Børre Bang; Dag Lukkassen

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APPLICATION OF HOMOGENIZATION THEORY RELATED TO STOKES FLOW IN POROUS MEDIA

Børre Bang and Dag Lukkassen, Narvik

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Abstract. We consider applications, illustration and concrete numerical treatments of some homogenization results on Stokes flow in porous media. In particular, we compute the global permeability tensor corresponding to an unidirectional array of circular fibers for several volume-fractions. A 3-dimensional problem is also considered.

Keywords: homogenization theory, Stokes flow, porous media, numerical experiments

MSC 2000: 76S05, 76D05, 76D07, 35B27

1. Introduction

The study of the low speed flow in porous media is important in a wide range of areas, including oil recovery, geothermal development, chemical and nuclear industries and civil engineering. A direct numerical treatment of such problems becomes cumbersome due to the rapid variations on the microscale level. However, when the characteristic size of the obstacles in the porous media ε is small as compared with the whole sample and the arrangement of obstacles is periodic with period equal to ε then it is possible to describe the macroscopic behavior by means of more sophisticated methods. One way to proceed is to use the *homogenization theory*, developed in the studies of partial differential equations for strongly heterogeneous problems.

Among mathematicians the most famous work in this field is Tartar's work on the Stokes equation with homogeneous Dirichlet data in periodic perforated domains (see [25]). This work contains for example a rigorous mathematical proof of Darcy's law and permits us to obtain a detailed description of microscopic as well as global phenomena; e.g., it is possible to prove the tensorial character of the permeability as well as its symmetry and positivity. Similar results for more general periodic cases are obtained by Ene and Polisevski in [10] and by Allaire in [1]. In [12] Lipton and

Avellaneda introduce an explicit characterization of the pressure extension introduced by Tartar in [25], and in [3], [8] and [9] Arbagast, Douglas, Hornung, Ene, Donato, and Saint Jean Paulin study porous media with a double periodicity. We also want to mention that Allaire and Mikelic (see [2] and [20]) treat some evolution cases and [22] an entirely new approach based on two-scale convergence developed by Allaire and Nandakumaran. Moreover, in [7] a generalization is developed for non-Newtonian flows in porous media by Mikelic and Bourgeat. Various periodic structures are considered in all these papers and the corresponding proofs of convergences are based on specific types of homogenization techniques. To our knowledge, the very recent paper of Beliaev and Kozlov [5] is the first of its kind where homogenization of Stokes equation in porous media has been rigorously treated without the traditional periodicity assumptions (see also [11]).

In this paper we focus on applications, illustration and concrete numerical treatments of some of the mathematical results mentioned above.

We start by presenting our model problem (see Section 2) and giving a short description of the homogenization theory for this problem (see Section 3). Whenever it is possible we give physical interpretations. We remark that this treatment of the homogenization theory yields the following:

- A concrete homogenization algorithm for computing the permeability tensor and solving the global problem.
- According to the theoretical results we have good control of the error estimates, stability and convergence questions.

It turns out that the key problem for computing the effective permeability tensor is to identify and solve a variational problem called the *cell problem*. Here, we choose to use a commercially available Computational Fluid Dynamics (CFD) package from Flow-Science, Los Alamos, called Flow3D, for solving this crucial problem (see Section 4). We utilize the homogenization algorithm to compute the global permeability tensor for unidirectional array of circular fibers for several volume-fractions. For the sake of illustration, we also present numerical results for some three dimensional problems. Moreover, we give a concrete example of a design problem where we utilize our numerical results.

Section 5 is reserved for concluding remarks and a final discussion of the homogenization method, our modeling and our numerical experiments. In particular we point out and compare some of the main difficulties regarding characterization of admissible effective moduli in the case of Stokes flow, elasticity and heat conduction.

2. The model problem

Let Ω be a smooth bounded and connected open subset of $\mathbb{R}^n$. For positive numbers ε we consider the following class of Navier-Stokes equations:

$$
\begin{aligned}
\nabla p^\varepsilon - \Delta u^\varepsilon &= f \quad \text{in } \Omega_\varepsilon, \\
u^\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon, \\
\nabla \cdot u^\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon.
\end{aligned} \tag{1}
$$

Here, Ω_ε is a smooth open subset of Ω which is εY-periodically distributed relative to a cell $Y \subset \mathbb{R}^n$. We also assume that Ω_ε has Lipschitz boundary and that $\partial\Omega_\varepsilon \subseteq \partial C_\varepsilon$, where C_ε is the union of all εY-cells entirely contained in Ω. In Figure 2.1 we illustrate the typical geometry of Ω_ε on ε-scale level.

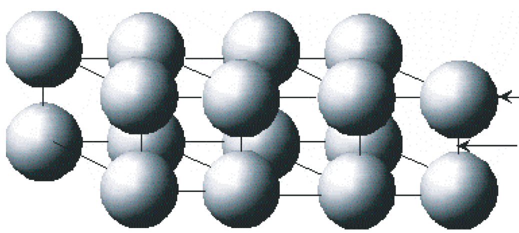


Figure 2.1. An example of a periodic structure. The obstacles (which in this case are spheres) are supported by a very small network (which is neglected).

It is a well known fact that if $f \in [\mathrm{L}^2(\Omega)]^n$, then for each $\varepsilon > 0$, the weak formulation of (1) possesses a unique solution $(u^\varepsilon, p^\varepsilon) \in [\mathrm{H}_0^1(\Omega_\varepsilon)]^n \times [\mathrm{L}^2(\Omega_\varepsilon) \setminus \mathbb{R}]$. A direct numerical treatment of this problem is practically impossible for small values of ε. However, by using the *homogenization procedure* in the next section, we can solve a corresponding *homogenized problem* which approximates, in a weak sense, our original problem (1) well for sufficiently small ε.

3. The homogenization procedure

Let Y_f (the fluid part of Y) and Y_s (the solid part of Y) denote the sets $Y \cap \Omega_1$ and Y/Y_f, respectively. The key step in the homogenization procedure is to solve the *cell problem*:

$$
\begin{aligned}
\nabla q^k - \Delta v^k &= e_k \quad \text{in } Y, \\
v^k & \quad \text{is } Y\text{-periodic}, \\
v^k &= 0 \quad \text{on } \partial Y_s, \\
\nabla \cdot v^k &= 0 \quad \text{in } Y
\end{aligned} \tag{2}
$$

for $k = 1, 2, \ldots, n$. The weak formulation of (2) possesses a unique solution $(v^k, q^k) \in [\mathrm{H}^1(Y_f)]^n \times [\mathrm{L}^2(Y_f)/\mathbb{R}]$ (see [1]). When v^k is found, we can compute the *permeability tensor* $\hat{A}$ defined by

$$\hat{A}_{ij} = \frac{1}{|Y|} \int_Y (v^j)_i \, \mathrm{d}x. \tag{3}$$

Here $|Y|$ denotes the measure (volume) of Y. We now define the homogenized problem (also called Darcy's law):

$$\begin{aligned} \nabla \cdot \hat{A}(f - \nabla p) &= 0 \quad \text{in } \Omega, \\ \hat{A}(f - \nabla p) \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4}$$

It is possible to prove that $\hat{A}$ is symmetric and positive definite (see e.g. [4]). This implies that (4) is an elliptic problem with a Neumann boundary condition which, accordingly, has a unique solution $p \in \mathrm{H}^1(\Omega)/\mathbb{R}$.

Let us define functions u and the extension $\tilde{u}^\varepsilon$ of u^ε to Ω as follows:

$$u = \hat{A}(f - \nabla p), \tag{5}$$

$$\tilde{u}^\varepsilon = \begin{cases} u_\varepsilon & \text{in } \Omega_\varepsilon, \\ 0 & \text{in } \Omega \setminus \Omega_\varepsilon. \end{cases}$$

We can also define a similar (though more complicated) extension $\tilde{p}^\varepsilon$ of p^ε to Ω (see e.g. [10]). The following homogenization result is crucial in the whole theory. For the proof we refer to Sanches-Palencia 1980.

Theorem 1. *As $\varepsilon \to 0$, then $\varepsilon^{-2}\tilde{u}^\varepsilon \rightharpoonup u$ weakly in $\left[\mathrm{L}^2(\Omega)\right]^n$ and $\tilde{p}^\varepsilon \to p$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ strongly.*

Remark 1. From a practical point of view Theorem 3 yields in particular the information that for any domain of $V \subset \Omega$, $\varepsilon^2 u$ and p approximate well the average values of $\tilde{u}^\varepsilon$ and $\tilde{p}^\varepsilon$ (taken over V), respectively, for small values of ε (compared with the size of Ω).

From the above we obtain that the model problem considered as an average problem can be solved by using the following homogenization procedure:

1. solve the cell problem (2) numerically;
2. insert the solution of the cell problem into (3);
3. find p by solving the homogenized problem (4) numerically;
4. compute u by (5).

R e m a r k 2. In the more general case (1) is of the form

$$\begin{aligned}\nabla p^\varepsilon - \nu\Delta w^\varepsilon &= f \quad \text{in } \Omega_\varepsilon,\\ w^\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon,\\ \nabla w^\varepsilon &= 0 \quad \text{in } \Omega_\varepsilon.\end{aligned}$$

Since ν is constant, the solution of this equation can be found directly from the solution of (1) simply by replacing "u^ε" by "νw^ε". Summing up, we find that generally, for small ε, the extension $\tilde{w}^\varepsilon$ of w^ε to Ω (the actual velocity) is approximately equal (in the weak sense) to $\hat{A}_{\text{glob}}(\varepsilon)(f - \nabla p)$, written

$$\tilde{w}^\varepsilon \overset{\text{weak}}{\simeq} \hat{A}_{\text{glob}}(\varepsilon)(f - \nabla p),$$

where

$$\hat{A}_{\text{glob}}(\varepsilon) = \frac{\varepsilon^2}{\nu}\hat{A} \tag{6}$$

(the "global" permeability).

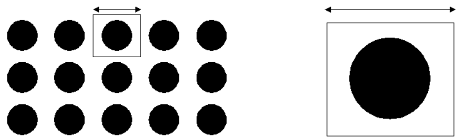
4. Numerical results

In this section we present some numerical results. We start by letting Y be the unit cube in $\mathbb{R}^2$ with center at origin and let Y_s be a disc of radius r with center at origin. This geometry yields a permeability tensor of the form $\hat{A} = \lambda I$, where $\lambda > 0$ (the permeability) and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easily seen that by varying the radius r between 0 and 1/2, λ takes all values between ∞ and 0, correspondingly. Hence, on macroscopic level, low speed flow in any isotropic porous medium may be modelled in this way. In order to compute λ we must go through the steps 1 and 2 of the procedure given in Section 3. We have chosen to use a commercially available Computational Fluid Dynamics (CFD) package from Flow-Science, Los Alamos, called Flow3D, for solving the cell problem. Flow 3D uses the Finite Difference Method and solves the full transient momentum equation, optionally coupled with turbulence models. Due to the fact that Flow3D always solves a time dependent equation, we put $v^k = 0$ and $q^k = 0$ at $t = 0$, where t denotes the time, and use the numerical solution v^k, q^k for $t = \infty$ as solutions for the (stationary) cell problem. For all concrete cases considered in our work v^k

and q^k converge rapidly and it turns out that the variations for $t > 1$ are negligible (see Figure 4.3). Therefore, the numerical solution v^k, q^k for $t = 1$ serves as a good approximation for the cell problem. The Y-cell is subdivided into a number of N^n Finite Difference Cells.



the actuall structure the Y-cell

Figure 4.1. Periodic distribution of circular discs.

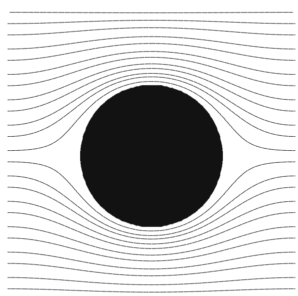


Figure 4.2. Solution of the cell problem. The velocity v^k is illustrated as streamlines for the case when $k = 1$ and $r = 0.25$.

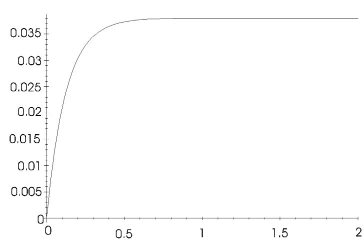


Figure 4.3. The estimated mean kinetic energy W as function of the time t for $r = 0.25$.

In the table below we have listed λ for some values of N for the case $r = 0.25$.

λ	0.025	0.023	0.021	0.021	0.021	0.020
N	10	20	40	60	80	100

In Figure 4.4 we have plotted λ as a function of r for two values of N. From the figure we see that with $N = 10$ the subdivision to computational cells becomes too coarse to get the necessary resolution for both the low and high values of r. As we see in the above table the $N = 40$ subdivision gives a good approximation of the permeability, at least for $r \in [0.1, 0.4]$.

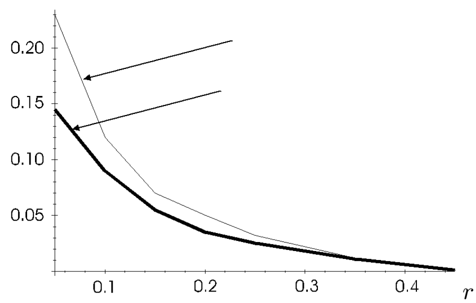


Figure 4.4. Numerical computations of the permeability in the two-dimensional case.

As the name indicates Flow3D is capable of handling three dimensional cases as well. In order to illustrate we have solved the cell problem numerically and computed the permeability λ for th[illegible] $\mathbb{R}^3$ with center at origin and Y_s is a sphere of radi[illegible] (see Figure 4.5).

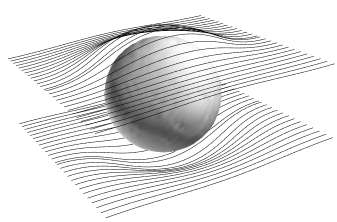


Figure 4.5. Solution of the three dimensional cell problem. The velocity v^k is illustrated as streamlines for the case when $k = 1$ and $r = 0.3$.

From the table below we see that $\lambda = 0.07$ serves as a good approximation in this case.

λ	0.058	0.067	0.068	0.070	0.071
N	10	20	30	40	50

4.1. A computational example.

In order to illustrate the usefulness of the above results we present an example. Let us consider a periodic and square symmetric distribution of circular fibres subjected to a flow of water whose direction is perpendicular to the fibre orientation (see Figure 4.6). The following data is given: The thickness of each fibre $d = 6 \cdot 10^{-4}\,\mathrm{m}$ and the viscosity of water $\nu = 1.7 \cdot 10^{-3}\,\mathrm{Ns/m^2}$. We want to find the distance ε between any two fibres such that the global permeability $\lambda_{\mathrm{glob}}(\varepsilon) = 1 \cdot 10^{-4}\,\mathrm{m^3/Ns}$.

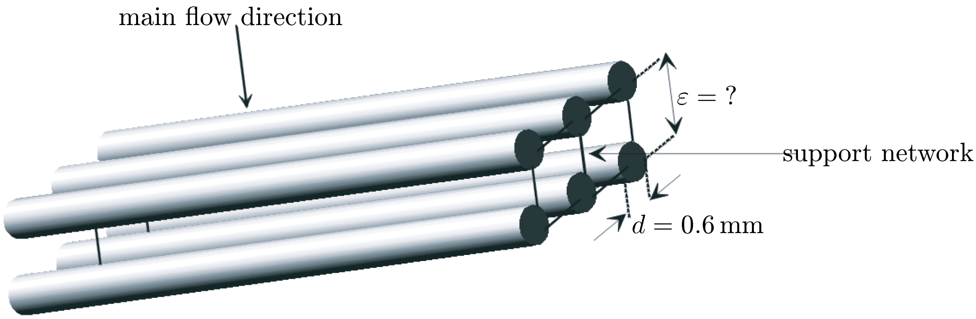


Figure 4.6. Periodic distribution of fibres.

From 6 we have that

$$\lambda_{\mathrm{glob}}(\varepsilon) = \frac{\varepsilon^2}{\nu}\lambda(\varepsilon),$$

where $\lambda(\varepsilon) = \lambda$ is the permeability corresponding to the relative radius $r = d/2\varepsilon = 3 \cdot 10^{-4}/\varepsilon$. Using the curve in Figure 4.4 we are now able to plot λ_{glob} as a function of the distance ε (see Figure 4.7). According to this plot we obtain that $\varepsilon \simeq 2 \cdot 10^{-3}\,\mathrm{m}$ serves as a god approximation to our problem.

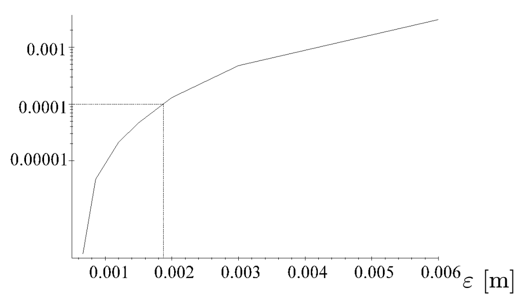


Figure 4.7. The global permeability as a function of the distance between the fibres.

5. Some final comments

The homogenization method is mathematically based and we therefore have good control of convergence questions (see Theorem 1). Moreover, this method enables us to extract a useful algorithm for solving the model problem.

In this paper we have used the homogenization method to solve a linear flow problem. However, it is important to observe that the method is not restricted to linear problems and can be applied to other partial differential equations as well (see e.g. [4, 11, 16, 23]). In particular, problems regarding computer aided applications of the homogenized method for concrete practical purposes in elasticity and heat conduction can be found (see [18, 19]).

In recent years, several bounds on effective moduli of non homogeneous problems have been discovered (see e.g. [13, 14, 15, 16, 17, 21] and the references therein). However, non-trivial bounds for the permeability tensor have not been found yet. For example, an interesting problem will be to characterize the set of all possible permeability tensors that can be generated for a given volume fraction $|Y_s| / |Y|$. Particularly, for two dimensional problems this set will principally look like the one illustrated in Figure 5.1, where λ_1 and λ_2 denote the eigenvalues of the permeability tensor $\tilde{A}$. It is interesting to note that in contrast to e.g. the two dimensional conductivity problem the laminate structure is not necessarily optimal with respect to Stokes flow. This is most easily seen for low volume fractions by using the following argument: due to the no-slip condition along the surfaces of the laminate it is obvious that the values of λ_1 and λ_2 must be lower than for example those corresponding to the structure consisting of one circular fiber in each period. Thus, even the points $(0, a)$ and $(a, 0)$ are nontrivial to determine.

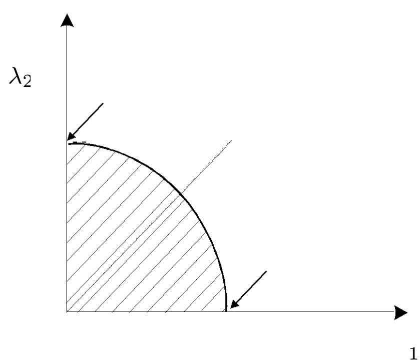


Figure 5.1. The set of admissable eigenvalues of the permeability tensor for a fixed volume-fraction.

The values of λ_1 and λ_2 computed numerically in this paper lie on the line of isotropy. An interesting question is how large these values are compared with the best possible ones. We intend to develop these ideas further in a forthcoming paper.

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Authors' addresses: *Børre Bang*, Department of Engineering Design and Production, HiN, Narvik Institute of Technology, N-8501 Narvik, Norway; *Dag Lukkassen*, Department of Mathematics, HiN, Narvik Institute of Technology, Narvik Technology A/S and NORUT Technology Ldt., N-8501 Narvik, Norway.