Sergey Korotov; Michal Křížek; Pekka Neittaanmäki

On the existence of strongly regular families of triangulations for domains with a piecewise smooth boundary

Applications of Mathematics, Vol. 44 (1999), No. 1, 33–42

Persistent URL: http://dml.cz/dmlcz/134404

ON THE EXISTENCE OF STRONGLY REGULAR FAMILIES OF TRIANGULATIONS FOR DOMAINS WITH A PIECEWISE SMOOTH BOUNDARY

Sergey Korotov[1], Michal Křížek[2], Pekka Neittaanmäki[1]

(Received March 30, 1998)

Abstract. In this note we present an algorithm for a construction of strongly regular families of triangulations for planar domains with a piecewise curved boundary. Some additional properties of the resulting triangulations are considered.

Keywords: strongly regular family, triangulation, planar domain

MSC 2000: 65N30

1. Introduction

There are two basic ways how to treat planar domains with a piecewise smooth boundary when solving partial differential equations by the finite element method. One way is to use isoparametric elements, see [2], or ideal curved triangular elements introduced by Zlámal in [12]. The second way is to approximate the curved boundary by a polygonal one. The strip between those boundaries can be neglected or decomposed into special biangular (triangular) finite elements, see [6]. Anyhow, we have to deal with the problem how to approximate the domain under consideration by a sequence of polygons and how to establish their triangulations having "reasonable" properties.

The aim of this paper is to give a constructive proof of existence of a strongly regular family of triangulations of domains with a piecewise smooth boundary. For

[1] The author was supported by the COMAS Graduate School at the University of Jyväskylä, Finland.

[2] The author was supported by Grant no. 201/98/1452 of the Grant Agency of the Czech Republic.

polygonal domains the existence of such a family can be easily obtained from a given triangulation by a bisection process, in which each triangle is divided into four congruent triangles by midlines. Another approach, which uses medians of triangles, is presented in [1] and [11]. We could also use Delaunay triangulations [4], even though we meet some technical problems near the boundary.

Let us point out that for a strongly regular family of triangulations the well-known inverse inequalities hold (see [2]), e.g.,

$$\|v_h\|_1 \leqslant \frac{C}{h}\|v_h\|_0 \quad \forall v_h \in V_h,$$

where V_h are finite element subspaces of the Sobolev space $H^1(\Omega)$, the symbol C stands for a constant independent of h, and $\|\cdot\|_k$ is the standard Sobolev norm. Inverse inequalities play an important role in proving convergence of the finite element method for various problems. But the existence of a strongly regular family for domains with a piecewise smooth boundary is "silently" assumed. To the authors' knowledge there has been no detailed proof until now. In [9], a strongly regular family of triangulations is constructed for a circle. Then, using a C^2-diffeomorphic mapping, a strongly regular family for any simply connected bounded domain with a smooth boundary is obtained. Another proof for the same class of domains is presented in [5].

We will obtain this result for the domains with piecewise curved boundaries (which is the most important case in practice) using a different technique. We divide the domain in question into a finite number of "quasi-triangles" with at most one curved side and then construct strongly regular families of triangulations of each quasi-triangle. Moreover, our algorithm generates piecewise quasi-uniform triangulations, which produce various superconvergence phenomena (see, e.g., [8]) when the finite element method is applied. Note that in [3], several algorithms for generating triangulations are presented, but nothing is mentioned there concerning the properties of a family of triangulations when the discretization parameter h tends to zero.

Throughout this paper we consider triangulations T_h which are sets of closed triangles having the standard properties (cf. [2]), i.e., any two triangles from T_h have a common edge or a common vertex or are disjoint. As usual, the parameter h characterizes the maximum length of all edges of all triangles in T_h.

Definition 2.1. Let Z be a bounded closed domain in $\mathbb{R}^2$, let $\mathscr{F} = \{T_h\}_{h\to 0}$ be a set of triangulations and let

$$Z_h = \bigcup_{K\in T_h} K.$$

The set $\mathscr{F}$ is said to be a *family of triangulations of* Z if for any $z \in Z$ there exists a sequence $\{z_h\}$, $z_h \in Z_h$ such that $z_h \to z$ as $h \to 0$ and for any convergent sequence $\{z_h\}$, $z_h \in Z_h$, there exists $z \in Z$ such that $z_h \to z$ as $h \to 0$.

According to Definition 2.1, $\mathscr{F}$ is a family of triangulations of Z if the associated polygons Z_h "converge" to Z. A similar definition has been introduced by Mosco in [10].

Definition 2.2. A family $\mathscr{F}$ of triangulations of a bounded closed domain $Z \subset \mathbb{R}^2$ is said to be *strongly regular* if there exist positive constants $\varkappa_1$, $\varkappa_2$, α_0, h_0 such that for all $h \in (0, h_0)$, all triangulations $T_h \in \mathscr{F}$ and all triangles $K \in T_h$ we have

$$\varkappa_1 h \leqslant l_K \leqslant \varkappa_2 h, \tag{2.1}$$

$$0 < \alpha_0 \leqslant \alpha_K, \tag{2.2}$$

where l_K is the length of any edge of K and α_K is any angle of K.

It is easy to see that this definition is equivalent to the inscribed ball condition (cf. [7]). Let us point out that the strongly regular family is sometimes also called quasi-uniform (cf. [2, p. 135]).

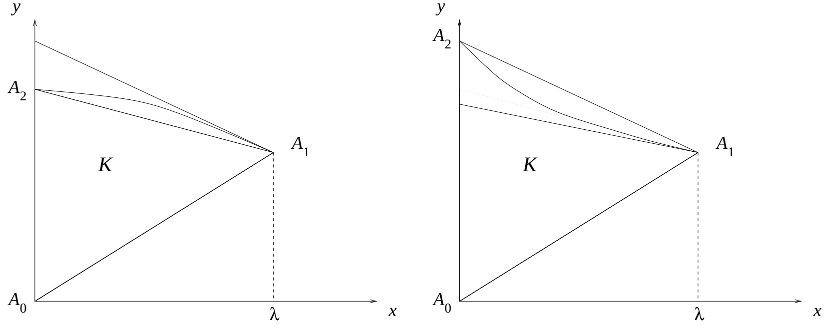

Figure 1

Definition 2.3. Let $\lambda > 0$, $f \in C^1[0,\lambda]$, $f(0) > 0$, $k_0 = f(\lambda)/\lambda$ and let there exist constants k_1, k_2 such that

$$k_0 > k_1 \geqslant k_2 \tag{2.3}$$

and

$$k_0\lambda + k_1(x-\lambda) \leqslant f(x) \leqslant k_0\lambda + k_2(x-\lambda), \quad x \in [0,\lambda], \tag{2.4}$$

i.e., the graph of f lies between two straight lines passing through the point $(\lambda, f(\lambda))$ (see Figure 1). Then the set

$$K = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant \lambda, \quad k_0 x \leqslant y \leqslant f(x)\}$$

is called a *quasi-triangle*.

Remark 2.4. A quasi-triangle K is thus a curved triangle with vertices $A_0 = (0,0)$, $A_1 = (\lambda, f(\lambda))$, $A_2 = (0, f(0))$ and at most one curved side A_1A_2 (see Figure 1). Suppose, moreover, that $f \in C^2[0,\lambda]$. If

$$f'' \leqslant 0 \quad \text{in } [0,\lambda] \tag{2.5}$$

then we may set $k_1 = (f(\lambda) - f(0))/\lambda$ and $k_2 = f'(\lambda)$, and if

$$f'' \geqslant 0 \quad \text{in } [0,\lambda] \quad \text{and} \quad f'(\lambda) < k_0, \tag{2.6}$$

then we may set $k_1 = f'(\lambda)$ and $k_2 = (f(\lambda) - f(0))/\lambda$ to satisfy (2.3) and (2.4).

3. Main theorems

Let $h = \lambda/n$ for a natural number n.

Theorem 3.1. *For any quasi-triangle there exists a strongly regular family of triangulations.*

Proof. We form polygonal approximations of K and their triangulations in the following manner: first, we divide the segment A_0A_1 by points $A_{00} \equiv A_0$, $A_{11}, \ldots, A_{nn} \equiv A_1$ into n segments of the same length (see Figure 2 with $n = 4$).

Let $A_{rn} = (rh, f(rh))$ for $r = 0, \ldots, n$, and let us divide the segments $A_{rr}A_{rn}$ by the points

$$A_{rq} = \Big(rh, k_0 rh + \frac{f(rh) - k_0 rh}{n-r}(q-r)\Big), \quad 0 \leqslant r \leqslant q < n, \tag{3.1}$$

into $n-r$ congruent smaller segments. The points $A_{r-1,q}$ and A_{rq}, also $A_{r-1,q-1}$ and A_{rq}, are joined by straight line segments for $1 \leqslant r \leqslant q \leqslant n$. As a result we get a triangulation of K_h consisting of the triangles $A_{r-1,q}A_{rq}A_{r,q+1}$ for $1 \leqslant r \leqslant q < n$ and the triangles $A_{r-1,q-1}A_{r-1,q}A_{rq}$ for $1 \leqslant r \leqslant q \leqslant n$ (see Figure 2). Uniform convergence of linear interpolants of f to f guarantees (see [2, p. 124]) that the proposed set of triangulations forms a family for $h \to 0$ as defined in Definition 2.1.

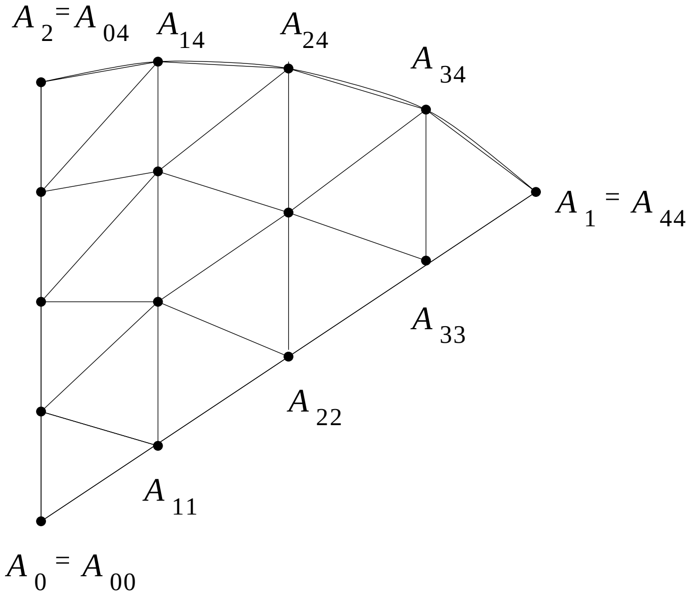

Figure 2

Now, we show that this family possesses properties (2.1) and (2.2). First consider the vertical segment $A_{rq}A_{r,q+1}$ for $0 \leqslant r < n$. Its length is given by

$$|A_{rq}A_{r,q+1}| = \frac{f(rh) - k_0 rh}{n-r}.$$

From (2.4) we have

$$k_0(\lambda - x) - k_1(\lambda - x) \leqslant f(x) - k_0 x \leqslant k_0(\lambda - x) - k_2(\lambda - x)$$

for $x \in [0,\ \lambda]$. Then, since $\lambda = nh$, we get

$$(k_0 - k_1)h \leqslant |A_{rq}A_{r,q+1}| \leqslant (k_0 - k_2)h. \tag{3.2}$$

Hence, we observe by (2.3) that

$$0 < C_2 h \leqslant |A_{rq}A_{r,q+1}| \leqslant C_3 h. \tag{3.3}$$

Second, we see that apparently

$$\frac{q-r}{n-r} \leqslant \frac{q-r+1}{n-r+1} \quad \text{for} \quad 0 \leqslant r \leqslant q \leqslant n \text{ and } r < n. \tag{3.4}$$

Consider the vector

$$\overrightarrow{A_{r-1,q}A_{rq}} = (h,\ \bar{y}), \tag{3.5}$$

where

$$\bar{y} = k_0 h + (q-r)\frac{f(rh) - k_0 rh}{n-r} - (q-r+1)\frac{f((r-1)h) - k_0(r-1)h}{n-r+1}. \tag{3.6}$$

Obviously, to get estimates like (3.3), we only have to estimate the second coordinate $\bar{y}$, expressed by (3.6). Using (3.6) and (3.4), we observe that if $\bar{y} \geqslant k_0 h$ then

$$\begin{aligned} \bar{y} &\leqslant k_0 h + \frac{q-r}{n-r}[f(rh) - k_0 rh - f((r-1)h) + k_0(r-1)h] \\ &\leqslant k_0 h + (f(rh) - f((r-1)h) - k_0 h) = f(rh) - f((r-1)h), \end{aligned} \tag{3.7}$$

since the expression in the square brackets [ ] is nonnegative. Similarly, if $\bar{y} \leqslant 0$ then

$$\bar{y} \geqslant k_0 h + \frac{q-r+1}{n-r+1}\{f(rh) - k_0 rh - f((r-1)h) + k_0(r-1)h\} \geqslant f(rh) - f((r-1)h),$$

since {} is nonpositive. From here and (3.7) we obtain

$$|\bar{y}| \leqslant |f(rh) - f((r-1)h)| \quad \text{for } \bar{y} \geqslant k_0 h \text{ or } \bar{y} \leqslant 0,$$

which leads, by the mean value theorem, to

$$|\bar{y}| \leqslant C_4 h, \tag{3.8}$$

where C_4 is a constant. For the remaining case $0 < \bar{y} < k_0 h$ we immediately get again (3.8) with $C_4 = k_0$. Finally, from (3.5) and (3.8) we obtain

$$h \leqslant |A_{r-1,q}A_{rq}| \leqslant C_5 h. \tag{3.9}$$

The same lower bound holds also for $|A_{r-1,q}A_{r,q+1}|$. Its upper bound follows from (3.3), (3.9) and the triangle inequality. Hence, (2.1) is valid.

From (3.9) we have

$$\sin \widehat{A_{r-1,q}A_{rq}A_{r,q+1}} \geqslant \frac{h}{C_5 h} = \frac{1}{C_5}. \tag{3.10}$$

Similar relations hold for the other angles, i.e., (2.2) is valid. □

R e m a r k 3.2. If $f \in C^2[0,\lambda]$ and K_h is the union of all triangles from the triangulation T_h proposed in the proof of Theorem 3.1 then

$$\max_{z\in\partial K} \operatorname{dist}(z,\partial K_h) \leqslant C_6 h^2. \tag{3.11}$$

This property follows immediately from the standard interpolation theory [2]. If K is a usual straight triangle then the triangulations T_h are uniform, i.e., any two adjacent triangles form a parallelogram.

The following theorem shows that under certain assumptions we can construct interior or exterior polygonal approximations of K and their triangulations forming a strongly regular family. This property is often required in many theoretical and practical problems.

Theorem 3.3. *Let $f \in C^2[0,\lambda]$, let K be the associated quasi-triangle and let* (2.5) *or* (2.6) *hold. Then there exist $h_0 > 0$ and a sequence of polygonal domains $\{K_h\}$ such that* (3.11) *holds, $A_0A_1 \cup A_0A_2 \subset \partial K_h$, and either $K \subset K_h$ or $K_h \supset K$ for all $h \in (0,h_0)$. For each K_h there exists a triangulation T_h such that* (2.1) *and* (2.2) *hold.*

P r o o f. Let f_h be a continuous piecewise linear function defined on $[0,\lambda]$ such that

$$f_h(0) = f(0), \quad f_h(\lambda) = f(\lambda) \tag{3.12}$$

and

$$|f_h(rh) - f(rh)| \leqslant C_1 h^2, \quad r = 0,\dots,n. \tag{3.13}$$

The graph of f_h and the segments A_0A_1 and A_0A_2 form the boundary of a polygonal approximation K_h of K (see Figure 3). To provide $K_h \supset K$ we just take $f_h \geqslant f$, and we choose $f_h \leqslant f$ to construct $K_h \subset K$. Concerning the interior approximation, we see that $k_0 rh < f_h(rh)$ whenever $r < n$ and $h \leqslant (k_0 - k_1)/C_1$, where C_1 is the constant from (3.13).

By (3.13) we can easily derive (3.11), since $f \in C^2[0,\lambda]$.

To check the validity of (2.1) and (2.2) is essentially the same as in Theorem 3.1 if we write f_h instead of f everywhere in its proof. Hence, A_{rq} are defined by (3.1), where f is replaced by f_h, etc. Then, in view of (3.13), we get similarly to (3.2) that

$$(k_0 - k_1)h - \frac{C_1h^2}{n-r} \leqslant |A_{rq}A_{r,q+1}| \leqslant (k_0 - k_2)h + \frac{C_1h^2}{n-r},$$

which again yields (3.3).

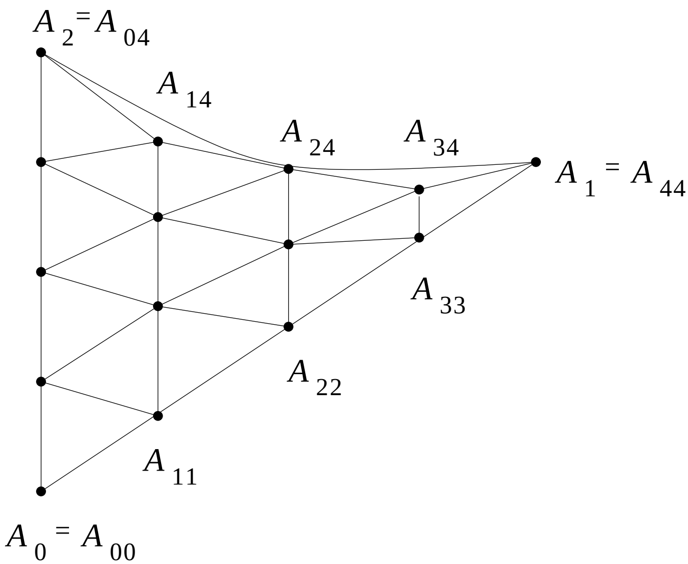

Figure 3

The triangle inequality

$$|\bar{y}| \leqslant |f_h(rh) - f(rh)| + |f(rh) - f((r-1)h)| + |f((r-1)h) - f_h((r-1)h)|$$

together with (3.13) again gives (3.9). □

Lemma 3.4. *Let Ω be a bounded planar domain with a boundary $\partial\Omega$ which consists of a finite number of differentiable arcs meeting at interior angles φ, where*

$$0 < \varphi_0 \leqslant \varphi \leqslant 2\pi - \varphi_0. \tag{3.14}$$

Then there exists a division of $\overline{\Omega}$ into a finite number of quasi-triangles with curved sides on $\partial\Omega$ only.

P r o o f. For simplicity we present only a sketch of the proof.

Let $\mathscr{T}$ be an arbitrary triangulation consisting of straight-line triangles such that $\Omega \subset \bigcup_{K\in\mathscr{T}} K$ ($\mathscr{T}$ can be, e.g., a uniform one), and let

$$P = \bigcup_{K\in\mathscr{T},\, K\subset\Omega} K \tag{3.15}$$

be non-empty. Hence, P is a closed polygon contained in Ω.

Now, consider the "strip" between ∂P and $\partial\Omega$. Obviously, connecting the nodes of ∂P with certain points of the boundary $\partial\Omega$ (e.g., along the normal to $\partial\Omega$ or using axes of exterior angles of P at the nodes of ∂P) and with end-points of smooth arcs forming $\partial\Omega$, we can divide the strip into a finite number of triangles with at most one curved side on $\partial\Omega$ only (see Figure 4). If not all of them are quasi-triangles according to Definition 2.3, then we divide all triangles from $\mathscr{T}$ by midlines, denote the new triangulation again by $\mathscr{T}$ and continue from (3.15).

If the triangulation $\mathscr{T}$ is sufficiently fine then conditions (2.3) and (2.4) are satisfied, since $\partial\Omega$ is piecewise differentiable and thus each differentiable arc is locally becoming straight as h tends to zero. Hence, the above division process has to finish in a finite number of steps. $\square$

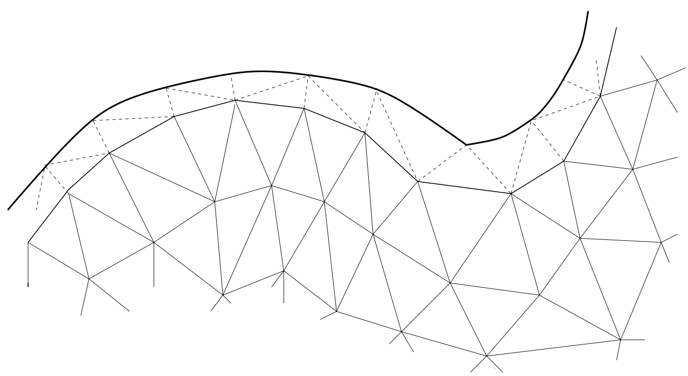

Figure 4

Note that there are several constructive proofs of Lemma 3.4. However, their detailed descriptions are always too technical.

Theorem 3.5. *Let Ω be a domain as described in Lemma 3.4. Then there exists a strongly regular family $\mathscr{F} = \{T_h\}_{h\to 0}$ of triangulations of $\overline{\Omega}$. If, moreover, $\partial\Omega$ is piecewise twice differentiable then*

$$\max_{x\in\partial\Omega} \operatorname{dist}(x, \partial\Omega_h) \leqslant C_7 h^2, \tag{3.16}$$

where $\Omega_h = \bigcup_{K\in T_h} K$.

P r o o f. If a division of $\overline{\Omega}$ into a finite number of quasi-triangles from Lemma 3.4 is built then for any natural number n we form triangulations as in Theorem 3.1. Obviously, all triangulations of all quasi-triangles can be taken together as a global triangulation of the whole domain $\overline{\Omega}$ in view of the way of construction.

If $\partial\Omega$ is piecewise twice differentiable the condition (3.16) follows directly from (3.11).

The only problem is to guarantee the nonexistence of common points for any two piecewise linear approximations of two neighbouring arcs, built outside Ω. However, this requirement is, obviously, fulfilled in view of (3.14) for sufficiently large n. $\square$

R e m a r k 3.6. If a quasi-triangle from the previous proof corresponds to a convex or concave function f then we may also utilize triangulations generated in Theorem 3.3 due to requirement (3.12).

Acknowledgement. The authors are indebted to Jan Brandts, Ivan Hlaváček (MÚ Prague) and Stanislav Koukal (VA Brno, Czech Republic) for valuable suggestions.

References

[1] *Bänsch, E.*: Local mesh refinement in 2 and 3 dimensions. Impact Comput. Sci. Engrg. *3* (1991), 181–191.
[2] *Ciarlet, P. G.*: Basic error estimates for elliptic problems, Handbook of Numer. Anal. (ed. P. G. Ciarlet, J. L. Lions). North-Holland, Amsterdam, 1991, pp. 17–351.
[3] *George, P. L.*: Automatic mesh generation. John Wiley & Sons, New York, 1991.
[4] *Joe, B.*: Delaunay triangular meshes in convex polygons. SIAM J. Sci. Stat. Comput. *7* (1986), 514–539.
[5] *Korneev, V. G.*: Schemes of the finite element method for high orders of accuracy. Leningrad University, Leningrad, 1977, (in Russian).
[6] *Koukal, S., Křížek, M.*: Curved affine quadratic finite elements. J. Comput. Appl. Math. *63* (1995), 333–339.
[7] *Křížek, M., Neittaanmäki, P.*: Mathematical and numerical modelling in electrical engineering. Theory and applications. Kluwer, Dordrecht, 1996.
[8] *Lin, Q., Xu, J. Ch.*: Linear finite elements with high accuracy. J. Comput. Math. *3* (1985), 115–133.
[9] *Matsokin, A. M.*: Automatic triangulation of domains with smooth boundaries for solving equations of the elliptic type. Preprint of Computer Center, 15, Novosibirsk, 1975, (in Russian).
[10] *Mosco, U.*: Convergence of convex sets and of solutions of variational inequalities. Adv. in Math. *3* (1969), 510–585.
[11] *Rosenberg, I. G., Stenger, F.*: A lower bound on the angle of triangles constructed by bisecting the longest side. Math. Comp. *29* (1975), 390–395.
[12] *Zlámal, M.*: Curved elements in the finite element method. SIAM J. Numer. Anal. *10* (1973), 229–240.

Authors' addresses: *S. Korotov, P. Neittaanmäki*, University of Jyväskylä, Department of Mathematics, P.O. Box 35, FIN-403 51 Jyväskylä, Finland, e-mail: `korotov@math.jyu.fi`, `pn@math.jyu.fi`; *M. Křížek*, Mathematical Institute, Academy of Sciences, Žitná 25, CZ-115 67 Prague 1, Czech Republic, e-mail: `krizek@math.cas.cz`.