup U_h(\partial\Omega)}\\
&\qquad + h^{1+\alpha}\,\|\boldsymbol{f}\|_{0,\frac{6}{5},\Omega})\,\|\boldsymbol{v}_h\|_{1,\Omega_h}.
\end{aligned}$$

Further, in view of (1.8) we have

$$\begin{aligned}
\left|\int_{\Gamma^N} \boldsymbol{\varphi}\cdot\bar{\boldsymbol{v}}_h^0\,\mathrm{d}\sigma - \int_{\Gamma_h^N} \boldsymbol{\varphi}\cdot\boldsymbol{v}_h\,\mathrm{d}\sigma\right| &\leqslant \left|\int_{\Gamma^N} \boldsymbol{\varphi}\cdot\bar{\boldsymbol{v}}_h^b\,\mathrm{d}\sigma\right| + \left|\int_{\Gamma_h^N} \boldsymbol{\varphi}\cdot\bar{\boldsymbol{v}}_h\,\mathrm{d}\sigma\right|\\
&\quad + \left|\int_{\Gamma^N} \boldsymbol{\varphi}\cdot(E_h\,\boldsymbol{v}_h - \bar{\boldsymbol{v}}_h)\,\mathrm{d}\sigma - \int_{\Gamma_h^N} \boldsymbol{\varphi}\cdot(E_h\,\boldsymbol{v}_h - \bar{\boldsymbol{v}}_h)\,\mathrm{d}\sigma\right|.
\end{aligned}$$

Using (4.2) and (4.3), we derive

$$\left|\int_{\Gamma^N} \boldsymbol{\varphi}\cdot\bar{\boldsymbol{v}}_h^0\,\mathrm{d}\sigma - \int_{\Gamma_h^N} \boldsymbol{\varphi}\cdot\boldsymbol{v}_h\,\mathrm{d}\sigma\right| \leqslant C\,(\|\boldsymbol{\varphi}\|_{0,\Gamma^N\cap U_h(\Gamma^D)} + h^\alpha\,\|\boldsymbol{\varphi}\|_{1,\widetilde{\Omega}})\,\|\boldsymbol{v}_h\|_{1,\Omega_h}.$$

Now, subtracting (2.3) with $\boldsymbol{v} = \bar{\boldsymbol{v}}_h^0|_\Omega$ from (3.7), we obtain using (4.1), (4.13) and (3.5) the inequality

$$|a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h - p)| \leqslant K_h\,\|\boldsymbol{v}_h\|_{1,\Omega_h}, \tag{6.8}$$

where $K_h \to 0$ for $h \to 0$. Let us define sets

$$\begin{aligned}
\mathbf{V}_h^b &= \{\widehat{\boldsymbol{v}}_h \in H^1(\Omega_h)^3;\ \widehat{\boldsymbol{v}}_h - \widetilde{\boldsymbol{u}}_{bh} \in \mathbf{V}_h\},\\
\mathcal{V}_h^b &= \{\widehat{\boldsymbol{z}}_h \in \mathbf{V}_h^b;\ b_h(\widehat{\boldsymbol{z}}_h, q_h) = 0 \quad \forall\, q_h \in \mathrm{Q}_h\}.
\end{aligned}$$

For any $\widehat{\boldsymbol{z}}_h \in \mathcal{V}_h^b$ we have $\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h \in \mathbf{V}_h$ and

$$b_h(\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h, p - p_h) = b_h(\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h, p - q_h) \qquad \forall\, q_h \in \mathrm{Q}_h.$$

Applying (4.4) we get

$$\begin{aligned}
C\,\|\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h}^2 \leqslant a_h(\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h, \boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h) &\leqslant (K_h + \sqrt{3}\,\|p - q_h\|_{0,\Omega_h}\\
&\quad + 2\,\nu\,\|\boldsymbol{u} - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h})\,\|\boldsymbol{u}_h - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h} \qquad \forall\, \widehat{\boldsymbol{z}}_h \in \mathcal{V}_h^b,\ q_h \in \mathrm{Q}_h,
\end{aligned}$$

which gives by the triangular inequality

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} \leqslant C\,(K_h + \|p - q_h\|_{0,\Omega_h} + \|\boldsymbol{u} - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h}) \quad \forall\, \widehat{\boldsymbol{z}}_h \in \mathcal{V}_h^b,\ q_h \in \mathrm{Q}_h. \tag{6.9}$$

Consider any $\widehat{\boldsymbol{v}}_h \in \mathbf{V}_h^b$. According to Lemma 4.1 from [5], p. 58, assumption A3 implies that there exists $\widetilde{\boldsymbol{v}}_h \in \mathbf{V}_h$ such that

$$b_h(\widetilde{\boldsymbol{v}}_h, q_h) = \int_{\Omega_h} q_h \operatorname{div}(\widehat{\boldsymbol{v}}_h - \boldsymbol{u}) \,\mathrm{d}x + \int_{\Omega_h \setminus \Omega} q_h \operatorname{div} \boldsymbol{u} \,\mathrm{d}x \qquad \forall\, q_h \in \mathrm{Q}_h$$

and

$$\beta \,\|\widetilde{\boldsymbol{v}}_h\|_{1,\Omega_h} \leqslant \sqrt{3}\,(|\widehat{\boldsymbol{v}}_h - \boldsymbol{u}|_{1,\Omega_h} + |\boldsymbol{u}|_{1,\Omega_h \setminus \Omega}).$$

Since $\operatorname{div} \boldsymbol{u} = 0$ in Ω (cf. Remark 2.1), we infer that $b_h(\widetilde{\boldsymbol{v}}_h, q_h) = -b_h(\widehat{\boldsymbol{v}}_h, q_h)$ for any $q_h \in \mathrm{Q}_h$. Setting $\widehat{\boldsymbol{z}}_h = \widetilde{\boldsymbol{v}}_h + \widehat{\boldsymbol{v}}_h$, we have $\widehat{\boldsymbol{z}}_h \in \mathcal{V}_h^b$ and

$$\|\boldsymbol{u} - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h} \leqslant \|\boldsymbol{u} - \widehat{\boldsymbol{v}}_h\|_{1,\Omega_h} + \|\widetilde{\boldsymbol{v}}_h\|_{1,\Omega_h} \leqslant C\,(\|\boldsymbol{u} - \widehat{\boldsymbol{v}}_h\|_{1,\Omega_h} + |\boldsymbol{u}|_{1,\Omega_h \setminus \Omega}),$$

which means that

$$\inf_{\widehat{\boldsymbol{z}}_h \in \mathcal{V}_h^b} \|\boldsymbol{u} - \widehat{\boldsymbol{z}}_h\|_{1,\Omega_h} \leqslant C\,(\|\boldsymbol{u} - \widehat{\boldsymbol{v}}_h\|_{1,\Omega_h} + |\boldsymbol{u}|_{1,\Omega_h \setminus \Omega}) \qquad \forall\, \widehat{\boldsymbol{v}}_h \in \mathbf{V}_h^b.$$

Thus, using (6.9), for any $\widehat{\boldsymbol{v}}_h \in \mathbf{V}_h^b$, $q_h \in \mathrm{Q}_h$ we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} \leqslant C\,(K_h + \|p - q_h\|_{0,\Omega_h} + \|\boldsymbol{u} - \widehat{\boldsymbol{v}}_h\|_{1,\Omega_h} + |\boldsymbol{u}|_{1,\Omega_h \setminus \Omega}),$$

which by (4.1), (3.4), (6.1) and (6.2) gives

$$\lim_{h \to 0} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega_h} = 0.$$

It follows from (6.8) that

$$|b_h(\boldsymbol{v}_h, p_h - p)| \leqslant \widetilde{K}_h \,\|\boldsymbol{v}_h\|_{1,\Omega_h} \qquad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h$$

with $\widetilde{K}_h \to 0$ and hence

$$|b_h(\boldsymbol{v}_h, p_h - q_h)| \leqslant (\widetilde{K}_h + \sqrt{3}\,\|p - q_h\|_{0,\Omega_h})\,\|\boldsymbol{v}_h\|_{1,\Omega_h} \quad \forall\, \boldsymbol{v}_h \in \mathbf{V}_h,\ q_h \in \mathrm{Q}_h.$$

Now, (3.3) implies that

$$\beta\,\|p_h - q_h\|_{0,\Omega_h} \leqslant \widetilde{K}_h + \sqrt{3}\,\|p - q_h\|_{0,\Omega_h} \qquad \forall\, q_h \in \mathrm{Q}_h$$

and we obtain by the triangular inequality

$$\|p - p_h\|_{0,\Omega_h} \leqslant C\,(\widetilde{K}_h + \|p - q_h\|_{0,\Omega_h}) \qquad \forall\, q_h \in \mathrm{Q}_h.$$

Hence $\|p - p_h\|_{0,\Omega_h} \to 0$ by Lemma 6.2. $\square$

7. Examples of finite element spaces satisfying A1–A3

In this section we give examples of finite element spaces which satisfy assumptions A1–A3 and are suitable for numerical solution of (1.1)–(1.5).

We denote by $\bar{\boldsymbol{n}}_h$ the piecewise linear interpolate of $\boldsymbol{n}|_{\Gamma^N} = \boldsymbol{m}|_{\Gamma^N}$, i.e., for any face $T' \subset \Gamma^N$, $\bar{\boldsymbol{n}}_h|_{\overline{T'}}$ is a linear function equal to $\boldsymbol{m}$ at the vertices of T'. Further, we denote

$$S_h^N = \{x \in \Gamma_h^N;\ x \text{ is a vertex of } \mathcal{T}_h \text{ or the midpoint of an edge of } \mathcal{T}_h\}$$

and introduce a function $\boldsymbol{n}_h^*\colon S_h^N \to \mathbb{R}^3$ satisfying for some $\alpha > 0$

$$|\boldsymbol{n}_h^*(x) - \bar{\boldsymbol{n}}_h(x)| \leqslant C\, h_T^{1+\alpha} \qquad \forall\, x \in S_h^N, \tag{7.1}$$

where T is any element containing the point x and C is independent of x, T and h. For $l = 1, 2$ we define spaces

$$\begin{aligned} \mathrm{V}_{l,h} &= \{v \in C(\overline{\Omega}_h);\ v|_T \in P_l(T)\ \ \forall\, T \in \mathcal{T}_h\},\\ \mathrm{M}_{l-1,h} &= \{v \in L^2(\Omega_h);\ v|_T \in P_{l-1}(T)\ \ \forall\, T \in \mathcal{T}_h\} \end{aligned}$$

and set

$$\begin{aligned} \boldsymbol{V}_{1,h} &= \{\boldsymbol{v} \in [\mathrm{V}_{1,h}]^3;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_h^D,\ (\boldsymbol{v}\cdot\boldsymbol{n}_h^*)(x) = 0 \text{ at any vertex } x \in \Gamma_h^N\},\\ \boldsymbol{V}_{2,h} &= \{\boldsymbol{v} \in [\mathrm{V}_{2,h}]^3;\ \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_h^D,\ (\boldsymbol{v}\cdot\boldsymbol{n}_h^*)(x) = 0\ \forall\, x \in S_h^N\}. \end{aligned}$$

Finally, we denote

$$\boldsymbol{\mathcal{P}}_h = \operatorname{span}\{p_{T'}\,\boldsymbol{n}_{T'}\}_{T' \not\subset \partial\Omega_h}, \qquad \boldsymbol{\mathcal{R}}_h = [\operatorname{span}\{r_T\}_{T\in\mathcal{T}_h}]^3,$$

where $p_{T'}$, $\boldsymbol{n}_{T'}$ were introduced in Example 3.1 and, for any $T \in \mathcal{T}_h$, the function $r_T \in H_0^1(T) \setminus \{0\}$ is any polynomial of degree four.

Theorem 7.1. *If $\mathbf{V}_h = \boldsymbol{V}_{1,h}$, then assumption* A1 *holds with $l_1 = 2$ and $\gamma_1 = \min\{1, \alpha + 1/2\}$. If $\mathbf{V}_h = \boldsymbol{V}_{2,h}$ and the extensions $\widetilde{\Gamma}_i$ of Γ_i, $i = K^D + 1, \dots, K$, are C^3 surfaces, then* A1 *also holds with $l_1 = 3$ and $\gamma_1 = \min\{3/2, \alpha + 1/2\}$.*

Proof. See [7], Section 6. □

Theorem 7.2. *If $\mathrm{Q}_h = \mathrm{M}_{0,h} \cap L_0^2(\Omega_h)$, then assumption* A2 *holds with $l_2 = \gamma_2 = 1$. If $\mathrm{Q}_h = \mathrm{M}_{1,h} \cap L_0^2(\Omega_h)$ or $\mathrm{Q}_h = \mathrm{V}_{1,h} \cap L_0^2(\Omega_h)$, then A2 holds with $l_2 = \gamma_2 = 2$.*

P r o o f. See [7], Section 6. □

Theorem 7.3. *Assumption* A3 *is satisfied if* $\mathrm{Q}_h = \mathrm{M}_{0,h} \cap L_0^2(\Omega_h)$ *and* $\mathbf{V}_h \supset [\mathrm{V}_{1,h} \cap H_0^1(\Omega_h)]^3 \oplus \boldsymbol{\mathcal{P}}_h$ *or if* $\mathrm{Q}_h = \mathrm{M}_{1,h} \cap L_0^2(\Omega_h)$ *and* $\mathbf{V}_h \supset [\mathrm{V}_{1,h} \cap H_0^1(\Omega_h)]^3 \oplus \boldsymbol{\mathcal{P}}_h \oplus \boldsymbol{\mathcal{R}}_h$ *or if* $\mathrm{Q}_h = \mathrm{V}_{1,h} \cap L_0^2(\Omega_h)$ *and* $\mathbf{V}_h \supset [\mathrm{V}_{1,h} \cap H_0^1(\Omega_h)]^3 \oplus \boldsymbol{\mathcal{R}}_h$. *If, for any* $T \in \mathcal{T}_h$, *at least one vertex of* T *lies in* Ω, *then* A3 *also holds for* $\mathrm{Q}_h = \mathrm{V}_{1,h} \cap L_0^2(\Omega_h)$ *and* $\mathbf{V}_h \supset [\mathrm{V}_{2,h} \cap H_0^1(\Omega_h)]^3$.

P r o o f. The basic ideas of the proof are the same as for $\Omega_h = \Omega$ (see e.g. [5] and [1]). □

Using the results given in this setion, we can set up several pairs of finite element spaces satisfying assumptions A1–A3. Six of them are presented in Table 1, where also the highest possible convergence orders are given to which these spaces can lead. The convergence order 1 is guaranteed if (7.1) holds with $\alpha \geqslant 1/2$, $\boldsymbol{f} \in L^3(\widetilde{\Omega})^3$, $E\,\widetilde{\boldsymbol{u}}_b \in H^2(\widetilde{\Omega})^3$ in (3.4), the convergence in (3.4) and (3.5) is linear and the weak solution of (1.1)–(1.5) satisfies $\boldsymbol{u} \in H^2(\Omega)^3$, $p \in H^1(\Omega)$. The convergence order 3/2 is guaranteed if $\alpha \geqslant 1$, $\boldsymbol{f} \in L^{12}(\widetilde{\Omega})^3$, $\boldsymbol{\varphi} \in W^{1,\infty}(\widetilde{\Omega})^3$, the extensions $\widetilde{\Gamma}_i$ of Γ_i, $i = K^D + 1, \ldots, K$, are C^3 surfaces, $E\,\widetilde{\boldsymbol{u}}_b \in H^3(\widetilde{\Omega})^3$ in (3.4), relations (3.4) and (3.5) hold with the convergence order 3/2 and $\boldsymbol{u} \in H^3(\Omega)^3$, $p \in H^2(\Omega)$. Proofs of these assertions can be found in [7]. We remark that the convergence order 3/2 is the best possible convergence order which can be obtained if a polyhedral approximation of the computational domain is used.

conv. order	$\mathbf{V}_h$	Q_h
1	$\boldsymbol{V}_{1,h} \oplus \boldsymbol{\mathcal{P}}_h$	$\mathrm{M}_{0,h} \cap L_0^2(\Omega_h)$
1	$\boldsymbol{V}_{1,h} \oplus \boldsymbol{\mathcal{R}}_h$	$\mathrm{V}_{1,h} \cap L_0^2(\Omega_h)$
1	$\boldsymbol{V}_{1,h} \oplus \boldsymbol{\mathcal{P}}_h \oplus \boldsymbol{\mathcal{R}}_h$	$\mathrm{M}_{1,h} \cap L_0^2(\Omega_h)$
3/2	$\boldsymbol{V}_{2,h} \oplus \boldsymbol{\mathcal{P}}_h \oplus \boldsymbol{\mathcal{R}}_h$	$\mathrm{M}_{1,h} \cap L_0^2(\Omega_h)$
3/2	$\boldsymbol{V}_{2,h} \oplus \boldsymbol{\mathcal{R}}_h$	$\mathrm{V}_{1,h} \cap L_0^2(\Omega_h)$
3/2	$\boldsymbol{V}_{2,h}$ [cf. Th. 7.3]	$\mathrm{V}_{1,h} \cap L_0^2(\Omega_h)$

Table 1. Examples of finite element spaces $\mathbf{V}_h$ and Q_h satisfying assumptions A1–A3.

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