A HEAT APPROXIMATION*

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Abstract. The Fourier problem on planar domains with time variable boundary is considered using integral equations. A simple numerical method for the integral equation is described and the convergence of the method is proved. It is shown how to approximate the solution of the Fourier problem and how to estimate the error. A numerical example is given.

Keywords: heat equation, boundary value problem, integral equations, numerical solution, boundary element method

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1. Introduction

Let $M \subset \mathbb{R}^2$ be a set of the form

$$M = \{ [x,t] \in \mathbb{R}^2 \mid t \in (a,b), x > \varphi(t) \}, \tag{1.1}$$

where φ is a continuous function of bounded variation on a compact interval $\langle a, b\rangle$. The Fourier problem on the domain M can be solved by means of a boundary integral equation. It was shown in [2] that the integral equation is solvable not only in the space of continuous functions but also in the space of bounded Baire functions. This result is connected with a proof of convergence of a simple numerical method for solving the integral equation mentioned. This numerical method leads to a very simple representation of an approximate solution of the Fourier problem on M.

With regard to the well known properties of the Weierstrass integral we can restrict our considerations to the case of zero initial condition (compare, for example,

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Remark 4.1, p. 428, in [2]). Thus we shall consider the following boundary value problem. Let g be a function continuous on $\langle a,b\rangle$, $g(a)=0$. The problem is to find a function h continuous and bounded on $\overline{M}$, caloric on M, that is h fulfils on M the heat equation

$$\frac{\partial h}{\partial t}=\frac{\partial^2 h}{\partial x^2},$$

and further $h(x,a)=0$ for $x\geqslant\varphi(a)$ and

$$h\bigl(\varphi(t),t\bigr)=g(t)$$

for $t\in\langle a,b\rangle$. Let us recall some basic notation and some assertions from [2] which we will need in the sequel.

Fix a compact interval $\langle a,b\rangle$ ($a,b\in\mathbb{R}^1$, $a<b$). By $\mathscr{C}(\langle a,b\rangle)$, $\mathscr{B}(\langle a,b\rangle)$ we mean the space of all continuous functions on $\langle a,b\rangle$ and the space of all bounded Baire functions on $\langle a,b\rangle$, respectively. Further denote

$$\mathscr{C}_0(\langle a,b\rangle)=\bigl\{\,f\in\mathscr{C}(\langle a,b\rangle)\bigm| f(a)=0\,\bigr\}.$$

All the spaces $\mathscr{C}(\langle a,b\rangle)$, $\mathscr{C}_0(\langle a,b\rangle)$, $\mathscr{B}(\langle a,b\rangle)$ are supposed to be endowed with the supremum norm. Thus these spaces are Banach spaces, $\mathscr{C}(\langle a,b\rangle)$ is a closed subspace of $\mathscr{B}(\langle a,b\rangle)$ and $\mathscr{C}_0(\langle a,b\rangle)$ is a closed subspace of $\mathscr{C}(\langle a,b\rangle)$ [and also of $\mathscr{B}(\langle a,b\rangle)$].

Further fix a continuous function φ on $\langle a,b\rangle$ with bounded variation (on $\langle a,b\rangle$) and denote

$$K=\bigl\{\,[\varphi(t),t]\bigm| t\in\langle a,b\rangle\,\bigr\}. \tag{1.2}$$

For $[x,t]\in\mathbb{R}^2$, $\alpha,r>0$, $\alpha<+\infty$, let $n_{x,t}(r,\alpha)$ stand for the number (finite or $+\infty$) of points of the set

$$K\cap\left\{[\xi,\tau]\in\mathbb{R}^2\;\middle|\;t-\tau=\left(\frac{\xi-x}{2\alpha}\right)^2,0<t-\tau<r\right\}.$$

It is known that for any $[x,t]\in\mathbb{R}^2$, $r>0$, the function $n_{x,t}(r,\alpha)$ is a measurable function of the variable $\alpha\in(0,+\infty)$. Further denote

$$V_K(r;x,t)=\int_0^{+\infty}\mathrm{e}^{-\alpha^2}n_{x,t}(r,\alpha)\,\mathrm{d}\alpha \tag{1.3}$$

(for $[x,t]\in\mathbb{R}^2$, $r>0$). Put $V_K(x,t)=V_K(+\infty;x,t)$. The function $V_K(\cdot,\cdot)$ is called the parabolic variation of the set (curve) K.

For $[x,t] \in \mathbb{R}^2$, $t > a$, let $\alpha_{x,t}$ stand for the function defined on the interval $\langle a, \min\{t,b\})$ by

$$\alpha_{x,t}(\tau) = \frac{x - \varphi(\tau)}{2\sqrt{t-\tau}}. \tag{1.4}$$

Let us recall that if φ has finite variation on $\langle a, b\rangle$ (and we suppose that) then $\alpha_{x,t}$ is of locally finite variation on $\langle a, \min\{t,b\})$ and

$$V_K(r; x, t) = \int_{\max\{a,t-r\}}^{\min\{t,b\}} \mathrm{e}^{-\alpha_{x,t}^2(\tau)} \, \mathrm{d}\bigl(\operatorname{var} \alpha_{x,t}(\tau)\bigr) \tag{1.5}$$

[whenever $\max\{a, t-r\} < \min\{t,b\}$, otherwise $V_K(r;x,t) = 0$]. For any fixed $r > 0$ the function $V_K(r;\cdot,\cdot)$ is lower-semicontinuous on $\mathbb{R}^2$ and finite on $\mathbb{R}^2 \setminus K$. Further recall that if

$$\sup_{[x,t]\in K} V_K(x,t) < +\infty, \tag{1.6}$$

then V_K is bounded on $\mathbb{R}^2$.

For $f \in \mathscr{B}(\langle a,b\rangle)$ the potential $Tf = T_K f$ is defined in the following way. For $[x,t] \in \mathbb{R}^2$, $t \leqslant a$, let $Tf(x,t) = 0$ and

$$Tf(x,t) = T_K(x,t) = \frac{2}{\sqrt{\pi}} \int_a^{\min\{t,b\}} f(\tau) \mathrm{e}^{-\alpha_{x,t}^2(\tau)} \, \mathrm{d}_\tau \alpha_{x,t}(\tau) \tag{1.7}$$

for $[x,t] \in \mathbb{R}^2$, $t > a$, if the integral on the right hand side exists and is finite. One can see easily that if $V_K(x,t) < +\infty$ then $Tf(x,t)$ is defined and

$$\bigl|Tf(x,t)\bigr| \leqslant \|f\|_{\mathscr{B}} \frac{2}{\sqrt{\pi}} V_K(x,t).$$

Since $V_K(x,t) < +\infty$ on $\mathbb{R}^2 \setminus K$ (assuming only that φ has finite variation on $\langle a,b\rangle$) the potential Tf is always defined at least on $\mathbb{R}^2 \setminus K$. On $\mathbb{R}^2 \setminus K$ the potential Tf is equal to a combination of a double and a single layer heat potentials and thus Tf solves on $\mathbb{R}^2 \setminus K$ the heat equation. Let us recall that the idea of investigating the potential $T_K f$ and the parabolic variation V_K was proposed by J. Král.

If the condition (1.6) is fulfilled then for any $f \in \mathscr{C}_0(\langle a,b\rangle)$ the potential $T_K f$ has "right hand side limits" and "left hand side limits" on K, that is for any $t \in \langle a,b\rangle$ and any $f \in \mathscr{C}_0(\langle a,b\rangle)$ the limits

$$\widetilde{T}_+ f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ t' \in \langle a,b\rangle, x' > \varphi(t')}} Tf(x',t'), \tag{1.8}$$

$$\widetilde{T}_- f(t) = \lim_{\substack{[x',t'] \to [\varphi(t),t] \\ t' \in \langle a,b\rangle, x' < \varphi(t')}} Tf(x',t') \tag{1.9}$$

exist and are finite. Clearly $\widetilde{T}_+ f(a) = 0 = \widetilde{T}_- f(a)$. Let us recall how to express the values of $\widetilde{T}_+ f(t), \widetilde{T}_- f(t)$ for $t \in (a,b\rangle$. It is known that if $[x,t] \in K$, $t > a$, $V_K(x,t) < +\infty$, then there is a limit (finite or infinite)

$$\alpha_{x,t}(t) = \lim_{\tau \to t-} \alpha_{x,t}(\tau). \tag{1.10}$$

If further G is the function defined on ${}^*\mathbb{R}^1$ by

$$G(t) = \begin{cases} 0, & t = -\infty, \\ \displaystyle\int_{-\infty}^{t} \mathrm{e}^{-x^2}\,\mathrm{d}x, & t > -\infty, \end{cases} \tag{1.11}$$

then for each $t \in (a,b\rangle$ and any $f \in \mathscr{C}_0(\langle a,b\rangle)$ the values $\widetilde{T}_+ f(t), \widetilde{T}_- f(t)$ can be expressed in the form

$$\widetilde{T}_+ f(t) = Tf\bigl(\varphi(t),t\bigr) + f(t)\left[2 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right], \tag{1.12}$$

$$\widetilde{T}_- f(t) = Tf\bigl(\varphi(t),t\bigr) - f(t)\frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr). \tag{1.13}$$

It is obvious that for any $f \in \mathscr{C}_0(\langle a,b\rangle)$ we have also $\widetilde{T}_+ f, \widetilde{T}_- f \in \mathscr{C}_0(\langle a,b\rangle)$ and the operators $\widetilde{T}_+$, $\widetilde{T}_-$ are linear,

$$\widetilde{T}_+ \colon \mathscr{C}_0(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle), \qquad \widetilde{T}_- \colon \mathscr{C}_0(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle).$$

Given $g \in \mathscr{C}_0(\langle a,b\rangle)$ suppose that $f \in \mathscr{C}_0(\langle a,b\rangle)$ is a solution of the equation

$$\widetilde{T}_+ f = g.$$

Then using (1.8) one can see that the potential Tf solves the Fourier problem on M [where M is given by (1.1)] with the zero initial condition and the boundary condition g (on K). Similarly the solution of the equation

$$\widetilde{T}_- f = g$$

is connected with the Fourier problem on the set

$$\bigl\{\, [x,t] \in \mathbb{R}^2 \;\big|\; t \in (a,b), x < \varphi(t) \,\bigr\}.$$

Conditions of solvability of the above equations in the space $\mathscr{C}_0(\langle a,b\rangle)$ were investigated in [4].

Let I denote the identity operator on $\mathscr{C}_0(\langle a,b\rangle)$ and put

$$\overline{T}_0 = \widetilde{T}_+ - I = \widetilde{T}_- + I$$

[see the equalities (1.12), (1.13)]. For $f \in \mathscr{C}_0(\langle a,b\rangle)$ we have then $\overline{T}_0 f(a) = 0$ and

$$\overline{T}_0 f(t) = Tf\bigl(\varphi(t),t\bigr) + f(t)\left[1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right]$$

for $t \in (a,b\rangle$. Operator $\overline{T}_0$ is a linear operator mapping $\mathscr{C}_0(\langle a,b\rangle)$ into itself. This operator can be easily extended from $\mathscr{C}_0(\langle a,b\rangle)$ onto $\mathscr{B}(\langle a,b\rangle)$ by putting $\overline{\mathcal{T}}_0 f(a) = 0$ [if $f \in \mathscr{B}(\langle a,b\rangle)$] and

$$\overline{\mathcal{T}}_0 f(t) = Tf\bigl(\varphi(t),t\bigr) + f(t)\left[1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right] \tag{1.14}$$

for $t \in (a,b\rangle$. Thus the operator $\overline{\mathcal{T}}_0$ is an extension of $\overline{T}_0$ from $\mathscr{C}_0(\langle a,b\rangle)$ to $\mathscr{B}(\langle a,b\rangle)$. It is shown in [2] that

$$\overline{\mathcal{T}}_0\colon\ \mathscr{B}(\langle a,b\rangle) \to \mathscr{B}(\langle a,b\rangle)$$

($\overline{\mathcal{T}}_0$ is linear, of course). Let $\mathcal{I}$ stand for the identity operator on $\mathscr{B}(\langle a,b\rangle)$. It is shown in [2] that if the condition

$$\lim_{r\to 0+}\ \sup_{t\in(a,b\rangle} \left[\frac{2}{\sqrt{\pi}} V_K\bigl(r;\varphi(t),t\bigr) + \left|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right|\right] < 1 \tag{1.15}$$

is fulfilled then for any $g \in \mathscr{B}(\langle a,b\rangle)$ the equation

$$(\overline{\mathcal{T}}_0 + \mathcal{I})f = g \tag{1.16}$$

[and also the equation $(\overline{\mathcal{T}}_0 - \mathcal{I})f = g$] has a unique solution $f \in \mathscr{B}(\langle a,b\rangle)$ (see Theorem 4.1 in [2]). Let us emphasize that if $g \in \mathscr{C}_0(\langle a,b\rangle)$ then the solution of the equation (1.16) belongs also to $\mathscr{C}_0(\langle a,b\rangle)$.

In the end let us recall the following assertion which we will use in the proof of convergence of a numerical method for the equation (1.16).

Proposition 1.1. *Let X be a Banach space, $X_0 \subset X$ its closed subspace. Let $Q, B\colon X \to X$ be bounded linear operators and suppose that B is compact, $\|Q\| < 1$ and*

$$B\colon\ X \to X_0, \qquad Q\colon\ X_0 \to X_0.$$

Let $H_n \subset X$ $(n = 1,2,3,\ldots)$ be subspaces in X and let

$$P_n\colon\ X \to H_n$$

be projections, $\|P_n\| = 1$, *and suppose that for each* $f \in X_0$

$$\big\|P_n f - f\big\| \to 0 \qquad \text{for } n \to \infty. \tag{1.17}$$

Further let B_n *be compact operators,* $B_n\colon X \to X_0$, *and suppose that* B_n *are collectively compact (which means that the set*

$$\big\{\, B_n f \;\big|\; n \in \mathbb{N}, f \in X, \|f\| \leqslant 1 \,\big\} \tag{1.18}$$

is relatively compact) and that for each $f \in X_0$

$$B_n f \to Bf \qquad \text{for } n \to \infty. \tag{1.19}$$

Consider the equations

$$(I - Q - B)u = f, \tag{1.20}$$

$$(I - QP_n - B_n)u_n = f, \tag{1.21}$$

where $f \in X$ *is given and* $u, u_n \in X$ *are unknown. Suppose that for each* $f \in X_0$ *the equation* (1.20) *has a unique solution in* X_0. *Then there is* n_0 *such that for each* $n > n_0$ *and each* $f \in X$ *the equation* (1.21) *is uniquely solvable in* X. *At the same time there are constants* c_1, c_2 *such that the corresponding solutions of* (1.20), (1.21) *satisfy the estimates*

$$\|u_n\| \leqslant c_1\|u\| \leqslant c_2\|f\|, \tag{1.22}$$

$$\|u - u_n\| \leqslant c_2\big(\|QP_n u - Qu\| + \|B_n u - Bu\|\big). \tag{1.23}$$

Proof of this assertion can be found, for example, in [6] (Lemma 2.3 in [6]).

2. Numerical solution

In this part we will use the some notation as in Part 1. Throughout this part let $\langle a, b\rangle \subset \mathbb{R}^1$ be a compact interval, $\varphi\colon \langle a, b\rangle \to \mathbb{R}^1$ a continuous function of bounded variation on $\langle a, b\rangle$. The set M is defined by (1.1) and K by (1.2). Throughout this part we will suppose that the condition (1.15) is fulfilled [note that then also the condition (1.6) is fulfilled].

We know that if $g \in \mathscr{C}_0(\langle a, b\rangle)$ and f is the solution of the equation

$$(\overline{\mathcal{T}}_0 + \mathcal{I})f = g, \tag{2.1}$$

then $f \in \mathscr{C}_0(\langle a,b\rangle)$ and the potential $T_K f$ solves the Fourier problem on M with zero initial condition and the boundary condition g on K [more precisely, on K we consider the boundary condition F defined for $[x,t] \in K$ by $F(x,t) = g(t)$].

Let us recall also the following notation from [2] (see Section 2.2 in [2]). For $\psi \in \mathscr{C}(\langle a,b\rangle)$, $r \geqslant 0$ ($r < +\infty$) define an operator ${}^r\mathcal{H}_\varphi^\psi$ on $\mathscr{B}(\langle a,b\rangle)$ by

$$(2.2) \qquad {}^r\mathcal{H}_\varphi^\psi f(t) = \begin{cases} 0 & \text{if } t \leqslant a+r, \\ \dfrac{2}{\sqrt{\pi}} \displaystyle\int_a^{t-r} f(\tau) \mathrm{e}^{-\alpha^2_{\psi(t),t}(\tau)} \, \mathrm{d}_\tau\bigl(\alpha_{\psi(t),t}(\tau)\bigr) & \text{if } t > a+r \end{cases}$$

for $f \in \mathscr{B}(\langle a,b\rangle)$, $t \in \langle a,b\rangle$ [the function $\alpha_{x,t}(\tau)$ is defined by (1.4)]. It is known that for each $\psi \in \mathscr{C}(\langle a,b\rangle)$ and any $r > 0$

$${}^r\mathcal{H}_\varphi^\psi \colon\ \mathscr{B}(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle)$$

and ${}^r\mathcal{H}_\varphi^\psi$ is a compact (linear) operator on $\mathscr{B}(\langle a,b\rangle)$ (see Lemma 2.1 in [2]).

For $r > 0$, $f \in \mathscr{B}(\langle a,b\rangle)$ denote further

$$(2.3) \qquad \overline{\mathcal{T}}_r f = {}^r\mathcal{H}_\varphi^\varphi f.$$

Thus if $r > 0$ then

$$(2.4) \qquad \overline{\mathcal{T}}_r \colon\ \mathscr{B}(\langle a,b\rangle) \to \mathscr{C}_0(\langle a,b\rangle)$$

and the operator $\overline{\mathcal{T}}_r$ is compact. Further it is known that if $\mathscr{D}$ is the unit ball in $\mathscr{B}(\langle a,b\rangle)$ then for each $t \in (a,b\rangle$ we have

$$(2.5) \qquad \sup_{f\in\mathscr{D}} \bigl[\overline{\mathcal{T}}_0 f(t) - \overline{\mathcal{T}}_r f(t)\bigr] = \frac{2}{\sqrt{\pi}} V_K\bigl(r;\varphi(t),t\bigr) + \left|1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right|$$

(see Lemma 2.3 in [2]).

Now let us consider the following simple approximate solution of the equation (2.1). We will suppose that $g \in \mathscr{C}_0(\langle a,b\rangle)$ [we know that then $f \in \mathscr{C}_0(\langle a,b\rangle)$]. We shall approximate the solution f of (2.1) by a piecewise constant function.

Given n natural, choose points t_i^n, $i = 0,1,\ldots,n$,

$$(2.6) \qquad a = t_0^n < t_1^n < \ldots < t_{n-1}^n < t_n^n = b.$$

Suppose that the nodes t_i^n are chosen in such a way that

$$(2.7) \qquad \lim_{n\to\infty} \max_{i=1,2,\ldots,n} (t_i^n - t_{i-1}^n) = 0;$$

one possibility is that we put

$$t_i^n = a + i\frac{b-a}{n}.$$

For $f \in \mathscr{B}(\langle a,b\rangle)$ define $P_n f \in \mathscr{B}(\langle a,b\rangle)$ such that we put $P_n f(a) = f(t_1^n)$ and

$$P_n f(t) = f(t_i^n) \qquad \text{for } t \in (t_{i-1}^n, t_i^n\rangle \tag{2.8}$$

$(i = 1, 2, \dots, n)$. Let H_n stand for the space of functions on $\langle a,b\rangle$ which are constant on the interval $\langle a, t_1^n\rangle$ and on the intervals $(t_{i-1}^n, t_i^n\rangle$ for $i = 2, 3, \dots, n$. Then the operator P_n is a projection of $\mathscr{B}(\langle a,b\rangle)$ onto H_n. Clearly $\|P_n\| = 1$ and [due to assumption (2.7)] for any $f \in \mathscr{C}_0(\langle a,b\rangle)$ we have

$$P_n f \to f \qquad \text{for } n \to \infty \tag{2.9}$$

in the sense of the norm in $\mathscr{B}(\langle a,b\rangle)$.

Instead of the equation (2.1) we will consider the equation

$$(\mathcal{I} + \overline{\mathcal{T}}_0 P_n)u_n = g. \tag{2.10}$$

First we will realize that for all sufficiently large n the equation (2.10) is uniquely solvable in $\mathscr{B}(\langle a,b\rangle)$ and that if $f \in \mathscr{C}_0(\langle a,b\rangle)$ is a solution of (2.1) [we suppose that $g \in \mathscr{C}_0(\langle a,b\rangle)$], u_n is the solution of (2.10) then

$$u_n \to f \qquad \text{for } n \to \infty \tag{2.11}$$

in the sense of the norm in $\mathscr{B}(\langle a,b\rangle)$ (that is $u_n \to f$ uniformly on $\langle a,b\rangle$). This fact follows from Proposition 1.1 in the following way. In Proposition 1.1 denote

$$X = \mathscr{B}(\langle a,b\rangle), \qquad X_0 = \mathscr{C}_0(\langle a,b\rangle).$$

Choose $r > 0$ such that

$$\sup_{t\in(a,b\rangle}\left[\frac{2}{\sqrt{\pi}}V_K\bigl(r;\varphi(t),t\bigr) + \left|1 - \frac{2}{\sqrt{\pi}}G\bigl(\alpha_{\varphi(t),t}(t)\bigr)\right|\right] < 1; \tag{2.12}$$

such an r exists due to the assumption (1.15). Further put

$$B = -\overline{\mathcal{T}}_r, \qquad Q = -(\overline{\mathcal{T}}_0 - \overline{\mathcal{T}}_r).$$

Then

$$B\colon\ X \to X_0, \qquad Q\colon\ X_0 \to X_0$$

(and $Q\colon X \to X$, of course). As we have seen above, B is a compact operator on X and it follows from (2.5) that [under the condition (2.12)] we have $\|Q\| < 1$. Put

$$B_n = BP_n.$$

Since B is compact and $\|P_n\| = 1$, the operators B_n are collectively compact. Since $P_n f \to f$ for $f \in X_0$, we have also

$$B_n f \to Bf$$

for $f \in X_0$. Now the equation (2.1) can be written in the form

$$(\mathcal{I} - Q - B)f = g \tag{2.13}$$

and the equation (2.10) in the form

$$(\mathcal{I} - QP_n - B_n)u_n = g. \tag{2.14}$$

We know that the equation (2.1), that is the equation (2.13), is uniquely solvable in X_0 for any $g \in X_0$ (it is even uniquely solvable in X for any $g \in X$—see Theorem 4.1 in [2]). Now it follows from Proposition 1.1 that there is n_0 such that for any n natural, $n > n_0$, the equation (2.14), that is the equation (2.10), is uniquely solvable in X (even for any $g \in X$). Now we want to prove that if f is the solution of the equation (2.10) then (2.11) is valid. Since $f \in X_0$ and $P_n f \to f$, we have

$$\|QP_n f - Qf\| = \big\|Q(P_n f - f)\big\| \to 0.$$

We have seen that $\|B_n f \to Bf\| \to 0$ and it follows from Proposition 1.1 [see inequality (1.23) in Proposition 1.1] that $u_n \to f$. This fact means that the numerical method considered converges.

Let us describe in detail the algorithm to which the numerical method leads. Let χ_1^n be the characteristic function of the interval $\langle a, t_1^n\rangle$ and for $i = 2, 3, \dots, n$ let χ_i^n be the characteristic function of $(t_{i-1}^n, t_i^n\rangle$. For $i, j = 1, 2, \dots, n$ put

$$A_{ij}^n = (\mathcal{I} + \overline{\mathcal{T}}_0)\chi_j^n(t_i^n) \tag{2.15}$$

and further

$$b_i^n = g(t_i^n), \tag{2.16}$$

$$x_i^n = u_n(t_i^n) \tag{2.17}$$

for $i = 1, 2, \ldots, n$. We want to show that solving the equation (2.10) [that is the equation (2.14)] is equivalent to solving the linear system of equations

$$\sum_{j=1}^{n} A_{ij}^n x_j^n = b_i^n, \qquad i = 1, 2, \ldots, n. \tag{2.18}$$

Suppose thus that n is so large that the equation (2.10) is uniquely solvable [in $\mathscr{B}(\langle a, b\rangle)$] and let u_n be the solution of this equation. For $f \in \mathscr{B}(\langle a, b\rangle)$ we have

$$P_n f = \sum_{i=1}^{n} f(t_i^n)\chi_i^n$$

and using the notation (2.17) we can write

$$P_n u_n = \sum_{i=1}^{n} x_i^n \chi_i^n.$$

Thus the equation (2.10) can be written in the form

$$u_n(t) + \sum_{j=1}^{n} x_j^n \overline{\mathcal{T}}_0 \chi_j^n = g(t). \tag{2.19}$$

Using the notation (2.16) we now get for $t = t_i^n$ $(i = 1, 2, \ldots, n)$

$$x_i^n + \sum_{j=1}^{n} x_j^n \overline{\mathcal{T}}_0 \chi_j^n(t_i^n) = b_i^n. \tag{2.20}$$

We have

$$A_{ij}^n = (\mathcal{I} + \overline{\mathcal{T}}_0)\chi_j^n(t_i^n) = \chi_j^n(t_i^n) + \overline{\mathcal{T}}_0 \chi_j^n(t_i^n)$$

and since

$$\chi_j^n(t_i^n) = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j \end{cases}$$

$(i, j = 1, 2, \ldots, n)$, we can write

$$A_{ij}^n = \begin{cases} \overline{\mathcal{T}}_0 \chi_j^n(t_i^n) & \text{for } i \neq j, \\ 1 + \overline{\mathcal{T}}_0 \chi_i^n(t_i^n) & \text{for } i = j. \end{cases} \tag{2.21}$$

Now it is seen that the system (2.20) is the same as the system (2.18). Thus we see that if u_n is the solution of the equation (2.10) then x_i^n given by (2.17) solves the system (2.18). Since $g \in \mathscr{C}_0(\langle a,b\rangle)$ was arbitrary, it means that the system (2.18) is solvable for any b_i^n (if n is sufficiently large) and thus the matrix

$$\mathsf{A} = \big(A_{ij}^n\big)$$

is regular.

Suppose now that n is sufficiently large and let x_i^n $(i = 1,2,\dots,n)$ be the solution of the system (2.18). If we put

$$u_n(t) = g(t) - \sum_{j=1}^{n} x_j^n \overline{\mathcal{T}}_0 \chi_j^n(t)$$

for $t \in \langle a,b\rangle$ then it follows from (2.19) that u_n solves the equation (2.10). Now we see that solving (2.10) is equivalent to solving (2.18), indeed.

Now let us show a simple way how to evaluate the coefficients A_{ij}^n of the linear system (2.18). Let us recall that for $f \in \mathscr{B}(\langle a,b\rangle)$, $t \in (a,b\rangle$ we have

$$\overline{\mathcal{T}}_0 f(t) = \frac{2}{\sqrt{\pi}} \int_a^t f(\tau) \mathrm{e}^{-\alpha^2_{\varphi(t),t}(\tau)} \,\mathrm{d}\alpha_{\varphi(t),t}(\tau) + f(t)\left[1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t),t}(t)\big)\right], \tag{2.22}$$

where for $\tau \in \langle a,t)$ the function $\alpha_{x,t}(\tau)$ is defined by (1.4) and

$$\alpha_{x,t}(t) = \lim_{\tau \to t-} \alpha_{x,t}(\tau). \tag{2.23}$$

It is seen easily that if $a \leqslant t_1 < t_2 \leqslant t \leqslant b$ then

$$\int_{t_1}^{t_2} \mathrm{e}^{-\alpha^2_{\varphi(t),t}(\tau)} \,\mathrm{d}\alpha_{\varphi(t),t}(\tau) = G\big(\alpha_{\varphi(t),t}(t_2)\big) - G\big(\alpha_{\varphi(t),t}(t_1)\big) \tag{2.24}$$

[if $t_2 = t$ then the value $\alpha_{\varphi(t),t}(t)$ is given by (2.23), of course]. For the evaluation of A_{ij}^n we use the equality (2.21). In the case $i < j$ we have $\chi_j^n(t) = 0$ for $t \in \langle a, t_i^n\rangle$ and it is seen from (2.21), (2.22) that then $A_{ij}^n = 0$. In the case $i > j$ we get from (2.21) and (2.22), (2.24) [and the fact that $\chi_j^n(t_i^n) = 0$]

$$\begin{aligned} A_{ij}^n = \overline{\mathcal{T}}_0 \chi_j^n(t_i^n) &= \frac{2}{\sqrt{\pi}} \int_{t_{j-1}^n}^{t_j^n} \mathrm{e}^{-\alpha^2_{\varphi(t_i^n),t_i^n}(\tau)} \,\mathrm{d}\alpha_{\varphi(t_i^n),t_i^n}(\tau) \\ &= \frac{2}{\sqrt{\pi}} \Big[G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_j^n)\big) - G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{j-1}^n)\big)\Big]. \end{aligned}$$

Now consider the case $i = j$. Then $\chi_i^n(t_i^n) = 1$ and we obtain

$$\begin{aligned}
A_{ii}^n &= 1 + \overline{\mathcal{T}}_0 \chi_i^n(t_i^n) \\
&= 1 + \frac{2}{\sqrt{\pi}} \int_{t_{i-1}^n}^{t_i^n} \mathrm{e}^{-\alpha^2_{\varphi(t_i^n),t_i^n}(\tau)} \, \mathrm{d}\alpha_{\varphi(t_i^n),t_i^n}(\tau) + \Big[1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_i^n)\big)\Big] \\
&= 1 + \frac{2}{\sqrt{\pi}} \Big[G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_i^n)\big) - G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)\big)\Big] + \Big[1 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_i^n)\big)\Big] \\
&= 2 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)\big).
\end{aligned}$$

Note that it is not necessary to know the value of the limit

$$\alpha_{\varphi(t_i^n),t_i^n}(t_i^n) = \lim_{\tau \to t_i^n -} \alpha_{\varphi(t_i^n),t_i^n}(\tau)$$

in order to evaluate A_{ii}^n. Further note that since the value $\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)$ is finite, we have

$$\frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)\big) < 2$$

and thus $A_{ii}^n \neq 0$ (even $A_{ii}^n > 0$) for all $i = 1, 2, \dots, n$. We can summarize that the values of the coefficients A_{ij}^n can be written in the form

$$A_{ij}^n = \begin{cases} 0 & \text{if } j > i, \\ 2 - \dfrac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)\big) & \text{if } j = i, \\ \dfrac{2}{\sqrt{\pi}} \Big[G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_j^n)\big) - G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{j-1}^n)\big)\Big] & \text{if } j < i. \end{cases} \tag{2.25}$$

We see that the matrix $\mathsf{A} = (A_{ij}^n)$ is triangular. We have seen above that it follows from Proposition 1.1 that A is regular for all sufficiently large n. Since $A_{ii}^n \neq 0$ we even see now that A is always regular.

It was shown previously that if f is the solution of (2.1) and u_n is the solution of (2.10) then [assuming $g \in \mathscr{C}_0(\langle a, b \rangle)$]

$$u_n \to f.$$

Since $f \in \mathscr{C}_0(\langle a, b \rangle)$ and the values x_i^n are of the form (2.17) we have also

$$P_n u_n = \sum_{i=1}^n x_i^n \chi_i^n \to f \qquad \text{for } n \to \infty \tag{2.26}$$

uniformly on $\langle a, b\rangle$ [(2.7) is still supposed]. Thus for a given n,

$$P_n u_n = \sum_{i=1}^{n} x_i^n \chi_i^n$$

can be regarded as an approximate solution of the integral equation (2.1). If f solves (2.1) then the potential $Tf = T_K f$ is the solution of the first boundary value problem for the heat equation on M with zero initial condition and the boundary condition g on K. Potential $T_K(P_n u_n)$ is an approximate solution of this boundary value problem [note that $T_K(P_n u_n)$ fulfils the zero initial condition exactly].

Let us show a simple way how to evaluate the potential $T_K(P_n u_n)$ on M using the values x_i^n. Given $t \in (a, b\rangle$, $x > \varphi(t)$, we first find n_0 integer such that

$$t_{n_0-1}^n < t \leqslant t_{n_0}^n \tag{2.27}$$

(then $1 \leqslant n_0 \leqslant n$). Since for $t \in (a, b\rangle$, $x > \varphi(t)$ we have

$$\alpha_{x,t}(t) = \lim_{\tau \to t-} \alpha_{x,t}(\tau) = +\infty$$

[and $G(+\infty) = \sqrt{\pi}$], using (2.24) we obtain for such x, t that

$$\begin{aligned} T_K(P_n u_n) &= \frac{2}{\sqrt{\pi}} \int_a^t P_n u_n(\tau) \mathrm{e}^{-\alpha_{x,t}^2(\tau)} \, \mathrm{d}\alpha_{x,t}(\tau) \\ &= \frac{2}{\sqrt{\pi}} \int_a^t \sum_{i=1}^{n_0} x_i^n \chi_i^n(\tau) \mathrm{e}^{-\alpha_{x,t}^2(\tau)} \, \mathrm{d}\alpha_{x,t}(\tau) \\ &= \frac{2}{\sqrt{\pi}} \sum_{i=1}^{n_0-1} x_i^n \int_{t_{i-1}^n}^{t_i^n} \mathrm{e}^{-\alpha_{x,t}^2(\tau)} \, \mathrm{d}\alpha_{x,t}(\tau) \\ &\quad + x_{n_0}^n \frac{2}{\sqrt{\pi}} \int_{t_{n_0-1}^n}^{t} \mathrm{e}^{-\alpha_{x,t}^2(\tau)} \, \mathrm{d}\alpha_{x,t}(\tau) \\ &= \frac{2}{\sqrt{\pi}} \sum_{i=1}^{n_0-1} x_i^n \Big[G\big(\alpha_{x,t}(t_i^n)\big) - G\big(\alpha_{x,t}(t_{i-1}^n)\big) \Big] \\ &\quad + x_{n_0}^n \frac{2}{\sqrt{\pi}} \Big[\sqrt{\pi} - G\big(\alpha_{x,t}(t_{n_0-1}^n)\big) \Big] \end{aligned} \tag{2.28}$$

(in the case $n_0 = 1$ the value of the sum $\sum_{i=1}^{n_0-1} \ldots$ is considered to be zero).

Now we can summarize the procedure into the following three steps:

Step 1. Choose nodes t_i^n such that (2.6), (2.7) is valid. In the subsequent calculation consider n to be fixed.

Step 2. Evaluate coefficients b_i^n by (2.16), coefficients A_{ij}^n by (2.25), and solve the linear system (2.18). Since this system has a triangular matrix with non-zero coefficients on the diagonal, it is very easy to solve it.

Step 3. For $[x,t] \in M$ the value of the approximate solution $T_K(P_n u_n)$ at $[x,t]$ of the boundary problem is given by (2.28).

Now examine the function $T_K(P_n u_n)$ in more detail. Clearly $P_n u_n \in \mathscr{B}(\langle a,b\rangle)$, the potential $T_K(P_n u_n)$ is caloric on M (i.e. fulfils the heat equation there) and for each $x > \varphi(a)$

$$\lim_{\substack{[y,\tau]\to[x,a]\\ [y,\tau]\in M}} T_K(P_n u_n)(y,\tau) = 0.$$

Thus the approximate solution $T_K(P_n u_n)$ fulfils the zero initial condition exactly (with the exception of the point $[\varphi(a),a]$).

Since $P_n u_n$ is constant on each interval $(t_{i-1}^n, t_i^n\rangle$, it follows from [3] (see Remark 2.4 in [3]) that for any $t \in (t_{i-1}^n, t_i^n)$ $(i = 1,2,\ldots,n)$ the limit

$$\lim_{\substack{[y,\tau]\to[\varphi(t),t]\\ \tau\in(a,b\rangle, y>\varphi(\tau)}} T_K(P_n u_n)(y,\tau)$$

exists and is finite; this limit exists and is finite also for $t = t_n^n = b$. Applying this assertion to the intervals $\langle a, t_i^n\rangle$ (instead of $\langle a,b\rangle$) we see that the limits

$$\lim_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ \tau\leqslant t_i^n, y>\varphi(\tau)}} T_K(P_n u_n)(y,\tau)$$

exist and are finite $(i = 1,2,\ldots,n)$. Further we have

$$\lim_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ \tau\leqslant t_i^n, y>\varphi(\tau)}} T_K(P_n u_n)(y,\tau) = T_K(P_n u_n)\big(\varphi(t_i^n), t_i^n\big) + P_n u_n(t_i^n)\left[2 - \frac{2}{\sqrt{\pi}} G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_i^n)\big)\right]. \tag{2.29}$$

Since $P_n u_n = \sum\limits_{j=1}^{n} x_j^n \chi_j^n$ we have

$$T_K(P_n u_n)\big(\varphi(t_i^n), t_i^n\big) = \sum_{j=1}^{n} x_j^n T_K \chi_j^n \big(\varphi(t_i^n), t_i^n\big).$$

Here $T_K\chi_j^n(\varphi(t_i^n), t_i^n) = 0$ for $j > i$ and

$$T_K\chi_j^n\big(\varphi(t_i^n), t_i^n\big) = \frac{2}{\sqrt{\pi}}\int_{t_{j-1}^n}^{t_j^n} \mathrm{e}^{-\alpha^2_{\varphi(t_i^n),t_i^n}(\tau)}\,\mathrm{d}\alpha_{\varphi(t_i^n),t_i^n}(\tau)$$
$$= \frac{2}{\sqrt{\pi}}\Big[G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_j^n)\big) - G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{j-1}^n)\big)\Big] = A_{ij}^n$$

for $j < i$ [see (2.25)]. In the case $j = i$ we have

$$T_K\chi_i^n\big(\varphi(t_i^n), t_i^n\big) = \frac{2}{\sqrt{\pi}}\int_{t_{i-1}^n}^{t_i^n} \mathrm{e}^{-\alpha^2_{\varphi(t_i^n),t_i^n}(\tau)}\,\mathrm{d}\alpha_{\varphi(t_i^n),t_i^n}(\tau)$$
$$= \frac{2}{\sqrt{\pi}}\Big[G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_i^n)\big) - G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)\big)\Big]$$

and thus

$$T_K(P_nu_n)\big(\varphi(t_i^n), t_i^n\big) = \sum_{j=1}^{i-1} x_j^n A_{ij}^n + x_i^n\frac{2}{\sqrt{\pi}}\Big[G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_i^n)\big) - G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)\big)\Big].$$

Now it follows from (2.29) that (for $i = 1, 2, \dots, n$)

$$\begin{aligned}
&\lim_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ \tau\leqslant t_i^n,\, y>\varphi(\tau)}} T_K(P_nu_n)(y,\tau)\\
&\quad= \sum_{j=1}^{i-1} x_j^n A_{ij}^n + x_i^n\frac{2}{\sqrt{\pi}}\big[G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_i^n)\big) - G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)\big)\big]\\
&\qquad\qquad + x_i^n\Big[2 - \frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_i^n)\big)\Big]\\
&\quad= \sum_{j=1}^{i-1} x_j^n A_{ij}^n + x_i^n\Big[2 - \frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t_i^n),t_i^n}(t_{i-1}^n)\big)\Big]\\
&\quad= \sum_{j=1}^{i} x_j^n A_{ij}^n
\end{aligned}$$

[see (2.25)]. As $A_{ij}^n = 0$ for $j > i$ it follows from (2.18), (2.16) that

$$\sum_{j=1}^{i} x_j^n A_{ij}^n = b_i^n = g(t_i^n)$$

and we see that for $i = 1, 2, \dots, n$

$$\lim_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ \tau\leqslant t_i^n,\, y>\varphi(\tau)}} T_K(P_nu_n)(y,\tau) = g(t_i^n). \tag{2.30}$$

Thus in this sense the approximate solution $T_K(P_n u_n)$ fulfils the boundary condition g at the points $[\varphi(t_i^n), t_i^n] \in K$ *exactly*—for $i < n$ we do not consider limits with respect to M but with respect to the set

$$M \cap \{\, [x,t] \in \mathbb{R}^2 \;\big|\; t \leqslant t_i^n \,\}.$$

Let us examine the expression (2.28) for the function $T_K(P_n u_n)$ more closely. Recall that the equality (2.28) holds for $t \in (a, b\rangle$, $x > \varphi(t)$ if n_0 is an integer, $1 \leqslant n_0 \leqslant n$ and such that (2.27) is fulfilled. But the right hand side of (2.28) has sense also if the condition $x > \varphi(t)$ is not fulfilled and even if $t > b$. Let us define a function h_n on $\mathbb{R}^2$ in the following way. For $[x,t] \in \mathbb{R}^2$, $t \leqslant a$ put $h_n(x,t) = 0$. Now let $[x,t] \in \mathbb{R}^2$, $t > a$. If $t \leqslant t_{n-1}^n$ let n_0 (integer) be such that

$$t_{n_0-1}^n < t \leqslant t_{n_0}^n;$$

if $t > t_{n-1}^n$ put $n_0 = n$. In both cases we define

$$\tag{2.31} \begin{aligned} h_n(x,t) = {} & \frac{2}{\sqrt{\pi}} \sum_{i=1}^{n_0-1} x_i^n \Big[G\big(\alpha_{x,t}(t_i^n)\big) - G\big(\alpha_{x,t}(t_{i-1}^n)\big) \Big] \\ & + x_{n_0}^n \frac{2}{\sqrt{\pi}} \Big[\sqrt{\pi} - G\big(\alpha_{x,t}(t_{n_0-1}^n)\big) \Big]. \end{aligned}$$

For $t \in (a, b\rangle$, $x > \varphi(t)$ we have $h_n(x,t) = T_K(P_n u_n)$, of course. Put further $x_0^n = 0$. Then for $t > a$ ($x \in \mathbb{R}^1$) the equality (2.31) can be written in the form

$$\tag{2.32} h_n(x,t) = 2x_{n_0}^n + \frac{2}{\sqrt{\pi}} \sum_{i=0}^{n_0-1} (x_i^n - x_{i+1}^n) G\big(\alpha_{x,t}(t_i^n)\big).$$

For $i = 0, 1, \dots, n-1$ put

$$\tag{2.33} q_i = -(x_i^n - x_{i+1}^n)$$

($x_0^n = 0$ all the time). Then

$$\tag{2.34} x_i^n = \sum_{j=0}^{i-1} q_j$$

for $i = 1, 2, \dots, n$ (formally also for $i = 0$). Define a function H on $\mathbb{R}^2$ by

$$\tag{2.35} H(x,t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ 2 - \dfrac{2}{\sqrt{\pi}} G\left(\dfrac{x}{2\sqrt{t}}\right) & \text{if } t > 0. \end{cases}$$

Let n_0 have the sense described above. It is seen from (2.32), (2.33), (2.34) that h_n can be written in the form

$$\begin{aligned} h_n(x,t) &= 2\sum_{i=0}^{n_0-1} q_i - \frac{2}{\sqrt{\pi}} \sum_{i=0}^{n_0-1} q_i G\bigl(\alpha_{x,t}(t_i^n)\bigr) \\ &= \sum_{i=0}^{n_0-1} q_i \Bigl[2 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{x,t}(t_i^n)\bigr)\Bigr] = \sum_{i=0}^{n_0-1} q_i H\bigl(x-\varphi(t_i^n), t-t_i^n\bigr). \end{aligned}$$

But $H(x,t) = 0$ for $t \leqslant 0$ and thus

$$h_n(x,t) = \sum_{i=0}^{n-1} q_i H\bigl(x-\varphi(t_i^n), t-t_i^n\bigr) \tag{2.36}$$

for all $[x,t] \in \mathbb{R}^2$.

On the set

$$\mathbb{R}^2 \setminus \bigl\{\, [x,0] \bigm| x \leqslant 0 \,\bigr\}$$

the function H is clearly continuous and caloric. At the points of the form $[y,0]$, $y \leqslant 0$, H is continuous with respect to the half-plane

$$\bigl\{\, [x,t] \in \mathbb{R}^2 \bigm| t \leqslant 0 \,\bigr\}$$

(it vanishes there). Hence we see that the function h_n possesses the following two properties:

(1) h_n is continuous and caloric on the set

$$\mathbb{R}^2 \setminus \bigcup_{i=0}^{n-1} \bigl\{\, [x,t_i^n] \bigm| x \leqslant \varphi(t_i^n) \,\bigr\}.$$

(2) At the points of the form $[y,t_i^n]$, $y \leqslant \varphi(t_i^n)$ the function h_n is continuous with respect to the set

$$\bigl\{\, [x,t] \in \mathbb{R}^2 \bigm| t \leqslant t_i^n \,\bigr\}$$

$(i = 0,1,\dots,n-1)$.

Especially h_n is continuous at the points $[\varphi(t_i^n), t_i^n]$ with respect to the half-plane $\{[x,t] \in \mathbb{R}^2 \mid t \leqslant t_i^n\}$ (at $[\varphi(t_n^n), t_n^n]$ the function h_n is even continuous with respect to $\mathbb{R}^2$). Since $h_n(x,t) = T_K(P_n u_n)$ for $[x,t] \in M$ it follows now from (2.30) that

$$h_n\bigl(\varphi(t_i^n), t_i^n\bigr) = g(t_i^n) \tag{2.37}$$

for $i = 1, 2, \ldots, n$ (this equality holds also for $i = 0$ but it is not interesting in this case). Now we see that the method of numerical solution of the first boundary value problem for the heat equation on M with zero initial condition and the boundary condition g on K [$g \in \mathscr{C}_0(\langle a, b\rangle)$] can be reformulated in the following surprisingly elementary way:

We seek an approximate solution h_n of the form (2.36) *such that h_n fulfils boundary conditions at points $[\varphi(t_i^n), t_i^n]$, that is* (2.37) *is valid for $i = 1, 2, \ldots, n$.*

The system of equations (2.18) was considered above for the unknowns x_j^n and coefficients q_i were written in the form (2.33). But the coefficients q_i can be determined directly from the conditions (2.37). It is seen easily that the conditions (2.37) lead to a system of linear equations with a triangular matrix. For $i = 1, 2, \ldots, n$ we have

$$\begin{aligned} g(t_i^n) = h_n\big(\varphi(t_i^n), t_i^n\big) &= \sum_{j=0}^{n-1} q_j H\big(\varphi(t_i^n) - \varphi(t_j^n), t_i^n - t_j^n\big) \\ &= \sum_{j=0}^{i-1} q_j H\big(\varphi(t_i^n) - \varphi(t_j^n), t_i^n - t_j^n\big). \end{aligned}$$

Especially for $i = 1$

$$g(t_1^n) = q_0 H\big(\varphi(t_1^n) - \varphi(t_0^n), t_1^n - t_0^n\big),$$

hence

$$q_0 = g(t_1^n) \big/ H\big(\varphi(t_1^n) - \varphi(t_0^n), t_1^n - t_0^n\big) \tag{2.38}$$

[clearly $H(\varphi(t_1^n) - \varphi(t_0^n), t_1^n - t_0^n) \neq 0$]. For $i = 1, 2, \ldots, n-1$ we obtain the recurrent formula

$$\begin{aligned} q_i = \Bigg[g(t_{i+1}^n) - \sum_{j=0}^{i-1} q_j H\big(\varphi(t_{i+1}^n) - \varphi(t_j^n), t_{i+1}^n - t_j^n\big) \Bigg] \\ \Bigg/ H\big(\varphi(t_{i+1}^n) - \varphi(t_i^n), t_{i+1}^n - t_i^n\big). \end{aligned} \tag{2.39}$$

As the approximation of the solution of the integral equation was piecewise constant one can not expect better convergence than linear. Note that we chose the piecewise constant approximation because the integral in the integral equation was Stieltjes (Lebesgue-Stieltjes). On the other hand, the algorithm to which the method leads is extremely simple and the approximate solution of the boundary value problem is actually caloric. Let us show a relatively simple estimation of the error for the approximate solution of the boundary value problem.

Let h_n be the approximate solution of the boundary value problem described above and let h be the exact solution of that problem, that is h is continuous on $\overline{M}$, h is caloric on M, $h(x,a)=0$ for $x \geqslant \varphi(a)$, and

$$h\big(\varphi(t),t\big)=g(t)$$

for $t \in \langle a,b\rangle$ [$g \in \mathscr{C}_0(\langle a,b\rangle)$ is the given boundary condition]. Denote

$$r=\sup_{t\in\langle a,b\rangle}\limsup_{\substack{[y,\tau]\to[\varphi(t),t]\\ [y,\tau]\in M}}\big|h_n(y,\tau)-h(y,\tau)\big|. \tag{2.40}$$

It follows from the maximum principle that

$$\big|h_n(x,t)-h(x,t)\big|\leqslant r \tag{2.41}$$

for any $[x,t]\in M$. Regarding the properties of h_n described above and the continuity of h with respect to $\overline{M}$ it is seen immediately that for $t\in\langle a,b\rangle$, $t\neq t_0^n, t_1^n,\dots,t_{n-1}^n$

$$\begin{aligned}\limsup_{\substack{[y,\tau]\to[\varphi(t),t]\\ [y,\tau]\in M}}\big|h_n(y,\tau)-h(y,\tau)\big| &= \Big|h_n\big(\varphi(t),t\big)-h\big(\varphi(t),t\big)\Big|\\ &= \Big|h_n\big(\varphi(t),t\big)-g(t)\Big|.\end{aligned} \tag{2.42}$$

Further, for $i=1,2,\dots,n$ we have [see (2.37) and the item (2) over (2.37)]

$$\limsup_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ [y,\tau]\in M,\tau\leqslant t_i^n}}\big|h_n(y,\tau)-h(y,\tau)\big|=0. \tag{2.43}$$

Given $i\in\{0,1,\dots,n-1\}$ consider $t\in(t_i^n,t_{i+1}^n\rangle$. Then (for any $x\in\mathbb{R}^1$)

$$\begin{aligned}h_n(x,t)-h\big(\varphi(t_i^n),t_i^n\big) &= h_n(x,t)-h_n\big(\varphi(t_i^n),t_i^n\big)\\ &=\sum_{j=0}^{i-1}q_jH\big(x-\varphi(t_j^n),t-t_j^n\big)\\ &\quad-\sum_{j=0}^{i-1}q_jH\big(\varphi(t_i^n)-\varphi(t_j^n),t_i^n-t_j^n\big)+q_iH\big(x-\varphi(t_i^n),t-t_i^n\big).\end{aligned} \tag{2.44}$$

Since the sum

$$\sum_{j=0}^{i-1}q_jH\big(x-\varphi(t_j^n),t-t_j^n\big) \tag{2.45}$$

is continuous at the point $[\varphi(t_i^n), t_i^n]$ (as a function of the variables x, t) it follows from (2.44) and the continuity of h that

$$\limsup_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ [y,\tau]\in M,\tau>t_i^n}} \left|h_n(y,\tau) - h(y,\tau)\right| = |q_i| \limsup_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ [y,\tau]\in M,\tau>t_i^n}} H\bigl(y - \varphi(t_i^n), \tau - t_i^n\bigr) \tag{2.46}$$

($H \geqslant 0$) and together with (2.43) we obtain that for $i = 0, 1, \dots, n-1$

$$\limsup_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ [y,\tau]\in M}} \left|h_n(y,\tau) - h(y,\tau)\right| = |q_i| \limsup_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ [y,\tau]\in M,\tau>t_i^n}} H\bigl(y - \varphi(t_i^n), \tau - t_i^n\bigr). \tag{2.47}$$

If $[y, \tau] \in M$ and $\tau > t_i^n$ then $y > \varphi(\tau)$ and

$$H\bigl(y - \varphi(t_i^n), \tau - t_i^n\bigr) < H\bigl(\varphi(\tau) - \varphi(t_i^n), \tau - t_i^n\bigr).$$

Since H is continuous on $\mathbb{R}^2 \setminus \{[x,0] \mid x \leqslant 0\}$ it is seen now that

$$\limsup_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ [y,\tau]\in M,\tau>t_i^n}} H\bigl(y - \varphi(t_i^n), \tau - t_i^n\bigr) = \limsup_{\tau\to t_i^n+} H\bigl(\varphi(\tau) - \varphi(t_i^n), \tau - t_i^n\bigr). \tag{2.48}$$

Using once more the equality (2.44) it follows from the continuity of h and of the sum (2.45) and from (2.47), (2.48) that

$$\begin{aligned}\limsup_{\substack{[y,\tau]\to[\varphi(t_i^n),t_i^n]\\ [y,\tau]\in M}} \left|h_n(y,\tau) - h(y,\tau)\right| &= |q_i| \limsup_{\tau\to t_i^n+} H\bigl(\varphi(\tau) - \varphi(t_i^n), \tau - t_i^n\bigr)\\ &= \limsup_{\tau\to t_i^n+} \Bigl|h_n\bigl(\varphi(\tau), \tau\bigr) - g(\tau)\Bigr|.\end{aligned} \tag{2.49}$$

Together with (2.42) we obtain the following simple expression for the error r:

$$r = \sup_{t\in\langle a,b\rangle} \Bigl|h_n\bigl(\varphi(t), t\bigr) - g(t)\Bigr|. \tag{2.50}$$

Finally, let us take notice of the fact that in any case $r < +\infty$. It follows from the continuity of h_n on $\overline{M}$ with the exception of the points $[\varphi(t_i^n), t_i^n]$ $(i = 0, 1, \dots, n-1)$ and the boundedness of H $(0 \leqslant H \leqslant 2)$. It is not hard to calculate numerically an approximate value of r in the form (2.50).

3. Numerical example

To illustrate the numerical method for solving the Fourier problem described above let us consider the following case. Put $a = 0$, $b = 4$, and on the interval $\langle 0, 4\rangle$ define a function φ by

$$\varphi(t) = \begin{cases} -2\sqrt{t} & \text{for } t \in \langle 0, 1/2\rangle, \\ -\dfrac{3\sqrt{2}}{4} - \dfrac{1}{2}\sqrt{|t-1|} & \text{for } t \in \langle 1/2, 3/2\rangle, \\ -4\sqrt{2}(2-t)^2 & \text{for } t \in \langle 3/2, 2\rangle, \\ \dfrac{1}{2}\sqrt{|t-5/2|} - \dfrac{\sqrt{2}}{4} & \text{for } t \in \langle 2, 3\rangle, \\ (t-3)^2 & \text{for } t \in \langle 3, 4\rangle. \end{cases}$$

For the sets

$$K = \{\, [\varphi(t), t] \mid t \in \langle 0, 4\rangle \,\},$$
$$M = \{\, [x, t] \in \mathbb{R}^2 \mid t \in (0, 4), x > \varphi(t) \,\}$$

see Fig. 1. It follows from Proposition 1 in [1] that for any $\varepsilon \in (0, 1/2)$ we have

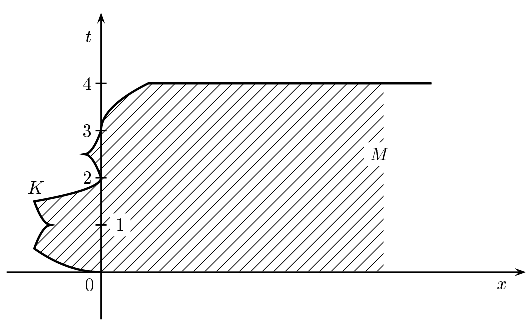


Figure 1.

$$\lim_{r\to 0+} \sup_{t\in S} \left[\frac{2}{\sqrt{\pi}} V_K\bigl(r; \varphi(t), t\bigr) + \left| 1 - \frac{2}{\sqrt{\pi}} G\bigl(\alpha_{\varphi(t),t}(t)\bigr) \right| \right] = 0,$$

where

$$S = \langle \varepsilon, 1-\varepsilon\rangle \cup \langle 1+\varepsilon, 5/2-\varepsilon\rangle \cup \langle 5/2+\varepsilon, 4\rangle.$$

It suffices to examine the parabolic variation on K near the points $t = 0$, 1, $5/2$. But it follows from examples in [1] that

$$\begin{aligned}\lim_{r\to 0+}\sup_{t\in\langle 0,4\rangle}&\left[\frac{2}{\sqrt{\pi}}V_K\big(r;\varphi(t),t\big)+\left|1-\frac{2}{\sqrt{\pi}}G\big(\alpha_{\varphi(t),t}(t)\big)\right|\right]\\&=\max\left\{\frac{2}{\sqrt{\pi}}\int_0^1 \mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha, 3\frac{2}{\sqrt{\pi}}\int_0^{1/4}\mathrm{e}^{-\alpha^2}\,\mathrm{d}\alpha\right\}\\&\approx \max\{0.842700793, 0.828979171\} = 0.842700793 < 1.\end{aligned}$$

Thus the condition for the convergence of the method is fulfilled.

We will solve the Fourier problem on M with zero initial condition and with a boundary condition g for which the (exact) solution is known. Let W stand for the heat kernel on $\mathbb{R}^2$, that is

$$W(x,t)=\begin{cases}\dfrac{1}{2\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4t}} & \text{if } t>0,\\ 0 & \text{if } t\leqslant 0.\end{cases}$$

On $\mathbb{R}^2$ define a function h by

$$\begin{aligned}h(x,t) &= 0.5W(x+1,t)+2W(x+2,t)-10W(x+2.6,t-0.9)\\&\quad+2W(x+2.5,t-1.3)-0.3W(x+0.8,t-2.49)+20W(x+2,t-2.8).\end{aligned}$$

Singular points of h do not belong to $\overline{M}$ and thus h is continuous (and bounded) on $\overline{M}$, h is caloric on M, and $h(x,0) = 0$. Putting

$$g(t) = h\big(\varphi(t),t\big)$$

for $t \in \langle 0,4\rangle$ the function h is the (exact) solution of the Fourier problem on M with zero initial condition and with the boundary condition g on K. For the graph of g see Fig. 2.

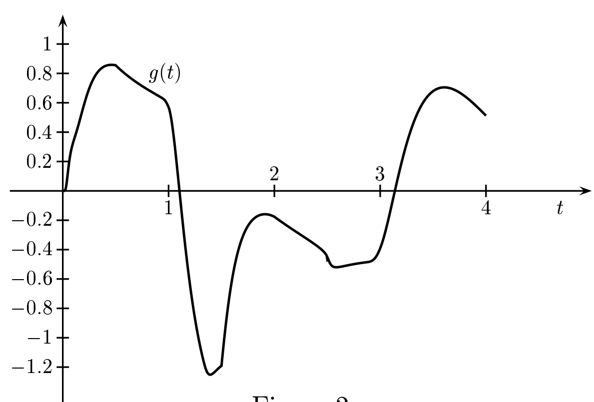


Figure 2.

Now we will look for the numerical solution of the first boundary value problem on M with zero initial condition and the boundary condition g on K using the scheme described above. For the sake of simplicity we will use an equidistant partition of the interval $\langle 0,4\rangle$, that is given n natural we consider nodes t_i^n of the form

$$t_i^n = i\frac{4}{n}$$

($i = 0,1,\dots,n$). The approximate solution h_n is given by (2.36), where the coefficients q_i are determined by the condition (2.37) and can be evaluated by the recurrent formula (2.39) [equivalently h_n can be expressed in the form (2.31), where the values x_i^n satisfy the system of equations (2.18)]. Let us note that for the evaluation of the error function $\frac{2}{\sqrt{\pi}}G$ we have used a procedure from the package *Numerical recipes* (in Pascal), see [9].

We have seen above that for $[x,t]\in M$ we have

$$\bigl|h_n(x,t) - h(x,t)\bigr| \leqslant r,$$

where

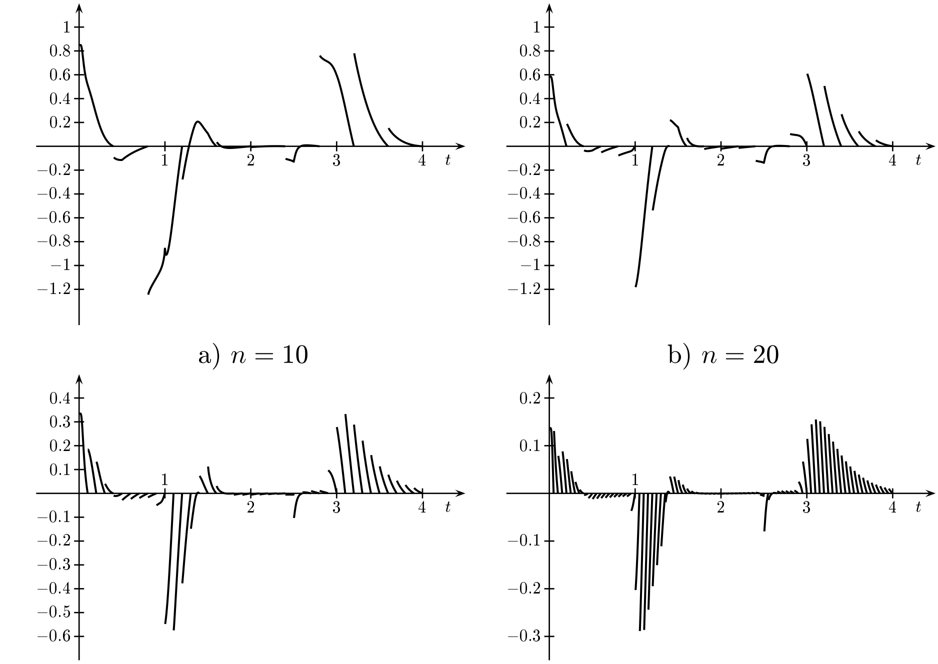
$$r = \sup_{t\in\langle a,b\rangle}\Bigl|h_n\bigl(\varphi(t),t\bigr) - g(t)\Bigr|. \tag{3.1}$$

In this connection we are interested in the graph of the function $h_n(\varphi(t),t)-g(t)$. For this graph for some values of n see Fig. 3. Take notice of the fact that $h_n(\varphi(t),t)-g(t)$ has discontinuities at points t_i^n for $i=0,1,\dots,n-1$. This is caused by the behaviour of the function H near the point $[0,0]$. Also take notice of that it seems (in our case) that for large n the function $h_n(\varphi(t),t)-g(t)$ is monotonous on each of the intervals $(t_i^n, t_{i+1}^n\rangle$ and vanishes at the nodes t_i^n [compare (2.37)]; in our case this is not true for $n=10$ and $n=20$. For a “typical” graph of $h_n(\varphi(t),t)-g(t)$ for large n see Fig. 4, where $n=1280$ and the graph is considered on two different intervals. The two intervals contain the points $t=1$ and $t=5/2$, respectively, at which the function φ has “edges”.

In this connection also the graph of the error $h_n(x,t)-h(x,t)$ for t fixed and x from an interval of the form $\langle\varphi(t),k\rangle$ may be interesting. In Fig. 5 the case $n=40$ for some different t is considered ($t=3.2$ is a node and the adjacent node is $t=3.3$).

As we have noted we have $|h_n(x,t)-h(x,t)|\leqslant r$ on M, where r is given by (3.1). For the graph of

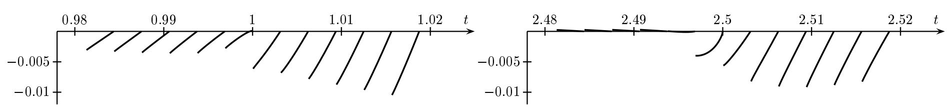
$$h_n\bigl(\varphi(t),t\bigr) - g(t) = h_n\bigl(\varphi(t),t\bigr) - h\bigl(\varphi(t),t\bigr)$$



a) $n = 10$ b) $n = 20$

c) $n = 40$ d) $n = 80$

Figure 3. Graph of $h_n(\varphi(t), t) - g(t)$.



a) b)

Figure 4. Graph of $h_n(\varphi(t), t) - g(t)$ for $n = 1280$.

for some not very large n see Fig. 3. In the case of large n, instead of the graph of $h_n(\varphi(t), t) - g(t)$ one can consider the graph of a function er which is linear on the intervals $\langle t_i^n, t_{i+1}^n \rangle$,

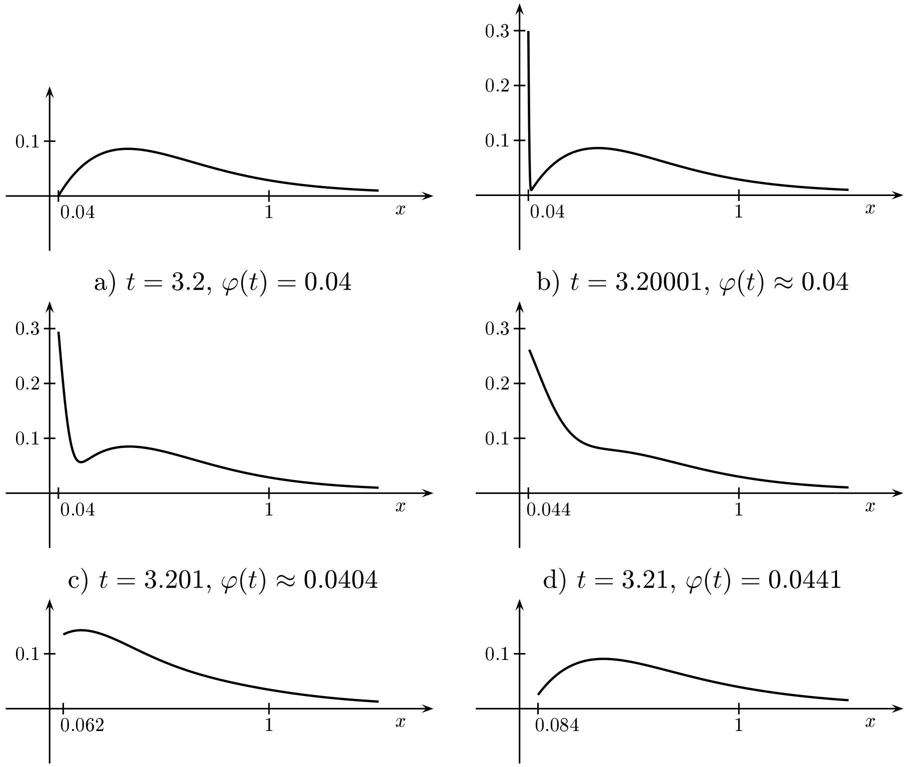
$$er(t_i^n) = \sup_{t \in (t_i^n, t_{i+1}^n)} \Big(h_n\big(\varphi(t), t\big) - g(t) \Big)$$

if

$$\left| \sup_{t \in (t_i^n, t_{i+1}^n)} \Big(h_n\big(\varphi(t), t\big) - g(t) \Big) \right| \geqslant \left| \inf_{t \in (t_i^n, t_{i+1}^n)} \Big(h_n\big(\varphi(t), t\big) - g(t) \Big) \right|$$

and

$$er(t_i^n) = \inf_{t \in (t_i^n, t_{i+1}^n)} \Big(h_n\big(\varphi(t), t\big) - g(t) \Big)$$

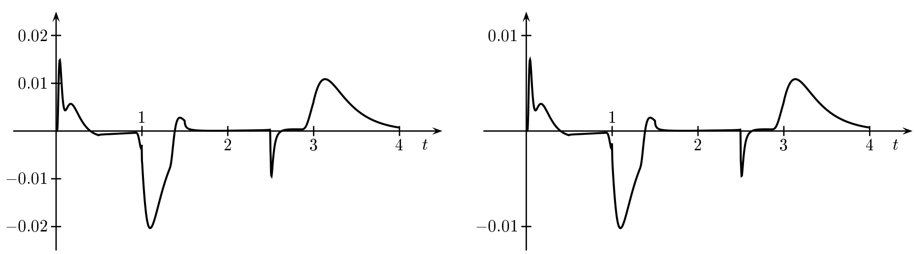


a) $t = 3.2$, $\varphi(t) = 0.04$ b) $t = 3.20001$, $\varphi(t) \approx 0.04$

c) $t = 3.201$, $\varphi(t) \approx 0.0404$ d) $t = 3.21$, $\varphi(t) = 0.0441$

e) $t = 3.25$, $\varphi(t) = 0.0625$ f) $t = 3.29$, $\varphi(t) = 0.0841$

Figure 5. Graph of $h_n(x,t) - h(x,t)$ for $x \in \langle \varphi(t), 1.5\rangle$ and t fixed.



a) $n = 1280$ b) $n = 2560$

Figure 6. Graph of er.

in the other case. For the graph of er for some n see Fig. 6. Compare Fig. 6 with d) in Fig. 3.

Let us note further that the value r can be evaluated approximately only by the values of the coefficients q_i. As we have seen in the "typical" situation, for large n the function $h_n(\varphi(t), t) - g(t)$ is monotonous on the intervals $(t_i^n, t_{i+1}^n\rangle$ and vanishes at t^n_{i+1}. If this is true then

$$\sup_{t\in(t^n_i,t^n_{i+1}\rangle}\Big|h_n\big(\varphi(t),t\big)-g(t)\Big| = \Big|\lim_{t\to t^n_i+}\Big(h_n\big(\varphi(t),t\big)-g(t)\Big)\Big|. \tag{3.2}$$

We have noted above that

$$\limsup_{t\to t^n_i+}\Big|h_n\big(\varphi(t),t\big)-g(t)\Big| = |q_i|\limsup_{t\to t^n_i+} H\big(\varphi(t)-\varphi(t^n_i),t-t^n_i\big) \tag{3.3}$$

[see the equality (2.49)]. As $0\leqslant H\leqslant 2$ we have

$$\Big|\lim_{t\to t^n_i+}\Big(h_n\big(\varphi(t),t\big)-g(t)\Big)\Big| \leqslant 2|q_i|. \tag{3.4}$$

If $h_n(\varphi(t),t)-g(t)$ is supposed to be really monotonous on the intervals $(t^n_i,t^n_{i+1}\rangle$ then we see that

$$r\leqslant 2\max_{i=0,1,\dots,n-1}|q_i|.$$

Note that if the limit

$$\lim_{t\to t^n_i+}\frac{\varphi(t)-\varphi(t^n_i)}{2\sqrt{t-t^n_i}} = a$$

exists then

$$\lim_{t\to t^n_i+} H\big(\varphi(t)-\varphi(t^n_i),t-t^n_i\big) = 2-\frac{2}{\sqrt{\pi}}G(a).$$

If, especially,

$$\lim_{t\to t^n_i+}\frac{\varphi(t)-\varphi(t^n_i)}{2\sqrt{t-t^n_i}} = 0, \tag{3.5}$$

then we obtain that

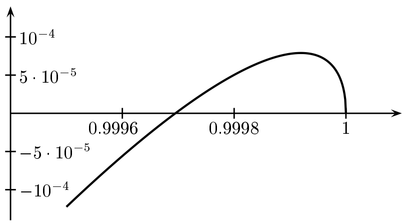
$$\Big|\lim_{t\to t^n_i+}\Big(h_n\big(\varphi(t),t\big)-g(t)\Big)\Big| = |q_i|.$$

Note that, in our case, (3.5) is not valid only at points 0, 1 and 5/2. Here

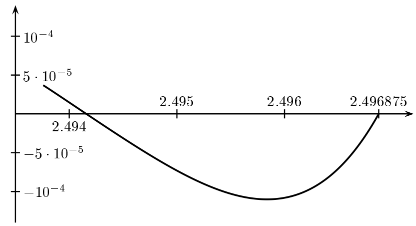
$$\lim_{t\to 0+}\frac{\varphi(t)-\varphi(0)}{2\sqrt{t}} = -1,\quad \lim_{t\to 1+}\frac{\varphi(t)-\varphi(1)}{2\sqrt{t-1}} = -\frac{1}{4},\quad \lim_{t\to 5/2+}\frac{\varphi(t)-\varphi(5/2)}{2\sqrt{t-5/2}} = \frac{1}{4},$$

hence

$$\lim_{t\to 0+} H\big(\varphi(t)-\varphi(0),t\big) = 2-\frac{2}{\sqrt{\pi}}G(-1)\approx 1.842700793,$$
$$\lim_{t\to 1+} H\big(\varphi(t)-\varphi(1),t-1\big) = 2-\frac{2}{\sqrt{\pi}}G(-\tfrac14)\approx 1.276326390,$$
$$\lim_{t\to 5/2+} H\big(\varphi(t)-\varphi(5/2),t-5/2\big) = 2-\frac{2}{\sqrt{\pi}}G(\tfrac14)\approx 0.723673610.$$



a)



b)

Figure 7. Graph of $h_n(\varphi(t), t) - g(t)$ for $n = 1280$.

We have supposed that for large n the function $h_n(\varphi(t), t) - g(t)$ is monotonous on all intervals between two adjacent nodes—as suggested by the graph of that function. But this is not true in general. In Fig. 4 the case $n = 1280$ was considered. In Fig. 7 the graph of $h_n(\varphi(t), t) - g(t)$ is shown on the interval $\langle 0.9995, 1\rangle$ (1 is a node) and on the interval $(2.49375, 2.496875\rangle$ (here both end-points of the interval are nodes); these intervals are subintervals of the intervals considered in Fig. 3. We see that the function considered is not monotonous on the given intervals and that for the second interval (3.2) is not valid—we have

$$\lim_{t \to 2.49375+} \Big(h_n\big(\varphi(t), t\big) - g(t) \Big) \approx 3.74 \cdot 10^{-5},$$

but

$$\sup_{t \in (2.49375, 2.496875\rangle} \Big| h_n\big(\varphi(t), t\big) - g(t) \Big| \approx 1.10 \cdot 10^{-4};$$

thus here even (3.4) is not valid.

Nevertheless, in the following table the approximate values of

$$r = \sup_{t \in \langle 0,4\rangle} \Big| h_n\big(\varphi(t), t\big) - g(t) \Big|$$

and of

$$q = \max_{i=0,1,\dots,n-1} |q_i|$$

are given for some different n.

n	10	20	40	80	160	320
r	1.2744	1.1939	0.57965	0.31137	0.16015	0.080942
q	1.2744	0.9354	0.57963	0.31135	0.16015	0.080938

n	640	1280	2560	5120
r	0.040595	0.020318	0.010164	$5.0833 \cdot 10^{-3}$
q	0.040593	0.020317	0.010164	$5.0832 \cdot 10^{-3}$

One can take notice of the fact that the convergence of the method is really only linear. Further, compare the values in the table for $n = 20$, b) in Fig. 3, equality (3.3), the fact that

$$\lim_{t\to 1+} H\bigl(\varphi(t)-\varphi(1), t-1\bigr) \approx 1.2763,$$

and that $0.9354 \cdot 1.2763 \approx 1.1939$.

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