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SUPERCONVERGENCE OF MIXED FINITE ELEMENT SEMI-DISCRETIZATIONS OF TWO TIME-DEPENDENT PROBLEMS

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Abstract. We will show that some of the superconvergence properties for the mixed finite element method for elliptic problems are preserved in the mixed semi-discretizations for a diffusion equation and for a Maxwell equation in two space dimensions. With the help of mixed elliptic projection we will present estimates global and pointwise in time. The results for the Maxwell equations form an extension of existing results. For both problems, our results imply that post-processing and a posteriori error estimation for the error in the space discretization can be performed in the same way as for the underlying elliptic problem.

Keywords: superconvergence, diffusion equation, Maxwell equations, mixed elliptic projection

MSC 2000: 65N30

1. Introduction

The main goal of this paper is to transfer the superconvergence results in [1, 2, 4] for the mixed finite element method for elliptic problems to some time-dependent problems that have an elliptic differential operator in space. Consequently, the discretization error in space for those time-dependent problems can then be estimated a posteriori in an asymptotically exact way, using the post-processing techniques for their elliptic counterparts described in the papers mentioned above.

For the transfer of superconvergence results we will use *mixed elliptic projection* of the exact solution, a technique similar to the one developed in the standard finite element context in [8]. We will define this projection and discuss some preliminaries in Section 2. Its use enables us to prove superconvergence for the semi-discrete approximations of the investigated time-dependent equations for those and only those mixed finite elements for which the discretized stationary problem is superconvergent. This includes, for the approximation of the vector field, the two lowest order mixed elements on regular families of uniform triangulations (cf. [1, 2]) and all the rectangular elements (cf. [4]). For the scalar function, this includes all order methods on regular triangulations (cf. [3]).

Key point is the proof that the difference between the elliptic projections and the semi-discrete approximations is of the same order as the superconvergence rate. Then, by means of a simple triangle inequality, the superconvergence for the elliptic projections (if present), easily transfers to the semi-discretisations. Vice versa, if the elliptic projections are not superconvergent, then neither are the semi-discretizations.

In Section 3, we will consider a diffusion equation which will be semi-discretized in space using mixed finite elements of Raviart-Thomas type. We prove that indeed, the semi-discretizations are superclose to the elliptic projections of the exact solutions.

In Section 4 we study the semi-discretisation of a Maxwell model problem in the plane using the Nédélec mixed elements (also called 'edge elements'). This system is suitable for mixed formulation and approximation and similar techniques as in Section 3 are used to prove superconvergence. The conclusion will be that for this problem too, superconvergence occurs exactly for those mixed elements for which the underlying corresponding elliptic equation allows superconvergence. This leads to a broad extension of the superconvergence results in [5, 6], where superconvergence is proved only for the lowest order method on rectangular partitions.

2. Preliminaries

2.1. Sobolev spaces. Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$. Denote the usual Sobolev spaces of order k by $H^k(\Omega)$ and suppose them normed and seminormed by $\|\cdot\|_k$ and $|\cdot|_k$. The differential operators

$$\operatorname{div}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{\partial}{\partial x}q_1 + \frac{\partial}{\partial y}q_2 \quad \text{and} \quad \operatorname{rot}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{\partial}{\partial x}q_2 - \frac{\partial}{\partial y}q_1 \tag{1}$$

give rise to the spaces $\mathbf{H}(\text{div};\Omega)$ and $\mathbf{H}(\text{rot};\Omega)$ of L^2 vector fields of which weak divergence and rotation respectively exist in $L^2(\Omega)$. Their subspaces of fields having—essentially—zero normal (tangential) component on $\partial\Omega$ we denote by $\mathbf{H}_0(\text{div};\Omega)$ and $\mathbf{H}_0(\text{rot};\Omega)$. The operators **grad** and **curl** are such that

$$\operatorname{div}\mathbf{grad} = -\operatorname{rot}\mathbf{curl} = \Delta. \tag{2}$$

2.2. Raviart-Thomas and Nédélec mixed finite element spaces. Here we recall the Raviart-Thomas mixed finite spaces $\mathbf{\Gamma}_h^k$ relative to a family $(\mathcal{T}_h)_h$ of triangulations of $\overline{\Omega}$. Instead of giving a formal definition (for which we refer to [7]) we would rather restrict ourselves to stating some characterizing properties. By $\mathcal{P}^k(\cdot)$

we mean the space of all polynomials of degree k and by W_h^k the space of all piecewise degree k polynomials over the given triangulation.

- $\mathbf{\Gamma}_h^k \subset \mathbf{H}(\mathrm{div};\Omega)$,
- $[W_h^k]^2 \subset \mathbf{\Gamma}_h^k \subset [W_h^{k+1}]^2$,
- $\mathrm{div}(\mathbf{\Gamma}_h^k) = W_h^k$,
- $\forall \mathbf{q}_h \in \mathbf{\Gamma}_h^k, \forall K \in \mathcal{T}_h, \mathbf{q}_h^{\mathrm{T}}\nu \in \mathcal{P}^k(\partial K)$ and is continuous across ∂K.

By pointwise rotation over $\pi/2$ of the vector fields in $\mathbf{\Gamma}_h^k$ one obtains the rotation-conforming Nédélec space $\mathbf{\Gamma}_h^{k\perp}$. Furthermore, set

$$\mathbf{\Gamma}_{0h}^k = \mathbf{\Gamma}_h^k \cap \mathbf{H}_0(\mathrm{div};\Omega) \quad \text{and} \quad \mathbf{\Gamma}_{0h}^{k\perp} = \mathbf{\Gamma}_h^{k\perp} \cap \mathbf{H}_0(\mathrm{rot};\Omega). \tag{3}$$

$\mathrm{P}_h^k\colon L^2(\Omega) \to W_h^k$ denotes the $L^2(\Omega)$-orthogonal projection on W_h^k and is therefore uniquely defined by

$$\forall w_h \in W_h^k\colon \quad (\mathrm{P}_h^k w, w_h) = (w, w_h). \tag{4}$$

2.3. Time dependency. Since our analysis of time-dependent problems aims to be pointwise in time as much as possible, we will usually write $u(t)$ when we mean the function $x \mapsto u(x,t)$ for some fixed $t \in T := [0, T']$. For later use we define the norms

$$|\!|\!| u |\!|\!|_{k,l,t} := \left(\int_0^t \|u(s)\|_k^l \,\mathrm{d}s \right)^{\frac{1}{l}}, \quad l = 1, 2. \tag{5}$$

The following useful equality holds for u smooth enough.

$$\left(\frac{\partial}{\partial t} u(t), u(t) \right) = \frac{1}{2} \frac{\partial}{\partial t} \|u(t)\|_0^2 = \|u(t)\|_0 \frac{\partial}{\partial t} \|u(t)\|_0, \tag{6}$$

which, by the Schwarz inequality, results in

$$\left| \frac{\partial}{\partial t} \|u(t)\|_0 \right| \leqslant \left\| \frac{\partial}{\partial t} u(t) \right\|_0. \tag{7}$$

All-time favourite in the analysis of time dependent partial differential equations is the Gronwall inequality, which we shall not use until Section 4. In this form, it is taken from [8].

Lemma 2.1. [Gronwall inequality] *Suppose f, g and h are piecewise continuous and nonnegative. Then if for all $t \in [0, T]$*

$$f(t) + h(t) \leqslant g(t) + \int_0^t f(s) \,\mathrm{d}s, \tag{8}$$

then

$$f(t) + h(t) \leqslant \mathrm{e}^t g(t). \tag{9}$$

2.4. Mixed elliptic projection. Let $b\colon \Omega \to \mathbb{R}$ be a non-negative function and $A\colon \Omega \to \mathrm{GL}(2)$ a function on Ω taking its values in the group $\mathrm{GL}(2)$ of invertible 2×2 matrices, satisfying

$$\exists \beta > 0\colon\ \forall \mathbf{q} \in [L^2(\Omega)]^2\colon\ (A^{-1}\mathbf{q}, \mathbf{q}) \geqslant \beta \|\mathbf{q}\|_0^2. \tag{10}$$

Given $u \in H^{k+2}(\Omega)$ for some non-negative integer k with trace g and given the vector field $\mathbf{p} = -A\,\mathbf{grad}\,u$, one can consider them to form the solution of the following Poisson problem (stated as a system of first order equations) with homogeneous Dirichlet boundary conditions and right-hand side $F := -\operatorname{div}(A\ \mathbf{grad}\,u) + bu$.

$$\mathbf{p} = -A\,\mathbf{grad}\,u, \quad \operatorname{div}\mathbf{p} + bu = F \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega. \tag{11}$$

Applying the mixed finite element method to this system in the usual way leads to approximations $u_h \in W_h^k$ for u and $\mathbf{p}_h \in \mathbf{\Gamma}_h^k$ for p, defined by

System 2.2. *Find $u_h \in W_h^k$ and $\mathbf{p}_h \in \mathbf{\Gamma}_h^k$ such that*

$$\begin{aligned} \forall \mathbf{q}_h \in \mathbf{\Gamma}_h^k &: (A^{-1}\mathbf{p}_h, \mathbf{q}_h) - (u_h, \operatorname{div}\mathbf{q}_h) = \left\langle g, \mathbf{q}_h^{\mathrm{T}}\nu \right\rangle_{\partial\Omega}, \\ \forall w_h \in W_h^k &: (\operatorname{div}\mathbf{p}_h, w_h) + (bu_h, w_h) \quad\ = (F, w_h). \end{aligned}$$

Starting from given u and $\mathbf{p}$ we have constructed approximations u_h and $\mathbf{p}_h$ in the mixed finite element spaces W_h^k and $\mathbf{\Gamma}_h^k$, respectively. This defines operators I_h^k and $\mathbf{I}_h^k$ as follows

$$\begin{aligned} \mathrm{I}_h^k &: H^{k+2}(\Omega) \to W_h^k\colon\ u \mapsto \mathrm{I}_h^k u := u_h, \\ \mathbf{I}_h^k &: \mathbf{H}(\mathrm{div};\Omega) \to \mathbf{\Gamma}_h^k\colon\ \mathbf{p} \mapsto \mathbf{I}_h^k\mathbf{p} := \mathbf{p}_h. \end{aligned} \tag{12}$$

In the sequel we will refer to these operators as *mixed elliptic projectors*. Slight changes in their definition would allow for other boundary conditions and for the Nédélec spaces to be included. One of the most important property of mixed elliptic projection is that it commutes with differentiation with respect to a parameter (in our case, time).

Lemma 2.3. *Let k be a non-negative integer and let $u\colon \Omega \times T \to \mathbb{R}$ be such that $u(t) \in H^{k+2}(\Omega)$ for all $t \in T$. Define $g(t) := \operatorname{trace}(u(t))$ and $f(t) := -\operatorname{div}(A\ \mathbf{grad}\,u)$. Then the elliptic projections of the four functions below are well-defined and*

$$\frac{\partial}{\partial t}\mathrm{I}_h^k u(t) = \mathrm{I}_h^k\Big(\frac{\partial}{\partial t}u\Big)(t) \quad \text{and} \quad \frac{\partial}{\partial t}\mathbf{I}_h^k\mathbf{p}(t) = \mathbf{I}_h^k\Big(\frac{\partial}{\partial t}\mathbf{p}\Big)(t). \tag{13}$$

P r o o f. One can check by differentiation of System 2.2 followed by changing the order of differentiation and integration that the pair

$$\Big(\frac{\partial}{\partial t}\mathbf{I}_h^k\mathbf{p}(t), \frac{\partial}{\partial t}\mathrm{I}_h^k u(t)\Big) \tag{14}$$

satisfies System 2.2 with F and g replaced by $\frac{\partial}{\partial t}F$, and $\frac{\partial}{\partial t}g$, respectively. By uniqueness of the solution, this proves the statement. □

2.5. Superconvergence of the elliptic projection. The elliptic projection, being just a mixed finite element discretization of an elliptic equation, is known to exhibit superconvergence in $\mathrm{I}_h^k u$. In fact, the following holds, under some assumptions on the smoothness of the coefficients of A and on b, and using regular families of triangulations (for the precise conditions we refer to [3])

$$\|\mathrm{P}_h^k u - \mathrm{I}_h^k u\|_0 \leqslant Ch^{k+2}\|u\|_{k+2}. \tag{15}$$

Both $\mathrm{P}_h^k u$ and $\mathrm{I}_h^k u$ converge to u with order h^{k+1} only, so in fact elliptic projection of u is a higher order perturbation of L^2 orthogonal projection on W_h^k. This important fact forms the basis for pointwise superconvergence in special points and also for easy post-processability of $\mathrm{I}_h^k u$ which in turn leads to a posteriori error estimation.

For the vector field elliptic projection $\mathbf{I}_h^k\mathbf{p}$ the same superconvergent bound holds with respect to the so called Fortin-interpolation $\boldsymbol{\Pi}_h^k\mathbf{p}$ of $\mathbf{p}$. Explicitly, for all rectangular elements (cf. [4]), as well as for the two lowest order methods on uniform triangular elements (cf. [1, 2]),

$$\|\boldsymbol{\Pi}_h^k\mathbf{p} - \mathbf{I}_h^k\mathbf{p}\|_0 \leqslant Ch^{k+2}\|\mathbf{p}\|_{k+2}. \tag{16}$$

In these situations, post-processing mechanisms are available which could be used for a posteriori error estimation. For the precise definition of Fortin interpolation we also refer to the papers mentioned above.

In the following two sections, we will prove that the difference between the elliptic projection of the exact solution of a time-dependent equation and the semi-discrete approximation is also of order h^{k+2}. This has as a consequence that for whenever the elliptic projection superconverges in the sense of (15) or (16), then so do the semi-discrete approximations by means of a simple triangle inequality.

Consider the following mixed formulation of a diffusion equation with given Dirichlet boundary conditions and initial condition.

$$\begin{aligned}
&\mathbf{p} = -A\,\mathbf{grad}\,u, \quad \frac{\partial}{\partial t}u + \operatorname{div}\mathbf{p} + bu = f \quad \text{in } \Omega \times T, \\
&u = g \quad \text{on } \partial\Omega \times T, \quad u(x,0) = U_0(x) \quad \text{on } \Omega.
\end{aligned} \tag{17}$$

Its semi-discrete formulation is as follows. The choice for the semi-discrete initial condition is made further on.

System 3.1. For all $t \in T$, find $u_h(t) \in W_h^k$ and $\mathbf{p}_h(t) \in \mathbf{\Gamma}_h^k$ such that

$$\begin{aligned}
\forall \mathbf{q}_h \in \mathbf{\Gamma}_h^k : &\quad (A^{-1}\mathbf{p}_h(t), \mathbf{q}_h) - (u_h(t), \operatorname{div}\mathbf{q}_h) &&= \left\langle g(t), \mathbf{q}_h^{\mathrm{T}}\nu \right\rangle_{\partial\Omega}, \\
\forall w_h \in W_h^k : &\quad (\operatorname{div}\mathbf{p}_h(t), w_h) + (bu_h(t), w_h) &&= (f - \tfrac{\partial}{\partial t}u_h(t), w_h).
\end{aligned}$$

In the analysis of this discretisation, we subtract the System 2.2 from 3.1 and test the resulting equations with the test functions

$$U_h(t) := u_h(t) - \mathrm{I}_h^k u(t) \in W_h^k \quad \text{and} \quad \mathbf{P}_h(t) := \mathbf{p}_h(t) - \mathbf{I}_h^k \mathbf{p}(t) \in \mathbf{\Gamma}_h^k. \tag{18}$$

Notice that these functions form the difference between elliptic projection and semi-discrete approximations and that it is exactly this difference that we want to prove supersmall. Some easy manipulations with the subtracted system lead to the equality

$$(A^{-1}\mathbf{P}_h(t), \mathbf{P}_h(t)) + \frac{1}{2}\frac{\partial}{\partial t}\|U_h(t)\|_0^2 + (bU_h(t), U_h(t)) = \left(\frac{\partial}{\partial t}(\mathrm{I}_h^k u - u)(t), U_h(t)\right).$$

The term in the right-hand side can be modified. By Lemma 2.3, we let $\frac{\partial}{\partial t}$ and I_h^k commute, and project $\frac{\partial}{\partial t}u$ on W_h^k. Then the Schwarz inequality yields

$$\begin{aligned}
&(A^{-1}\mathbf{P}_h(t), \mathbf{P}_h(t)) + \frac{1}{2}\frac{\partial}{\partial t}\|U_h(t)\|_0^2 + (bU_h(t), U_h(t)) \\
&\qquad \leqslant \left\| (\mathrm{I}_h^k - P_h^k)\Big(\frac{\partial}{\partial t}u\Big)(t) \right\|_0 \|U_h(t)\|_0
\end{aligned} \tag{19}$$

with, as an immediate consequence of the non-negativity of the first term and third term,

$$\frac{\partial}{\partial t}\|U_h(t)\|_0 \leqslant \left\| (\mathrm{I}_h^k - P_h^k)\Big(\frac{\partial}{\partial t}u\Big)(t) \right\|_0. \tag{20}$$

Taking as initial condition for System 3.1, $u_h(x,0) := \mathrm{I}_h^k U_0(x)$, integration of inequality (20) gives

$$\|U_h(t)\|_0 \leqslant \left\| (\mathrm{I}_h^k - P_h^k)\Big(\frac{\partial}{\partial t}u\Big) \right\|_{0,1,t}. \tag{21}$$

To obtain an estimate for $\mathbf{P}_h$, we use the ellipticity of A^{-1} in (10) and integrate (19) from zero to t using (21). Deleting the irrelevant positive terms obtained in the left-hand side, we end up with

$$\begin{aligned}
\beta \, \|\mathbf{P}_h\|_{0,2,t}^2 &\leqslant \int_0^t \left\| (\mathrm{I}_h^k - P_h^k)\Big(\frac{\partial}{\partial s}u\Big)(s) \right\|_0 \|U_h(s)\|_0 \,\mathrm{d}s \\
&\leqslant \int_0^t \left\| (\mathrm{I}_h^k - P_h^k)\Big(\frac{\partial}{\partial s}u\Big)(s) \right\|_0 \left\| (\mathrm{I}_h^k - P_h^k)\Big(\frac{\partial}{\partial s}u\Big)(s) \right\|_{0,1,s} \mathrm{d}s \\
&\leqslant \frac{1}{2} \left\| (\mathrm{I}_h^k - P_h^k)\Big(\frac{\partial}{\partial t}u\Big) \right\|_{0,1,t}^2.
\end{aligned}$$

Our analysis results in the following theorem.

Theorem 3.2. *The semi-discrete approximations $(u_h(t), \mathbf{p}_h(t))$ of the problem (17) are superclose to the elliptic projections of the exact solutions in the sense that*

$$\|\mathbf{P}_h\|_{0,2,t} + \|U_h(t)\|_0 \leqslant C h^{k+2} \left(\left\| \frac{\partial}{\partial t}u \right\|_{k+2,1,t} + \|u(t)\|_{k+2} \right). \tag{22}$$

Thus $u_h(t)$ and $\mathbf{p}_h(t)$ superconverge if and only if the elliptic projections do so.

Remark 3.3. The a priori estimates for the mixed method are only of (optimal) order h^{k+1}, so Theorem 3.2 is indeed a superconvergence result.

4. A Maxwell problem

Consider the following simple Maxwell problem in the plane. Since it is a second order problem in time, the analysis is different than for the diffusion equation of the previous section.

$$\begin{gathered}
\frac{\partial}{\partial t}\mathbf{E} = \mathbf{curl}\, H, \quad \frac{\partial}{\partial t}H = -\operatorname{rot}\mathbf{E} \quad \text{in } \Omega \times T \\
(\mathbf{curl}\, H)^{\mathrm{T}} \tau = 0 \text{ on } \partial\Omega \times T, \quad \mathbf{E}(x,0) = \mathbf{E}_0(x), \quad H(x,0) = H_0(x).
\end{gathered} \tag{23}$$

Notice that, when changing the order of time and space differentiations is allowed, H satisfies a wave equation

$$\frac{\partial^2}{\partial t^2} H = -\operatorname{rot}(\mathbf{curl}\, H) = \Delta H \quad \text{in } \Omega \times T, \tag{24}$$

with the following initial and boundary conditions

$$(\mathbf{curl}\, H)^T \tau = 0 \quad \text{on } \partial\Omega, \times T, \quad H(x,0) = H_0(x), \quad \frac{\partial}{\partial t} H(x,0) = -\operatorname{rot} \mathbf{E}_0(x).$$

The semi-discretization (System 4.1 below) has been thoroughly analysed in two as well as three space dimensions in [5, 6]. It is formulated in terms of rotation and curl, which explains the use of Nédélec instead of Raviart-Thomas spaces and also the use a corresponding rotation and curl formulation of the elliptic projection given in System 4.2.

System 4.1. For all $t \in T$, find $H_h(t) \in W_h^k$ and $\mathbf{E}_h(t) \in \mathbf{\Gamma}_{0h}^{k\perp}$ such that

$$\begin{aligned} \forall \mathbf{q}_h \in \mathbf{\Gamma}_{0h}^{k\perp} &: (\tfrac{\partial}{\partial t}\mathbf{E}_h(t), \mathbf{q}_h) &&= (H_h(t), \operatorname{rot} \mathbf{q}_h), \\ \forall w_h \in W_h^k &: (\operatorname{rot} \mathbf{E}_h(t), w_h) &&= (\tfrac{\partial}{\partial t} H_h(t), w_h). \end{aligned}$$

The elliptic projection will this time be used with homogeneous Neumann boundary conditions. The projectors for this Maxwell problem are given by

System 4.2. For all $t \in T$, find $\mathrm{I}_h^k H(t) \in W_h^k$ and $\mathbf{I}_h^k \frac{\partial}{\partial t}\mathbf{E}(t) \in \mathbf{\Gamma}_{0h}^{k\perp}$ such that

$$\begin{aligned} \forall \mathbf{q}_h \in \mathbf{\Gamma}_{0h}^{k\perp} &: (\mathbf{I}_h^k \tfrac{\partial}{\partial t}\mathbf{E}(t), \mathbf{q}_h) &&= (\mathrm{I}_h^k H(t), \operatorname{rot} \mathbf{q}_h), \\ \forall w_h \in W_h^k &: (\operatorname{rot} \mathbf{I}_h^k \tfrac{\partial}{\partial t}\mathbf{E}(t), w_h) &&= (\tfrac{\partial^2}{\partial t^2} H(t), w_h). \end{aligned}$$

The result of Lemma 2.3 (with the corresponding minor modifications), still holds for this new situation. Define

$$\Phi_h(t) = \mathrm{I}_h^k H(t) - H_h(t) \quad \text{and} \quad \Psi_h(t) = \mathbf{E}_h(t) - \mathbf{I}_h^k \mathbf{E}(t) \tag{25}$$

and take as initial conditions for the system 4.1

$$H_h(0) = \mathrm{I}_h^k H_0 \quad \text{and} \quad \mathbf{E}_h(0) = \mathbf{I}_h^k \mathbf{E}_0. \tag{26}$$

Lemma 4.3. *Conditions* (26) *for System* 4.1 *imply* $\Phi_h(0) = \Psi_h(0) = 0$ *and*

$$\Big(\frac{\partial}{\partial t}\Phi_h\Big)(0) = (\mathrm{I}_h^k - \mathrm{P}_h^k)\Big(\frac{\partial}{\partial t} H\Big)(0) \quad \text{and} \quad \Big(\frac{\partial}{\partial t}\Psi_h\Big)(0) = 0. \tag{27}$$

P r o o f. By the first equation of both System 4.1 and 4.2, both for $t = 0$, and using the first of the initial conditions in (26) and Lemma 2.3,

$$\Big(\frac{\partial}{\partial t}\mathbf{E}_h\Big)(0) = \mathbf{I}_h^k\Big(\frac{\partial}{\partial t}\mathbf{E}\Big)(0) = \Big(\frac{\partial}{\partial t}\mathbf{I}_h^k\mathbf{E}\Big)(0), \quad \text{so} \quad \Big(\frac{\partial}{\partial t}\Psi_h\Big)(0) = 0.$$

Recall that $\mathbf{I}_h^k\mathbf{E}_0(x)$ is the vector field solution of 4.2 with $\frac{\partial^2}{\partial t^2}H$ replaced by $\frac{\partial}{\partial t}H$. The second initial condition in (26) implies $\operatorname{rot}\mathbf{E}_h(0) = \operatorname{rot}\mathbf{I}_h^k\mathbf{E}_0(x)$, and by the second equations of 4.1 and the second of System 4.2 with $\frac{\partial^2}{\partial t^2}H$ replaced by $\frac{\partial}{\partial t}H$, this results in

$$\Big(\frac{\partial}{\partial t}H_h\Big)(0) = \mathrm{P}_h^k\Big(\frac{\partial}{\partial t}H\Big)(0).$$

This proves the first of the two statements in (27). $\square$

Differentiate both equations in System 4.1 and the first of System 4.2 in time and subtract system 4.1 from 4.2. Testing the subtracted system with

$$w_h = \frac{\partial}{\partial t}\Phi_h(t) \quad \text{and} \quad \mathbf{q}_h = \frac{\partial}{\partial t}\Psi_h(t) \tag{28}$$

leads after some easy manipulations to the equality

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\Big\|\frac{\partial}{\partial t}\Phi_h(t)\Big\|_0^2 + \Big\|\frac{\partial}{\partial t}\Psi_h(t)\Big\|_0^2\right) = \Big(\frac{\partial^2}{\partial t^2}(\mathrm{I}_h^k H - H)(t), \frac{\partial}{\partial t}\Phi_h(t)\Big). \tag{29}$$

So, integration of (29) from zero to t followed by applying the Schwarz inequality and the inequality $|2ab| \leqslant a^2 + b^2$ results in

$$\begin{aligned}\Big\|\frac{\partial}{\partial t}\Phi_h(t)\Big\|_0^2 + \Big\|\frac{\partial}{\partial t}\Psi_h(t)\Big\|_0^2 &= 2\int_0^t \Big(\frac{\partial^2}{\partial^2 s}(\mathrm{I}_h^k H - H), \frac{\partial}{\partial s}\Phi_h\Big)(s)\,\mathrm{d}s + \Big\|\Big(\frac{\partial}{\partial t}\Phi_h\Big)(0)\Big\|_0^2 \\ &\leqslant \int_0^t \Big\|(\mathrm{I}_h^k - \mathrm{P}_h^k)\Big(\frac{\partial^2}{\partial^2 s}H\Big)(s)\Big\|_0^2\,\mathrm{d}s + \int_0^t \Big\|\frac{\partial}{\partial s}\Phi_h(s)\Big\|_0^2\,\mathrm{d}s + \Big\|\Big(\frac{\partial}{\partial t}\Phi_h\Big)(0)\Big\|_0^2.\end{aligned}$$

Applying the Gronwall inequality (Lemma 2.1) leads to

$$\Big\|\frac{\partial}{\partial t}\Phi_h(t)\Big\|_0^2 + \Big\|\frac{\partial}{\partial t}\Psi_h(t)\Big\|_0^2 \leqslant \mathrm{e}^t\left(\Big|\Big|\Big|(\mathrm{I}_h^k - \mathrm{P}_h^k)\Big(\frac{\partial^2}{\partial t^2}H\Big)\Big|\Big|\Big|_{0,2,t}^2 + \Big\|\Big(\frac{\partial}{\partial t}\Phi_h\Big)(0)\Big\|_0^2\right).$$

This is only an estimate for the first time derivatives of the functions of interest. To obtain estimates for those functions themselves, we apply the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ to the left-hand side and the inequality $a^2 + b^2 \leqslant (|a|+|b|)^2$ to the right-hand side. Next, we take square roots on both sides. Then we apply (7) to the left-hand side and end up with

$$\frac{\partial}{\partial t}\left(\|\Phi_h(t)\|_0 + \|\Psi_h(t)\|_0\right) \leqslant \sqrt{2}\mathrm{e}^{t/2}\left(\Big|\Big|\Big|(\mathrm{I}_h^k - \mathrm{P}_h^k)\Big(\frac{\partial^2}{\partial t^2}H\Big)\Big|\Big|\Big|_{0,2,t} + \Big\|\Big(\frac{\partial}{\partial t}\Phi_h\Big)(0)\Big\|_0\right).$$

Integration from zero to t (by parts on the right-hand side) gives, taking into account the semi-discrete initial conditions,

$$\begin{aligned}(30)\qquad &\|\Phi_h(t)\|_0 + \|\Psi_h(t)\|_0 + 2\sqrt{2}\int_0^t \mathrm{e}^{s/2}\frac{\partial}{\partial s}\left|\!\left|\!\left|(\mathrm{I}_h^k - \mathrm{P}_h^k)\Big(\frac{\partial^2}{\partial^2 s}H\Big)\right|\!\right|\!\right|_{0,2,s}\,\mathrm{d}s\\ &\leqslant 2\sqrt{2}\mathrm{e}^{s/2}\left(\left|\!\left|\!\left|(\mathrm{I}_h^k - \mathrm{P}_h^k)\Big(\frac{\partial^2}{\partial^2 s}H\Big)\right|\!\right|\!\right|_{0,2,s} + \left\|\Big(\frac{\partial}{\partial s}\Phi_h\Big)(0)\right\|_0\right)\Bigg|_0^t.\end{aligned}$$

Since the norm $|\!|\!|\cdot|\!|\!|_{0,2,s}$ is monotonely increasing in s, the integral in the left-hand side is non-negative, so we can leave it away. Substitution of the integration limits zero and t in the right-hand side of (30) gives

$$\|\Phi_h(t)\|_0+\|\Psi_h(t)\|_0 \leqslant 2\sqrt{2}\mathrm{e}^{t/2}\left|\!\left|\!\left|(\mathrm{I}_h^k - \mathrm{P}_h^k)\Big(\frac{\partial^2}{\partial t^2}H\Big)\right|\!\right|\!\right|_{0,2,t} + 2\sqrt{2}(\mathrm{e}^{t/2}-1)\left\|\Big(\frac{\partial}{\partial t}\Phi_h\Big)(0)\right\|_0.$$

So, essentially, this bound grows in time like $\mathrm{e}^{t/2}$, but is pointwise still a superconvergence result in space. Summarizing, we can state the following theorem.

Theorem 4.4. *The semi-discrete approximations* $(H_h(t), \mathbf{E}_h(t))$ *of problem* (23) *are superclose to the elliptic projections of the exact solutions in the sense that*

$$\|\Phi_h(t)\|_0 \leqslant C\mathrm{e}^{\frac{1}{2}t}h^{k+2}\left(\left|\!\left|\!\left|\frac{\partial^2}{\partial t^2}H\right|\!\right|\!\right|_{k+2,2,t} + \left\|\Big(\frac{\partial}{\partial t}H\Big)(0)\right\|_{k+2} + \|H(t)\|_{k+2}\right)$$

and

$$\|\Psi_h(t)\|_0 \leqslant C\mathrm{e}^{\frac{1}{2}t}h^{k+2}\left|\!\left|\!\left|\frac{\partial^2}{\partial t^2}H\right|\!\right|\!\right|_{k+2,2,t} + C(\mathrm{e}^{\frac{1}{2}t}-1)h^{k+2}\left\|\Big(\frac{\partial}{\partial t}H\Big)(0)\right\|_{k+2}$$

so that superconvergence in $(H_h(t), \mathbf{E}_h(t))$ *will occur if and only if the elliptic projections superconverge.*

Proof. Follows immediately from the results above, (15) and Lemma 4.3. □

Remark 4.5. The a priori estimates for $H_h(t)$ and $\mathbf{E}_h(t)$ are only of (optimal) order h^{k+1}, so the result of Theorem 4.4 is a superconvergence result.

Consequently, whenever the post-processing mechanisms developed for the elliptic equation in [1, 2, 4] are successfully applicable, they are successfully applicable for the time dependent Maxwell equations as well.

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