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SOLUTION OF THE ROBIN PROBLEM FOR THE LAPLACE EQUATION

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Abstract. For open sets with a piecewise smooth boundary it is shown that we can express a solution of the Robin problem for the Laplace equation in the form of a single layer potential of a signed measure which is given by a concrete series.

Keywords: Laplace equation, Robin problem, single layer potential

MSC 2000: 31B10, 35J05, 35J25

Suppose that $G \subset \mathbb{R}^m$ $(m > 2)$ is an open set with a non-void compact boundary ∂G. Fix a nonnegative element λ of $\mathscr{C}'(\partial G)$ (= the Banach space of all finite signed Borel measures with support in ∂G with the total variation as a norm) and suppose that the single layer potential $\mathscr{U}\lambda$ is bounded and continuous on ∂G. Here

$$\mathscr{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y)\,\mathrm{d}\nu(y),$$

where $\nu \in \mathscr{C}'(\partial G)$,

$$h_x(y) = (m-2)^{-1}A^{-1}|x-y|^{2-m},$$

A is the area of the unit sphere in $\mathbb{R}^m$. It was shown in [24] that $\mathscr{U}\lambda$ is bounded and continuous on ∂G if and only if

$$\lim_{r\to 0+} \sup_{y\in\partial G} \int_{\mathscr{U}(y;r)} h_y(x)\,\mathrm{d}\lambda(x) = 0,$$

where $\mathscr{U}(x;r) = \{y \in \mathbb{R}^m;\ |y-x| < r\}$. According to [14], Lemma 2.18 this is true if there are constants $\alpha > m-2$ and $k > 0$ such that $\lambda(\mathscr{U}(x;r)) \leqslant kr^{\alpha}$ for all $x \in \mathbb{R}^m$ and all $r > 0$.

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If G has a smooth boundary, $u \in \mathscr{C}^1(\operatorname{cl} G)$ is a harmonic function on G and

$$\frac{\partial u}{\partial n} + fu = g \text{ on } \partial G$$

where $f, g \in \mathscr{C}(\partial G)$ (= the space of all bounded continuous functions on ∂G equipped with the maximum norm) and n is the exterior unit normal of G, then for $\varphi \in \mathscr{D}$ (= the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$) we have

$$(1) \qquad \int_{\partial G} \varphi g \, \mathrm{d}\mathscr{H}_{m-1} = \int_G \nabla \varphi \cdot \nabla u \, \mathrm{d}\mathscr{H}_m + \int_{\partial G} \varphi f u \, \mathrm{d}\mathscr{H}_{m-1}.$$

Here $\mathscr{H}_k$ is the k-dimensional Hausdorff measure normalized such that $\mathscr{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$. If we denote by $\mathscr{H}$ the restriction of $\mathscr{H}_{m-1}$ onto ∂G and by $N^G u$ the distribution

$$(2) \qquad \langle \varphi, N^G u \rangle = \int_G \nabla \varphi \cdot \nabla u \, \mathrm{d}\mathscr{H}_m$$

then (1) has the form

$$(3) \qquad N^G u + f u \mathscr{H} = g \mathscr{H}.$$

Here $N^G u$ is a characterization in the sense of distributions of the normal derivative of u.

The formula (3) motivates our definition of the solution of the Robin problem for the Laplace equation

$$(4) \qquad \begin{aligned} \Delta u &= 0 \text{ in } G, \\ N^G u + u\lambda &= \mu, \end{aligned}$$

where $\mu \in \mathscr{C}'(\partial G)$ (compare [14], [23]). From now on $G \subset \mathbb{R}^m$ is a general open set with a non-void compact boundary ∂G.

We introduce in $\mathbb{R}^m$ the fine topology, i.e. the weakest topology in which all superharmonic functions in $\mathbb{R}^m$ are continuous (see [3]). This topology is stronger than ordinary topology. Since the set of fine isolated points of $\operatorname{cl} G$ is polar (see [3], Chapter VII, §6, §4) and λ does not charge polar sets ([17], Chapter II, §1 and p. 222) λ-a.a. points x of $\operatorname{cl} G$ are in the fine closure of $\operatorname{cl} G \setminus \{x\}$.

If u is a harmonic function on G such that

$$(5) \qquad \int_H |\nabla u| \, \mathrm{d}\mathscr{H}_m < \infty$$

for all bounded open subsets H of G we define the weak normal derivative $N^G u$ of u as a distribution

$$\langle \varphi, N^G u\rangle = \int_G \nabla\varphi \cdot \nabla u \, \mathrm{d}\mathscr{H}_m$$

for $\varphi \in \mathscr{D}$.

Let $\mu \in \mathscr{C}'(\partial G)$. Now we formulate the Robin problem for the Laplace equation (4) as follows: Find a function $u \in L^1(\lambda)$ on $\operatorname{cl} G$, the closure of G, harmonic on G and fine continuous in λ-a.a. points of ∂G for which $|\nabla u|$ is integrable over all bounded open subsets of G and $N^G u + u\lambda = \mu$.

As in [25] we will look for a solution of the Robin problem in the form of the single layer potential $\mathscr{U}\nu$, where $\nu \in \mathscr{C}'(\partial G)$. We will prove that if G has a smooth boundary or $m = 3$ and G has a piecewise-smooth boundary then there is a solution of the Robin problem with the boundary condition μ if and only if $\mu(\partial H) = 0$ for all bounded components H of $\operatorname{cl} G$ for which $\lambda(\partial H) = 0$. In this case we can express the solution in the form of the single layer potential $\mathscr{U}\nu$ where ν is given by a concrete series.

Notation. $\mathscr{C}'_c(\partial G)$ will stand for the subspace of those $\mu \in \mathscr{C}'(\partial G)$ for which there exists a continuous function $\mathscr{U}_c\mu$ on $\mathbb{R}^m$ coinciding with $\mathscr{U}\mu$ on $\mathbb{R}^m \setminus \partial G$. It was shown in [27] that if $\nu \in \mathscr{C}'(\partial G)$ and the restriction of $\mathscr{U}\nu$ onto ∂G is finite and continuous then $\mathscr{U}\nu$ is finite and continuous in $\mathbb{R}^m$ and $\nu \in \mathscr{C}'_c(\partial G)$. For example $\lambda \in \mathscr{C}'_c(\partial G)$.

Lemma 1. *Let $\nu \in \mathscr{C}'(\partial G)$, $\mu \in \mathscr{C}'_c(\partial G)$. Suppose that $\mu = \lambda$ or $\mathscr{H}_m(\partial G) = 0$. Then $\mathscr{U}\nu$ is harmonic on G, finite and fine continuous at $|\mu|$-a.a. points of ∂G, $\mathscr{U}\nu \in L^1(\lambda)$ and $|\nabla u|$ is integrable over all bounded open subsets of G. Here $|\mu| = \mu^+ + \mu^-$, where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ. If*

$$c_\lambda = \sup_{x\in\partial G} \mathscr{U}\lambda(x) \tag{6}$$

then

$$\int_{\partial G} |\mathscr{U}\nu| \, \mathrm{d}\lambda \leqslant c_\lambda \|\nu\|, \tag{7}$$

where $\|\nu\|$ is the total variation of ν. If $\nu \in \mathscr{C}'_c(\partial G)$ then $\mathscr{U}_c\nu = \mathscr{U}\nu$ at $|\mu|$-a.a. points.

Proof. $\mathscr{U}\nu$ is a harmonic function on G such that (5) holds for $u = \mathscr{U}\nu$ and all bounded open subsets H of G (see [14], Remark on p. 9). Because $\mathscr{U}\nu^+$, $\mathscr{U}\nu^-$ are superharmonic functions they are continuous with respect to the fine topology. Put $M = \{x \in \partial G;\ \mathscr{U}|\nu|(x) = \infty\}$. Then $\mathscr{U}\nu$ is finite and continuous with respect to the fine topology on $\operatorname{cl} G \setminus M$. Moreover, if $\nu \in \mathscr{C}'_c(\partial G)$ then $\mathscr{U}_c \nu = \mathscr{U}\nu$ on $\operatorname{cl} G \setminus M$. Since M is polar its Newtonian capacity is null (see [17], Chapter III, §1 and p. 222). Since μ has a finite energy by [21], Lemma 6 and [17], Theorem 1.20 the measure $|\mu|$ has a finite energy as well and $|\mu|(M) = 0$ by [17], Theorem 2.1

$$\int_{\partial G} |\mathscr{U}\nu| \,\mathrm{d}\lambda \leqslant \int_{\partial G} \mathscr{U}|\nu| \,\mathrm{d}\lambda = \int_{\partial G} \mathscr{U}\lambda \; d|\nu| \leqslant c_\lambda \|\nu\|.$$

□

R e m a r k 1. Let $\nu \in \mathscr{C}'(\partial G)$. We have seen that for λ-a.a. points $x \in \partial G$ we have $\mathscr{U}|\nu|(x) < \infty$. Fix such a point. Fix $\alpha > 1$ and denote $P_\alpha(x) = \{z \in G;\ |z - x| \leqslant \alpha \operatorname{dist}(z, \partial G)\}$, where $\operatorname{dist}(z, \partial G) = \inf\{|z - y|;\ y \in \partial G\}$. Suppose that $x \in \operatorname{cl} P_\alpha(x)$. Then

(8) $$\lim_{z \in P_\alpha(x), z \to x} \mathscr{U}\nu(z) = \mathscr{U}\nu(x).$$

P r o o f. Fix $\varepsilon > 0$. Since $\mathscr{U}|\nu|(x) < \infty$ there is $r > 0$ such that

$$\int_{\partial G \cap \mathscr{U}(x;r)} h_x(y) \,\mathrm{d}|\nu| < \frac{\varepsilon}{4}(\alpha + 1)^{2-m}.$$

Since

$$|y - x| \leqslant |y - z| + |x - z| \leqslant (\alpha + 1)|y - z|$$

for $z \in P_\alpha(x), y \in \partial G$, we have

$$\int_{\partial G \cap \mathscr{U}(x;r)} h_z(y) \,\mathrm{d}|\nu| \leqslant (\alpha + 1)^{m-2} \int_{\partial G \cap \mathscr{U}(x;r)} h_x(y) \,\mathrm{d}|\nu| < \frac{\varepsilon}{4}.$$

Since

$$z \mapsto \int_{\partial G \setminus \mathscr{U}(x;r)} h_z(y) \,\mathrm{d}\nu$$

is a continuous function in x there is $\delta > 0$ such that for $z \in \mathscr{U}(x;\delta)$ we have

$$\left| \int_{\partial G \setminus \mathscr{U}(x;r)} h_z(y) \,\mathrm{d}\nu - \int_{\partial G \setminus \mathscr{U}(x;r)} h_x(y) \,\mathrm{d}\nu \right| < \frac{\varepsilon}{2}$$

and thus for $z \in \mathscr{U}(x;\delta) \cap P_\alpha(x)$ we have $|\mathscr{U}\nu(x) - \mathscr{U}\nu(z)| < \varepsilon$. □

R e m a r k 2. If ∂G is a finite set then $\lambda = 0$. Suppose now that ∂G is an infinite set. Choose a simple sequence $\{x_n\} \subset \partial G$ such that x_n converges to x_0 as $n \to \infty$. Choose a sequence $\{a_n\}$ of positive numbers such that

$$\sum_{n=1}^{\infty} a_n |x_0 - x_n|^{2-m} < \infty.$$

If we put

$$\nu(M) = \sum_{x_n \in M} a_n$$

then $\mathscr{U}\nu(x_0) < \infty$ but $\mathscr{U}\nu(x_n) = \infty$ for all integer numbers n. Using the lower-semicontinuity of $\mathscr{U}\nu$ we obtain that

$$\mathscr{U}\nu(x_0) < \limsup_{x \in G, x \to x_0} \mathscr{U}\nu(x) = \infty$$

in spite of $\mathscr{U}\nu(x_0)$ being finite.

R e m a r k 3. It was shown in [14] that $N^G \mathscr{U}\nu \in \mathscr{C}'(\partial G)$ for each $\nu \in \mathscr{C}'(\partial G)$ if and only if $V^G < \infty$, where

$$V^G = \sup_{x \in \partial G} v^G(x),$$

$$v^G(x) = \sup \left\{ \int_G \nabla \varphi \cdot \nabla h_x \, \mathrm{d}\mathscr{H}_m ;\ \varphi \in \mathscr{D}, |\varphi| \leqslant 1, \operatorname{spt} \varphi \subset \mathbb{R}^m \setminus \{x\} \right\} \text{ for } x \in \mathbb{R}^m.$$

There are more geometrical characterizations of $v^G(x)$ which ensure $V^G < \infty$ for G convex or for G with $\partial G \subset \bigcup_{i=1}^{k} L_i$, where L_i are $(m-1)$-dimensional Ljapunov surfaces (i.e. of class $\mathscr{C}^{1+\alpha}$). Denote by

$$\partial_e G = \{x \in \mathbb{R}^m ;\ \bar{d}_G(x) > 0, \bar{d}_{\mathbb{R}^m \setminus G}(x) > 0\}$$

the essential boundary of G where

$$\bar{d}_M(x) = \limsup_{r \to 0_+} \frac{\mathscr{H}_m(M \cap \mathscr{U}(x;r))}{\mathscr{H}_m(\mathscr{U}(x;r))}$$

is the upper density of M at x. Then

$$v^G(x) = \frac{1}{A} \int_{\partial \mathscr{U}(0;1)} n(\theta, x) \, \mathrm{d}\mathscr{H}_{m-1}(\theta),$$

where $n(\theta, x)$ is the number of all points of $\partial_e G \cap \{x + t\theta;\ t > 0\}$ (see [5]). This expression is a modification of a similar expression in [14]. As a consequence we see that $V^G \leqslant \frac{1}{2}$ if G is convex. Since $v^G(x) \leqslant V^G + \frac{1}{2}$ by [14],Theorem 2.16, we see that if

$$\partial G \subset \bigcup_{i=1}^{n} \partial G_i$$

and $G_1, \dots, G_n$ are convex then $V^G \leqslant n$.

Let us recall another characterization of $v^G(x)$ using the notion of an interior normal in Federer's sense. If $z \in \mathbb{R}^m$ and θ is a unit vector such that the symmetric difference of G and the half-space $\{x \in \mathbb{R}^m;\ (x - z) \cdot \theta > 0\}$ has m-dimensional density zero at z then $n^G(z) = \theta$ is termed *the interior normal of G at z in Federer's sense*. (The symmetric difference of B and C is equal to $(B \setminus C) \cup (C \setminus B)$.) If there is no interior normal of G at z in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$. The set $\{y \in \mathbb{R}^m;\ |n^G(y)| > 0\}$ is called the reduced boundary of G and will be denoted by $\widehat{\partial} G$. Clearly $\widehat{\partial} G \subset \partial_e G$.

If $\mathscr{H}_{m-1}(\partial_e G)$, the perimeter of G, is finite then $\mathscr{H}_{m-1}(\partial_e G \setminus \widehat{\partial} G) = 0$ (see [6], Theorem 4.5.6) and

$$v^G(x) = \int_{\widehat{\partial} G} |n^G(y) \cdot \nabla h_x(y)| \,\mathrm{d}\mathscr{H}_{m-1}(y)$$

for each $x \in \mathbb{R}^m$ (see [14], Lemma 2.15).

Lemma 2. *$N^G(\mathscr{U}\nu) + (\mathscr{U}\nu)\lambda \in \mathscr{C}'(\partial G)$ for each $\nu \in \mathscr{C}'(\partial G)$ if and only if $V^G < \infty$. If $V^G < \infty$ then $\tau\colon \nu \mapsto N^G(\mathscr{U}\nu) + (\mathscr{U}\nu)\lambda$ is a bounded linear operator on $\mathscr{C}'(\partial G)$ and $\|\tau\| \leqslant V^G + 1 + c_\lambda$. (If we want to emphasize that τ depends on G we will write τ^G instead of τ.)*

Proof. Lemma 1 yields that $\nu \mapsto (\mathscr{U}\nu)\lambda$ is a bounded linear operator on $\mathscr{C}'(\partial G)$ with a norm majorized by c_λ. The rest is a conclusion of [14], Theorem 1.13. □

Remark 4. Lemma 2 was proved in [23] under more general conditions.

Remark 5. We will assume that $V^G < \infty$ and $\partial G = \partial(\mathrm{cl}\, G)$.Then for each $x \in \mathbb{R}^m$ there exists

$$d_G(x) = \lim_{r \to 0_+} \frac{\mathscr{H}_m(\mathscr{U}(x; r) \cap G)}{\mathscr{H}_m(\mathscr{U}(x; r))}$$

(see [14], Lemma 2.9). According to [14], Observation 1.5, Proposition 2.8 and Lemma 2.15 we have

$$N^G \mathscr{U}\nu(M) = \int_M d_G(x)\,\mathrm{d}\nu(x) - \int_{\partial G} \int_{(\partial G \cap M)} n^G(y) \cdot \nabla h_x(y)\,\mathrm{d}\mathscr{H}_{m-1}(y)\,\mathrm{d}\nu(x)$$

for each $\nu \in \mathscr{C}'(\partial G)$ and a Borel set M. (This relation holds even if $\partial G \neq \partial(\mathrm{cl}\, G)$.)

If we denote for $f \in \mathscr{C}(\partial G)$ (= the space of all bounded continuous function on ∂G equipped with the maximum norm) and $x \in \partial G$

$$W^G f(x) = d_G(x) f(x) - \int_{\partial G} f(y) n^G(y) \cdot \nabla h_x(y)\,\mathrm{d}\mathscr{H}_{m-1}(y),$$
$$V f(x) = \mathscr{U}(f\lambda)(x)$$

then W^G, V are bounded linear operators on $\mathscr{C}(\partial G)$ and $N^G \mathscr{U} \colon \nu \mapsto N^G(\mathscr{U}\nu)$ is the dual operator of W^G and τ is the dual operator of $(W^G + V)$ (see [14], Proposition 2.5, Proposition 2.20, [24], Proposition 9 and [23], Proposition 8). V is a compact operator on $\mathscr{C}(\partial G)$ by [24], Proposition 9. Since $\tau - N^G \mathscr{U}$ is the dual operator of V, it is compact, too (see [32], Chapter IV, Theorem 4.1). If τ is a Fredholm operator then $N^G \mathscr{U}$ and W^G are Fredholm operators, too (see [32], Chapter V, Theorem 3.1, Chapter VII, Theorem 3.5) and $\mathrm{cl}\, G$ has finitely many components by [21], Lemma 3.

Lemma 3. *Let* $\mathrm{cl}\, G$ *have finitely many components. Let* $\mu \in \mathscr{C}'(\partial G)$ *for which there is a solution of the Robin problem with the boundary condition* μ *(i.e. there exists a harmonic function* u *for which* $N^G u + u\lambda = \mu$*). Then* $\mu(\partial H) = 0$ *for each bounded component* H *of* $\mathrm{cl}\, G$ *such that* $\lambda(\partial H) = 0$.

P r o o f. Let H be a bounded component of $\mathrm{cl}\, G$ such that $\lambda(\partial H) = 0$. Choose $\varphi \in \mathscr{D}$ such that $\varphi = 1$ on H and $\varphi = 0$ on $\mathrm{cl}\, G \setminus H$. Then

$$\mu(\partial H) = \langle \varphi, N^G u + u\lambda \rangle = \int_G \nabla u \cdot \nabla \varphi \,\mathrm{d}\mathscr{H}_m + \int_{\partial G} u\varphi \,\mathrm{d}\lambda = 0.$$

□

N o t a t i o n. Let L be a linear space over the field of real numbers. We will denote by $\hat{}L$ the set of all elements of the form $x + iy$ where $x, y \in L$. If the sum of two elements of $\hat{}L$ and the multiplication of an element of $\hat{}L$ by a complex number are defined in the obvious way then $\hat{}L$ becomes a linear space over the field of complex numbers. Let Q be a linear operator acting on L. The same symbol will denote the extension of Q to $\hat{}L$ defined by $Q(x + \mathrm{i}y) = Q(x) + \mathrm{i}Q(y)$. If an operator Q on L possesses an inverse operator Q^{-1}, then the extension of Q^{-1} to $\hat{}L$ is an inverse operator (on $\hat{}L$) of the extension of Q to $\hat{}L$.

If Q is a bounded linear operator on the complex space L we denote by $\sigma(Q)$ the spectrum of Q. We denote by $\Phi(Q)$ the set of all complex number α for which $\alpha I - Q$ is Fredholm, where I is the identity operator. We denote by $\Omega(Q)$ the unbounded component of $\Phi(Q)$.

Lemma 4. *$\mathscr{H}_m(\{x \in \partial G;\ d_G(x) \neq 0\}) = 0$. If there is a one-to-one sequence $\{x_n\} \subset \partial G$ such that*

$$\alpha = \lim_{n\to\infty} d_G(x_n),$$

then $\alpha \notin \Omega(\tau)$. If moreover $d_G(x_n) = \alpha$ for each n then $\alpha \notin \Phi(\tau)$. In particular, $\frac{1}{2} \notin \Phi(\tau)$. If τ is a Fredholm operator then the set $\{x \in \partial G;\ d_G(x) = 0\}$ is finite and $\mathscr{H}_m(\partial G) = 0$.

Proof. Since G has a finite perimeter, $\mathscr{H}_{m-1}(\widehat{\partial} G) < \infty$ and $\mathscr{H}_{m-1}(\{x \in \partial G;\ 0 < d_G(x) < 1\} \setminus \widehat{\partial} G) = 0$ by [6], Theorem 4.5.6. Denote $M_1 = \{x \in \partial G;\ d_G(x) = 1\}$. Since $d_{\mathbb{R}^m \setminus G}(x) = 0$ for each $x \in M_1 \subset \mathbb{R}^m \setminus G$ we obtain $\mathscr{H}_m(M_1) = 0$ by [34], Theorem 1.3.8 (or [18], Theorem 29.2).

Fix $x \in \partial G$, $\nu \in \mathscr{C}'(\partial G)$. Then

$$(N^G \mathscr{U} \nu - d_G(x)\nu)(\{x\}) = \int_{\partial G \cap \{x\}} [d_G(y) - d_G(x)]\,\mathrm{d}\nu(y) - \int_{\partial G} \int_{\{x\}} n^G(z) \cdot \nabla h_y(z)\,\mathrm{d}\mathscr{H}_{m-1}(z)\,\mathrm{d}\nu(y) = 0$$

and $(d_G(x) I - N^G \mathscr{U})(\mathscr{C}'(\partial G)) \subset \{\mu \in \mathscr{C}'(\partial G);\ \mu(\{x\}) = 0\}$.

Suppose now that there is a one-to-one sequence $\{x_n\} \subset \partial G$ such that

$$\alpha = \lim_{n\to\infty} d_G(x_n).$$

If $d_G(x_n) = \alpha$ for each n then $\operatorname{codim}(N^G \mathscr{U} \nu - \alpha I)(\mathscr{C}'(\partial G)) = \infty$ and $\alpha \notin \Phi(N^G \mathscr{U}) = \Phi(\tau)$ (see Remark 5 and [32], Chapter V, Theorem 3.1). Suppose now that the sequence $d_G(x_n)$ is one-to-one. Then $d_G(x_n)$, $\alpha \in \sigma(N^G \mathscr{U})$. Since all points of $\sigma(N^G \mathscr{U}) \cap \Omega(N^G \mathscr{U})$ are isolated points of $\sigma(N^G \mathscr{U})$ by [12], Satz 51.4, we obtain $\alpha \notin \Omega(N^G \mathscr{U}) = \Omega(\tau)$ (see Remark 5 and [32], Chapter V, Theorem 3.1).

Since $\partial G = \partial(\mathbb{R}^m \setminus \operatorname{cl} G)$ we have $\mathscr{H}_{m-1}(\widehat{\partial} G) > 0$ by Isoperimetric Lemma (see [14], p. 50) and $\frac{1}{2} \notin \Phi(\tau)$. □

Definition. *We will say that W is Plemelj's operator if W is a bounded linear operator acting on $\widetilde{\mathscr{C}}(\partial G)$ whose dual W' maps $\widetilde{\mathscr{C}}'_c(\partial G)$ into itself and*

$$\mu \in \widetilde{\mathscr{C}}'_c(\partial G) \Longrightarrow W(\mathscr{U}_c \mu) = \mathscr{U}_c(W' \mu).$$

Lemma 5. *If $\mathscr{H}_m(\partial G) = 0$ then $W^G + V$ is Plemelj's operator.*

P r o o f. W^G is Plemelj's operator by Plemelj's exchange theorem ([14], p. 68). Let $\mu \in \mathscr{C}'_c(\partial G)$. Since $(\mathscr{U}_c\mu)^+$, $(\mathscr{U}_c\mu)^-$ are bounded functions on ∂G and $\mathscr{U}\lambda$ is bounded and continuous on ∂G, $\mathscr{U}((\mathscr{U}_c\mu)^+\lambda)$ and $\mathscr{U}((\mathscr{U}_c\mu)^-\lambda)$ are bounded and continuous on ∂G by [24], Proposition 6. Regularity principle ([17],Theorem 1.7) yields that $\mathscr{U}((\mathscr{U}_c\mu)^+\lambda)$, $\mathscr{U}((\mathscr{U}_c\mu)^-\lambda)$ are finite continuous functions in $\mathbb{R}^m$. The function $\mathscr{U}((\mathscr{U}\mu)\lambda) = \mathscr{U}((\mathscr{U}_c\mu)\lambda) = \mathscr{U}((\mathscr{U}_c\mu)^+\lambda) - \mathscr{U}((\mathscr{U}_c\mu)^-\lambda)$ is continuous by Lemma 1. Thus $V\mu = (\mathscr{U}\mu)\lambda \in \mathscr{C}'_c(\partial G)$ and $V(\mathscr{U}_c\mu) = \mathscr{U}((\mathscr{U}_c\mu)\lambda) = \mathscr{U}(V'\mu) = \mathscr{U}_c(V'\mu)$. □

Since the condition $\mathscr{H}_m(\partial G) = 0$ plays no role in the proof of Lemma 4.5 in [14] the following lemma holds:

Lemma 6. *Let $\mu_n \in \mathscr{C}'_c(\partial G)$ $(n = 1, 2, \ldots)$, $\sum \|\mu_n\| < \infty$, $\sum \|\mathscr{U}_c\mu_n\| < \infty$. Then $\mu = \sum \mu_n \in \mathscr{C}'_c(\partial G)$ and*

$$\mathscr{U}_c\mu = \sum_n \mathscr{U}_c\mu_n.$$

Lemma 7. *Let W be Plemelj's operator. Then all operators $(W + \alpha I)$ with $|\alpha| > \|W\|$ have Plemelj's inverses. If $(W + \beta I)^{-1}$ is Plemelj's operator with $\|(W + \beta I)^{-1}\| \leqslant K$ then also all operators $(W + \gamma I)$ with $|\gamma - \beta| < 1/K$ possess Plemelj's inverses.*

P r o o f. The proof is the same as the proof of Lemma 4.6 in [14], where we substitute T by W and T_γ by $W + \gamma I$. □

Lemma 8. *Let W be Plemelj's operator. All operators $(W - \gamma I)$ with $\gamma \in \Omega(W) \setminus \sigma(W)$ possess inverses that are Plemelj's.*

P r o o f. According to [12], Satz 51.4 the set $\Omega(W) \cap \sigma(W)$ is isolated in $\Omega(W)$. Now we use the proof of Lemma 4.7 in [14] where we replace the operator T_γ by the operator $W - \gamma I$. □

Lemma 9. *Suppose that $f_1, \ldots, f_q \in \widetilde{\mathscr{C}}(\partial G)$ are linearly independent. Then there exist $\mu_1, \ldots, \mu_q \in \widetilde{\mathscr{C}}'_c(\partial G)$ such that*

$$\langle f_i, \mu_j \rangle = \delta_{ij} \text{ (= Kronecker's symbol)}, \; 1 \leqslant i, j \leqslant q.$$

P r o o f. The proof is the same as the proof of Lemma 4.9 in [14]. □

Lemma 10. *If p is a positive integer, W is Plemelj's operator and $\gamma \in \Omega(W)$ then any $\mu \in \widetilde{\mathscr{C}}'(\partial G)$ satisfying the homogeneous equation*

$$(W' - \gamma I)^p \mu = 0$$

necessarily belongs to $\widetilde{\mathscr{C}}_c'(\partial G)$.

P r o o f. It suffices to suppose that $\gamma \in \sigma(W' - \gamma I)$. The resolvents of the operators $(W - \lambda I)$, $(W - \lambda I)'$ have poles at γ and these poles are of the same order, say p_0 (cf. [12], Satz 51.4, Theorem 51.1, Satz 50.2). Now we use the proof of Theorem 4.10 in [14] where we replace the operator T_α by the operator $(W - \alpha I)$. $\square$

Lemma 11. *Let $\mathscr{H}_m(\partial G) = 0$, $0 \neq \mu \in \widetilde{\mathscr{C}}_c'(\partial G)$, $\alpha \in \mathbb{C}$, $(\tau - \alpha I)\mu = 0$. Then $\alpha \geqslant 0$. If $\alpha = 0$ then $\mathscr{U}\mu$ is locally constant on G and $\mathscr{U}_c\mu = 0$ on each component H of $\operatorname{cl} G$ for which $\lambda(\partial H) \neq 0$.*

P r o o f. Denote by $\overline{\mu}$ the complex conjugate of μ. According to [21], Lemma 7 we have

$$\alpha \int_{\partial G} \mathscr{U}_c\overline{\mu}\,\mathrm{d}\mu = \int_{\partial G} \mathscr{U}_c\overline{\mu}\,\mathrm{d}(\tau(\mu)) = \int_{\partial G} \mathscr{U}_c\overline{\mu}\; dN^G\mathscr{U}\mu + \int_{\partial G} |\mathscr{U}_c\mu|^2\,\mathrm{d}\lambda$$
$$= \int_G |\nabla \mathscr{U}\mu|^2 + \int_{\partial G} |\mathscr{U}_c\mu|^2\,\mathrm{d}\lambda.$$

By Lemma 1, [21], Lemma 6, [17], Theorem 1.20, Theorem 1.15 we obtain

$$\int_{\partial G} \mathscr{U}_c\overline{\mu}\,\mathrm{d}\mu = \int_{\partial G} \overline{\mathscr{U}\mu}\,\mathrm{d}\mu = \int_{\mathbb{R}^m} |\nabla \mathscr{U}\mu|^2 > 0.$$

So we obtain

$$\alpha = \left[\int_{\mathbb{R}^m} |\nabla \mathscr{U}\mu|^2\right]^{-1} \left[\int_G |\nabla \mathscr{U}\mu|^2 + \int_{\partial G} |\mathscr{U}_c\mu|^2\,\mathrm{d}\lambda\right] \geqslant 0.$$

If $\alpha = 0$ then $\mathscr{U}\mu$ is locally constant on G and

$$\int_{\partial G} |\mathscr{U}_c\mu|^2\,\mathrm{d}\lambda = 0.$$

Since $\mathscr{U}_c\mu$ is constant on each component of $\operatorname{cl} G$ we obtain $\mathscr{U}_c\mu = 0$ on each component H of $\operatorname{cl} G$ for which $\lambda(\partial H) \neq 0$. $\square$

Lemma 12. *Let $\mathscr{H}_m(\partial G)=0$, $\mu,\nu\in\widetilde{\mathscr{C}}'_c(\partial G)$, $\tau(\mu)=0$, $\tau(\nu)=\mu$. Then $\mu=0$.*

Proof. We can suppose that $\mu,\nu\in\mathscr{C}'_c(\partial G)$. According to Lemma 1 and [21], Lemma 7 we have

$$0=\Bigg[\int_{\partial G}\mathscr{U}_c\mu\;dN^G\mathscr{U}\nu+\int_{\partial G}(\mathscr{U}_c\mu)(\mathscr{U}\nu)\,\mathrm{d}\lambda\Bigg]-\Bigg[\int_{\partial G}\mathscr{U}_c\nu\;dN^G\mathscr{U}\mu+\int_{\partial G}(\mathscr{U}_c\nu)(\mathscr{U}\mu)\,\mathrm{d}\lambda\Bigg]$$
$$=\int_{\partial G}\mathscr{U}_c\mu\,\mathrm{d}\tau(\nu)-\int_{\partial G}\mathscr{U}_c\nu\,\mathrm{d}\tau(\mu)=\int_{\partial G}\mathscr{U}_c\mu\,\mathrm{d}\mu=\int_{\partial G}\mathscr{U}\mu\,\mathrm{d}\mu.$$

So $\mu=0$ by [17], Theorem 1.15, [21], Lemma 6 and [17], Theorem 1.20. □

Lemma 13. *Let $0\in\Omega(\tau),\nu,\mu\in\mathscr{C}'(\partial G),\tau(\nu)=\mu$. Then $\mu\in\mathscr{C}'_c(\partial G)$ if and only if $\nu\in\mathscr{C}'_c(\partial G)$. If $\mu\in\mathscr{C}'_c(\partial G)$ then $\mathscr{U}_c\mu\in(W^G+V)(\mathscr{C}(\partial G))$.*

Proof. If $\nu\in\mathscr{C}'_c(\partial G)$ then $\tau(\nu)\in\mathscr{C}'_c(\partial G)$ by Lemma 4 and Lemma 5.

Now let $\mu\in\mathscr{C}'_c(\partial G)$. We prove that $\mathscr{U}_c\mu\in(W^G+V)(\widetilde{\mathscr{C}}(\partial G))$. If $\sigma\in\operatorname{Ker}\tau$ then $\sigma\in\widetilde{\mathscr{C}}'_c(\partial G)$ by Lemma 10. The number of components of $\operatorname{cl}G$ is finite by Remark 5. Denote by $H_1,\dots,H_k$ all bounded components of $\operatorname{cl}G$ for which $\lambda(\partial H_i)=0$. Lemma 11 yields that there are $c_1,\dots,c_j$ such that

$$\mathscr{U}_c\sigma=c_i\text{ on }H_i, i=1,\dots,k,$$
$$\mathscr{U}_c\sigma=0\text{ on }\operatorname{cl}G\setminus\bigcup_{i=1}^k H_i.$$

Let $\varphi\in\mathscr{D}$ be such that $\varphi=\mathscr{U}_c\sigma$ on $\operatorname{cl}G$. Using Lemma 1 and Fubini's theorem we obtain

$$\int_{\partial G}\mathscr{U}_c\mu\,\mathrm{d}\sigma=\int_{\partial G}\mathscr{U}\mu\,\mathrm{d}\sigma=\int_{\partial G}\mathscr{U}\sigma\,\mathrm{d}\mu=\int_{\partial G}\mathscr{U}_c\sigma\,\mathrm{d}\mu=\sum_{i=1}^k c_i\mu(\partial H_i)$$
$$=\int_{\partial G}\varphi\,\mathrm{d}\mu=\langle\varphi,\tau(\nu)\rangle=\int_G\nabla\mathscr{U}_c\sigma\cdot\nabla\mathscr{U}\nu\,\mathrm{d}\mathscr{H}_m+\int_{\partial G}(\mathscr{U}_c\nu)(\mathscr{U}_c\sigma)\,\mathrm{d}\lambda=0.$$

Since $(W^G+V)(\widetilde{\mathscr{C}}(\partial G))$ is closed because (W^G+V) is a Fredholm operator we conclude that $\mathscr{U}_c\mu\in(W^G+V)(\widetilde{\mathscr{C}}(\partial G))$ by [33], Chapter VII, §5.

Since $\operatorname{Ker}\tau\cap\tau(\widetilde{\mathscr{C}}'(\partial G))=\emptyset$ by Lemma 4, Lemma 5, Lemma 10 and Lemma 12 and $\operatorname{codim}\tau(\widetilde{\mathscr{C}}'(\partial G))=\dim\operatorname{Ker}\tau$ because τ is a Fredholm operator with index 0, the space $\widetilde{\mathscr{C}}'(\partial G)$ is the direct summ of $\operatorname{Ker}\tau$ and $\tau(\widetilde{\mathscr{C}}'(\partial G))$. So there are $\nu_1\in\tau(\widetilde{\mathscr{C}}'(\partial G))$ and $\nu_2\in\operatorname{Ker}\tau$ such that $\nu=\nu_1+\nu_2$. Lemma 10 yields that

$\nu_2 \in \widehat{\mathscr{C}}'_c(\partial G)$. Denote by $\tilde{\tau}$ the restriction of τ onto $\tau(\widehat{\mathscr{C}}'(\partial G))$. Then $\tilde{\tau}$ is invertible. According to [12], Satz 51.4 there is $\delta > 0$ such that for $0 < |\alpha| < \delta$ the operator $(\tau - \alpha I)$ is invertible. Since $(\tau - \alpha I)(\operatorname{Ker} \tau) \subset \operatorname{Ker} \tau$, $(\tau - \alpha I)\tau(\widehat{\mathscr{C}}'(\partial G)) \subset \tau(\widehat{\mathscr{C}}'(\partial G))$, $(\tilde{\tau} - \alpha I)$ is invertible for $|\alpha| < \delta$ and $(\tilde{\tau} - \alpha I)^{-1}$ is the restriction of $(\tau - \alpha I)^{-1}$ onto $\tau(\widehat{\mathscr{C}}'(\partial G))$ for $\alpha \neq 0$. Denote by $\tilde{W}$ the restriction of $(W^G + V)$ onto $(W^G + V)(\widehat{\mathscr{C}}(\partial G))$. We obtain in an analogicous way that $(\tilde{W} - \alpha I)$ is invertible for $|\alpha| < \delta$ and $(\tilde{W} - \alpha I)^{-1}$ is the restriction of $(W^G + V - \alpha I)^{-1}$ onto $(W^G + V)(\widehat{\mathscr{C}}(\partial G))$ for $\alpha \neq 0$. Put

$$K = \sup_{|\alpha| \leqslant \frac{1}{2}\delta} \max\big(\|(\tilde{\tau} - \alpha I)^{-1}\|, \|(\tilde{W} - \alpha I)^{-1}\|\big).$$

Choose α such that $0 < |\alpha| < \min(\frac{1}{2}\delta, K^{-1})$. Then

$$\tilde{\tau}^{-1} = \sum_{k=0}^{\infty} (-\alpha)^k \big[(\tilde{\tau} - \alpha I)^{-1}\big]^{k+1}.$$

Thus

$$\nu_1 = \tilde{\tau}^{-1}(\mu) = \sum_{k=0}^{\infty} (-\alpha)^k \big[(\tilde{\tau} - \alpha I)^{-1}\big]^{k+1} \mu.$$

Put $\mu_n = (-\alpha)^n [(\tilde{\tau} - \alpha I)^{-1}]^{n+1} \mu$. Then $\|\mu_n\| \leqslant (|\alpha| K)^n K \|\mu\|$ and $\sum \|\mu_n\| \leqslant \infty$. Since $\mu \in \mathscr{C}'_c(\partial G)$, Lemma 8, Lemma 5 and Lemma 4 yield that $\mu_n = (-\alpha)^n [(\tau - \alpha I)^{-1}]^{n+1} \mu \in \mathscr{C}'_c(\partial G)$ and $\mathscr{U}_c \mu_n = (-\alpha)^n [(W^G + V - \alpha I)^{-1}]^{n+1} \mathscr{U}_c \mu$.

Since $\mathscr{U}_c \mu \in (W^G + V)(\widehat{\mathscr{C}}(\partial G))$ we have

$$\|\mathscr{U}_c \mu_n\| = \|(-\alpha)^n [(\tilde{W} - \alpha I)^{-1}]^{n+1} \mathscr{U}_c \mu\| \leqslant (|\alpha| K)^n K \|\mathscr{U}_c \mu\|$$

and $\nu_1 = \sum \mu_n \in \widehat{\mathscr{C}}'_c(\partial G)$ by Lemma 6. □

Theorem 1. *Let $0 \in \Omega(\tau), \mu \in \widehat{\mathscr{C}}'(\partial G)$. Then there is a harmonic function u on G which is a solution of the Robin problem*

$$N^G u + u\lambda = \mu, \tag{9}$$

if and only if $\mu \in \mathscr{C}'_0(\partial G)$ ($=$ the space of such $\nu \in \widehat{\mathscr{C}}'(\partial G)$ that $\nu(\partial H) = 0$ for each bounded component H of $\operatorname{cl} G$ for which $\lambda(\partial H) = 0$). If $\mu \in \mathscr{C}'_0(\partial G)$ then there is a unique $\nu \in \widehat{\mathscr{C}}'_0(\partial G)$ such that

$$\tau(\nu) = \mu \tag{10}$$

and for this ν the single layer potential $\mathcal{U}\nu$ is a solution of (9)*. Moreover, $\nu \in \widetilde{\mathcal{C}}'_c(\partial G)$ if and only if $\mu \in \widetilde{\mathcal{C}}'_c(\partial G)$.*

P r o o f. According to Remark 5, $\operatorname{cl} G$ has finitely many components. If for $\mu \in \widetilde{\mathcal{C}}'(\partial G)$ there is a solution of the Robin problem (9) then $\mu \in \mathcal{C}'_0(\partial G)$ by Lemma 3. Since $\mathcal{U}\nu$ solves (9) for $\mu = \tau(\nu)$ we have $\tau(\widetilde{\mathcal{C}}'(\partial G)) \subset \mathcal{C}'_0(\partial G)$. Denote by $H_1, \ldots, H_j$ all bounded components of $\operatorname{cl} G$ for which $\lambda(\partial H_i) = 0$. Since $\operatorname{codim} \mathcal{C}'_0(\partial G) = j$ and τ is a Fredholm operator with index 0 (see [12], Satz 51.1) it suffices to prove that $\operatorname{codim} \tau(\widetilde{\mathcal{C}}'(\partial G)) = \dim \operatorname{Ker} \tau \leqslant j$. By Lemma 4, Lemma 5 and Lemma 10 we have $\operatorname{Ker} \tau \subset \widetilde{\mathcal{C}}'_c(\partial G)$. Lemma 11 yields that for $\mu \in \operatorname{Ker} \tau$ there are $c_1, \ldots, c_j$ such that

$$\mathcal{U}_c\mu = c_i \text{ on } H_i, i = 1, \ldots, j,$$

$$\mathcal{U}_c\mu = 0 \text{ on } \operatorname{cl} G \setminus \bigcup_{i=1}^{j} H_i.$$

If $c_1 = c_2 = \ldots = c_j = 0$ then

$$\int_{\partial G} \mathcal{U}\mu \, \mathrm{d}\mu = \int_{\partial G} \mathcal{U}_c\mu \, \mathrm{d}\mu = 0$$

by virtue of Lemma 1, and $\mu = 0$ by [21], Lemma 6, [17], Theorem 1.20, Theorem 1.15. Thus $\dim \operatorname{Ker}(\tau) \leqslant j$.

Since $\operatorname{Ker} \tau \cap \tau(\widetilde{\mathcal{C}}'(\partial G)) = \emptyset$ by Lemma 4, Lemma 5, Lemma 10 and Lemma 12 and $\operatorname{codim} \tau(\widetilde{\mathcal{C}}'(\partial G)) = \dim \operatorname{Ker} \tau$, the space $\widetilde{\mathcal{C}}'(\partial G)$ is the direct sum of $\operatorname{Ker} \tau$ and $\tau(\widetilde{\mathcal{C}}'(\partial G)) = \mathcal{C}'_0(\partial G)$. So $\tau(\mathcal{C}'_0(\partial G)) = \mathcal{C}'_0(\partial G)$ and τ is injective on $\mathcal{C}'_0(\partial G)$. The rest is a consequence of Lemma 13. $\square$

R e m a r k 6. Let $\mu \in \mathcal{C}'(\partial G)$. If

$$\lim_{r \to 0+} \sup_{y \in \partial G} \int_{\mathcal{U}(y;r)} h_y(x) \; d|\mu|(x) = 0,$$

then $\mathcal{U}\mu$ is a finite continuous function in $\mathbb{R}^m$ and thus $\mu \in \mathcal{C}'_c(\partial G)$ ([24]). Now suppose that C is such a constant that $\mathcal{H}(\mathcal{U}(x;r)) \leqslant Cr^{m-1}$ for each $x \in \mathbb{R}^m$, $r > 0$, where $\mathcal{H}$ is the restriction of $\mathcal{H}_{m-1}$ onto $\widehat{\partial} G$. (This condition is true for $C = Am(m+2)^m(V^G + \frac{1}{2})r^{m-1}$ by [14], Corollary 2.17.) Fix p, $m-1 < p \leqslant \infty$. Put $q = \frac{p}{p-1}$ if $p < \infty$, $q = 1$ if $p = \infty$. If $\mu = f\mathcal{H}$, where $f \in L^p(\mathcal{H})$ then

$$\|\mu\| \leqslant (\mathcal{H}(\partial G))^{1/q} \|f\|_p \leqslant \left[C(\operatorname{diam} \partial G)^{(m-1)}\right]^{1/q} \|f\|_p \tag{11}$$

by the Schwarz inequality, where

$$\|f\|_p = \left\{ \int\limits_{\partial G} |f|^p \, d\mathscr{H} \right\}^{1/p} \quad \text{for } p < \infty,$$

$\|f\|_p$ is the $\mathscr{H}$-supremum of $|f|$ for $p = \infty$. Fix $z \in \mathbb{R}^m$, $R > 0$. Then using the Schwarz inequality we obtain

$$\int\limits_{\mathscr{U}(z;R)} h_z(x)|f(x)| \, d\mathscr{H}(x) \leqslant A^{-1}(m-2)^{-1} \Bigg[\int\limits_{\mathscr{U}(z;R)} |z-x|^{q(2-m)} \, d\mathscr{H}(x) \Bigg]^{1/q} \|f\|_p$$

$$\leqslant A^{-1}(m-2)^{-1} R^{2-m} \left[\sum_{k=0}^{\infty} 2^{(k+1)q(m-2)} \mathscr{H}\big(\mathscr{U}(z; 2^{-k}R) \setminus \mathscr{U}(z; 2^{-(k+1)}R)\big) \right]^{1/q} \|f\|_p$$

$$\leqslant A^{-1}(m-2)^{-1} R^{2-m} \left[C R^{m-1} \sum_{k=0}^{\infty} 2^{(k+1)q(m-2)-k(m-1)} \right]^{1/q} \|f\|_p$$

$$\leqslant A^{-1}(m-2)^{-1} R^{2-m} 2^{m-2} [1 - 2^{q(m-2)-(m-1)}]^{-1/q} R^{(m-1)/q} C^{1/q} \|f\|_p.$$

Continuity of $\mathscr{U}\mu$ is an easy consequence of this inequality and thus $\mu \in \mathscr{C}'_c(\partial G)$. Since

$$\sup_{x \in \mathbb{R}^m} \mathscr{U}|\mu|(x) \leqslant \sup_{x \in \partial G} \mathscr{U}|\mu|(x)$$

by the maximum principle (see [17], p. 91), we obtain

$$\text{(12)} \qquad \sup_{x \in \mathbb{R}^m} \mathscr{U}|\mu|(x) \leqslant C^{1/q} 2^{m-2} A^{-1}(m-2)^{-1} \frac{(\operatorname{diam} \partial G)^{(m-1)/q+2-m}}{[1 - 2^{q(m-2)-(m-1)}]^{1/q}} \|f\|_p.$$

E x a m p l e 1. Let $1 \leqslant p < m-1$. Since $\partial G = \partial(\operatorname{cl} G) \neq \emptyset$, Isoperimetric Lemma ([14], p. 50) yields that $\mathscr{H}_{m-1}(\widehat{\partial} G) > 0$. Fix $z \in \widehat{\partial} G$. Put $f(y) = |y-z|^{-\alpha}$ where $1 < \alpha < \frac{m-1}{p}$. Since

$$\mathscr{H}(\mathscr{U}(z;r)) \leqslant Am(m+2)^m (V^G + 1/2) r^{m-1}$$

for each $r > 0$ by [14], Corollary 2.17, we obtain

$$\int |f|^p \, d\mathscr{H} \leqslant \sum_{k=0}^{\infty} (2^{-k-1} \operatorname{diam} G)^{-p\alpha} \mathscr{H}\big(\mathscr{U}(z; 2^{-k}(\operatorname{diam} G)) \setminus \mathscr{U}(z; 2^{-k-1}(\operatorname{diam} G))\big)$$

$$\leqslant \sum_{k=0}^{\infty} Am(m+2)^m (V^G + \tfrac{1}{2}) 2^{p\alpha} [2^{-k}(\operatorname{diam} G)]^{m-1-p\alpha} < \infty,$$

so $f \in L^p(\mathscr{H})$. Since there is $\beta > 0$ such that for each $r < \operatorname{diam} G$

$$\mathscr{H}(\mathscr{U}(z;r)) \geqslant \beta r^{m-1}$$

by Isoperimetric Lemma ([14], p. 50),

$$\begin{aligned}
&\mathscr{U}(f\mathscr{H})(z)\\
&\geqslant \frac{1}{(m-2)A}\sum_{k=0}^{\infty}(2^{-k}\operatorname{diam} G)^{-\alpha-m+2}\mathscr{H}\big(\mathscr{U}(z;2^{-k}(\operatorname{diam} G))\setminus\mathscr{U}(z;2^{-k-1}(\operatorname{diam} G))\big)\\
&\geqslant \frac{(\operatorname{diam} G)^{-\alpha-m+2}}{(m-2)A}\sum_{k=1}^{\infty}\mathscr{H}\big(\mathscr{U}(z;2^{-k}(\operatorname{diam} G))\big)[2^{k(\alpha+m-2)}-2^{(k-1)(\alpha+m-2)}]\\
&\geqslant \frac{(\operatorname{diam} G)^{\alpha+m-2}}{(m-2)A}\sum_{k=1}^{\infty}\beta\big[2^{-k}(\operatorname{diam} G)\big]^{m-1}2^{k(\alpha+m-2)}(1-2^{-(\alpha+m-2)})=\infty.
\end{aligned}$$

Since $\mathscr{U}(f\mathscr{H})$ is a lower semicontinuous function ([17], Theorem 1.3) we have $f\mathscr{H} \notin \mathscr{C}'_c(\partial G)$.

Lemma 14. *Let* $0 \in \Omega(\tau)$. *Then*

$$\inf_{x\in\partial G} d_G(x) > 0. \tag{13}$$

Let λ be absolutely continuous with respect to $\mathscr{H}$, the restriction of $\mathscr{H}_{m-1}$ onto $\widehat{\partial}G$. Let $\nu,\mu \in \mathscr{C}'(\partial G)$ and $\tau(\nu) = \mu$. Then ν is absolutely continuous with respect to $\mathscr{H}$ if and only if μ is absolutely continuous with respect to $\mathscr{H}$.

P r o o f. If there is $x \in \partial G$ such that $d_G(x) = 0$ then $N^G\mathscr{U}(\widetilde{\mathscr{C}}'(\partial G)) \subset \{\varrho \in \widetilde{\mathscr{C}}'(\partial G);\ \varrho(\{x\}) = 0\}$. Let H be the component of $\operatorname{cl} G$ such that $x \in H$. Since $\partial G = \partial(\operatorname{cl} G) \neq \emptyset$ there is $y \in \partial H \setminus \{x\}$. Then $\delta_x - \delta_y \notin N^G\mathscr{U}(\widetilde{\mathscr{C}}'(\partial G))$ which is a contradiction with Theorem 1. (δ_x means the Dirac measure concentrated at the point x.) Lemma 4 yields the relation (13). So ν is absolutely continuous with respect to $\mathscr{H}$ if and only if μ is absolutely continuous with respect to $\mathscr{H}$ by [23], Proposition 12. □

Lemma 15. *Let τ be a Fredholm operator and $\alpha > 0$ and $\sigma(\tau)\cap\{\beta \in \mathbb{C};\ |\beta-\alpha| \geqslant \alpha\} \subset \{0\}$. Then there are constants $c \in \langle 1,\infty)$, $q \in (0,1)$ such that for each $\mu \in \mathscr{C}'_0(\partial G)$ and integer number n*

$$\left\|\left(\frac{\tau-\alpha I}{\alpha}\right)^n\mu\right\| \leqslant Cq^n\|\mu\|. \tag{14}$$

If $\mu \in \mathscr{C}_0'(\partial G)$ then there is a unique $\nu \in \mathscr{C}_0'(\partial G)$ such that $\tau(\nu) = \mu$. This ν is given by

$$\nu = \sum_{n=0}^{\infty} \left(-\frac{\tau - \alpha I}{\alpha} \right)^n \frac{\mu}{\alpha}. \tag{15}$$

The single layer potential $\mathscr{U}\nu$ is a solution of the Robin problem $N^G u + u\lambda = \mu$.

P r o o f. Since $r_{\text{ess}}(\frac{1}{\alpha}\tau - I) \equiv \sup\{|\beta|;\ \beta \in \mathbb{C} \setminus \Phi(\frac{1}{\alpha}\tau - I)\} < 1$ there are $c \in \langle 1, \infty), q \in (0,1)$ such that (14) holds for each $\mu \in \mathscr{C}_0'(\partial G)$ by Lemma 4, Lemma 5, Lemma 10, Lemma 12, Theorem 1 and [21], Proposition 3. The series (15) converges and ν given by (15) satisfies

$$\left(\frac{\tau - \alpha I}{\alpha} \right) \nu + I\nu = \frac{\mu}{\alpha}.$$

Thus $\tau(\nu) = \mu$ and we can use Theorem 1. □

R e m a r k 7. If L is a bounded linear operator on the complex Banach space X we denote by $\|L\|_{\text{ess}}$ the essential norm of L, i.e. the distance of L from the space of all compact linear operators on X. The essential radius of L is defined by

$$r_{\text{ess}} L = \lim_{n \to \infty} (\|L^n\|_{\text{ess}})^{1/n}.$$

According to [12], Satz 51.8, [7] we have

$$r_{\text{ess}}(L) = \sup_{\lambda \in \mathbb{C} \setminus \Omega(L)} |\lambda| = \inf_p p_{\text{ess}}(L),$$

where p ranges over all norms equivalent to $\|\ \|$. Thus if there is $\alpha \in \mathbb{C}$ such that $r_{\text{ess}}(\tau - \alpha I) < |\alpha|$ then $0 \in \Omega(\tau)$ and we can use Theorem 1. Some sufficient conditions for $r_{\text{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$ are known . But it is a question whether there is G such that $0 \in \Omega(\tau)$ and $r_{\text{ess}}(\tau - \frac{1}{2}I) \geqslant \frac{1}{2}$ under our supposition $\partial G = \partial(\operatorname{cl} G)$. If we omit the condition $\partial G = \partial(\operatorname{cl} G)$ we obtain such a set putting $G = \mathbb{R}^m \setminus K$ where K is an arbitrary compact set of null Lebesgue measure. For such G we have $V^G = 0$ and if we put $\lambda = 0$ we obtain $\tau = N^G\mathscr{U} = I$ and thus $\sigma(\tau) = \{1\}, 0 \in \Omega(\tau)$ and $r_{\text{ess}}(\tau - \frac{1}{2}I) = \frac{1}{2}$.

It is well-known that the condition $r_{\text{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$ is fulfilled for sets with a smooth boundary (of class $C^{1+\alpha}$) (see [15]) and for convex sets (see [26]). R. S. Angell, R. E. Kleinman, J. Král and W. L. Wendland proved that rectangular domains (i.e. formed from rectangular parallelepipeds) in $\mathbb{R}^3$ have this property (see [2], [16]).

A. Rathsfeld showed in [29], [30] that polyhedral cones in $\mathbb{R}^3$ have this property. (By a polyhedral cone in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface (i.e. every point of $\partial\Omega$ has a neighbourhood in $\partial\Omega$ which is homeomorphic to $\mathbb{R}^2$) and $\partial\Omega$ is formed by a finite number of plane angles. By a polyhedral open set with bounded boundary in $\mathbb{R}^3$ we mean an open set Ω whose boundary is locally a hypersurface and $\partial\Omega$ is formed by a finite number of polygons.) N. V. Grachev and V. G. Maz'ya obtained independently an analogous result for polyhedral open sets with bounded boundary in $\mathbb{R}^3$ (see [11]). (Let us note that there is a polyhedral set in $\mathbb{R}^3$ which has not a locally Lipschitz boundary.) In [20] it was shown that the condition $r_{\mathrm{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$ has a local character. As a conclusion we obtain that this condition is fullfiled for $G \subset \mathbb{R}^3$ such that for each $x \in \partial G$ there are $r(x) > 0$, a domain D_x which is polyhedral or smooth or convex or a complement of a convex domain and a diffeomorphism $\psi_x\colon \mathscr{U}(x; r(x)) \to \mathbb{R}^3$ of class $C^{1+\alpha}$, where $\alpha > 0$, such that $\psi_x(G \cap \mathscr{U}(x; r(x))) = D_x \cap \psi_x(\mathscr{U}(x; r(x)))$. V. G. Maz'ya and N. V. Grachev proved this condition for several types of sets with "piecewise-smooth" boundary in the general Euclidean space (see [8]–[10]).

If we have $r_{\mathrm{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$ and $\partial G \neq \partial(\mathrm{cl}\, G)$ we can use this theory, too. Denote by $\mathscr{I}$ the set of all isolated points of ∂G. Then $\mathscr{I}$ is finite by [21], Lemma 1 and for $\tilde{G} = G \cup \mathscr{I}$ we have $\partial\tilde{G} = \partial(\mathrm{cl}\, G)$. Let now $\mu \in \mathscr{C}'(\partial\tilde{G})$. We denote by μ_r the restriction of μ onto $\partial\tilde{G} (\subset \partial G)$ and by μ_s the restriction of μ onto $\mathscr{I}$. The set cl $G = \mathrm{cl}\, \tilde{G}$ has finitely many components (see Remark 5) and a necessary condition for the existence of a solution of the Robin problem for G with the boundary condition μ is that $\mu(\partial H) = 0$ for each bounded component H of cl $G = \mathrm{cl}\, \tilde{G}$ such that $\lambda(\partial H) = 0$. Suppose that this condition is fulfilled. Let now $\nu \in \mathscr{C}'(\partial G)$. Since $N^G \mathscr{U} \nu_s = \nu_s$ and $(\mathscr{U}\nu_s)\lambda \in \mathscr{C}'(\partial\tilde{G})$, the necessary condition for $\tau^G \nu = \mu$ leads to the equation $\tau^{\tilde{G}}(\nu_r) = \mu_r - (\mathscr{U}\mu_s)\lambda$. Let now H be a bounded component of $\mathrm{cl}\, \tilde{G}$ such that $\lambda(\partial H) = 0$. Since $\mu(\partial H) = 0$ we have

$$\mu_r(\partial H) - \int_{\partial H} (\mathscr{U}\mu_s)\lambda = -\mu_s(\partial H) - \int_{\partial H} (\mathscr{U}\mu_s)\lambda = -(\tau^G \mu_s)(\partial H) = 0.$$

Theorem 1 yields that there is $\nu_r \in \mathscr{C}'(\partial\tilde{G})$ for which $\tau^{\tilde{G}}(\nu_r) = \mu_r - (\mathscr{U}\mu_s)\lambda$.

Theorem 2. *Let* $r_{\mathrm{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$ *(see Remark 7). For* $\lambda \equiv 0$ *put* $\alpha_0 = \frac{1}{2}$*, for* $\lambda \not\equiv 0$ *put* $\alpha_0 = \frac{1}{2}(V^G + 1 + c_\lambda)$*. Then for each* $\alpha > \alpha_0$ *there are constants* $d_\alpha \in \langle 1, \infty)$*,* $q_\alpha \in (0,1)$ *such that for each* $\mu \in \mathscr{C}'_0(\partial G)$ *and a natural number* n

$$\left\| \left(\frac{\tau - \alpha I}{\alpha} \right)^n \mu \right\| \leqslant d_\alpha q_\alpha^n \|\mu\|. \tag{16}$$

If $\mu \in \mathscr{C}_0'(\partial G)$ then there is a unique $\nu \in \mathscr{C}_0'(\partial G)$ such that $\tau(\nu) = \mu$ and this ν is given by

$$\nu = \sum_{n=0}^{\infty} \Big(- \frac{\tau - \alpha I}{\alpha} \Big)^n \frac{\mu}{\alpha}. \tag{17}$$

The single layer potential $\mathscr{U}\nu$ is a solution of the Robin problem $N^G u + u\lambda = \mu$. If $\lambda \equiv 0$ then

$$\nu = \mu + \sum_{j=0}^{\infty} \big[-(2\tau - I)\big]^j [2I - 2\tau]\mu.$$

P r o o f. Put $C = \mathbb{R}^m \setminus \operatorname{cl} G$. Since $\mathscr{H}_m(\partial G) = 0$ by Lemma 4, $V^C = V^G < \infty$ and $N^C\mathscr{U} = I - N^G\mathscr{U}$ (see Remark 5). Thus $\sigma(\tau) \cap \{\beta;\ |\beta - \frac{1}{2}| \geqslant \frac{1}{2}\} \subset \langle 0, 2\alpha_0\rangle$ by Lemma 2, Lemma 4, Lemma 5, Lemma 10 and Lemma 11. If $\alpha > \alpha_0$ then $\sigma(\tau) \cap \{\beta;\ |\beta - \alpha| \geqslant \alpha\} \subset \langle 0, 2\alpha_0\rangle \cap \{\beta;\ |\beta - \alpha| \geqslant \alpha\} = \{0\}$ because $\{\beta;\ |\beta - \frac{1}{2}| \geqslant \frac{1}{2}\} \supset \{\beta;\ |\beta - \alpha| \geqslant \alpha\}$. The rest is a consequence of Lemma 15 and [21], Theorem 1. □

Corollary 1. *Let $r_{\text{ess}}(\tau - \frac{1}{2}I) < \frac{1}{2}$. Then $\mathscr{H}_{m-1}(\partial G) < \infty$, $\mathscr{H}_{m-1}(\partial G - \widehat{\partial} G) = 0$, $0 < \inf\{d_G(x);\ x \in \partial G\} \leqslant \sup\{d_G(x);\ x \in \partial G\} < 1$. Suppose that $\lambda = f\mathscr{H}$ where $f \in L^1(\mathscr{H})$. If we denote for $h \in {}^\frown L^1(\mathscr{H})$, $x \in \partial G$*

$$Th(x) = \frac{1}{2}h(x) - \int_{\widehat{\partial} G} h(y) n^G(x) \cdot \nabla h_y(x) \, \mathrm{d}\mathscr{H}(y) + \mathscr{U}(h\mathscr{H})(x) f(x)$$

then $Th \in {}^\frown L^1(\mathscr{H})$ and $T\colon\ h \mapsto Th$ is a bounded linear operator on ${}^\frown L^1(\mathscr{H})$. Let α_0 have the same sense as in Theorem 2. Then for each $\alpha > \alpha_0$ there are constants $d_\alpha \in \langle 1, \infty)$, $q_\alpha \in (0,1)$ such that for each natural number n and $g \in {}^\frown L^1(\mathscr{H})$, for which $(g\mathscr{H}) \in \mathscr{C}_0'(\partial G)$, we have

$$\Big\| \Big(\frac{T - \alpha I}{\alpha} \Big)^n g \Big\| \leqslant d_\alpha q_\alpha^n \|g\|. \tag{18}$$

Let $g \in L^1(\mathscr{H})$ and suppose that $g\mathscr{H} \in \mathscr{C}_0'(\partial G)$. Then there is a unique $h \in {}^\frown L^1(\mathscr{H})$ such that $g\mathscr{H} = \tau(h\mathscr{H})$ and $h\mathscr{H} \in \mathscr{C}_0'(\partial G)$. The function h is given by the series

$$h = \sum_{n=0}^{\infty} \Big(\frac{\alpha I - T}{\alpha} \Big)^n \frac{g}{\alpha}. \tag{19}$$

If $f \equiv 0$ then

$$h = g + \sum_{j=0}^{\infty} \big[-(2T - I)\big]^j [2I - 2T] g.$$

P r o o f. Denote $C = \mathbb{R}^m \setminus \operatorname{cl} G$. Since $\mathscr{H}_m(\partial G) = 0$ by Lemma 4 we have $N^G\mathscr{U} + N^C\mathscr{U} = I$ (see Remark 5). The assumption and Remark 5 yield that $0 \in \Omega(N^G\mathscr{U}) \cap \Omega(N^C\mathscr{U})$. Lemma 14 yields that

$$0 < \inf_{x\in\partial G} d_G(x) \leqslant \sup_{x\in\partial G} d_G(x) < 1.$$

Thus $\mathscr{H}_{m-1}(\partial G) < \infty$, $\mathscr{H}_{m-1}(\partial G - \widehat{\partial} G) = 0$ by [6], Theorem 4.5.6. The rest is a consequence of Theorem 2 and Lemma 14. □

Corollary 2. *Let $r_{\rm ess}(\tau - \frac{1}{2}I) < \frac{1}{2}$, $\mu \in \mathscr{C}_0'(\partial G)$. Then there is $\nu \in \widetilde{\mathscr{C}}_c'(\partial G)$ such that $\tau(\nu) = \mu$ if and only if $\mu \in \widetilde{\mathscr{C}}_c'(\partial G)$. If $\mu \in \widetilde{\mathscr{C}}_c'(\partial G)$ then $\nu \in \widetilde{\mathscr{C}}_c'(\partial G)$ for each $\nu \in \widetilde{\mathscr{C}}'(\partial G)$ such that $\tau(\nu) = \mu$. Let α_0 have the same sense as in Theorem 2. Then for each $\alpha > \alpha_0$ there are constants $d \in \langle 1, \infty)$, $q \in (0,1)$ depending only on G and α such that for $\mu \in \mathscr{C}_0'(\partial G) \cap \widetilde{\mathscr{C}}_c'(\partial G)$,*

$$\mu_n = \left(-\frac{\tau - \alpha I}{\alpha}\right)^n \frac{\mu}{\alpha}, \quad u_n = \mathscr{U}_c(\mu_n), \qquad n = 0,1,2,\dots \tag{20}$$

we have

$$\sup_{x\in \operatorname{cl} G} |u_n(x)| \leqslant dq^n \sup_{x\in\partial G} |\mathscr{U}_c\mu|. \tag{21}$$

Thus

$$\sum_{n=0}^{\infty} u_n = \mathscr{U}_c\nu \tag{22}$$

where ν is given by (17) and the series in (22) converges absolutely and uniformly on $\operatorname{cl} G$ to the continuous solution $\mathscr{U}_c\nu$ of the Robin problem $N^G u + U\lambda = \mu$. Define on $\widetilde{\mathscr{C}}_c'(\partial G)$ a norm p by

$$p(\mu) = \|\mu\| + \sup_{x\in\partial G} |\mathscr{U}_c\mu|. \tag{23}$$

Then $\widetilde{\mathscr{C}}_c'(\partial G)$ is a Banach space with respect to the norm p. The operator τ maps $\widetilde{\mathscr{C}}_c'(\partial G)$ into $\widetilde{\mathscr{C}}_c'(\partial G)$ and is bounded with respect to the norm p. If $\mu \in \widetilde{\mathscr{C}}_c'(\partial G) \cap \mathscr{C}_0'(\partial G)$ then the series (17) converges with respect to the norm p.

If $m-1 < s \leqslant \infty$ then there is a constant d_s such that for each $\mu = g\mathscr{H} \in \mathscr{C}_0'(\partial G)$, where $g \in L^s(\mathscr{H})$, we have

$$\sup_{x\in \operatorname{cl} G} |u_n(x)| + \|\mu_n\| \leqslant d_s q^n \|g\|_s$$

where u_n *is given by* (20) ($\mu \in \mathscr{C}_0'(\partial G)$) *and for* $\nu \in \mathscr{C}_0'(\partial G) \cap \mathscr{C}_c'(\partial G)$ *given by* (17) *we have*

$$\sup_{x \in \operatorname{cl} G} |\mathscr{U}\nu(x)| + \|\nu\| \leqslant d_s \|g\|_s.$$

If $\lambda \equiv 0$ *then analogous results hold for* $\mu_0 = (3I - 2N^G\mathscr{U})\mu$,

$$\mu_n = (I - 2N^G\mathscr{U})^n (2I - 2N^G\mathscr{U})\mu, \ n \in \mathbb{N}.$$

P r o o f. Lemma 13 yields that there is $\nu \in \widetilde{\mathscr{C}}_c'(\partial G)$ such that $\tau(\nu) = \mu$ if and only if $\mu \in \widetilde{\mathscr{C}}_c'(\partial G)$. Let $\mu \in \widetilde{\mathscr{C}}_c'(\partial G) \cap \mathscr{C}_0'(\partial G)$. Then $\mathscr{U}_c\mu \in (W+V)(\widetilde{\mathscr{C}}(\partial G))$ by Lemma 13. Fix $\alpha > \alpha_0$. In the proof of Theorem 2 it was shown that $\sigma(\tau) \cap \{\beta;\ |\beta - \alpha| \geqslant \alpha\} \subset \{0\}$. Since τ is the dual operator of $(W+V)$ (see Remark 5) we have $\sigma(W+V) \cap \{\beta;\ |\beta-\alpha| \geqslant \alpha\} \subset \{0\}$ by [12], Satz 44.2. Since τ is a Fredholm operator with index 0 and $\operatorname{Ker}\tau^2 = \operatorname{Ker}\tau$ by Lemma 4, Lemma 5, Lemma 10 and Lemma 12, the operator $(W+V)$ is Fredholm with index 0 and $\operatorname{Ker}(W+V)^2 = \operatorname{Ker}(W+V)$ by [32], Chapter VII, Theorem 3.5 and [12], Satz 27.1. [21], Proposition 3 yields that there are constants $M \in \langle 1, \infty)$, $q \in (0,1)$ such that for each $f \in (W+V)(\widetilde{\mathscr{C}}(\partial G))$ and each natural number n

$$\big\|[\alpha^{-1}(W+V-\alpha I)]^n f\big\| \leqslant Mq^n \|f\|.$$

Lemma 4 and Lemma 5 yield that $\mu_n \in \widetilde{\mathscr{C}}_c'(\partial G)$ and

$$u_n = \mathscr{U}_c\mu_n = \mathscr{U}_c\Big(-\frac{\tau - \alpha I}{\alpha}\Big)^n \frac{\mu}{\alpha} = \Big[\frac{1}{\alpha}(-W - V + \alpha I)\Big]^n \frac{\mathscr{U}_c\mu}{\alpha}.$$

Thus we obtain the estimate (21) by Lemma 13 while Lemma 6 yields the relation (22).

Let $\lambda \equiv 0$. Put $C = \mathbb{R}^m \setminus \operatorname{cl} G$. Since $\mathscr{H}_m(\partial G) = 0$ by Lemma 4, $V^C = V^G < \infty$ and $N^C\mathscr{U} = I - N^G\mathscr{U}$ (see Remark 5) and $r_{\text{ess}}(N^C\mathscr{U} - \frac{1}{2}I) = r_{\text{ess}}(N^G\mathscr{U} - \frac{1}{2}I) < \frac{1}{2}$. Thus $\sigma(W) \cap \{\beta;\ |\beta - \frac{1}{2}|\} \subset \{0; 1\}$, $\operatorname{Ker} W^2 = \operatorname{Ker} W$, $\operatorname{Ker}(W - I)^2 = \operatorname{Ker}(W - I)$. [21], Proposition 3 yields that there are constants $M \in \langle 1, \infty)$, $q \in (0,1)$ such that for each $f \in (W+V)(\widetilde{\mathscr{C}}(\partial G))$ and each natural number n

$$\big\|(I - 2W)^n (2I - 2W) f\big\| \leqslant Mq^n \|f\|.$$

Lemma 4 and Lemma 5 yield that $\mu_n \in \widetilde{\mathscr{C}}_c'(\partial G)$ and

$$u_0 = \mathscr{U}_c\mu_0 = \mathscr{U}_c(3I - 2N^G\mathscr{U})\mu = (3I - 2W)\mathscr{U}_c\mu,$$

$$u_n = \mathscr{U}_c\mu_n = \mathscr{U}_c(I - 2N^G\mathscr{U})^n(2I - 2N^G\mathscr{U})\mu = (I - 2W)^n(2I - 2W)\mathscr{U}_c\mu.$$

Thus we obtain the estimate (21) by Lemma 13 while Lemma 6 yields the relation (22).

The rest is a consequence of Remark 6. □

R e m a r k 8. Suppose $r_{\rm ess}(\tau - \frac{1}{2}I) < \frac{1}{2}$. If $\lambda \equiv 0$ we put $\alpha_0 = \frac{1}{2}$. If $C = \mathbb{R}^m \backslash \operatorname{cl} G$ has a bounded component then $N^C\mathscr{U}(\,\widehat{\ }\mathscr{C}'(\partial G)) \neq \widehat{\ }C'(\partial G)$ by Theorem 1 and there is $\mu \in \operatorname{Ker}(N^C\mathscr{U}), \mu \neq 0$. Since $N^C\mathscr{U} + N^G\mathscr{U} = I$ we have $N^G\mathscr{U}\mu = \mu$. The series (17) diverges for $\alpha = \frac{1}{2}$. So, our choice of α_0 in Theorem 2 is the best possible. Now, let $\lambda \not\equiv 0$. It is a question whether it is possible to choose a better λ_0 than $\frac{1}{2}(V^G + 1 + c_\lambda)$ in Theorem 2. But it is necessary to put $\lambda_0 \geqslant \frac{1}{2}c_\lambda$ as the following example shows. Let G be bounded. Then there is a positive measure $\mu \in \mathscr{C}'(\partial G)$ such that $\mathscr{U}\mu = 1$ on G (see [17], Chapter II, §1). Since $d_G(x) > 0$ for each $x \in \partial G$ by Corollary 1 and $\mathscr{U}\nu$ is fine continuous we obtain $\mathscr{U}\nu \equiv 1$ on $\operatorname{cl} G$ by [3], Chapter VII, §2. Put $\lambda = c\mu$ for $c > 0$. Then $c_\lambda = c$, $\tau(\mu) = \lambda = c\mu$. The series (17) diverges for $\alpha = \frac{1}{2}c_\lambda$.

E x a m p l e 2. Put $G = \{[x_1, x_2, x_3];\ |x_1| < 1, |x_2| < 1, -1 < x_3 < 0\} \cup \{[t, ty_2, ty_3];\ 0 < t < 1, \frac{1}{3} < |y_2| < \frac{2}{3}, 0 \leqslant y_3 < \frac{1}{3}\} \subset \mathbb{R}^3$. Let f, g be continuous functions on ∂G. Suppose that f is nonnegative and if $f \equiv 0$ then

$$\int_{\partial G} g = 0.$$

We would like to find a solution of the problem

$$\Delta u = 0 \text{ in } G,$$
$$\frac{\partial u}{\partial n} + fu = g \text{ on } \widehat{\partial} G.$$

Notice that G has not a locally Lipschitz boundary, so we cannot use the theory for Lipschitz domains. In fact, the boundary of G is not a graph of a function in a neigbourhood of the point $[0,0,0]$. Let θ be a unit vector. If there is $\delta > 0$ such that each line with the direction θ intersects $\partial G \cap \mathscr{U}([0,0,0];\delta) \cap \{[x_1, x_2, x_3];\ x_2 > 0\}$ in at most one point then $\theta \in \{[t, ty_2, ty_3];\ t \in \mathbb{R}, \frac{1}{3} < y_2 < \frac{2}{3}\}$. If there is $\delta > 0$ such that each line with the direction θ intersects $\partial G \cap \mathscr{U}([0,0,0];\delta) \cap \{[x_1, x_2, x_3];\ x_2 < 0\}$ in at most one point then $\theta \in \{[t, ty_2, ty_3];\ t \in \mathbb{R}, -\frac{2}{3} < y_2 < -\frac{1}{3}\}$. So there is no unit vector θ nor a positive number δ such that each line with the direction θ intersects $\partial G \cap \mathscr{U}([0,0,0];\delta)$ in at most one point.

The open set G is not a domain with a locally Lipschitz boundary but it is a polyhedral domain. Instead of the original problem we can solve the problem

$$\Delta u = 0 \text{ in } G, \tag{24}$$
$$N^G u + u(f\mathscr{H}) = g\mathscr{H}.$$

Since G is the union of three convex sets, we have $V^G \leqslant 3$ (see Remark 3). Denote

$$c_f = \sup_{x \in \partial G} f(x).$$

Since $\mathscr{H}(\mathscr{U}(x;r)) \leqslant 12\pi r^2$ for each $x \in \mathbb{R}^m$, $r > 0$, because ∂G is a subset of the union of 12 planes, we have (see Remark 6)

$$\frac{1}{2}(V^G + 1 + c_{f\mathscr{H}}) < 2 + 24c_f.$$

If $\alpha > 2 + 24c_f$ put

$$h = \sum_{n=0}^{\infty} \left(\frac{\alpha I - T}{\alpha}\right)^n \frac{g}{\alpha}.$$

Then $\mathscr{U}(h\mathscr{H})$ is a continuous function in $\mathbb{R}^3$ which is a solution of the problem (24) (see Remark 7, Corollary 1 and Corollary 2).

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