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A NOTE ON BOUNDS FOR NON-LINEAR MULTIVALUED HOMOGENIZED OPERATORS

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Abstract. In this paper we study the behaviour of maximal monotone multivalued highly oscillatory operators. We construct Reuss-Voigt-Wiener and Hashin-Shtrikmann type bounds for the minimal sections of G-limits of multivalued operators by using variational convergence and convex analysis.

Keywords: multivalued operators, highly oscillatory operators, Reuss-Voigt-Wiener bounds, Hashin-Shtrikman bounds

MSC 2000: 35Q35, 35J20

Introduction

We are interested in constructing upper and lower bounds for the *effective* or *homogenized* properties of heterogeneous structures which are governed by non-linear equations of monotone type with highly oscillatory behaviour. We consider the case when the differential operators are allowed to be multivalued. By assuming periodic oscillation, well-known results from the theories of G-convergence and Γ-convergence guarantee the existence of a unique homogenized limit operator with no oscillatory behaviour. Our goal is to find upper and lower bounds for this homogenized operator. Since it most likely is multivalued we consider the Yosida approximants (which are single-valued). This allows us to define non-linear versions of the standard variational principles, for the Yosida approximants. By using the fact that the Yosida approximant passes nicely to the unique minimal section of the homogenized operator we can argue as in the linear case and obtain bounds for the minimal sections of the homogenized operator.

A strong motivation for considering multivalued operators comes from non-linear elasticity and plasticity theory in heterogeneous structures. In linear elasticity one of the cornerstones is a constitutive relation (Hooke type law) between the stress tensor σ and the strain tensor e of the form

$$\sigma = Ce.$$

If we allow plastic deformation and hardening effects to be present there is no longer a unique relation between the stress and the strain tensors and by letting φ be the density of the stored potential energy in the structure one ends up with a more general relation between stress and strain

$$\sigma \in \partial\varphi(e),$$

where $\partial\varphi$ is the subdifferential of the energy density function φ. In general, the subdifferential is a multivalued operator. In the problem of constructing bounds for linear heterogeneous structures the amount of contributions is enormous. The problem goes back to the nineteenth century when e.g. Maxwell and Voigt contributed with their results. In the early 1960's a couple of famous papers by Hashin and Shtrikman [11] and [12] and a fundamental paper by Hill [13] really contributed to considerable progress in this problem. Hashin and Shtrikman established upper and lower bounds for an arbitrary mixing of two isotropic well-ordered materials via a now classical variational principle, named the Hashin-Shtrikman (H-S) variational principle. The bounds they derived are called the H-S bounds, or sometimes also the trace bounds. For many cases in e.g. linear heat conduction or linear elasticity the H-S bounds have been proved to be optimal. This means that they are sharp, but also that any effective value inbetween them corresponds to a composite, i.e. to a mixture of given different properties in a given portion. To give the reader a sample of more recent contributions we list in chronological order, without having any ambition of being complete, the papers [28], [29], [9], [16], [25], [8], [3], [14], [17], [26] and [1]. We also mention the survey papers [10] and [27].

For the corresponding non-linear problems it has been proved by using variational principles that the bounds for the linear case have their natural generalization to the monotone case. See e.g. the series of papers by Ponte-Castaneda [18] and [19], Suquet [21], Talbot and Willis [23] and [24] and Willis [30] and [31], where H-S type bounds for the effective overall energy response have been proved. For recent results on sharp bounds for the homogenized p-Laplacian we refer to Lukkassen et al. [15].

In the present paper we define the effective energies associated with multivalued operators by using well-known G- and Γ-convergence results for sequences of periodic monotone operators and the corresponding convex functionals, respectively. We then establish bounds for the associated homogenized operators by using duality arguments for convex functions. By introducing a comparison energy density function, in an analogous way as for the linear case we can also establish variational principles and the corresponding bounds of H-S type for monotone (possibly multivalued) maps.

The paper is organized in the following way: In Section 1 we collect some basic notation, definitions and standard results from non-linear functional analysis and in Section 2 we recall some basic homogenization results for non-linear elliptic problems. The main results of the paper, upper and lower bounds for the non-linear homogenized operator, are presented in Section 3. Concerning the fundamentals of variational convergence we refer to the books [2] and [6].

1. Some preliminaries

Let us briefly recall some basic definitions about nonlinear operators in Banach spaces that we will need. For more details we refer to e.g. [32]. Consider two linear spaces X and Y. An element of the product space $X \times Y$ is written by $[x, y]$ for $x \in X$ and $y \in Y$. A *multivalued* operator A from X to Y will be viewed as a subset of $X \times Y$. We define

$$Ax = \{y \in Y\colon\ [x, y] \in A\},\ D(A) = \{x \in X\colon\ Ax \neq \emptyset\},$$
$$R(A) = \cup_{x \in D(A)} Ax,\ A^{-1} = \{[y, x]\colon\ [x, y] \in A\}.$$

If, for every $x \in X$, the set Ax contains exactly one element of Y, then we say that A is *single-valued*.

Let X be a real Banach space and let X' be its dual. By $\langle\cdot,\cdot\rangle_X$ we denote the duality pairing between X' and X. A subset $A \subseteq X \times X'$ is called *monotone* (*strictly monotone*) if

$$\langle y_2 - y_1, x_2 - x_1\rangle \geqslant 0\ (> 0)$$

for any $[x_1, y_1] \in A$ and $[x_2, y_2] \in A$. A monotone subset $A \subseteq X \times X'$ is called *maximal monotone* if for every $[x, y] \in X \times X'$ for which

$$\langle y - \eta, x - \xi\rangle \geqslant 0\ \forall [\xi, \eta] \in A$$

we have $[x, y] \in A$.

Definition. Let X be a reflexive Banach space and let $F\colon X \to\ ]-\infty, +\infty]$ be a convex and lower semicontinuous function. The *subdifferential* ∂F of F is the (possibly) multivalued map from X into X' given by

$$\partial F(x) = \{f \in X'\colon\ \langle f, z - x\rangle_X \leqslant F(z) - F(x)\,\forall z \in X\}.$$

The subdifferential of the convex function $x \mapsto \frac{1}{2}\|x\|_X^2$ is called the *duality map* from X into X'. We denote it by J. By the Hahn-Banach theorem J is nonempty and if X is strictly convex, then J is single-valued.

Definition. For the operator $A\colon X \to X^*$ we define the *resolvent* and *Yosida approximant* of index $\lambda > 0$ as

$$R_\lambda^A(x) = (I + \lambda A)^{-1}(x)$$

and

$$A_\lambda(x) = \frac{1}{\lambda} J(x - R_\lambda^A(x)),$$

respectively, where I is the identity map.

Definition. Let $\varphi\colon X \to]-\infty, +\infty]$ be a convex and lower semicontinuous function, where (X, τ) is a locally convex topological space. The conjugate φ^* to φ is defined via the Legendre-Young-Fenchel transform

$$\varphi^*(y) = \sup_{x \in X}\{\langle y, x\rangle - \varphi(x)\}$$

for all $y \in X'$. If φ is convex and lower semicontinuous, then

$$\varphi(z) = \varphi^{**}(z) = \sup_{y \in X'}\{\langle z, y\rangle - \varphi^*(y)\} \tag{1.1}$$

for all $z \in X$. Moreover, if $\partial\psi$ is the subdifferential of a convex function ψ, then $\partial\psi^* = (\partial\psi)^{-1}$.

Proposition 1. *Let X be a reflexive Banach space such that X and its dual are strictly convex. Further, let $\varphi\colon X \to]-\infty, +\infty]$ be a convex and lower semicontinuous function. For every $\lambda > 0$, let $(\partial\varphi)_\lambda$ be the Yosida approximant to the maximal monotone operator $\partial\varphi$. The sequence $((\partial\varphi)_\lambda)$ satisfies: For every $x \in D(\partial\varphi)$*

$$(\partial\varphi)_\lambda(x) \to (\partial\varphi)_0(x) = \inf_{y \in X^*}\{y \in \partial\varphi(x)\}. \tag{1.2}$$

Proof. We refer to Proposition 3.56 in [2]. $\square$

Let Y be the unit cube in $\mathbb{R}^n$. By $V(Y)$ we will denote the subset of $W^{1,p}(Y)$ of all functions with mean value zero over Y which take the same trace on the opposite faces of Y, and by $V^*(Y)$ the subset of $(L^{p'}(Y))^n$ of $\mathbb{R}^n$-valued maps σ such that $\operatorname{div}\sigma = 0$, σ has mean value zero over Y and (σ, n) takes opposite values on the opposite faces of Y, where n denotes the usual outer normal. Throughout the paper Ω is a bounded open set in $\mathbb{R}^n$ and, if nothing else is said, we denote $\mathrm{V} = \mathrm{W}_0^{1,p}(\Omega)$ and the dual $\mathrm{V}' = \mathrm{W}^{-1,p'}(\Omega)$. $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$ and $|\cdot|$ the usual Euclidean norm in $\mathbb{R}^n$.

Let us consider the monotone boundary value problem

$$\begin{cases} -\operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon}, Du_\varepsilon\Big)\Big) \ni f \text{ in } \Omega \\ u_\varepsilon \in V, \end{cases} \tag{2.1}$$

where $a(y,\xi)$ is Y-periodic with respect to $y \in \mathbb{R}^n$ and satisfies certain measurability, growth and coercivity conditions. More precisely, a is assumed to belong to the class $M_{\mathbb{R}^n}\mathbb{R}^n$ (see [4]). It is shown in [5] that by sending $\varepsilon \to 0$ in (2.1), the sequence of solutions (u_ε) converges in the weak topology of V to the solution u to the corresponding homogenized problem

$$\begin{cases} -\operatorname{div}(b(Du)) \ni f \text{ in } \Omega \\ u \in V, \end{cases}$$

where b is a multivalued, maximal monotone and non-oscillating map. Our problem is to establish upper and lower bounds for the map b. We will restrict ourselves to maps $a(y,\xi) \in M_\Omega(\mathbb{R}^n)$ which are of the form

$$a(x,\xi) = \partial_\xi \psi(x,\xi), \tag{2.2}$$

where $\partial_\xi\psi(x,\xi)$ denotes the subdifferential of the function $\psi\colon \Omega \times \mathbb{R}^n \to [0,+\infty[$ which is measurable in (x,ξ) and convex in ξ. Then it follows that the operator $A\colon V \to V'(\Omega)$ given by

$$Au = -\operatorname{div}(a(x, Du(x)))$$

is the subdifferential of the functional $\Psi\colon V \to [0,+\infty]$ given by

$$\Psi(u,\Omega) = \int_\Omega \psi(x, Du(x))\,\mathrm{d}x, \tag{2.3}$$

which is proper, lower semicontinuous and convex. Thus, the solution of (2.1) is also the unique minimum of the following problem:

$$\min_{u_\varepsilon \in V} \Big\{ \int_\Omega \psi\Big(\frac{x}{\varepsilon}, Du_\varepsilon(x)\Big)\,\mathrm{d}x - \langle f, u_\varepsilon\rangle \Big\}. \tag{2.4}$$

The G-convergence of the sequence (a_ε) is closely related to the Γ-convergence of the corresponding sequence of functionals (Ψ_ε). This is carried out for a general class of cyclically monotone operators in [7] and will not be repeated here. Let us collect the results needed in the following two propositions:

Proposition 2. *Consider a sequence of functions $\psi_\varepsilon\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty[$, such that $\psi_\varepsilon(x,\xi)$ is Lebesgue measurable and Y-periodic in x and convex in ξ and satisfies*

$$c_1(|\xi|^p - 1) \leqslant \psi_\varepsilon(x,\xi) \leqslant c_2(|\xi|^p + 1)$$

for some strictly positive constants c_1 and c_2. Define, for an open bounded set Ω in $\mathbb{R}^n$, a sequence of functionals

$$\Psi_\varepsilon(u) = \int_\Omega \psi_\varepsilon(x, Du)\,\mathrm{d}x = \int_\Omega \psi\Big(\frac{x}{\varepsilon}, Du\Big)\,\mathrm{d}x,$$

and let τ denote the strong topology on $L^p(\Omega)$. Then there exists a functional $\overline{\Psi}$ such that the sequence (Ψ_ε) $\Gamma(\tau)$-converges to $\overline{\Psi}$, where

$$\overline{\Psi}(u) = \int_\Omega \overline{\psi}(Du)\,\mathrm{d}x, \tag{2.5}$$

and, for every $\xi \in \mathbb{R}^n$,

$$\overline{\psi}(\xi) = \min_{v\in V(Y)} \int_Y \psi(x, \xi + Dv)\,\mathrm{d}x. \tag{2.6}$$

Moreover, the sequence of minimizers (u_ε) to (2.4) converges in the weak topology of V to the minimizer u to the minimization problem

$$\min_{u\in V}\Big\{\int_\Omega \overline{\psi}(Du)\,\mathrm{d}x - \langle f, u\rangle\Big\}. \tag{2.7}$$

Proof. See e.g. [6]. □

Proposition 3. *Suppose that the sequences (Ψ_ε) and (ψ_ε) satisfy the hypotheses of Proposition 2. Let $a_\varepsilon(x,\xi) = \partial_\xi\psi_\varepsilon(x,\xi)$ for each $\varepsilon > 0$. Then the sequence (a_ε) G-converges to the map b, where $b(\xi) = \partial_\xi\overline{\psi}(\xi)$.*

Proof. Combining Proposition 2 with Remark 2.9 and Theorem 3.2 in [7] the proof follows. □

In [2, p. 288], Attouch proved the following:

Proposition 4. *Let $\overline{\psi}$ be defined as in (2.6). Then its conjugate function $\overline{\psi}^*$ is convex and lower semicontinuous and for every $\xi \in \mathbb{R}^n$ it is given by*

$$\overline{\psi}^*(\xi) = \min_{\sigma\in V^*(Y)} \int_Y \psi^*(x, \xi + \sigma)\,\mathrm{d}x.$$

3.1. Elementary bounds

Let us start by assuming in the sequel that the function ψ, in addition to the properties from the previous section, is positively homogeneous of degree p, i.e. that

$$\psi(kx) = k^p \psi(x)$$

for every $k > 0$ and every x. By combining Proposition 3 with (2.6) we can define a variational principle for the Yosida approximants b_λ, $\lambda > 0$, of the homogenized operator b. Let us fix $\xi \in \mathbb{R}^n$ and put, for $\lambda > 0$,

$$(b_\lambda(\xi), \xi) = \inf_{v \in V(Y)} \left\{ \int_Y ((\partial_\xi \psi)_\lambda(x, \xi + Dv(x)), \xi + Dv(x))\, \mathrm{d}x \right\}. \tag{3.1}$$

Using Proposition 1, we can send $\lambda \to 0$. This yields a variational principle for the minimal section b_0 of b via the energy over the cube Y

$$(b_0(\xi), \xi) = \inf_{v \in V(Y)} \left\{ \int_Y ((\partial_\xi \psi)_0(x, \xi + Dv(x)), \xi + Dv(x))\, \mathrm{d}x \right\}. \tag{3.2}$$

We now consider the Yosida approximant $(b^{-1})_\lambda$ of the inverse b^{-1} of b. Because of Proposition 4 we can define

$$((b^{-1})_\lambda(\xi), \xi) = \inf_{\sigma \in V^*(Y)} \left\{ \int_\Omega \big(((\partial_\xi \psi)^{-1})_\lambda(x, \xi + \sigma(x)), \xi + \sigma(x)\big)\, \mathrm{d}x \right\}. \tag{3.3}$$

Sending $\lambda \to 0$ and using Proposition 1 again we obtain the dual variational principle via the dual definition of the energy

$$((b^{-1})_0(\xi), \xi) = \inf_{\sigma \in V^*(Y)} \left\{ \int_\Omega \big(((\partial_\xi \psi)^{-1})_0(x, \xi + \sigma(x)), \xi + \sigma(x)\big)\, \mathrm{d}x \right\}. \tag{3.4}$$

We can now show that $b_0 = (\partial_\xi \overline{\psi})_0$ satisfies the following generalized Reuss-Voigt-Wiener upper and lower bounds:

$$(b_0(\xi), \xi) \leqslant \int_Y ((\partial_\xi \psi)_0(x, \xi), \xi)\, \mathrm{d}x. \tag{3.5}$$

and

$$((b^{-1})_0(\xi), \xi) \leqslant \int_Y \big(((\partial_\xi \psi)^{-1})_0(x, \xi), \xi\big)\, \mathrm{d}x, \tag{3.6}$$

where (3.6) actually yields a lower bound on b_0 by duality. First we consider the variational principle (3.2) and observe that any choice of $v \in V(Y)$ yields an upper bound on b_0. We choose $v \equiv 0$ and obtain (3.5). Similarly, by considering the dual variational principle (3.4), we observe that the admissible choice $\sigma = \overline{\sigma} \in V^*(Y)$, where $\overline{\sigma}$ denotes the mean value of σ over Y, gives the lower bound (3.6).

Remark 1. If b is single-valued then $b_0 = b$ and all subscript zeros can be dropped.

Remark 2. For the special case when $p = 2$ and a is linear and of the form $a(x,\xi) = \alpha(x)\xi$ we recover the classical Reuss-Voigt-Wiener bounds. This follows immediately, since (3.5) becomes

$$(b\xi,\xi) \leqslant \int_Y (\alpha(x)\xi,\xi)\,\mathrm{d}x,$$

which can also be expressed as

$$b \leqslant \int_Y \alpha(x)\,\mathrm{d}x \cdot I,$$

where I denotes the unit matrix in $\mathbb{R}^n$. Analogously (3.6) becomes

$$(b^{-1}\xi,\xi) \leqslant \int_Y (\alpha^{-1}(x)\xi,\xi)\,\mathrm{d}x,$$

or equivalently

$$b \geqslant \left(\int_Y \alpha^{-1}(x)\,\mathrm{d}x\right)^{-1} \cdot I.$$

For the case $a(x,\xi) = \alpha(x)|\xi|^{p-1}\xi$, i.e. the p-Laplacian, we recover the Reuss-Voigt-Wiener bounds earlier proved in [15].

3.2. Hashin-Shtrikman type bounds

The Hashin-Shtrikman (H-S) variational principle, which led to the proof of upper and lower so called H-S bounds for linear heterogeneous structures was developed in [12] and [13]. The extension of the H-S variational principle to the non-linear monotone case is due to a pioneering work by Willis [30]. Thereafter, the theory has been unified via a series of papers by e.g. Ponte Castaneda, Suquet, Talbot-Willis and Willis, see the introduction. In all these papers the construction of bounds is based on controllability of the overall energy via variational principles for the energy density ψ. We here contribute with a non-linear version of the H-S variational principle based on the homogenized operator b.

Following the standard approach we introduce a comparison function $\hat{\psi}\colon\ \Omega\times\mathbb{R}^n \to \mathbb{R}$, which has the same qualitative properties as the function ψ (see Proposition 2). In addition we assume that there exists at least one point $\xi_0 \in \mathbb{R}^n$ such that both $\psi(\xi_0)$ and $\hat{\psi}(\xi_0)$ are finite. We now use the Moreau-Rockafellar theorem, see Theorem 47B in [32, p. 389], and subtract and add the quantity $((\partial\hat{\psi})_\lambda(x,\xi+Dv),\xi+Dv)$ to (3.1):

$$(b_\lambda(\xi),\xi) = \inf_{v\in V(Y)} \left\{ \int_Y \big[\big((\partial_\xi(\psi-\hat{\psi})_\lambda(x,\xi+Dv(x)),\xi+Dv(x)\big) \right.$$

$$\left. + \big((\partial_\xi\hat{\psi})_\lambda(x,\xi+Dv(x)),\xi+Dv(x)\big)\big]\,\mathrm{d}x \right\}.$$

Passing to the limit, sending $\lambda \to 0$, we obtain

$$(b_0(\xi),\xi) = \inf_{v\in V(Y)} \left\{ \int_Y \big[\big((\partial_\xi(\psi-\hat{\psi})_0(x,\xi+Dv(x)),\xi+Dv(x)\big) \right.$$

$$\left. + \big((\partial_\xi\hat{\psi})_0(x,\xi+Dv(x)),\xi+Dv(x)\big)\big]\,\mathrm{d}x \right\}.$$

If we further assume that the first term in the last integral is positive we can apply the Legendre-Young-Fenchel transform to it and obtain, for some admissible Y-periodic, $\mathbb{R}^n$-valued function η,

$$(b_0(\xi),\xi) = \inf_{v\in V(Y)} \sup_\eta \left\{ \int_Y [(\xi+Dv(x),\eta(x)) - \big(((\partial_\xi(\psi-\hat{\psi})^{-1})_0(x,\eta(x)),\eta(x)\big) \right.$$
$$\left. + \big((\partial_\xi\hat{\psi})_0(x,\xi+Dv(x),\xi+Dv(x)\big)]\,\mathrm{d}x \right\}.$$

If we instead insert the comparison functional $\hat{\psi}$ into (3.3) we arrive, after a limit passage in λ and analogous reasoning as above, at

$$((b^{-1})_0(\xi),\xi) = \inf_{\sigma\in V^*(Y)} \sup_\mu \left\{ \int_Y [(\xi+\sigma(x),\mu(x)) - ((\partial_\xi(\psi-\hat{\psi})_0(x,\mu(x)),\mu(x)) \right.$$
$$\left. + \big(((\partial_\xi\hat{\psi})^{-1})_0(x,\xi+\sigma(x),\xi+\sigma(x)\big)]\,\mathrm{d}x \right\},$$

where μ is an admissible Y-periodic $\mathbb{R}^n$-valued function. The variational principles (3.7) and (3.8) are generalizations of the classical H-S variational principles. By convexity we can now interchange the order of inf and sup and obtain, for any admissible η and μ,

$$(b_0(\xi),\xi) \geqslant \inf_{v\in V(Y)} \left\{ \int_Y \big[(\xi+Dv(x),\eta(x)) - \big(((\partial_\xi(\psi-\hat{\psi})^{-1})_0(x,\eta(x)),\eta(x)\big) \right.$$

$$\left. + ((\partial_\xi\hat{\psi})_0(x,\xi+Dv(x),\xi+Dv(x))]\,\mathrm{d}x \right\}$$

and

$$((b^{-1})_0(\xi),\xi) \geqslant \inf_{\sigma\in V^*(Y)} \left\{ \int_Y [(\xi+\sigma(x),\mu(x)) - ((\partial_\xi(\psi-\hat{\psi})_0(x,\mu(x)),\mu(x)) \right.$$
$$\left. + \big(((\partial_\xi\hat{\psi})^{-1})_0(x,\xi+\sigma(x),\xi+\sigma(x))\big] \,\mathrm{d}x \right\}.$$

If we now take the supremum over the minimal sections $\hat{\psi}_0$ of all admissible comparison functions $\hat{\psi}$ we obtain

$$(b_0(\xi),\xi) \geqslant \sup_{\hat{\psi}_0} \inf_{v\in V(Y)} \left\{ \int_Y \big[(\xi+Dv(x),\eta(x)) - \big(((\partial_\xi(\psi-\hat{\psi})^{-1})_0(x,\eta(x)),\eta(x)\big) \right.$$
$$\left. + ((\partial_\xi\hat{\psi})_0(x,\xi+Dv(x),\xi+Dv(x))\big] \,\mathrm{d}x \right\}$$

and

$$((b^{-1})_0(\xi),\xi) \geqslant \sup_{\hat{\psi}_0} \inf_{\sigma\in V^*(Y)} \left\{ \int_Y [(\xi+\sigma(x),\mu(x)) - ((\partial_\xi(\psi-\hat{\psi})_0(x,\mu(x)),\mu(x)) \right.$$
$$\left. + \big(((\partial_\xi\hat{\psi})^{-1})_0(x,\xi+\sigma(x),\xi+\sigma(x))\big] \,\mathrm{d}x \right\},$$

which are the general forms of H-S type lower and upper bounds, respectively.

Remark 3. By letting $\eta\equiv 0$ in (3.9) we obtain for the constitutive relation an analogous lower bound as the one introduced in Ponte-Castaneda [18] for the corresponding effective energy functional, see also Talbot-Willis [23] and [24].

Remark 4. One has the freedom to choose the comparison functional in a suitable way, to obtain more explicit expressions for the bounds. This has been performed for instance in the study of effective behaviour of composite materials. In this case ψ and $\hat{\psi}$ are erergy density functions for certain composite materials. One possibility which has been used by e.g. Ponte-Castaneda and Talbot-Willis is to choose the comparison material with energy density $\hat{\psi}$ to be linear but with the same microstructure as the material whose effective behaviour is to be bounded.

Remark 5. In the present study we have considered scalar-valued functions u and used Γ-convergence results for such functions. The results on Γ-convergence remain valid also in the vector-valued case but one needs to change the notation a bit. This means that the results presented can be stated in an elasticity setting, see [22], and be seen as a complement to the works by e.g. Ponte Castaneda and Talbot and Willis.

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