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ON ONE MATHEMATICAL MODEL OF CREEP IN SUPERALLOYS

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Abstract. In [Sv[1]] a new micromechanical approach to the prediction of creep flow in composites with perfect matrix/particle interfaces, based on the nonlinear Maxwell viscoelastic model, taking into account a finite number of discrete slip systems in the matrix, has been suggested; high-temperature creep in such composites is conditioned by the dynamic recovery of the dislocation structure due to slip/climb motion of dislocations along the matrix/particle interfaces. In this article the proper formulation of the system of PDE's generated by this model is presented, some existence results are obtained and the convergence of Rothe sequences, applied in the specialized software CDS, is studied.

Keywords: Strain and stress distributions in superalloys, high-temperature creep, viscoelasticity, interface diffusion, PDE's of evolution, method of discretization in time, Rothe sequences

MSC 2000: 73F05, 73F15

1. Introduction

The creep resistance of composites (metal reinforced by hard particles) depends strongly on the active diffusion processes on the interface between the matrix and particles. Since in standard software packages all such processes are neglected, the conventional time-dependent strain and stress analysis at high temperature (e.g. making use of some "user supplied material description" in the ABAQUS finite element code as in [Pe]) gives no satisfactory results. To overcome this difficulty, in 1992–97 at the Institute of Physics of Materials of the Academy of Sciences of the Czech Republic in Brno the non-commercial PC software CDS (abbreviation for "creep with diffusion and sliding") in C++ language for the study of high-temperature phenomena in materials consisting of several phases has been developed; its functions have been demonstrated e.g. in [Va[1]]. One interesting application of this software to the design of metal matrix composites (where the particles are added into the matrix during solidification) has been published in [Va]. In this article we pay attention to the case of superalloys (where the particles precipitate in the matrix during the heat treatment of the material).

The physical background of the processes of elastic deformation, slip in the matrix, deposition of dislocation on matrix/particle interfaces and recovery of dislocation structure is discussed in details in [Sv^1] (making use of some ideas from [Sv]), including the confrontation with other approaches to similar problems in the literature and software. The approach of [Sv^1] leads to the formulation of 5 types of equations (to these types we will refer later):

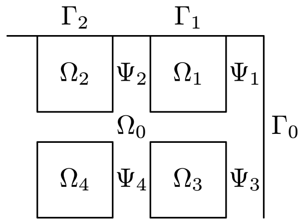
(A) equations of the principle of virtual displacement rates for materials consisting of two phases,
(B) constitutive equations for stress components, making use of the serial Maxwell model with one linear elastic and one nonlinear viscous part,
(C) compatibility conditions for geometrical configuration on the matrix/particle interfaces,
(D) equations for the kinetics of the displacement between the matrix and the particle,
(E) equations for the evolution of the dislocation density.

Our aim is to formulate correctly this system of PDE's of evolution and to verify its solvability and convergence of sequences of Rothe functions, using the method of discretization in time.

2. Virtual configurations and stresses

To make the resulting system of equations as simple as possible, we shall formulate in an explicit way only the equations of type (A) and (B), containing two types of unknown abstract functions—stresses σ and displacement rates v mapping a time interval $I = \{t \in \mathbb{R}_+ : 0 \leqslant t \leqslant \tau\}$ with $\tau \in \mathbb{R}_+$ into the corresponding function spaces V of virtual displacement rates and S of virtual stresses in a deformable body, respectively. The spaces V and S must be defined carefully to include the equations of type (C)—in other words, to respect both the support boundary conditions and the special slip structure of deformation with discontinuities in matrix/particle interface configuration.

In the 3-dimensional Euclidean space $\mathbb{R}^3$ let us consider a domain Ω, occupied by a deformable body, containing subdomains $\Omega_1, \ldots, \Omega_N$—mutually separated (with positive distances) elastic particles, and $\Omega_0 = \Omega \setminus (\overline{\Omega}_1 \cup \ldots \cup \overline{\Omega}_N)$—the creeping matrix. We decompose also the boundary of elastic particles $\partial\Omega_i = \Gamma_i \cup \Psi_i$ where $\Psi_i = \partial\Omega_i \setminus \partial\Omega$ and $\Gamma_i = \partial\Omega_i \setminus \Psi_i$ ($i \in \{1, \ldots, N\}$); for the creeping matrix we introduce similarly $\Gamma_0 = \partial\Omega \setminus (\Gamma_1 \cup \ldots \cup \Gamma_N)$. One very simple example of the material structure of this type, analyzed in [Sv[1]], is shown in the figure:



The typical size of particles, observed in laboratories for Ni-based superalloys (see e.g. the photographs in [De], p. 290), is about 10^{-6} m; matrix channels are usually several times thinner.

In the whole article we will apply the standard notation of Sobolev spaces from [Ma] and of spaces of Bochner integrable abstract functions from [Ga]. Moreover, we will assume such geometry of the domains that the following properties are satisfied:

(i) The imbedding of $W_2^1(\Omega_i, \mathbb{R})$ into $L_2(\Omega_i, \mathbb{R})$ is compact ("Rellich theorem", see [Ma], p. 204).

(ii) All functions from $W_2^1(\Omega_i, \mathbb{R})$ have their traces in $L_2(\partial\Omega_i, \mathbb{R})$; moreover, if $i \neq 0$ then the mapping of $W_2^1(\Omega_i, \mathbb{R})$ into $L_2(\Psi_i, \mathbb{R})$ is compact ("trace theorem", see [Ma], p. 222).

(iii) The condition of coerciveness of strains on Ω_i in the sense of [Ne], p. 79, is preserved.

Following the physical considerations from [Sv[1]], we shall assume that creep in Ω_0 is active only in a finite number M of slip systems, characterized by slip directions $a^k \in \mathbb{R}^3$ and normals to slip planes $c^k \in \mathbb{R}^3$ for $k \in \{1, \ldots, M\}$. (In the plane-strain simplification applied in the numerical example presented in [Sv[1]] only 2 slip systems are considered; in [Sv] 12 slip systems for cuboidal particles have been distinguished.) It will be useful to introduce tensors b^k of order 2 such that $b^k_{ij} = \frac{1}{2}(a^k_i c^k_j + a^k_j c^k_i)$ for every $i, j \in \{1, 2, 3\}$, too.

Let

$$\Lambda \in \prod_{j=0}^{N} L_\infty(\Gamma_j, \mathbb{R}^{3\times 3})$$

(symbol $\prod$ is reserved for Cartesian products), which means

$$\Lambda = (\Lambda_0, \ldots, \Lambda_N) \quad \text{with} \quad \Lambda_0 \in L_\infty(\Gamma_0, \mathbb{R}^{3\times 3}), \ldots, \Lambda_N \in L_\infty(\Gamma_N, \mathbb{R}^{3\times 3}),$$

be given support characteristics; if $x \in \Omega_i$ then $\Lambda(x)$ can be understood as $\Lambda_i(x)$ for the corresponding $i \in \{0, \ldots, N\}$. Let us emphasize that these characteristics are not variable in time. Moreover, for every fixed x the image $\Lambda(x)$ can be represented by a certain real square matrix of order 3; in extreme cases the regular one generates Dirichlet boundary conditions, the zero one Neumann boundary conditions.

Similarly for every $v \in \widetilde{V}$ where

$$\widetilde{V} = \prod_{j=0}^{N} C^{\infty}(\overline{\Omega}_j, \mathbb{R}^3),$$

which means

$$v = (v_0,\ \ldots, v_N) \quad \text{with} \quad v_0 \in C^{\infty}(\overline{\Omega}_0, \mathbb{R}^3),\ \ldots,\ v_N \in C^{\infty}(\overline{\Omega}_N, \mathbb{R}^3),$$

if $x \in \Omega_i$ then $v(x)$ can be understood as $v_i(x)$ for the corresponding $i \in \{0, \ldots, N\}$ and a certain $\mathrm{D}\, v(x)$ can be constructed from the differences $v_i - v_0$ on Ψ_i with $i \neq 0$. Let us notice that this approach makes it possible to define

$$\mathrm{D}\, v \in \prod_{j=1}^{N} L_2(\Psi_j, \mathbb{R}^3)$$

even if v is an arbitrary element of the Sobolev space

$$\breve{V} = \prod_{j=0}^{N} W_2^1(\Omega_j, \mathbb{R}^3)$$

only; this follows from the assumption (ii). The same argument yields

$$\Lambda v \in \prod_{j=0}^{N} L_2(\Gamma_j, \mathbb{R}^3)$$

($\Lambda(x)v(x)$ can be always evaluated by multiplying a matrix $\Lambda_i(x)$ by a vector $v_i(x)$ in the standard way for any fixed $x \in \Gamma_i$).

The realistic model of strain evolution must include all possible movements of total dislocation loops in each direction c^k (for the detailed physical analysis see [Sv[1]] again). Let U_{ik} be the set of all plane curves generated as boundaries of non-empty 2-dimensional cuts of Ω_i by parallel planes

$$\{x \in \mathbb{R}^3 :\ x.\, c^k = \xi\}$$

(dot symbols for scalar products in $\mathbb{R}^3$ are applied) for any $\xi \in \mathbb{R}$ and a certain $i \in \{1, \ldots, N\}$. The evolution of deformation of Ω can be then quantified using the rate of displacement between the corresponding points of Ω in the actual and reference (initial) configurations. We can define the *space of virtual displacement rates* V as the completion of the set of all $v \in \widetilde{V}$ with the properties

$$\forall\, i \in \{1, \ldots, N\}\ \forall\, k \in \{1, \ldots, M\}\ \forall\, \Upsilon \in U_{ik}\ \forall\, x, y \in \Upsilon \quad | \quad (\mathrm{D}\, v(x) - \mathrm{D}\, v(y)).\, a^k = 0$$

and

$$\forall\, x \in \partial\Omega \quad | \quad \Lambda(x) v(x) \text{ is a zero point of } \mathbb{R}^3$$

in the norm of $\breve{V}$.

The configuration changes, forced by external loads (that will be specified in the next section), are closely connected with the redistribution of stresses in time. Unfortunately, the constitutive relations between strains (and strain rates) and stresses, based on the elastic deformation and on the creep flow in the matrix and modified by the matrix/particle interface diffusion, are not trivial; therefore (unlike the pure theory of elasticity) we have to introduce the *space of virtual stresses*

$$S = \prod_{j=0}^{N} L_2(\Omega_j, \mathbb{R}^{3\times 3}_{\mathrm{sym}}).$$

Evidently, $\breve{V}$, V, and S are Hilbert spaces.

3. Equations of type (A)

We shall assume that the strain and stress development in a deformable body in time is the consequence of superposition of the following 4 types of external forces and strains:

(a) surface loads

$$\gamma \in W_2^2(I, B), \quad \text{where} \quad B = \prod_{j=0}^{N} L_2(\Gamma_j, \mathbb{R}^3),$$

(b) volume loads (body forces)

$$\varphi \in L_2(I, H), \quad \text{where} \quad H = \prod_{j=0}^{N} L_2(\Omega_j, \mathbb{R}^3),$$

(c) prescribed strains (e.g. those generated by temperature changes or support motions)

$$\vartheta \in W_2^1(I, S),$$

(d) forces caused by length changes of dislocation loops (for details see [Sv[1]]) for particular slip systems

$$\eta^1, \ldots, \eta^M \in W_2^2(I, Q), \quad \text{where} \quad Q = \prod_{j=1}^{N} L_2(\Psi_j, \mathbb{R}),$$

recalling that $W_2^1(I, X)$ is a space of all abstract functions $v \in L_2(I, X)$ mapping the time interval I to a certain Hilbert space X such that $\dot{v} \in L_2(I, X)$, too (dotted symbols are used for time derivatives). The forces (a), (b) and (d) will be incorporated into the principle of virtual displacement rates, the strains (c) will appear in the constitutive equations in the next section.

In the above introduced Hilbert spaces we will apply the brief notation for scalar products

$$(.,.) \text{ in } H, \qquad \langle .,. \rangle \text{ in } B, \qquad [.,.] \text{ in } S, \qquad \lfloor .,. \rfloor \text{ in } Q$$

and for norms

$$|.|_H \text{ in } H, \qquad |.|_B \text{ in } B, \qquad |.|_S \text{ in } S, \qquad |.|_Q \text{ in } Q, \qquad \|.\| \text{ in } V.$$

For every $k \in \{0, \ldots, M\}$ let

$$q^k \in \prod_{j=1}^{N} L_2(\Psi_j, \mathbb{R})$$

be the magnitudes of (a priori unknown) contact loads in a^k directions on matrix/particle interfaces. Moreover, let

$$\varrho \in \prod_{j=0}^{N} L_\infty(\Omega_j, \mathbb{R}_+)$$

be the material density (needed for the inertia forces $\varrho\dot{v}$). Then we are ready to formulate the variational *principle of virtual displacement rates* at any time $t \in I$

$$(1) \quad \forall \widetilde{v} \in V \quad | \quad (\widetilde{v}, \varrho\dot{v}) + [\varepsilon(\widetilde{v}), \sigma] + \lfloor \mathrm{D}\,\widetilde{v}.\, a^k, q^k \rfloor = (\widetilde{v}, \varphi) + \langle \widetilde{v}, \gamma \rangle + \lfloor \mathrm{D}\,\widetilde{v}.\, a^k, \eta^k \rfloor\,,$$

where the Einstein summation rule for indices k is applied in scalar products and the symbol $\varepsilon(.)$ is reserved for the well-known small deformation tensor from the linearized theory of elasticity, $\varepsilon(\widetilde{v}) = \frac{1}{2}\left(\nabla\widetilde{v} + (\nabla\widetilde{v})^T\right)$.

Now it is necessary to involve the equations of types (D) and (E). Let us define

$$S_1 = \prod_{j=1}^{N} L_2(\Omega_j, \mathbb{R}), \qquad T = \prod_{j=1}^{N} L_\infty(\Psi_j, \mathbb{R}_+).$$

Let us select an arbitrary k-th slip system for $k \in \{1, \dots, M\}$. Let ϱ^k (unlike the material density ϱ) be the dislocation density on Ψ_k corresponding to such a system. If at time $t = 0$ some dislocation density $\varrho_0^k \in Q$ is a priori known then by [Sv[1]] the values $\varrho^k(t)$ at every time $t \in I$ can be computed from the relation

$$\varrho^k(t) = \varrho_0^k + \int_0^t G(f(\sigma(t') : b^k), \mathrm{D}\, v(t').\, a^k)\, \mathrm{d}t'$$

(here a binary operator : means the sum of products of all corresponding elements of 2 tensors of order 2) where

$$G\colon\ Q \times Q \ \to\ Q$$

is a bounded mapping continuous in both variables and

$$f\colon\ S_1 \ \to\ Q$$

is a compact and weakly continuous mapping; this is the equation of type (E). (In general we cannot replace f by the standard trace operator, as no traces to some elements from $L_2(\Omega_j, \mathbb{R})$ with $j \in \{1, \dots, N\}$ may exist.) Finally, for a mapping

$$\beta\colon\ Q \ \to\ T$$

such that $g_1 \leqslant \beta(\widetilde{\varrho}) \leqslant g_2$ for any $\widetilde{\varrho} \in Q$ with certain $g_1, g_2 \in \mathbb{R}_+$ $(g_1 \leqslant g_2)$ Lipschitz continuous with a constant $\kappa \in \mathbb{R}_+$ in the sense

$$\forall\, \widetilde{\varrho}, \widehat{\varrho} \in Q \quad | \quad |\beta(\widetilde{\varrho}) - \beta(\widehat{\varrho})|_T \leqslant \kappa\, |\widetilde{\varrho} - \widehat{\varrho}|_Q$$

the equations of type (D) can be considered in the form

$$q^k = \beta(\varrho^k)\, \mathrm{D}\, v.\, a^k.$$

Substituting these expressions into (1) we obtain

$$\begin{aligned} \forall\, \widetilde{v} \in V \quad | \quad & (\widetilde{v}, \varrho\dot{v}) + [\varepsilon(\widetilde{v}), \sigma] + \left\lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \beta(\varrho^k)\, \mathrm{D}\, v.\, a^k \right\rfloor \\ & = (\widetilde{v}, \varphi) + \langle \widetilde{v}, \gamma \rangle + \left\lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \eta^k \right\rfloor . \end{aligned} \tag{2}$$

For the complete analysis of strain and stress distribution in time we need to know the initial values $v_0 \in V$, $\sigma_0 \in S$ of the abstract functions v and σ (their required properties will be made precise in the next section) and the initial values $\dot{v}_0$, $\dot{\sigma}_0$ of their derivatives, too. It is easy to see that they cannot be chosen in an arbitrary way. Let us suppose that there $\varphi_\star \in H$ exists such that

$$\forall\, \widetilde{v} \in V \quad | \quad (\widetilde{v}, \varphi_\star) = [\varepsilon(\widetilde{v}), \sigma_0] + \langle \widetilde{v}, \gamma(0)\rangle + \lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \beta(\varrho_0^k)\, \mathrm{D}\, v_0.\, a^k - \eta^k(0) \rfloor$$

holds. Then it is possible to set

$$\dot{v}_0 = \frac{\varphi(0) - \varphi_\star}{\varrho}.$$

The analogy of the resulting relation

$$\begin{aligned} \forall\, \widetilde{v} \in V \quad | \quad & (\widetilde{v}, \varrho\dot{v}_0) + [\varepsilon(\widetilde{v}), \sigma_0] + \lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \beta(\varrho_0^k)\, \mathrm{D}\, v_0.\, a^k \rfloor \\ & = (\widetilde{v}, \varphi(0)) + \langle \widetilde{v}, \gamma(0)\rangle + \lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \eta^k(0) \rfloor \end{aligned} \tag{3}$$

with (2) is obvious.

4. Equations of type (B)

The principle of virtual displacement rates from the previous section assumes no special constitutive strain-stress relations (it is evident that σ has to be calculated using $\varepsilon(v)$, but no such algorithm is available). Therefore it is necessary to formulate them separately in the equations of type (B). Equations presented in this and the following sections without additional comments should be valid on Ω_i for any $i \in \{0, \dots, N\}$; this will be not repeated.

We shall start from the Maxwell viscoelastic model (for the classification of rheological models see [Ko], p. 143, variational principles are discussed in [Vs], p. 593). For simplicity we will suppose that there is no viscous component for $i = 0$. Then for any time $t \in I$ we have

$$\varepsilon(v) = A\dot{\sigma} - \vartheta, \tag{4}$$

where the a priori strains ϑ have been mentioned in the preceding section (as type (c) loads) and

$$A \in \prod_{j=0}^{N} L_\infty(\Omega_j, \mathbb{R}^{(3\times3)\times(3\times3)})$$

are conventional stiffness characteristics from the Hooke law that are symmetrical in the sense

$$\forall\, \widetilde{\sigma}, \widehat{\sigma} \in S \quad | \quad [\widetilde{\sigma}, A\widehat{\sigma}] = [\widehat{\sigma}, A\widetilde{\sigma}]$$

and, moreover, positive definite in the sense

$$\forall\, \widetilde{\sigma} \in S \quad | \quad [\widetilde{\sigma}, A\widetilde{\sigma}] \geqslant \alpha\, |\widetilde{\sigma}|_S^2$$

where the constant $\alpha \in \mathbb{R}_+$ is independent of the choice of $\widetilde{\sigma} \in S$.

For $i \neq 0$ the dominant creep flow cannot be ignored. The resulting creep strain rate can be studied as a superposition of the strain rates corresponding to particular slip systems; in [Sv^1] it has been suggested in the form

$$\forall\, \widetilde{\sigma} \in S \quad | \quad F(\widetilde{\sigma}) = \sum_{k=1}^{M} \psi^k \varphi(\widetilde{\sigma}.\, b^k, \varkappa^k, \varkappa_\star^k) b^k$$

with creep factors

$$\psi^k \in \prod_{j=0}^{N} L_\infty(\Omega_j, \mathbb{R})$$

and two control stress levels

$$\varkappa^k, \varkappa_\star^k \in \prod_{j=0}^{N} L_\infty(\Omega_j, \mathbb{R})$$

(first of them are the known "Orowan stresses") such that $0 \leqslant \psi^k$ and $0 \leqslant \varkappa^k \leqslant \varkappa_\star^k$ everywhere on Ω_j for any $j \in \{0, \ldots, N\}$ ($\psi^k = \varkappa^k = \varkappa_\star^k = 0$ can be set in the special case $i = 0$) and φ are real functions of three real arguments (the so-called "Norton power-law creep functions", $0 \leqslant y_1 \leqslant y_2$)

$$\varphi(y, y_1, y_2) = \begin{cases} 0 & \text{for } 0 \leqslant y \leqslant y_1, \\ (y - y_1)^n & \text{for } y_1 \leqslant y \leqslant y_2, \\ (y_2 - y_1)^n + n(y_2 - y_1)^{n-1}(y - y_2) & \text{for } y_2 \leqslant y, \\ -\varphi(-y, y_1, y_2) & \text{for } y \leqslant 0 \text{ recurrently;} \end{cases}$$

in this definition the constant real creep exponent $n \geqslant 1$ occurs. For fixed y_1, y_2 and every real $y, \widetilde{y}$ the mean value theorem gives

$$\varphi(y, y_1, y_2) - \varphi(\widetilde{y}, y_1, y_2) = (y - \widetilde{y}) \frac{\mathrm{d}\varphi}{\mathrm{d}y}(\widehat{y}, y_1, y_2)$$

where there exists a real $\widehat{y}$ such that

$$\min(y,\widetilde{y}) \leqslant \widehat{y} \leqslant \max(y,\widetilde{y});$$

our special choice of φ forces

$$0 \leqslant \frac{\mathrm{d}\varphi}{\mathrm{d}y}(\widehat{y}, y_1, y_2) \leqslant n(y_2 - y_1)^{n-1},$$

which guarantees the validity of the (not strict) monotony condition

$$\forall\, \widetilde{\sigma}, \widehat{\sigma} \in S \quad | \quad [\widetilde{\sigma} - \widehat{\sigma}, F(\widetilde{\sigma}) - F(\widehat{\sigma})] \geqslant 0.$$

More generally, we will assume the existence of a mapping

$$F\colon \prod_{j=0}^{N} L_2(\Omega_j, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \to \prod_{j=0}^{N} L_2(\Omega_j, \mathbb{R}^{3\times3}_{\mathrm{sym}})$$

Lipschitz continuous in the sense

$$\forall\, \widetilde{\sigma}, \widehat{\sigma} \in S \quad | \quad |F(\widetilde{\sigma}) - F(\widehat{\sigma})|_S \leqslant \zeta\, |\widetilde{\sigma} - \widehat{\sigma}|_S$$

with a constant $\zeta \in \mathbb{R}_+$. (Let us notice that no monotony or weak continuity property is required here; from [Ap], pp. 63 and 103, we know that no nonlinear mappings in $L_2(\Omega, \mathbb{R})$ (and clearly in S as well) can be weakly continuous.) Using the mapping F we can write (as a generalization of (4)) for every time $t \in I$ the *constitutive equation*

$$\varepsilon(v) = A\dot{\sigma} + F(\sigma) - \vartheta. \tag{5}$$

Obviously the validity of (5) must be preserved also in the case $t = 0$. This influences the choice of an admissible initial status. We assume that $\dot{\sigma}_0$ satisfies (5) at time $t = 0$,

$$\varepsilon(v_0) = A\dot{\sigma}_0 + F(\sigma_0) - \vartheta(0). \tag{6}$$

Let us remark that (5) and (6) can be easily transformed into a form similar to (2) and (3)

$$\forall\, \widetilde{\sigma} \in S \quad | \quad [\widetilde{\sigma}, \varepsilon(v)] - [\widetilde{\sigma}, A\dot{\sigma}] - [\widetilde{\sigma}, F(\sigma)] = -\,[\widetilde{\sigma}, \vartheta]$$

for every $t \in I$ and

$$\forall\, \widetilde{\sigma} \in S \quad | \quad [\widetilde{\sigma}, \varepsilon(v_0)] - [\widetilde{\sigma}, A\dot{\sigma}_0] - [\widetilde{\sigma}, F(\sigma_0)] = -\,[\widetilde{\sigma}, \vartheta(0)]\,;$$

such conversions in proofs of lemmas and theorems will be left to the reader. We can summarize that we have reached the following problem (subscript L is used to denote Lipschitz continuous functions):

Problem 4.1. *For given $v_0 \in V$, $\sigma_0 \in S$ and $\dot v_0 \in V$, $\dot\sigma_0 \in S$ preserving* (3) *and* (6) *find $v \in C_L(I,H) \cap L_\infty(I,V)$ and $\sigma \in C_L(I,S)$ such that $\dot v \in L_\infty(I,H)$ and $\dot\sigma \in L_\infty(I,S)$ and* (2) *and* (5) *are satisfied.*

Our main aim will be to verify the following assertion:

Theorem 4.2. *Problem* 4.1 *has a solution.*

P r o o f. This will be based on several lemmas from Section 5, where the method of discretization in time (construction of Rothe sequences) will be applied to simplify Problem 4.1; it will be completed in Section 6. $\square$

5. Discretization in time

In this section we will make use of the standard technique of discretization in time (no special discretization in $\mathbb{R}^3$ for the design of sequences of "approximate solutions" will be needed in this article). Let m be a (fixed) positive integer. In addition to an interval I we will consider also intervals $I_s = \{t \in I\colon\ (s-1)h < t \leqslant sh\}$ for $h = \tau/m$ and $s \in \{1, \ldots, m\}$; for $s > 1$ we will use also the notation $I_s^2 = I_s \cup I_{s-1}$. Let X be a Hilbert space (supplied with a norm $|.|_X$) and $u_0, \ldots, u_m$ its arbitrary elements. Let us denote

$$\begin{aligned} \delta u_s &= \frac{u_s - u_{s-1}}{h}, \\ \delta^2 u_s &= \frac{\delta u_s - \delta u_{s-1}}{h} \quad \text{for} \quad s > 1, \end{aligned}$$

and construct the Rothe functions

$$\begin{aligned} \bar u^m(t) &= \begin{cases} u_0 & \text{for} \quad t = 0, \\ u_s & \text{for} \quad t \in I_s, \end{cases} \\ \breve u^m(t) &= \begin{cases} u_0 & \text{for} \quad t = 0, \\ u_{s-1} & \text{for} \quad t \in I_s, \end{cases} \\ u^m(t) &= \begin{cases} u_0 & \text{for} \quad t = 0, \\ u_s + (t - sh)\delta u_s = u_{s-1} + (t - (s-1)h)\delta u_s & \text{for } t \in I_s. \end{cases} \end{aligned}$$

Let us recall that $\dot{u}^m(t)$ denotes the time derivative of u^m at a time t; if $t = sh$ then the derivative (which does not exist in general) must be substituted by the left derivative, $\dot{u}^m(0)$ will be handled as a certain given element of X. For all $t \in I$ we have

$$(7) \qquad \max\left(|u^m(t) - \breve{u}^m(t)|_X\,, |u^m(t) - \bar{u}^m(t)|_X\,, |u^m(t) - \breve{u}^m(t)|_X\right) \leqslant h\,|\dot{u}^m(t)|_X$$

(moreover $\bar{u}^m(t) - \breve{u}^m(t) = h\dot{u}^m(t)$ for $t > 0$). For $u \in C(I, X)$ we define (similarly to $\bar{u}^m(t)$)

$$\widehat{u}^m(t) = \begin{cases} u(0) & \text{for} \quad t = 0, \\ u(sh) & \text{for} \quad t \in I_s \end{cases}$$

(in all situations with no danger of misunderstanding we will write u_0 and u_s instead of $u(0)$ and $u(sh)$). Then for every $t \in I$

$$(8) \qquad \{\widehat{u}^m(t)\}_{m=1}^{\infty} \text{ has a strong limit } u(t) \text{ in } X.$$

Thus for any $s \in \{1, \ldots, m\}$ we can estimate

$$(9) \qquad h\sum_{r=1}^{s} |\delta u_r|_X^2 = h^{-1}\sum_{r=1}^{s} \left|\int_{I_r} \dot{u}(t)\,\mathrm{d}t\right|_X^2 \leqslant \sum_{r=1}^{s} \int_{I_r} |\dot{u}(t)|_X^2 \;\mathrm{d}t = \int_I |\dot{u}(t)|_X^2 \;\mathrm{d}t$$

provided $u \in W_2^1(I, X)$,

$$(10) \qquad |\delta u_s|_X^2 = h^{-2}\left|\int_{I_s} \dot{u}(t)\,\mathrm{d}t\right|_X^2 \leqslant \operatorname*{ess\,sup}_{t \in I} |\dot{u}(t)|_X^2$$

provided $u \in W_\infty^1(I, X)$ and

$$(11) \qquad \begin{aligned} h\sum_{r=2}^{s} |\delta^2 u_r|_X^2 &= h^{-3}\sum_{r=2}^{s} \left|\int_{I_r}\int_{t'-h}^{t'} \ddot{u}(t')\,\mathrm{d}t'\,\mathrm{d}t\right|_X^2 \\ &\leqslant h^{-3}\sum_{r=2}^{s}\left(\int_{I_r}\int_{I_r^2} |\ddot{u}(t')|_X \;\mathrm{d}t'\,\mathrm{d}t\right)^2 \\ &\leqslant 2\sum_{r=1}^{s}\int_{I_r^2} |\ddot{u}(t')|_X^2 \;\mathrm{d}t' = 2\int_I |\ddot{u}(t)|_X^2 \;\mathrm{d}t \end{aligned}$$

provided $u \in W_2^2(I, X)$.

Now we will try to replace (2) by

$$(12) \qquad \begin{aligned} \forall\, \widetilde{v} \in V \quad | \quad & (\widetilde{v}, \varrho\dot{v}^m) + [\varepsilon(\widetilde{v}), \bar{\sigma}^m] + \left\lfloor \mathrm{D}\,\widetilde{v}.\, a^k, \beta(\breve{\varrho}^{km})\,\mathrm{D}\,\bar{v}^m.\, a^k \right\rfloor \\ &= (\widetilde{v}, \widehat{\varphi}^m) + \langle \widetilde{v}, \widehat{\gamma}^m \rangle + \left\lfloor \mathrm{D}\,\widetilde{v}.\, a^k, \widehat{\eta}^{km} \right\rfloor \end{aligned}$$

applying the notation

$$\breve{\varrho}^{km} = \varrho_0^k + \beta\Big(\int_0^t G(f(\breve{\sigma}^m(t') : b^k), \mathrm{D}\,\breve{v}^m(t').\,a^k)\,\mathrm{d}t'\Big),$$

and (5) by

$$\varepsilon(\bar{v}^m) = A\dot{\sigma}^m + F(\bar{\sigma}^m) - \widehat{\vartheta}^m. \tag{13}$$

This trick leads to the decomposition of an original system of PDE's of evolution, step-by-step for $s \in \{1, \ldots, m\}$, into m systems without any time variables: (12) yields

$$\begin{aligned}\forall\,\widetilde{v} \in V \quad | \quad & (\widetilde{v}, \varrho\delta v_s) + [\varepsilon(\widetilde{v}), \sigma_s] + \big\lfloor \mathrm{D}\,\widetilde{v}.\,a^k, \beta(\varrho_{s-1}^k)\,\mathrm{D}\,v_s.\,a^k\big\rfloor \\ & = (\widetilde{v}, \varphi_s) + \langle\widetilde{v}, \gamma_s\rangle + \big\lfloor \mathrm{D}\,\widetilde{v}.\,a^k, \eta_s^k\big\rfloor\end{aligned} \tag{14}$$

with the notation

$$\varrho_s^k = \varrho_0^k + h\sum_{r=1}^{s} G(f(\sigma_r : b^k), \mathrm{D}\,v_r.\,a^k)$$

and (13) similarly

$$\varepsilon(v_s) = A\delta\sigma_s + F(\sigma_s) - \vartheta_s. \tag{15}$$

We can finally formulate

Problem 5.1. *For given $v_0, \ldots, v_{s-1} \in V$ and $\sigma_0, \ldots, \sigma_{s-1} \in S$ find $v_s \in V$ and $\sigma_s \in S$ such that* (14) *and* (15) *are satisfied.*

To analyze the limit procedure leading from (12) and (13) to (2) and (5), we need the following results:

Theorem 5.2. *Problem* 5.1 *has a solution.*

Theorem 5.3. *If* (3) *and* (6) *hold then there exists such a $\Theta \in \mathbb{R}_+$ that*

$$\max\left(\|v_s\|, |\sigma_s|_S, |\delta v_s|_H, |\delta\sigma_s|_S\right) \leqslant \Theta \tag{16}$$

independently of the choice of $m \in \{1, 2, \ldots\}$ and $s \in \{1, \ldots, m\}$.

The remaining extensive part of this section contains the necessary proofs. The technique of approximation of nonlinear operators by linear ones is applied to carry out the proofs in the way compatible with the algorithms incorporated in CDS software.

Proof of Theorem 5.2. For $j \in \{1, 2, \ldots\}$ let us consider a linearized iterative scheme with (14) replaced by

$$\forall \widetilde{v} \in V \quad | \quad \left(\widetilde{v}, \varrho v_s^j\right) + h \left[\varepsilon(\widetilde{v}), \sigma_s^j\right] + h \left\lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \beta(\varrho_{s-1}^k)\, \mathrm{D}\, v_s^j.\, a^k \right\rfloor \\ = h\, (\widetilde{v}, \varphi_s) + h\, \langle \widetilde{v}, \gamma_s \rangle + h \left\lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \eta_s^k \right\rfloor + (\widetilde{v}, \varrho v_{s-1}) \tag{17}$$

for certain $v_s^j \in V$ and the corresponding $\sigma_s^j \in S$ calculated from the modification of (15)

$$A\sigma_s^j = h\varepsilon(v_s^j) + h\vartheta_s - hF(\sigma_s^{j-1}) + A\sigma_{s-1}. \tag{18}$$

Substituting σ_s^j from (18) into (17) for $j = 1$ we obtain

$$\forall \widetilde{v} \in V \quad | \quad b_s(\widetilde{v}, v_s^1) = f_s(\widetilde{v})$$

with

$$b_s(\widetilde{v}, v_s^1) = \left(\widetilde{v}, \varrho v_s^1\right) + h^2 \left[\varepsilon(\widetilde{v}), A^{-1}\varepsilon(v_s^1)\right] + h \left\lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \beta(\varrho_{s-1}^k)\, \mathrm{D}\, v_s^1.\, a^k \right\rfloor$$

and

$$f_s(\widetilde{v}) = h\, (\widetilde{v}, \varphi_s) + h\, \langle \widetilde{v}, \gamma_s \rangle + h \left\lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \eta_s^k \right\rfloor + (\widetilde{v}, \varrho v_{s-1}) \\ - h^2 \left[\varepsilon(\widetilde{v}), A^{-1}\vartheta_s\right] - h\, [\varepsilon(\widetilde{v}), \sigma_{s-1}] - h \left[\varepsilon(\widetilde{v}), A^{-1}F(\sigma_{s-1})\right].$$

Clearly f_s is a linear functional on V (we have required the existence of traces in the sense of (ii)) and b_s can be identified with a bounded bilinear form on V. Moreover, we have

$$\forall \widetilde{v} \in V \quad | \quad b_s(\widetilde{v}, \widetilde{v}) \geqslant (\widetilde{v}, \varrho \widetilde{v}) + h^2 \left[\varepsilon(\widetilde{v}), A^{-1}\varepsilon(\widetilde{v})\right] \geqslant \chi \min(1, h^2)\, \|\widetilde{v}\|^2$$

where the existence of some $\chi \in \mathbb{R}_+$ independent of h follows from the property (iii) in the form

$$\forall \widetilde{v} \in V \quad | \quad \omega\, \|\widetilde{v}\|^2 \leqslant |\widetilde{v}|_H^2 + |\varepsilon(\widetilde{v})|_S^2 \tag{19}$$

with a certain fixed $\omega \in \mathbb{R}_+$. Thus the Lax-Milgram theorem (see [Yo], p. 134) guarantees the existence of $v_s^1 \in V$; consequently $\sigma_s^1 \in S$ can be calculated directly from (15). Because of complications with possible $F(\sigma_s^{j-1}) \neq F(\sigma_{s-1})$ similar arguments are not available for $j > 1$. In this case let us consider the difference between two consecutive equations (17)

$$\forall \widetilde{v} \in V \quad | \quad \left(\widetilde{v}, \varrho \Delta v_s^j\right) + h \left[\varepsilon(\widetilde{v}), \Delta\sigma_s^j\right] + h \left\lfloor \mathrm{D}\, \widetilde{v}.\, a^k, \beta(\varrho_{s-1}^k)\, \mathrm{D}\, \Delta v_s^j.\, a^k \right\rfloor = 0 \tag{20}$$

and (18)

$$\varepsilon(\Delta v_s^j) = A\Delta\sigma_s^j + h\left(F(\sigma_s^{j-1}) - F(\sigma_s^{j-2})\right); \tag{21}$$

for the sake of brevity the notation

$$\Delta v_s^j = v_s^j - v_s^{j-1}, \qquad \Delta\sigma_s^j = \sigma_s^j - \sigma_s^{j-1}$$

is used here. Putting $\widetilde{v} = \Delta v_s^j$ into (20) and substituting $\varepsilon(\Delta v_s^j)$ from (21) into (20) we obtain

$$\begin{aligned}&\left(\Delta v_s^j, \varrho\Delta v_s^j\right) + \left[\Delta\sigma_s^j, A\Delta\sigma_s^j\right] + h\left\lfloor \mathrm{D}\,\Delta v_s^j.\,a^k, \beta(\varrho_{s-1}^k)\,\mathrm{D}\,\Delta v_s^j.\,a^k\right\rfloor\\ &= -h\left[\Delta\sigma_s^j, F(\sigma_s^{j-1}) - F(\sigma_s^{j-2})\right],\end{aligned}$$

which implies the existence of $K \in \mathbb{R}_+$ independent of m such that the estimates

$$\begin{aligned}\left|\Delta v_s^j\right|_H^2 + \left|\Delta\sigma_s^j\right|_S^2 &\leqslant Kh\left|\Delta\sigma_s^j\right|_S\left|\Delta\sigma_s^{j-1}\right|_S \leqslant \frac{1}{2}\left|\Delta\sigma_s^j\right|_S^2 + \frac{1}{2}(Kh)^2\left|\Delta\sigma_s^{j-1}\right|_S^2\\ &\leqslant (Kh)^2\left|\Delta\sigma_s^{j-1}\right|_S^2 \leqslant (Kh)^{2j-2}\left|\Delta\sigma_s^1\right|_S^2\end{aligned}$$

are true. Consequently, for an arbitrary integer $i > j$ and any fixed $h < 1/K$ the inequalities

$$\begin{aligned}\left|v_s^i - v_s^j\right|_H &\leqslant \left|\Delta v_s^{j+1}\right|_H + \ldots + \left|\Delta v_s^i\right|_H \leqslant \left((Kh)^j + \ldots + (Kh)^{i-1}\right)\left|\Delta\sigma_s^1\right|_S\\ &\leqslant \frac{(Kh)^j}{1-Kh}\left|\Delta\sigma_s^1\right|_S,\\ \left|\sigma_s^i - \sigma_s^j\right|_S &\leqslant \left|\Delta\sigma_s^{j+1}\right|_S + \ldots + \left|\Delta\sigma_s^i\right|_S \leqslant \left((Kh)^j + \ldots + (Kh)^{i-1}\right)\left|\Delta\sigma_s^1\right|_S\\ &\leqslant \frac{(Kh)^j}{1-Kh}\left|\Delta\sigma_s^1\right|_S\end{aligned}$$

hold which ensures that

$$\begin{aligned}&\{v_s^j\}_{j=1}^{\infty} \text{ is a Cauchy sequence in } H,\\ &\{\sigma_s^j\}_{j=1}^{\infty} \text{ is a Cauchy sequence in } S.\end{aligned}$$

Moreover, from (18) we obtain

$$\varepsilon(v_s^i) - \varepsilon(v_s^j) = A(\sigma_s^i - \sigma_s^j) - h\left(F(\sigma_s^{i-1}) - F(\sigma_s^{j-1})\right),$$

which with respect to the continuity of F implies

$$\{\varepsilon(v_s^j)\}_{j=1}^{\infty} \text{ is a Cauchy sequence in } S;$$

therefore the coerciveness of strains (19) gives

$$\{v_s^j\}_{j=1}^{\infty} \text{ is a Cauchy sequence in } V.$$

Thus the completeness of V and S guarantees the existence of such $v_s \in V$ and $\sigma_s \in S$ that

$$\{v_s^j\}_{j=1}^{\infty} \text{ has a strong limit } v_s \text{ in } V,$$
$$\{\sigma_s^j\}_{j=1}^{\infty} \text{ has a strong limit } \sigma_s \text{ in } S.$$

Then v_s and σ_s must satisfy (14) and by virtue of the continuity of F also (15). □

Lemma 5.4. *There exists $\Theta_0 \in \mathbb{R}_+$ such that*

$$|\varepsilon(v_s)|_S^2 \leqslant \Theta_0 \left(1 + |\sigma_s|_S^2 + |\delta\sigma_s|_S^2\right) \tag{22}$$

independently of the choice of $m \in \{1, 2, \ldots\}$ and $s \in \{1, \ldots, m\}$.

Proof. Since $\vartheta \in L_\infty(I, S)$ (this is evidently weaker than the above assumed $\vartheta \in W_2^1(I, S)$), ϑ_s must be bounded in S independently of s, m. Thus the assertion follows from (15). □

Remark. In accordance with the property (iii) a possible alternative form of (22) is

$$\|v_s\|^2 \leqslant \Theta_0 \left(1 + |v_s|_H^2 + |\sigma_s|_S^2 + |\delta\sigma_s|_S^2\right)$$

(see the estimate (19)).

Lemma 5.5. *There exists $\Theta_1 \in \mathbb{R}_+$ such that*

$$|v_s|_H^2 + |\sigma_s|_S^2 \leqslant \Theta_1 \left(1 + h \sum_{r=1}^{s} \left(|\delta v_r|_H^2 + |\delta\sigma_r|_S^2\right)\right) \tag{23}$$

independently of the choice of $m \in \{1, 2, \ldots\}$ and $s \in \{1, \ldots, m\}$.

Proof. This is an immediate consequence of the inequality

$$\begin{aligned} |v_s|_H^2 + |\sigma_s|_S^2 &\leqslant \left(|v_0|_H + h \sum_{r=1}^{s} |\delta v_r|_H\right)^2 + \left(|\sigma_0|_S + h \sum_{r=1}^{s} |\delta\sigma_r|_S\right)^2 \\ &\leqslant 2\,|v_0|_H^2 + 2\,|\sigma_0|_S^2 + 2sh \sum_{r=1}^{s} |\delta v_r|_H^2 + 2sh \sum_{r=1}^{s} |\delta\sigma_r|_S^2 \end{aligned}$$

which guarantees that the constant

$$\Theta_1 = 2 \max\left(|v_0|_H^2 + |\sigma_0|_S^2, \tau\right)$$

always exists. □

R e m a r k. An alternative (more "physical") way how to obtain some estimate of this type is to put $\widetilde{v} = v_s$ into (14) with $\varepsilon(v_s)$ substituted from (15). The rest of the proof (construction of a priori estimates) can be done analogously to that of Lemma 5.6. (Such approach is applied e.g. in [Ka[1]].)

Lemma 5.6. *If* (3) *and* (6) *are preserved then there exists* $\Theta_2 \in \mathbb{R}_+$ *such that*

$$
\begin{aligned}
(24)\qquad & |\delta v_s|_H^2 + |\delta\sigma_s|_S^2 \\
& \leqslant \Theta_2 \Big(1 + |v_s|_H^2 + |\sigma_s|_S^2 + h \sum_{r=1}^{s} \big(|v_r|_H^2 + |\delta v_r|_H^2 + |\sigma_r|_S^2 + |\delta\sigma_r|_S^2 \big) \Big)
\end{aligned}
$$

independently of the choice of $m \in \{1, 2, \ldots\}$ *and* $s \in \{1, \ldots, m\}$.

P r o o f. For $s > 1$ let us calculate differences between two consecutive equations (14) and (15). From (14) we obtain

$$
\begin{aligned}
(25)\qquad \forall \widetilde{v} \in V \quad | \quad & h\left(\widetilde{v}, \varrho\delta^2 v_s\right) + h\left[\varepsilon(\widetilde{v}), \delta\sigma_s\right] + h\left\lfloor \mathrm{D}\,\widetilde{v}.\, a^k, \beta(\varrho_s^k)\,\mathrm{D}\,\delta v_s.\, a^k \right\rfloor \\
& = h\left(\widetilde{v}, \delta\varphi_s\right) + h\left\langle \widetilde{v}, \delta\gamma_s\right\rangle + h\left\lfloor \mathrm{D}\,\widetilde{v}.\, a^k, \delta\eta_s^k \right\rfloor \\
& + \left\lfloor \mathrm{D}\,\widetilde{v}.\, a^k, \left(\beta(\varrho_s^k) - \beta(\varrho_{s-1}^k)\right)\mathrm{D}\, v_{s-1}.\, a^k \right\rfloor
\end{aligned}
$$

and from (15) similarly

$$
(26)\qquad h\varepsilon(\delta v_s) = hA\delta^2\sigma_s + F(\sigma_s) - F(\sigma_{s-1}) - h\delta\vartheta_s .
$$

To admit the case $s = 1$, we can see from (3) and (6) that it is natural to consider $\delta v_0 = \dot{v}_0$ and $\delta\sigma_0 = \dot{\sigma}_0$; thus no modifications in (25) or (26) are needed. Putting $\widetilde{v} = \delta v_s$ into (25) with $\varepsilon(\delta v_s)$ substituted from (26) we obtain

$$
\begin{aligned}
& h\left(\delta v_s, \varrho\delta^2 v_s\right) + h\left[\delta\sigma_s, A\delta^2\sigma_s\right] + h\left[\delta\sigma_s, \frac{F(\sigma_s) - F(\sigma_{s-1})}{h}\right] \\
& \quad + h\left\lfloor \mathrm{D}\,\delta v_s.\, a^k, \beta(\varrho_s^k)\,\mathrm{D}\,\delta v_s.\, a^k \right\rfloor \\
& = h\left(\delta v_s, \delta\varphi_s\right) + h\left\langle \delta v_s, \delta\gamma_s\right\rangle + h\left\lfloor \mathrm{D}\,\delta v_s.\, a^k, \delta\eta_s^k \right\rfloor + h\left[\delta\sigma_s, \delta\vartheta_s\right] \\
& \quad + \left\lfloor \mathrm{D}\,\delta v_s.\, a^k, \left(\beta(\varrho_s^k) - \beta(\varrho_{s-1}^k)\right)\mathrm{D}\, v_{s-1}.\, a^k \right\rfloor .
\end{aligned}
$$

The same is evidently true as well for any $r \in \{1, \ldots, s\}$ instead of s. Summing over all such equations we obtain

$$\begin{aligned}
&\frac{1}{2}(\delta v_s, \varrho \delta v_s) - \frac{1}{2}(\delta v_0, \varrho \delta v_0) + \frac{h^2}{2}\sum_{r=1}^{s}\left(\delta^2 v_r, \varrho \delta^2 v_r\right) + \frac{1}{2}[\delta\sigma_s, A\delta\sigma_s] - \frac{1}{2}[\delta\sigma_0, A\delta\sigma_0] \\
&\quad + \frac{h^2}{2}\sum_{r=1}^{s}\left[\delta^2\sigma_r, A\delta^2\sigma_r\right] + h\sum_{r=1}^{s}\left\lfloor \mathrm{D}\,\delta v_r.\, a^k, \beta(\varrho_r^k)\,\mathrm{D}\,\delta v_r.\, a^k\right\rfloor \\
&= h\sum_{r=1}^{s}(\delta v_r, \delta\varphi_r) + h\sum_{r=1}^{s}\langle \delta v_r, \delta\gamma_r\rangle - h\sum_{r=1}^{s}\left\lfloor \mathrm{D}\,\delta v_r.\, a^k, \delta\eta_r^k\right\rfloor + h\sum_{r=1}^{s}[\delta\sigma_r, \delta\vartheta_r] \\
&\quad - h\sum_{r=1}^{s}\left[\delta\sigma_r, \frac{F(\sigma_r) - F(\sigma_{r-1})}{h}\right] + \sum_{r=1}^{s}\left\lfloor \mathrm{D}\,\delta v_r.\, a^k, \left(\beta(\varrho_r^k) - \beta(\varrho_{r-1}^k)\right)\mathrm{D}\, v_{r-1}.\, a^k\right\rfloor .
\end{aligned}$$

With regard to the Abel summation technique

$$\begin{aligned}
h\sum_{r=1}^{s}\langle \delta v_r, \delta\gamma_r\rangle &= \langle v_s, \delta\gamma_s\rangle - \langle v_0, \delta\gamma_0\rangle - h\sum_{r=1}^{s}\left\langle v_{r-1}, \delta^2\gamma_{r-1}\right\rangle , \\
h\sum_{r=1}^{s}\left\lfloor \mathrm{D}\,\delta v_r.\, a^k, \delta\eta_r^k\right\rfloor &= \left\lfloor \mathrm{D}\, v_s.\, a^k, \delta\eta_s^k\right\rfloor - \left\lfloor \mathrm{D}\, v_0.\, a^k, \delta\eta_0^k\right\rfloor - h\sum_{r=1}^{s}\left\lfloor \mathrm{D}\, v_{r-1}.\, a^k, \delta^2\eta_{r-1}^k\right\rfloor
\end{aligned}$$

with formal notation $\delta\gamma_0 = \dot\gamma(0)$ and $\delta\eta_0^k = \dot\eta^k(0)$ for every $\nu \in \mathbb{R}_+$ the estimates (with the same constant $\lambda \in \mathbb{R}_+$ as in the proof of Lemma 5.5)

$$\begin{aligned}
h\sum_{r=1}^{s}(\delta v_r, \delta\varphi_r) &\leqslant \nu h\sum_{r=1}^{s}|\delta v_r|_H^2 + \frac{h}{4\nu}\sum_{r=1}^{s}|\delta\varphi_r|_H^2 , \\
\langle v_s, \delta\gamma_s\rangle &\leqslant \nu\,|v_s|_H^2 + \nu\,|\varepsilon(v_s)|_S^2 + \frac{\lambda}{4\nu}\,|\delta\gamma_s|_B^2 , \\
-\langle v_0, \delta\gamma_0\rangle &\leqslant \nu\,|v_0|_H^2 + \nu\,|\varepsilon(v_0)|_S^2 + \frac{\lambda}{4\nu}\,|\delta\gamma_0|_B^2 , \\
-h\sum_{r=1}^{s}\left\langle v_{r-1}, \delta^2\gamma_r\right\rangle &\leqslant \nu h\sum_{r=1}^{s}|v_{r-1}|_H^2 + \nu h\sum_{r=1}^{s}|\varepsilon(v_{r-1})|_S^2 + \frac{\lambda h}{4\nu}\sum_{r=1}^{s}\left|\delta^2\gamma_r\right|_B^2 ,
\end{aligned}$$

$$\lfloor \mathrm{D}\, v_s .\, a^k, \delta\eta^k_s \rfloor \leqslant \nu\, |v_s|^2_H + \nu\, |\varepsilon(v_s)|^2_S + \frac{\lambda}{4\nu} \sum_{k=1}^{M} \left|\delta\eta^k_s\right|^2_Q ,$$

$$-\lfloor \mathrm{D}\, v_0 .\, a^k, \delta\eta^k_0 \rfloor \leqslant \nu\, |v_0|^2_H + \nu\, |\varepsilon(v_0)|^2_S + \frac{\lambda}{4\nu} \sum_{k=1}^{M} \left|\delta\eta^k_0\right|^2_Q ,$$

$$-h \sum_{r=1}^{s} \lfloor \mathrm{D}\, v_{r-1} .\, a^k, \delta^2\eta^k_r \rfloor \leqslant \nu h \sum_{r=1}^{s} |v_{r-1}|^2_H + \nu h \sum_{r=1}^{s} |\varepsilon(v_{r-1})|^2_S + \frac{\lambda h}{4\nu} \sum_{r=1}^{s} \sum_{k=1}^{M} \left|\delta^2\eta^k_r\right|^2_Q ,$$

$$h \sum_{r=1}^{s} [\delta\sigma_r, \delta\vartheta_r] \leqslant \nu h \sum_{r=1}^{s} |\delta\sigma_r|^2_S + \frac{h}{4\nu} \sum_{r=1}^{s} |\delta\vartheta_r|^2_S ,$$

$$-h \sum_{r=1}^{s} \left[\delta\sigma_r, \frac{F(\sigma_r) - F(\sigma_{r-1})}{h}\right] \leqslant \zeta h \sum_{r=1}^{s} |\delta\sigma_r|^2_S ,$$

and from the estimate based on the Lipschitz continuity

$$\left|\beta(\varrho^k_r) - \beta(\varrho^k_{r-1})\right|_T \leqslant \kappa \left|\varrho^k_r - \varrho^k_{r-1}\right|_Q = \kappa h \left|G(f(\sigma_r : b^k), \mathrm{D}\, v_r .\, a^k)\right|_Q \leqslant \kappa g_2 h$$

also

$$\begin{aligned} &-\sum_{r=1}^{s} \lfloor \mathrm{D}\, \delta v_r .\, a^k, \big(\beta(\varrho^k_r) - \beta(\varrho^k_{r-1})\big)\, \mathrm{D}\, v_{r-1} .\, a^k \rfloor \\ &\leqslant \nu h \sum_{r=1}^{s} \left|\mathrm{D}\, \delta v_r .\, a^k\right|^2_Q + \frac{\kappa g_2 h}{4\nu} \sum_{r=1}^{s} \left|\mathrm{D}\, v_r .\, a^k\right|^2_Q \\ &\leqslant \nu h \sum_{r=1}^{s} \left|\mathrm{D}\, \delta v_r .\, a^k\right|^2_Q + \frac{\lambda\kappa g_2 h}{4\nu} \sum_{r=1}^{s} |v_r|^2_H + \frac{\lambda\kappa g_2 h}{4\nu} \sum_{r=1}^{s} |\varepsilon(v_r)|^2_S \end{aligned}$$

are valid. The a priori known terms on the right-hand side can be estimated using the choices $X = H$, $u = \varphi$ in (9), $X = S$, $u = \vartheta$ in (9), $X = B$, $u = \gamma$ in (10), $X = Q$, $u = \eta^k$ in (10) for an arbitrary $k \in \{1, \ldots, M\}$, $X = B$, $u = \gamma$ in (11), and $X = Q$, $u = \eta^k$ in (11) for an arbitrary $k \in \{1, \ldots, M\}$. Moreover,

$$h \sum_{r=1}^{s} \lfloor \mathrm{D}\, \delta v_r .\, a^k, \beta(\varrho^k_r)\, \mathrm{D}\, \delta v_r .\, a^k \rfloor \geqslant g_1 h \sum_{r=1}^{s} \left|\mathrm{D}\, \delta v_r .\, a^k\right|^2_Q$$

is an evident property of the mapping β. Making use of these facts we obtain

$$\begin{aligned} |\delta v_s|^2_H + |\delta\sigma_s|^2_S &\leqslant \nu\, |\varepsilon(v_s)|^2_S \\ &\quad + \Theta_2 \Big(1 + |v_s|^2_H + h \sum_{r=1}^{s} \Big(|v_r|^2_H + |\delta v_r|^2_H + |\varepsilon(v_r)|^2_H + |\delta\sigma_r|^2_S\Big)\Big) \end{aligned}$$

for an arbitrary $\nu \in \mathbb{R}_+$ and some $\Theta_2 \in \mathbb{R}_+$ (dependent on ν only); to derive (24) it now remains only to apply Lemma 5.4 for some $\nu < \Theta_0^{-1}$. □

Lemma 5.7. *There exists $\mu_\star \in \mathbb{R}_+$ (independent of m) such that for any $\mu_0, \mu_1, \ldots, \mu_m \in \mathbb{R}_+$ ($m \in \{1, 2, \ldots\}$, $h = \tau/m$) the assumption*

$$\forall\, s \in \{1, \ldots, m\} \quad | \quad \mu_s \leqslant \mu_0 \Bigl(1 + h \sum_{r=1}^{s} \mu_r\Bigr) \tag{27}$$

guarantees that

$$\forall\, s \in \{1, \ldots, m\} \quad | \quad \mu_s \leqslant \mu_\star$$

is true for m sufficiently large.

Proof. This is one special discrete version of the Gronwall lemma (its slightly modified form can be found in [Ka], p. 29). Its verification is simple: it is sufficient to prove

$$\forall\, s \in \{1, \ldots, m\} \quad | \quad \mu_s \,(1 - \mu_0 h)^s \leqslant \mu_0 \tag{28}$$

for any m sufficiently large and the above assumed $\mu_0, \mu_1, \ldots, \mu_m$ (in more detail: because the estimate

$$\lim_{m \to \infty} \mu_0 \,(1 - \mu_0 h)^{-s} \leqslant \lim_{m \to \infty} \mu_0 \,(1 - \mu_0 h)^{-\frac{\tau}{h}} = \mu_0 e^{\mu_0 \tau}$$

can be established using the classical L'Hospital rule, the existence of some $\mu_\star$ is evident for a large m). For $s = 1$ the proof is trivial. For $s > 1$ (applying induction) we have

$$\begin{aligned}
\mu_s \,(1 - \mu_0 h) &\leqslant \mu_0 + \mu_0 h \sum_{r=1}^{s-1} \mu_r \leqslant \mu_0 + \mu_0 h \sum_{r=1}^{s-1} \frac{1}{(1 - \mu_0 h)^r} \\
&= \mu_0 \left(1 + \frac{\mu_0 h}{1 - \mu_0 h} \cdot \frac{1 - \frac{1}{(1-\mu_0 h)^{s-1}}}{1 - \frac{1}{(1-\mu_0 h)}}\right) = \frac{\mu_0}{(1 - \mu_0 h)^{s-1}}
\end{aligned}$$

which (for $h = \tau/m < \mu_0^{-1}$) yields (28). □

Proof of Theorem 5.3. For any $s \in \{1, \ldots, M\}$ let us set

$$\mu_s = |v_s|_H^2 + |\sigma_s|_S^2 + |\delta v_s|_H^2 + |\delta \sigma_s|_S^2 \,.$$

We can apply Lemma 5.5 and Lemma 5.6. The sum of (23) and (24) multiplied by an arbitrary $\nu \in \mathbb{R}_+$ gives

$$\begin{aligned}
&(\Theta_1 - \nu\Theta_2) \left(|v_s|_H^2 + |\sigma_s|_S^2\right) + \nu\Theta_2 \left(|\delta v_s|_H^2 + |\delta\sigma_s|_S^2\right) \\
&\leqslant (\Theta_1 + \nu\Theta_2)\Bigl(1 + h \sum_{r=1}^{s} \left(|\delta v_r|_H^2 + |\delta\sigma_r|_S^2\right)\Bigr) + \nu\Theta_2 h \sum_{r=1}^{s} \left(|v_r|_H^2 + |\sigma_r|_S^2\right).
\end{aligned}$$

If $\nu < \Theta_1/\Theta_2$ then this inequality yields (27) with

$$\mu_0 = \frac{\Theta_1 + \nu\Theta_2}{\min(\Theta_1 - \nu\Theta_2, \nu\Theta_2)}.$$

Thus we obtain the estimate $\mu_s \leqslant \Theta$ for a certain $\Theta \in \mathbb{R}_+$ from Lemma 5.7. Its generalization (16) follows from Lemma 5.4 (see also the corresponding remark). $\square$

6. Existence and convergence results

The estimates from the previous section will be used in the analysis of convergence of Rothe sequences and the properties of their weak limits.

Lemma 6.1. *Let X, Y be two Hilbert spaces such that $X = Y$ or such that the imbedding of X into Y is compact. Let us consider Rothe sequences in the sense of definitions from the beginning of the previous section. Let us suppose that*

$$\{u^m\}_{m=1}^\infty \text{ is bounded in } C(I, X),$$
$$\{\dot{u}^m\}_{m=1}^\infty \text{ is equibounded as a sequence of mappings of } I \text{ into } Y.$$

Then there exists some $u \in C_L(I, Y) \cap L_\infty(I, X)$ with $\dot{u} \in L_\infty(I, Y)$ such that up to subsequences

$$\{u^m(t)\}_{m=1}^\infty, \{\bar{u}^m(t)\}_{m=1}^\infty, \{\breve{u}^m(t)\}_{m=1}^\infty \text{ have a weak limit } u(t) \text{ in } X \text{ for every } t \in I,$$
$$\{\dot{u}^m\}_{m=1}^\infty \text{ has a weak limit } \dot{u} \text{ in } L_2(I, Y).$$

In particular,

$$\{u^m\}_{m=1}^\infty \text{ has a strong limit } \dot{u} \text{ in } C(I, Y)$$

if $X \neq Y$.

P r o o f. Let us notice that directly from the definitions of Rothe sequences the inequality

$$\max_{t\in I} |\bar{u}^m(t)|_X \leqslant \max_{t\in I} |u^m(t)|_X$$

(alternatively $\bar{u}$ can be replaced by $\breve{u}$ everywhere in this proof) follows for any $m \in \{1, 2, \dots\}$. Then by the Eberlein-Shmul'yan theorem (see [Yo], p. 201) applied separately to each fixed $t \in I$, there exist such $\widehat{u}(t), \bar{u}(t) \in X$ that up to subsequences (whose choice need not be the same at different times t)

$$\{u^m(t)\}_{m=1}^\infty \text{ has a weak limit } \widehat{u}(t) \text{ in } X \text{ for every } t \in I,$$
$$\{\bar{u}^m(t)\}_{m=1}^\infty \text{ has a weak limit } \bar{u}(t) \text{ in } X \text{ for every } t \in I.$$

If $X \neq Y$ then the compactness of the imbedding of X into Y implies up to subsequences again

$$\{u^m(t)\}_{m=1}^\infty \text{ has a strong limit } \widehat{u}(t) \text{ in } Y \text{ for every } t \in I,$$
$$\{\bar{u}^m(t)\}_{m=1}^\infty \text{ has a strong limit } \bar{u}(t) \text{ in } Y \text{ for every } t \in I.$$

In general,

$$\{u^m(t)\}_{m=1}^\infty \text{ has a weak limit } \widehat{u}(t) \text{ in } Y \text{ for every } t \in I,$$
$$\{\bar{u}^m(t)\}_{m=1}^\infty \text{ has a weak limit } \bar{u}(t) \text{ in } Y \text{ for every } t in\ I,$$

which together with the consequence of (7)

$$|\widehat{u}(t) - \bar{u}(t)|_Y^2 = \lim_{m\to\infty} (\widehat{u}(t) - \bar{u}(t), u^m(t) - \bar{u}^m(t))_Y = 0$$

proves $\widehat{u} = \bar{u}$ on I. To derive better convergence, we use two modifications of the Arzelà-Ascoli theorem from [Ka], p. 24. (Complete proof of the Arzelà-Ascoli theorem can be found in [Yo], p. 125, and [Ku], pp. 35 and 42.) From the estimate

$$|u^m(t) - u^m(t')|_Y \leqslant |t - t'| \max_{t''\in I} |\dot{u}^m(t'')|_Y$$

for all $t, t' \in I$ we see that

$$\{u^m\}_{m=1}^\infty \text{ is equicontinuous in } C(I, Y).$$

If $X = Y$ then by the first modification of the Arzelà-Ascoli theorem there exists $u \in L_\infty(I, X)$ such that up to a subsequence

$$\{u^m(t)\}_{m=1}^\infty \text{ has a weak limit } u(t) \text{ in } X \text{ for every } t \in I;$$

else by the second modification of the same theorem there exists $u \in C(I, Y) \cap L_\infty(I, X)$ such that up to a subsequence

$$\{u^m(t)\}_{m=1}^\infty \text{ has a weak limit } u(t) \text{ in } X \text{ for a.e. } t \in I,$$
$$\{u^m\}_{m=1}^\infty \text{has a strong limit } u(t) \text{ in } C(I, Y).$$

In both cases $u = \widehat{u}$ a.e. on I. Let us notice that $L_2(I, Y)$ is a Hilbert space, too (see [Ga], p. 135). Therefore by the Riesz representation theorem there exists $u' \in L_2(I, Y)$ such that up to a subsequence

$$\{\dot{u}^m\}_{m=1}^\infty \text{ has a weak limit } u' \text{ in } L_2(I, Y).$$

Using the basic properties of the Bochner integral (see [Ga], p. 126) we obtain

$$\begin{aligned}&\left|u(t)-u_0-\int_0^t u'(t')\,\mathrm{d}t'\right|_Y^2\\&=\lim_{m\to\infty}\left(u(t)-u_0-\int_0^t u'(t')\,\mathrm{d}t',u^m(t)-u_0-\int_0^t \dot u^m(t')\,\mathrm{d}t'\right)_Y=0\end{aligned}$$

for a. e. $t\in I$; this shows $\dot u=u'$ a. e. on I. Finally, for all $t,t'\in I$ we have

$$\begin{aligned}|u(t)-u(t')|_Y^2&=\lim_{m\to\infty}(u(t)-u(t'),u^m(t)-u^m(t'))_Y\\&=\lim_{m\to\infty}\left(u(t)-u(t'),\int_{t'}^t \dot u^m(t'')\,\mathrm{d}t''\right)_Y\\&\leqslant |u(t)-u(t')|_Y\,|t-t'|\lim_{m\to\infty}\max_{t''\in I}|\dot u^m(t'')|_Y\\&\leqslant |t-t'|^2\left(\lim_{m\to\infty}\max_{t''\in I}|\dot u^m(t'')|_Y\right)^2,\end{aligned}$$

which yields the Lipschitz continuity of $u\in C(I,Y)$. Since $u,\dot u\in L_2(I,Y)$, u must be strongly differentiable for a. e. $t\in I$ (see [Ka], p. 24), which implies

$$|\dot u(t)|_Y=\left|\lim_{t'\to t}\frac{u(t)-u(t')}{t-t'}\right|_Y\leqslant\lim_{t'\to t}\frac{|u(t)-u(t')|_Y}{|t-t'|}\leqslant\lim_{m\to\infty}\max_{t''\in I}|\dot u^m(t'')|_Y\,;$$

hence $\dot u\in L_\infty(I,Y)$. □

R e m a r k. Most arguments of the preceding proof (namely for $X\neq Y$) are taken from [Ka], p. 25. One can see that X and Y may be reflexive Banach (not only Hilbert) spaces; this generalization is left to the reader because it is not needed in our analysis.

P r o o f o f T h e o r e m 4.2. From Theorem 5.3 we obtain

$\{v^m\}_{m=1}^\infty$ is bounded in $C(I,V)$,
$\{\dot v^m\}_{m=1}^\infty$ is equibounded as a sequence of mappings of I into H,
$\{\sigma^m\}_{m=1}^\infty$ is bounded in $C(I,S)$,
$\{\dot\sigma^m\}_{m=1}^\infty$ is equibounded as a sequence of mappings of I into S.

The property (i) (see Section 2) implies the compactness of the imbedding of V into H. Therefore Lemma 6.1 for $X=Y=S$ and for $X=V$ and $Y=H$ guarantees the existence of $v \in C_L(I,H) \cap L_\infty(I,V)$ and $\sigma \in C_L(I,S)$ such that $\dot v \in L_\infty(I,H)$ and $\dot\sigma \in L_\infty(I,S)$ and up to subsequences

$$\begin{aligned}\{v^m(t)\}_{m=1}^\infty, \{\bar v^m(t)\}_{m=1}^\infty, \{\breve v^m(t)\}_{m=1}^\infty &\text{ have a common weak limit } v(t) \text{ in } V,\\ \{\sigma^m(t)\}_{m=1}^\infty, \{\bar\sigma^m(t)\}_{m=1}^\infty, \{\breve\sigma^m(t)\}_{m=1}^\infty &\text{ have a common weak limit } \sigma(t) \text{ in } S\end{aligned}$$

for any $t \in I$ and

$$\begin{aligned}\{\dot v^m\}_{m=1}^\infty &\text{ has a weak limit } \dot v \text{ in } L_2(I,H),\\ \{\dot\sigma^m\}_{m=1}^\infty &\text{ has a weak limit } \dot\sigma \text{ in } L_2(I,S),\\ \{v^m\}_{m=1}^\infty &\text{ has a strong limit } v \text{ in } C(I,H).\end{aligned}$$

Let us choose an arbitrary $t \in I$. We can write e.g. for every $k \in \{1, \dots, M\}$

$$\{\breve\sigma^m(t)\colon\, b^k\}_{m=1}^\infty \text{ has a weak limit } \sigma(t)\colon\, b^k \text{ in } S_1$$

and up to a subsequence

$$\{f(\breve\sigma^m(t)\colon\, b^k)\}_{m=1}^\infty \text{ has a strong limit } f(\sigma(t) : b^k) \text{ in } Q$$

due to the compactness of the mapping f. By virtue of the property (ii) clearly

$$\{\mathrm{D}\, \bar v^m(t).\, a^k\}_{m=1}^\infty \text{ has a weak limit } \mathrm{D}\, v(t).\, a^k \text{ in } Q,$$

but using the definition

$$Q^* = \prod_{j=1}^N L_2(\Psi_j, \mathbb{R}^3)$$

simultaneously the property (ii) guarantees the existence of such $v_*(t) \in Q^*$ that up to a subsequence

$$\{\bar v^m(t)\}_{m=1}^\infty \text{ has a strong limit } v_*(t) \text{ in } Q^*$$

and consequently

$$\{\mathrm{D}\, \bar v^m(t).\, a^k\}_{m=1}^\infty \text{ has a strong limit } \mathrm{D}\, v_*(t).\, a^k \text{ in } Q;$$

this proves $v(t) = v_*(t)$. Repeating this consideration with $\bar v^m$ substituted by $\breve v^m$ we obtain the same result. The continuity properties of G yield

$$\begin{gathered}\big\{G(f(\breve\sigma^m(t) : b^k), \mathrm{D}\, \breve v^m(t).\, a^k)\big\}_{m=1}^\infty \text{ has a strong limit}\\ G(f(\sigma(t) : b^k), \mathrm{D}\, v(t).\, a^k) \text{ in } Q\end{gathered}$$

and therefore

$$\{\beta(\breve{\varrho}^{km}(t))\}_{m=1}^{\infty} \text{ has a strong limit } \beta(\varrho^k(t)) \text{ in } T.$$

Comparing (12) and (13) with (2) and (5) and taking into consideration the above convergence results together with (8) for approximations of a priori known abstract functions we conclude that our v and σ (generated by an artificial time-discretized approximation scheme) satisfy (2) and (5) with the initial conditions (3) and (6) if

$$\{F(\bar{\sigma}^m)\}_{m=1}^{\infty} \text{ has a weak limit } F(\sigma) \text{ in } L_2(I, S)$$

(which is not trivial because F is allowed not to be weakly continuous). As a consequence of (7) and the Lipschitz continuity of F the last assumption can be reformulated in the form

$$\{F(\sigma^m)\}_{m=1}^{\infty} \text{ has a weak limit } F(\sigma) \text{ in } L_2(I, S),$$

too. We shall verify that this is satisfied. Clearly it is sufficient to prove

$$\{\sigma^m\}_{m=1}^{\infty} \text{ has a strong limit } \sigma \text{ in } C(I, S).$$

Let us introduce the notation

$$u^{ml} = u^m - u^l$$

for arbitrary elements u^m, u^l with $m, l \in \{1, 2, \ldots\}$ of some sequences in the corresponding Hilbert spaces. The difference between two equations (12) with the test function $v = \bar{v}^{ml}$ is

$$\begin{aligned}
&\left(\bar{v}^{ml}, \varrho\dot{v}^{ml}\right) + \left[\varepsilon(\bar{v}^{ml}), \bar{\sigma}^{ml}\right] + \left\lfloor \mathrm{D}\,\bar{v}^{ml}.\, a^k, \left(\beta(\breve{\varrho}^{km})\,\mathrm{D}\,\bar{v}^m - \beta(\breve{\varrho}^{kl})\,\mathrm{D}\,\bar{v}^l\right).\, a^k \right\rfloor \\
&= \left(\bar{v}^{ml}, \widehat{\varphi}^{ml}\right) + \left\langle \bar{v}^{ml}, \widehat{\gamma}^{ml} \right\rangle + \left\lfloor \mathrm{D}\,\bar{v}^{ml}.\, a^k, \widehat{\eta}^{kml} \right\rfloor
\end{aligned}$$

and the difference between two equations (13) similarly

$$\varepsilon(\bar{v}^{ml}) = A\dot{\sigma}^{ml} + F(\bar{\sigma}^m) - F(\bar{\sigma}^l) - \widehat{\vartheta}^{ml},$$

which together yields

$$\begin{aligned}
&\left(v^{ml}, \varrho\dot{v}^{ml}\right) + \left(\bar{v}^{ml} - v^{ml}, \varrho\dot{v}^{ml}\right) \\
&\quad + \left[\sigma^{ml}, A\dot{\sigma}^{ml}\right] + \left[\bar{\sigma}^{ml} - \sigma^{ml}, A\dot{\sigma}^{ml}\right] + \left[\bar{\sigma}^{ml}, F(\bar{\sigma}^m) - F(\bar{\sigma}^l)\right] \\
&\quad + \left\lfloor \mathrm{D}\,\bar{v}^{ml}.\, a^k, \beta(\breve{\varrho}^{kl})\,\mathrm{D}\,\bar{v}^{ml} \right\rfloor + \left\lfloor \mathrm{D}\,\bar{v}^{ml}.\, a^k, \left(\beta(\breve{\varrho}^{km}) - \beta(\breve{\varrho}^{kl})\right)\mathrm{D}\,\bar{v}^m \right\rfloor \\
&= \left(\bar{v}^{ml}, \widehat{\varphi}^{ml}\right) + \left\langle \bar{v}^{ml}, \widehat{\gamma}^{ml} \right\rangle + \left\lfloor \mathrm{D}\,\bar{v}^{ml}.\, a^k, \widehat{\eta}^{kml} \right\rfloor + \left[\bar{\sigma}^{ml}, \widehat{\vartheta}^{ml}\right].
\end{aligned}$$

Let us integrate this equation in time. Passing to the limit with $m, l \to \infty$ the right-hand side vanishes completely as a consequence of the convergence properties (8) and of the boundedness (16) from Theorem 5.3. Moreover, by (7) for an arbitrary $t \in I$ we have

$$\begin{aligned} \lim_{m,l\to\infty} \left|\bar{v}^{ml} - v^{ml}\right|_H &\leqslant \lim_{m\to\infty} \left|\bar{v}^{m} - v^{m}\right|_H + \lim_{l\to\infty} \left|\bar{v}^{l} - v^{l}\right|_H = 0, \\ \lim_{m,l\to\infty} \left|\bar{\sigma}^{ml} - \sigma^{ml}\right|_S &\leqslant \lim_{m\to\infty} \left|\bar{\sigma}^{m} - \sigma^{m}\right|_S + \lim_{l\to\infty} \left|\bar{\sigma}^{l} - \sigma^{l}\right|_S = 0; \end{aligned} \tag{29}$$

this permits to ignore the second and the fourth additive terms on the left-hand side. The sixth term is always non-negative and the seventh can be omitted by virtue of our strong convergence result

$$\lim_{m,l\to\infty} \left|\beta(\breve{\varrho}^{km}) - \beta(\breve{\varrho}^{kl})\right|_T = 0$$

for any $k \in \{1, \ldots, M\}$. It remains only

$$\begin{aligned} &\frac{1}{2} \lim_{m,l\to\infty} \left(\left(v^{ml}, \varrho v^{ml}\right) + \left[\sigma^{ml}, A\sigma^{ml}\right]\right) \\ &\leqslant \lim_{m,l\to\infty} \int_0^t \left[\bar{\sigma}^{ml}(t'), F(\bar{\sigma}^{l}(t')) - F(\bar{\sigma}^{m}(t'))\right] \,\mathrm{d}t', \end{aligned}$$

which implies

$$\lim_{m,l\to\infty} \left|\sigma^{ml}\right|_S^2 \leqslant \Theta_* \int_0^t \lim_{m,l\to\infty} \left|\bar{\sigma}^{ml}(t')\right|_S^2 \,\mathrm{d}t'$$

for a certain $\Theta_* \in \mathbb{R}_+$ (in more detail: the first positive additive term on the left-hand side can be left out, the total right-hand side can be estimated using the Lipschitz continuity of F). Let us notice that here $\bar{\sigma}^m$ can be substituted by σ^m with no loss of generality (see (29) again). Then the standard "continuous" Gronwall lemma (see [Gr], p. 52, and [Ka], p. 28) gives

$$\lim_{m,l\to\infty} \left|\sigma^{ml}\right|_S^2 = 0,$$

which by virtue of the completeness of S (every Cauchy sequence in S must have a strong limit which cannot differ from the weak one) yields the required convergence property. $\square$

Theorem 4.2 gives no answer to the question how many solutions Problem 4.1 may have. The source of crucial difficulties is a rather wide definition of f and G; this we will see in Lemma 6.2. Under a certain additional growth assumption on these mappings Theorem 6.3 guarantees that Problem 4.1 has a unique solution. (We need no new growth assumption on F here; the Lipschitz continuity from the analysis of equations of type (B) will be sufficient.)

In the following proofs let (v, σ) and $(\bar{v}, \bar{\sigma})$ be two solutions of Problem 4.1. All needed quantities "with bars" can be calculated using the same formulae as for the original ones—e.g.

$$\bar{\varrho}^k(t) = \varrho_0^k + \int_0^t G(f(\bar{\sigma}(t') : b^k), \mathrm{D}\, \bar{v}(t').\, a^k)$$

for any $k \in \{1, \ldots, M\}$. Let us denote $\Delta v = v - \bar{v}$ and $\Delta\sigma = \sigma - \bar{\sigma}$.

Lemma 6.2. *The couples (v, σ), $(\bar{v}, \bar{\sigma})$ satisfy the equation*

$$\begin{aligned}
(30) \qquad & \frac{1}{2}(\Delta v, \varrho\Delta v) + \frac{1}{2}[\Delta\sigma, A\Delta\sigma] + \int_0^t [\Delta\sigma(t'), F(\sigma(t')) - F(\bar{\sigma}(t'))] \,\mathrm{d}t' \\
& + \int_0^t \lfloor \mathrm{D}\,\Delta v(t').\, a^k, \beta(\bar{\varrho}^k(t'))\,\mathrm{D}\,\Delta v(t').\, a^k \rfloor \,\mathrm{d}t' \\
& = \int_0^t \lfloor \mathrm{D}\,\Delta v(t').\, a^k, \big(\beta(\bar{\varrho}^k(t')) - \beta(\varrho^k(t'))\big)\,\mathrm{D}\, v(t').\, a^k \rfloor \,\mathrm{d}t'
\end{aligned}$$

at every time $t \in I$.

Proof. Let us choose a certain time $t \in I$. Subtracting the equation (2) with quantities "with bars" from the original one for $\widetilde{v} = \Delta v$ we obtain

$$(\Delta v, \varrho\Delta\dot{v}) + [\varepsilon(\Delta v), \Delta\sigma] + \lfloor \mathrm{D}\,\Delta v.\, a^k, \beta(\varrho^k)\,\mathrm{D}\, v.\, a^k - \beta(\bar{\varrho}^k)\,\mathrm{D}\,\bar{v}.\, a^k \rfloor = 0.$$

In a similar way the difference between two equations (5) gives

$$\varepsilon(\Delta v) = A\Delta\dot{\sigma} + F(\sigma) - F(\bar{\sigma});$$

this can be easily substituted into the preceding equation. The same is valid for any time t' belonging to the closed interval from 0 to t. Then the time integration from 0 to t yields (30). $\square$

Theorem 6.3. *Let $\varpi \in \mathbb{R}_+$ be such that the growth condition*

$$(31) \quad \forall \widetilde{\varsigma}, \widehat{\varsigma} \in S_1 \, \forall \widetilde{q}, \widehat{q} \in Q \quad | \quad |G(f(\widetilde{\varsigma}), \widetilde{q}) - G(f(\widehat{\varsigma}), \widehat{q})|_Q \leqslant \varpi \left(|\widetilde{\varsigma} - \widehat{\varsigma}|_{S_1} + |\widetilde{q} - \widehat{q}|_Q \right)$$

is satisfied. Then Problem 4.1 has a unique solution.

P r o o f. Let us consider an arbitrary time $t \in I$. We shall start from the estimate

$$\begin{aligned}
&\lfloor \mathrm{D}\,\Delta v.\, a^k, \big(\beta(\bar\varrho^k) - \beta(\varrho^k)\big)\,\mathrm{D}\, v.\, a^k \rfloor \\
&\leqslant \lfloor \mathrm{D}\,\Delta v.\, a^k, \beta(\bar\varrho^k)\,\mathrm{D}\,\Delta v.\, a^k \rfloor + \frac{1}{4}\left\lfloor \mathrm{D}\, v.\, a^k, \frac{|\beta(\bar\varrho^k) - \beta(\varrho^k)|}{\beta(\bar\varrho^k)}\,\mathrm{D}\, v.\, a^k \right\rfloor \\
&\leqslant \lfloor \mathrm{D}\,\Delta v.\, a^k, \beta(\bar\varrho^k)\,\mathrm{D}\,\Delta v.\, a^k \rfloor + \frac{1}{4g_1^2}\sum_{k=1}^{M} \big|\mathrm{D}\, v.\, a^k\big|_Q^2 \cdot \\
&\cdot \kappa^2 \left| \int_0^t \Big(G\big(f(\sigma(t').\, b^k), \mathrm{D}\, v(t').\, a^k\big) - G\big(f(\bar\sigma(t').\, b^k), \mathrm{D}\,\bar v(t').\, a^k\big)\Big)\,\mathrm{d}t' \right|_Q^2 \\
&\leqslant \lfloor \mathrm{D}\,\Delta v.\, a^k, \beta(\bar\varrho^k)\,\mathrm{D}\,\Delta v.\, a^k \rfloor + \frac{1}{4g_1^2}\sum_{k=1}^{M} \big|\mathrm{D}\, v.\, a^k\big|_Q^2 \cdot \\
&\cdot \kappa^2 \tau \int_0^t \Big|\Big(G\big(f(\sigma(t').\, b^k), \mathrm{D}\, v(t').\, a^k\big) - G\big(f(\bar\sigma(t').\, b^k), \mathrm{D}\,\bar v(t').\, a^k\big)\Big)\Big|_Q^2\,\mathrm{d}t'
\end{aligned}$$

which can be (with respect to the boundedness of v in V and the "trace property" (i)) converted using the growth condition (31) into the form

$$\begin{aligned}
&\lfloor \mathrm{D}\,\Delta v.\, a^k, \big(\beta(\bar\varrho^k) - \beta(\varrho^k)\big)\,\mathrm{D}\, v.\, a^k \rfloor \\
&\leqslant \kappa_\star \Big(\int_0^t \big|\Delta\sigma(t') : b^k\big|_Q^2\,\mathrm{d}t' + \int_0^t \big|\mathrm{D}\,\Delta v(t').\, a^k\big|_Q^2\,\mathrm{d}t' \Big)
\end{aligned}$$

for a certain fixed $\kappa_\star \in \mathbb{R}_+$. An immediate consequence of the Lipschitz continuity of F is

$$-\int_0^t [\Delta\sigma(t'), F(\sigma(t')) - F(\bar\sigma(t'))]\,\mathrm{d}t' \leqslant \zeta \int_0^t |\Delta\sigma(t')|_S\,\mathrm{d}t'.$$

From Lemma 6.2 we obtain a constant $\Theta_\star \in \mathbb{R}_+$ such that

$$|\Delta v|_H^2 + \Phi(\Delta\sigma) \leqslant \Theta_\star \int_0^t \Big(|\Delta v(t')|_H^2 + \Phi(\Delta\sigma(t'))\Big)\,\mathrm{d}t'$$

where

$$\Phi(\Delta\sigma) = |\Delta\sigma|_S^2 + \int_0^t |\Delta\sigma(t')|_S^2\,\mathrm{d}t'.$$

Then the standard Gronwall lemma (the same as in the proof of Theorem 4.2 in this section) gives

$$|\Delta v|_H^2 + \Phi(\Delta\sigma) = 0,$$

which implies $v = \bar v$ in V and $\sigma = \bar\sigma$ in S. □

We can summarize: we have verified solvability of the problem of time evolution of strain and stress distributions in superalloy composites under some physically motivated assumptions and simplifications. This conclusion corresponds to the "reasonable" numerical results obtained for special material structures from CDS software; some of them can be found in [Sv¹]. Nevertheless, several improvements in this approach, both in the geometrical description of structured deformation and in the analysis of significant physical processes controlling the strain and stress evolution, are needed; possible perspectives of this research have been discussed in [Va²].

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