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MATHEMATICAL MODELS OF SUSPENSION BRIDGES

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Abstract. In this work we try to explain various mathematical models describing the dynamical behaviour of suspension bridges such as the Tacoma Narrows bridge. Our attention is concentrated on the derivation of these models, an interpretation of particular parameters and on a discussion of their advantages and disadvantages. Our work should be a starting point for a qualitative study of dynamical structures of this type and that is why we have a closer look at the models, which have not been studied in literature yet. We are also trying to find particular conditions for unique solutions of some models.

Keywords: dynamical behaviour, suspension bridges, Tacoma Narrows bridge

MSC 2000: 35B10, 70K30, 73K05

1. Introduction

One of the most problematic and not fully explained areas of mathematical modelling involves nonlinear dynamical systems, especially systems with the so called jumping nonlinearity. It can be seen that its presence brings unexpected difficulties into the whole problem and very often it is a cause of nonuniqueness of solutions.

A suspension bridge is an example of such a dynamical system. The nonlinear aspect is caused by the presence of supporting cable stays, which restrain the movement of the center span of the bridge in a downward direction, but have no influence on its behaviour in the opposite direction.

The fact that we deal with a serious and topical problem is demonstrated for example by the collapse of the Tacoma Narrows suspension bridge (1940), which came unexpectedly into large-scale oscillations, followed by the destruction of the whole structure. So it would be very contributive to determine under what conditions a similar situation cannot occur, and find out safe parameters of the bridge construction.

In the second chapter, we show various possibilities how to model the behaviour of suspension bridges and give a survey of known facts. Then—in the third chapter—we introduce our own results concerning existence and uniqueness of time-periodic solutions of two chosen models—a single beam and a beam coupled with a vibrating string by nonlinear cables.

2. Survey of mathematical models and known results

2.1. One-dimensional model of a suspension bridge

One of the easiest ways how to model the behaviour of a suspension bridge is to describe it as a vibrating one-dimensional beam with simply supported ends. In the first step, we do not have to take into account the other two dimensions because proportions of the bridge in these dimensions are very small in comparison with its length and so can be omitted (see Figure 1). If we neglect also the influence of the towers and side parts, we can use the mentioned model of a simply supported one-dimesional beam.

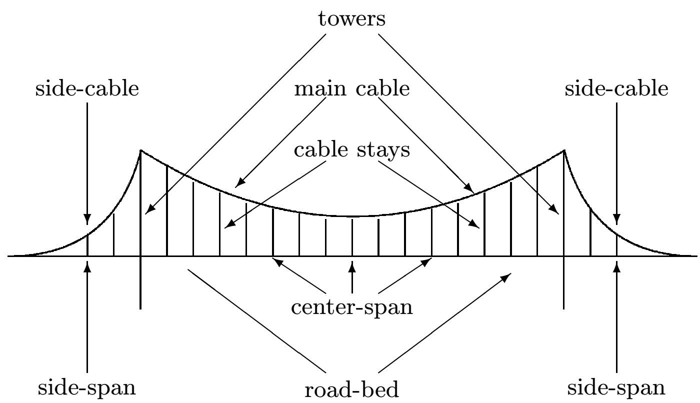

Figure 1. The main ingredients in a model of a one-dimensional suspension bridge.

Our problem will be divided into two basic cases—with and without a damping term.

2.1.1. Damped model—the first idealization

Let us consider a vibrating beam with simply supported ends. It is subjected to the gravitation force, to the external periodic force (e.g. due to the wind) and in the opposite direction to the restoring force of the cable stays. The construction holding these stays is taken as a solid and immovable object.

Our system is illustrated in Figure 2.

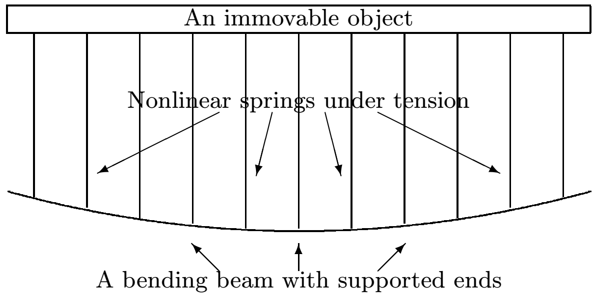

Figure 2. The simplest model of a suspension bridge—the bending beam with simply supported ends, held by nonlinear cables, which are fixed on an immovable construction.

The displacement $u(x,t)$ of this beam is described by a nonlinear partial differential equation

$$m\frac{\partial^2 u(x,t)}{\partial t^2} + EI\frac{\partial^4 u(x,t)}{\partial x^4} + b\frac{\partial u(x,t)}{\partial t} = -\kappa u^+(x,t) + W(x) + \varepsilon f(x,t), \tag{1}$$

with the boundary conditions

$$\begin{aligned} &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \\ &u(x, t+2\pi) = u(x,t), \quad -\infty < t < \infty,\ x \in (0,L). \end{aligned} \tag{2}$$

The meaning of the particular parameters used in the equation is the following:

m	mass per unit length of the bridge
E	Young's modulus
I	moment of inertia of the cross section
b	damping coefficient
κ	stiffness of the cables (spring constant)
W	weight per unit length of the bridge
εf	external time-periodic forcing term (due to the wind)
L	length of the center-span of the bridge

As we can see from the equation (1) and the boundary conditions (2), we are describing vibrations of a beam of length L, with simply supported ends. Its deflection $u(x,t)$ at the point x and at time t is measured in the downward direction. The first term in the equation represents an inertial force, the second term is an elastic force and the last term on the left hand side describes a viscous damping. On the right hand side, we have the influence of the cable stays, the gravitation force and the external force due to the wind (we assume it to be time-periodic). The cable stays can be taken as one-sided springs obeying Hooke's law, with a restoring force proportional to the displacement if they are stretched, and with no restoring force if they are compressed. This fact is described by the expression κu^+, where $u^+ = \max\{0, u\}$ and κ is a coefficient which characterizes the stiffness of the cable stays.

We have not considered the inertial effects of the rotation motion (in the plane xu) in the equation since they are usually omitted.

This model was introduced in a paper [6] by A. C. Lazer and P. J. McKenna and has been used as the starting point for study of suspension bridges in all cited works by the other authors. It does not describe exactly the behaviour of a suspension bridge but on the other hand it is reasonably simple and applicable.

For further considerations, it would be useful to transform the equation (by making a change of the scale of the variable x and dividing by the mass m) to the form

$$
\begin{aligned}
&u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + k u^+ = W(x) + \varepsilon f(x,t),\\
&u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0,\\
&u(x, t+2\pi) = u(x,t), \quad -\infty < t < \infty,\ x \in (0,\pi),
\end{aligned} \tag{3}
$$

where $\alpha^2 = \frac{EI}{m}\left(\frac{\pi}{L}\right)^4 \neq 0$ and $\beta = \frac{b}{m} > 0$. (We use the same symbols for rescaled W, ε and f.)

Besides the mentioned paper by A. C. Lazer and P. J. McKenna, who described this model in detail but did not deal with it in this form, we can find contributive results e.g. in a work [2] by P. Drábek. He proved the existence of at least one generalized solution of the equation (3) for an arbitrary right hand side. In case there is no external force (it means no wind), the bridge achieves a unique position (called the equilibrium) determined only by its weight $W(x)$. Under some special assumptions on $W(x)$, the paper [2] shows that in case of small external disturbances, there is always a solution "near" to the equilibrium. If we assume that $W(x) = W_0 \sin x$ and the periodic function $f(x,t)$ is of a special form then there is another solution which is in a certain sense "far" from this position.

However, there are still many open questions left.

2.1.2. Damped model—the second idealization

Another possible but a little more complicated process is not to consider the construction holding the cable stays as an immovable object, but to treat it as a vibrating string, coupled with the beam of the roadbed by nonlinear cable stays (see Figure 3).

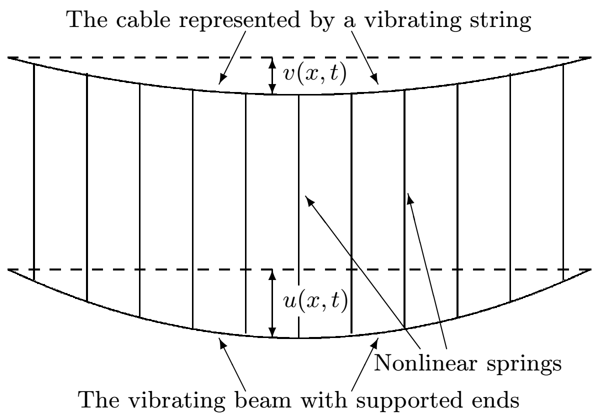

Figure 3. A more complicated model of a one-dimensional suspension bridge—the coupling of the main cable (a vibrating string) and the roadbed (a vibrating beam) by the stays treated as nonlinear springs.

Instead of one equation, we have now a system of two connected equations in the form

$$
\begin{aligned}
&m_1 v_{tt} - T v_{xx} + b_1 v_t - \kappa(u-v)^+ = W_1 + \varepsilon f_1(x,t), \\
&m_2 u_{tt} + EI u_{xxxx} + b_2 u_t + \kappa(u-v)^+ = W_2 + \varepsilon f_2(x,t),
\end{aligned}
\tag{4}
$$

with boundary conditions

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = v(0,t) = v(L,t) = 0,$$

where $v(x,t)$ measures the displacement of the vibrating string representing the main cable and $u(x,t)$ means—as in the previous section—the displacement of the bending beam standing for the roadbed of the bridge. Both functions are considered to be periodic in the time variable. The nonlinear stays connecting the beam and the string pull the cable down, hence we have the minus sign at $k(u-v)^+$ in the first equation, and hold the roadbed up, therefore we consider the plus sign at the same term in the second equation.

We can find a description of this model again in A. C. Lazer and P. J. McKenna [6], but these authors consider the right hand sides in a rather purer form. In the first equation, they neglect the weight of the string W_1, and on the other hand, in the second equation, they ignore the external force $\varepsilon f_2(x,t)$. However, nobody (as far as we know) has treated this model in detail yet.

2.1.3. Non-damped model—the first idealization

This section is quite analogous to the previous ones, we only omit the damping terms in all equations. This is rather unrealistic, but the equations will be much simpler and we will be able to deal with them more easily. Another advantage of this approach is the fact that the model in this case has a variational structure and enables us to use not only topological methods, but also variational principles for its investigation.

Let us consider the partial differential equation describing the motion of a non-damped beam with simply supported ends

$$mu_{tt} + EIu_{xxxx} + \kappa u^{+} = W(x) + \varepsilon f(x,t), \tag{5}$$

with the boundary conditions

$$\begin{aligned} &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \\ &u(x,t+2\pi) = u(x,t), \quad -\infty < t < \infty,\ x \in (0,L). \end{aligned} \tag{6}$$

We use the same symbols as in the previous parts, and after the same transformation as in Section 2.1.1 we get

$$\begin{aligned} &u_{tt} + \alpha^2 u_{xxxx} + ku^{+} = W(x) + \varepsilon f(x,t), \\ &u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0, \\ &u(x,t+2\pi) = u(x,t), \quad -\infty < t < \infty,\ x \in (0,\pi). \end{aligned} \tag{7}$$

This case was studied by W. Walter and P. J. McKenna in the paper [7]. Under an additional assumption $\alpha = 1$, they proved the following theorem:

Theorem 2.1. *Let $W(x) \equiv W_0$ (positive constant) and let $f(x,t)$ be a function even and π-periodic in the time variable t and symmetric in the space variable x about $\frac{\pi}{2}$. Then, if $0 < k < 3$, the equation (7) has a unique periodic solution of the period π, which corresponds to small oscillations about the equilibrium. If $3 < k < 15$, the equation has in addition another periodic solution with a large amplitude.*

In other words, this theorem says that strengthening the stays, which means increasing the coefficient k, can paradoxly lead to the destruction of the bridge.

Unfortunately, we do not know whether the corresponding result holds as well for a model where damping is present, or whether the restriction $k < 15$ can be removed.

2.1.4. Non-damped model—the second idealization

We can again omit the damping term in the system of partial differential equations describing the behaviour of a suspension bridge with a movable main cable holding the nonlinear cable stays. The corresponding equations can be written in the following way:

$$
\begin{aligned}
&m_1 v_{tt} - T v_{xx} - \kappa(u-v)^+ = W_1 + \varepsilon f_1(x,t),\\
&m_2 u_{tt} + EI u_{xxxx} + \kappa(u-v)^+ = W_2 + \varepsilon f_2(x,t),\\
&u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = v(0,t) = v(L,t) = 0,
\end{aligned}
\tag{8}
$$

where $u(x,t)$ and $v(x,t)$ are again periodic in time.

However, also in this case, there are no works known which would treat the system (8) and its solutions.

2.2. Reduction to the system of ODEs by discretization in the space variable

The complexity of the models described in the previous chapters led naturally to an effort to simplify the equations even more. One of the possible attitudes is to transform the partial differential equation into a simpler system of ordinary differential equations by using discretization in the space variable. We can again consider two cases—with and without the damping term.

2.2.1. Damped model

We replace the function $u(x,t)$, which represents the displacement of the bridge, by a vector

$$\vec{u}(t) = [u_1(t), u_2(t), \dots, u_N(t)]^{\mathrm{T}}.$$

If we use the spatial discretization by finite differences of the equation (1) with the boundary conditions (2) (the model of a suspension bridge described by Lazer and McKenna), we obtain a system of ordinary differential equations

$$\vec{u}'' + \beta\vec{u}' + \mathbf{A}\vec{u} + k\vec{u}^+ = \vec{p}(t). \tag{9}$$

$\mathbf{A}$ is a symmetric matrix of order N

$$
\mathbf{A} = \frac{EI}{m}\delta^{-4}
\begin{pmatrix}
5 & -4 & 1 & & & & \\
-4 & 6 & -4 & 1 & & & \\
1 & -4 & 6 & -4 & 1 & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & 1 & -4 & 6 & -4 & 1 \\
 & & & 1 & -4 & 6 & -4 \\
 & & & & 1 & -4 & 5
\end{pmatrix}
\tag{10}
$$

with the constant $\delta = \frac{L}{N+1}$, N is the number of division points, $\vec{p}(t)$ stands for the discretized vector of external forces $W(x) + \varepsilon f(x,t)$ and the symbol $\vec{u}^+$ represents the vector

$$[u_1^+, u_2^+, \ldots, u_N^+]^{\mathrm{T}}.$$

This simplified model of a suspension bridge can be found e.g. in the paper [1] by J. M. Alonso and R. Ortega. The most interesting conclusion they came to is a theorem, which says the following.

Theorem 2.2. *If the condition*

$$k < \beta^2 + 2\alpha\beta, \tag{11}$$

where $\alpha = \sqrt{\frac{EI}{m}}\left(\frac{\pi}{L}\right)^2$, *holds then there exists* $N_0 \in \mathbb{N}$ *such that if* $N \geqslant N_0$ *then the discretization of a suspension bridge equation* (9) *has a unique bounded solution that is exponentially asymptotically stable in the large.*

This result has a similar sense as the theorem stated in Section 2.1.3—the more flexible the cable stays are, the better the situation is and the oscillations of the bridge cannot be too high.

Unfortunately, we do not have any idea whether the condition (11) is not too restrictive and whether this conclusion holds for the original, non-discretized equation.

2.2.2. Non-damped model

This case is again a direct analogue of the previous one. We only consider the damping coefficient β equal to zero.

But it is easy to see that we cannot use the results stated above (we would get $k < 0$, which contradicts the fact that the coefficient k must be positive).

Unfortunately, we do not know any papers which treat this problem. However, if we considered a system with a sufficiently small damping term, this model would be applicable and we could interpret any contingent results in a reasonable way.

2.3. Reduction to the ODE on the basis of a special form of the right hand side

Another possible way how to simplify the complicated model described by the partial differential equation is to consider the right hand side of a special form

$$W_0 \sin x + \varepsilon f(t) \sin x \tag{12}$$

and eliminate thus the space variable from the whole equation. If we assume that the expected solution can be written in an analogous form

$$u(x,t) = y(t)\sin x, \tag{13}$$

(which expresses a natural assumption that the response of the bridge has the same form as the external force), we get—after substituting into the original equation—an ordinary equation.

The assumption that the weight of the bridge has a special form $W_0 \sin x$ is not very realistic but it simplifies the whole problem in a pleasant way. On the other hand, there is no reason why not to consider the bridge to be excited by a force which is expressed as a function $f(t)\sin x$.

Our problem can be again divided into two cases—with and without the damping term.

2.3.1. Damped model—the first idealization

We will use the Lazer and McKenna model described by the equation (3). After substituting the right hand side (12) and the supposed form of the solution (13) into the equation (3), we get a reduced model

$$y'' + \beta y' + \alpha^2 y + ky^+ = W_0 + \varepsilon f(t). \tag{14}$$

We use again the same symbols and so the constant α means the expression $\sqrt{\frac{EI}{m}\left(\frac{\pi}{L}\right)^2}$. The function $y(t)$ is supposed to be periodic in time.

This equation was studied especially by J. Glover, A. C. Lazer and P. J. McKenna (see [5]). We will briefly recall their results.

First of all, the authors rewrote the equation using simple transformations into the equivalent form

$$y'' + \varepsilon cy' + by^+ - ay^- = 1 + \varepsilon g(t), \tag{15}$$

where $b = \alpha^2 + k$, $a = \alpha^2$ and the symbol y^- means $max\{0, -y\}$. Then they proved that with a mild nondegeneracy condition on the function $g(t)$ and with ε and c sufficiently small, there exist large amplitude solutions of the equation (15), which are asymptotically stable and close to a translated solution u_0 of the equation

$$u'' + bu^+ - au^- = 1.$$

2.3.2. Damped model—the second idealization

The same reduction can be done also for the more complicated model of a suspension bridge described by the system of two equations (4) with the right hand sides of Lazer and McKenna. After a simple transformation, we get

$$
\begin{aligned}
z'' + \beta_1 z' + \alpha_1^2 z - k_1(y-z)^+ &= \varepsilon g(t), \\
y'' + \beta_2 y' + \alpha_2^2 y + k_2(y-z)^+ &= W_0,
\end{aligned}
\tag{16}
$$

where y a z are periodic functions in t.

However, the system stays very complicated even after this simplification. This follows e.g. from the fact that we have not found any published results in this area yet.

2.3.3. Non-damped model—the first idealization

If we consider a zero damping term in the reduced models stated above, the situation changes considerably.

If we omit the damping term in the equation (14), we get

$$
y'' + \alpha^2 y + k y^+ = W_0 + \varepsilon f(t), \tag{17}
$$

which can be transformed into

$$
y'' + b y^+ - a y^- = 1 + \varepsilon g(t). \tag{18}
$$

This problem is mentioned again in the paper [6] by A. C. Lazer and P. J. McKenna, where we can find the following results:

If $b \neq n^2$ (for n a positive integer), we can explicitly write down a 2π-periodic solution of the equation (18)*:*

$$
y = \frac{1}{b} + \varepsilon y_1(t).
$$

Here y_1 represents the 2π-periodic solution of the equation $y'' + by = g(t)$. This is an obvious and expected solution: the external force $1+\varepsilon g(t)$ produces a displacement $\frac{1}{b}$ and a small oscillation about this new equilibrium of order of magnitude ε. Moreover, if $n^2 < a, b < (n+1)^2$, then this is a unique solution of period 2π.

However, if the difference between a and b is large, then additional oscillatory solutions exist, and their order of magnitude is that of the exciting constant, which is in our case equal to 1.

2.3.4. Non-damped model—the second idealization

The last reduced model we consider here is a model described by the system of equations (16) (again with the simplified right hand sides of Lazer and McKenna), but this time without the damping term:

$$
\begin{aligned}
z'' + \alpha_1^2 z - k_1(y-z)^+ &= \varepsilon g(t), \\
y'' + \alpha_2^2 y + k_2(y-z)^+ &= W_0.
\end{aligned}
\tag{19}
$$

This is the only studied case in which the displacement of the main cable holding the cable stays was considered. We can find it again in the paper [6] by A. C. Lazer and P. J. McKenna. They proved under an additional assumption that k_2 and ε are sufficiently small that the equations (19) have periodic solutions with large and small amplitudes. The large amplitude solutions were of the following form: y (the displacement of the roadbed) close to the equilibrium, and z (the motion of the cable) large. This is the phenomenon called "galloping cables".

2.4. Two-dimensional model of a suspension bridge

So far we have considered only one-dimensional models of a suspension bridge. But we can choose a more general approach. Namely, not to restrict the problem to the simplified model, but include also another dimension and the torsional oscillations in its direction, which are certainly not quite negligible. (They are said to be one of the direct causes of the destruction of the above mentioned Tacoma Narrows bridge.)

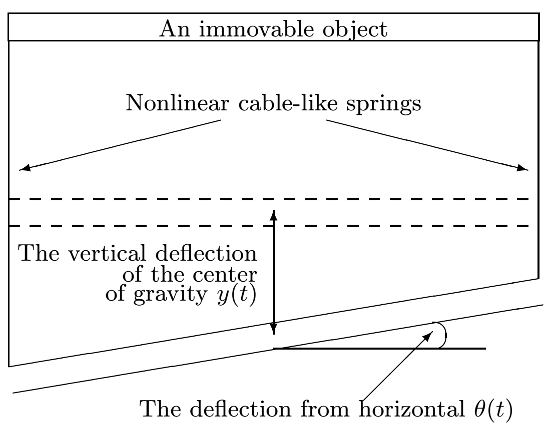

Figure 4. Cross section of the two-dimensional model of a suspension bridge.

The two-dimensional suspension bridge can be modelled (in the simplest way) as a long narrow vibrating plate, coupled at its side with the main cables by the nonlinear cable stays. The unknown functions are the displacement $y(x,t)$ measured at the centre of gravity, and $\theta(x,t)$, which measures the angle of the bar from the horizontal—see Figure 4.

The solution which has a large y component and a small θ component would be primarily a vertical motion, whereas the solution with a small y component and a large θ component would be primarily a torsional motion.

This model is mentioned in the paper [6] by A. C. Lazer and P. J. McKenna, but the authors themselves admit that it is a very complicated model and no particular results are known even under the simplifying assumption that the unknown functions do not depend on the space variable x.

3. Main results—existence and uniqueness of the solution

As we can see from the previous survey of known results, the main problem is to prove the existence of the solutions of particular models and find out the conditions under which the solution is unique and stable. Practically it means that we are looking for conditions which guarantee that the bridge cannot exhibit large-scale oscillations and cannot be destructed by any wind of an arbitrary power. We have tried to summarize and clear up these problems for two one-dimensional models—the first considering the bridge as a single beam supported by nonlinear springs, and the second describing the bridge as a beam coupled with a string by nonlinear cables.

3.1. The first case—a single beam

As we have stated above, we model the suspension bridge as a one-dimensional beam with simply supported ends, which is held by nonlinear springs hanging on an immovable construction. This situation is described by the boundary value problem

$$\begin{aligned} &mu_{tt} + EIu_{xxxx} + bu_t + \kappa u^+ = h(x,t),\\ &u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0,\\ &u(x,t+2\pi) = u(x,t), \quad -\infty < t < \infty,\ x \in (0,L), \end{aligned} \tag{20}$$

which can be transformed (as in Section 2.1.1) into a new form

$$\begin{aligned} &u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + ku^+ = h(x,t),\\ &u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0,\\ &u(x,t+2\pi) = u(x,t), \quad -\infty < t < \infty,\ x \in (0,\pi) \end{aligned} \tag{21}$$

(see A. C. Lazer, P. J. McKenna [6]).

3.1.1. Preliminaries

Let us denote by $\Omega = (0, \pi) \times (0, 2\pi)$ the domain considered, by $H = L_2(\Omega)$ the usual Hilbert space with the corresponding L_2-norm

$$\|u(x,t)\| = \left[\int_\Omega |u(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}}$$

and by $\mathscr{D}$ the set of all smooth functions satisfying the boundary conditions from equation (21). Now we can generalize the notion of a classical solution by which we mean a continuous function with continuous derivatives up to the fourth order with respect to x and up to the second order with respect to t in the set $[0, \pi] \times [0, 2\pi]$, satisfying the boundary value problem (21), and define the so called generalized solution of (21).

Definition 3.1. A function $u(x,t) \in H$ is called a generalized solution of the boundary value problem (21) if and only if the integral identity

$$\int_\Omega u(v_{tt} + \alpha^2 v_{xxxx} - \beta v_t)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega (h - ku^+)v\,\mathrm{d}x\,\mathrm{d}t$$

holds for all $v \in \mathscr{D}$.

R e m a r k 3.1. It is obvious that if the problem (21) has a classical solution, than it is also the generalized solution. But the properties required of a classical solution are too strong and the integral identity stated above can be satisfied by a more general function.

Let us consider a Sobolev space $\tilde{H} = H + \mathrm{i}H$. As the set

$$\{\mathrm{e}^{int} \sin mx;\ n \in \mathbb{Z}, m \in \mathbb{N}\}$$

forms a complete orthogonal system in this space, each function $u(x,t)$ can be represented by the Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} u_{nm}\mathrm{e}^{int} \sin mx. \tag{22}$$

Moreover, we have $\sum_n \sum_m |u_{nm}|^2 < \infty$, and $u_{-nm} = \bar{u}_{nm}$ (see P. Drábek [2]).

3.1.2. The linear beam equation

First of all, we will treat the solvability of the equation

$$u_{tt} + \alpha^2 u_{xxxx} + \beta u_t - \lambda u = h. \tag{23}$$

If we define a generalized solution of this equation in an analogous way as in Definition 3.1, then the following lemma is an easy consequence of the expansion (22).

Lemma 3.1. *If u_{nm} and h_{nm} are the corresponding Fourier coefficients of the functions u and h, then the equation* (23) *has a generalized solution if and only if*

$$(-n^2+\alpha^2 m^4+\mathrm{i}\beta n-\lambda)u_{nm}=h_{nm} \tag{24}$$

holds for all $n\in\mathbb{Z}$, $m\in\mathbb{N}$.

If we denote

$$L(u)=u_{tt}+\alpha^2 u_{xxxx}+\beta u_t$$

the linear operator, and put

$$N_\lambda=\{(m,n)\in\mathbb{N}\times\mathbb{Z}\,;\ \alpha^2 m^4-n^2-\lambda=0\},$$

$$S=\{\lambda\in\mathbb{R};\ N_\lambda\neq\emptyset\},$$
$$\sigma=\{\lambda\in\mathbb{R};\ \lambda=\alpha^2 q^4, q\in\mathbb{N}\},$$

then σ is the set of eigenvalues of the operator L, and $\sigma\subset S$ holds. Further, we can rewrite the equation (23) into a new form

$$L(u)-\lambda u=h$$

and formulate the following theorem (see P. Drábek [2]).

Theorem 3.2. *Let $\lambda\in\mathbb{R}$. Then for an arbitrary $h\in H$ the equation* (23) *has a unique generalized solution $u\in H$ if and only if*

$$\lambda\notin\sigma.$$

If $\lambda\notin\sigma$, then there exists a mapping

$$T_\lambda\colon\ H\to H,\quad T_\lambda\colon\ h\to u$$

with the following properties:
(i) *T_λ is linear and $\mathscr{R}(T_\lambda)\subset C(\overline{\Omega})$;*
(ii) *T_λ is compact from H into $C(\overline{\Omega})$ (and thus from H into H) and for its norm we have*

$$\|T_\lambda\|\leqslant\frac{1}{\max\{\operatorname{dist}(\lambda,S),\min\{\beta,\operatorname{dist}(\lambda,\sigma)\}\}}$$
$$=\frac{1}{\min\{\operatorname{dist}(\lambda,\sigma),\max\{\beta,\operatorname{dist}(\lambda,S)\}\}}.$$

P r o o f. I. ($\Rightarrow$) Let $h \in H$ be arbitrary and let there exist a generalized solution $u \in H$ of the equation (23). Then according to Lemma 3.1, the equation

$$(-n^2 + \alpha^2 m^4 + \mathrm{i}\beta n - \lambda) u_{nm} = h_{nm}$$

must have a solution for all $m \in \mathbb{N}$, $n \in \mathbb{Z}$. Hence it follows that

$$\lambda \neq \alpha^2 m^4,$$

which means

$$\lambda \notin \sigma.$$

II. ($\Leftarrow$) Let $\lambda \notin \sigma$ and $h \in H$. Let us define an auxiliary function k_λ by the expression

$$k_\lambda \colon (x,t) \to \sum_{m,n} (-n^2 + \alpha^2 m^4 + \mathrm{i}\beta n - \lambda)^{-1} \mathrm{e}^{\mathrm{i}nt} \cos mx.$$

It is clear that thanks to the assumption $\lambda \notin \sigma$ and $\lambda \in \mathbb{R}$ this function is well defined. If we make an odd extension of the function h (or u) in the space variable x to the interval $\langle -\pi, \pi \rangle$, we can put

$$u(x,t) = \frac{1}{2\pi^2} (k_\lambda * h),$$

where

$$k_\lambda * h \colon (x,t) \to \int_0^{2\pi} \int_{-\pi}^{\pi} k_\lambda(x-\xi, t-\tau) h(\xi,\tau) \,\mathrm{d}\xi \,\mathrm{d}\tau.$$

If we use this expression, we obtain

$$\begin{aligned} u(x,t) = \frac{1}{2\pi^2} \int_0^{2\pi} \int_{-\pi}^{\pi} &\Big[\sum_{m,n} (-n^2 + \alpha^2 m^4 + \mathrm{i}\beta n - \lambda)^{-1} \mathrm{e}^{\mathrm{i}n(t-\tau)} \cos m(x-\xi) \Big] \\ &\times \Big[\sum_{r,s} h_{rs} \mathrm{e}^{\mathrm{i}r\tau} \sin s\xi \Big] \,\mathrm{d}\xi \,\mathrm{d}\tau \\ = \frac{1}{2\pi^2} &\sum_{m,n} \sum_{r,s} (-n^2 + \alpha^2 m^4 + \mathrm{i}\beta n - \lambda)^{-1} h_{rs} \\ &\times \Big[\mathrm{e}^{\mathrm{i}nt} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(r-n)\tau} \,\mathrm{d}\tau \int_{-\pi}^{\pi} \cos m(x-\xi) \sin s\xi \,\mathrm{d}\xi \Big]. \end{aligned}$$

Moreover, thanks to the orthogonality of the basis functions on the domain $\langle -\pi, \pi \rangle \times \langle 0, 2\pi \rangle$, the following relations are satisfied:

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(r-n)\tau} \,\mathrm{d}\tau = \begin{cases} 2\pi & \text{for} r = n, \\ 0 & \text{for} r \neq n, \end{cases}$$

as well as

$$\int_{-\pi}^{\pi} \cos m(x-\xi) \sin s\xi \,\mathrm{d}\xi = \begin{cases} \pi \sin mx & \text{for} m = s, \\ 0 & \text{for} m \neq s. \end{cases}$$

This means that

$$u(x,t) = \sum_{m,n} (-n^2 + \alpha^2 m^4 + \mathrm{i}\beta n - \lambda)^{-1} h_{nm} \mathrm{e}^{int} \sin mx$$

and $u_{nm} = (-n^2 + \alpha^2 m^4 + \mathrm{i}\beta n - \lambda)^{-1} h_{nm} = (k_\lambda)_{nm} h_{nm}$ for $m \in \mathbb{N}$, $n \in \mathbb{Z}$. So according to Lemma 3.1, u is a unique generalized solution of the equation (23).

An operator T_λ will be defined in the following way:

$$T_\lambda(h) = \frac{1}{2\pi^2} (k_\lambda * h) . \tag{25}$$

Obviously, this operator is linear, and if we define u equal to zero outside the set $\langle -\pi, \pi \rangle \times \langle 0, 2\pi \rangle$, we have the following estimate for $\delta_1, \delta_2 \in \mathbb{R}$:

$$\begin{aligned} &|u(x+\delta_1, t+\delta_2) - u(x,t)| \\ &= \frac{1}{2\pi^2} \left| \int_0^{2\pi} \int_{-\pi}^{\pi} [k_\lambda(x+\delta_1-\xi, t+\delta_2-\tau) - k_\lambda(x-\xi, t-\tau)] \, h(\xi,\tau) \,\mathrm{d}\xi \,\mathrm{d}\tau \right| \\ &\leqslant \frac{1}{\sqrt{2}\pi^2} \|h\| \left(\int_0^{2\pi} \int_{-\pi}^{\pi} |k_\lambda(x+\delta_1-\xi, t+\delta_2-\tau) - k_\lambda(x-\xi, t-\tau)|^2 \,\mathrm{d}\xi \,\mathrm{d}\tau \right)^{\frac{1}{2}} . \end{aligned}$$

As $k_\lambda \in L_2(\Omega)$, the function u is continuous and thus $\mathscr{R}(T_\lambda) \subset C(\overline{\Omega})$. Moreover, T_λ maps any bounded subset of the space H onto a bounded subset of uniformly continuous functions in the space $C(\overline{\Omega})$. The Arzelà-Ascoli theorem implies that the mapping $T_\lambda \colon H \to C(\overline{\Omega})$ is compact. As $C(\overline{\Omega}) \subset H$, the operator $T_\lambda \colon H \to H$ is compact as well.

The estimate of the norm $\|T_\lambda\|$ can be obtained by an easy calculation:

$$\begin{aligned} &\|T_\lambda(h)\| = \|u\| \\ &= \left[\int_\Omega \left| \sum_{m,n} (\mathrm{i}\beta n + \alpha^2 m^4 - n^2 - \lambda)^{-1} h_{nm} \mathrm{e}^{int} \sin mx \right|^2 \mathrm{d}x \,\mathrm{d}t \right]^{\frac{1}{2}} \\ &\leqslant \max_{m,n} \frac{1}{|\mathrm{i}\beta n + \alpha^2 m^4 - n^2 - \lambda|} \underbrace{\left[\int_\Omega \left| \sum_{m,n} h_{nm} \mathrm{e}^{int} \sin mx \right|^2 \mathrm{d}x \,\mathrm{d}t \right]^{\frac{1}{2}}}_{\|h\|} \end{aligned}$$

and thus

$$
\begin{aligned}
\|T_\lambda\| &\leqslant \max_{m,n} \frac{1}{\sqrt{\beta^2 n^2 + (\alpha^2 m^4 - n^2 - \lambda)^2}} \\
&\leqslant \frac{1}{\max\{\operatorname{dist}(\lambda, S), \min\{\beta, \operatorname{dist}(\lambda, \sigma)\}\}} \\
&= \frac{1}{\min\{\operatorname{dist}(\lambda, \sigma), \max\{\beta, \operatorname{dist}(\lambda, S)\}\}}.
\end{aligned}
$$

□

3.1.3. Banach contraction theorem

Now we turn our attention to the equation (21) and deal with its solvability.

As zero is not an eigenvalue of the operator L, we can rewrite this equation—in accordance with the previous paragraph—into an equivalent form

$$u = T_0(-ku^+ + W + \varepsilon f). \tag{26}$$

If we assume (with respect to the real values of the parametres) that $\alpha^2 \leqslant 1$, we have the following estimate for the norm of the operator T_0:

$$\|T_0\| \leqslant \max_{m\in\mathbb{N}, n\in\mathbb{Z}} \frac{1}{\sqrt{\beta^2 n^2 + (\alpha^2 m^4 - n^2)^2}} \leqslant \frac{1}{\min\{\alpha^2, \sqrt{\beta^2 + (\alpha^2 m_0^4 - 1)^2}\}} = K_0,$$

where m_0 solves the problem

$$\min_{m\in\mathbb{N}} |\alpha^2 m^4 - 1|.$$

If we want to find out conditions for the existence of a unique solution, it is suitable to use the *Banach contraction theorem*, which reads as follows:

Theorem 3.3. *Let the operator $G\colon H \to H$ be a contraction, i.e. there exists $c \in (0, 1)$ such that*

$$\|G(u) - G(v)\| \leqslant c\|u - v\| \quad \forall u, v \in H.$$

Then there exists a unique u_0 such that

$$G(u_0) = u_0.$$

In our case $G(u) = T_0(-ku^+ + W + \varepsilon f)$ and

$$\begin{aligned}\|G(u) - G(v)\| &= \|T_0(W + \varepsilon f - ku^+) - T_0(W + \varepsilon f - kv^+)\| \\ &= \|T_0(kv^+ - ku^+)\| \\ &\leqslant k\|T_0\|\|v^+ - u^+\| \\ &\leqslant kK_0\|v - u\|.\end{aligned}$$

If we require the operator G to be a contraction, the condition

$$0 < kK_0 < 1$$

must be satisfied, and thus

$$0 < \frac{k}{\min\{\alpha^2, \sqrt{\beta^2 + (\alpha^2 m_0^4 - 1)^2}\}} < 1.$$

Hence, if we put again $k = \frac{\kappa}{m}$, a sufficient condition for the existence of a unique solution of our boundary value problem has the final form

$$\boxed{\kappa < m \min\left\{\alpha^2, \sqrt{\beta^2 + (\alpha^2 m_0^4 - 1)^2}\right\}.} \tag{27}$$

3.1.4. A comparison with a discrete model

In the case of a discrete model (9), the uniqueness condition had the form

$$k < \beta^2 + 2\beta\alpha$$

(see R. M. Alonso, R. Ortega [1]). If instead of (27) we consider the roughest estimate which is

$$k < \min\{\alpha^2, \beta\}, \tag{28}$$

we can make the following discussion.

(i) If the condition

$$\alpha \in \left(0; \beta + \sqrt{2}\beta\right\rangle \cup \left\langle \frac{1-\beta}{2}; \infty\right)$$

is satisfied, which means (in an equivalent formulation)

$$\beta \in \left\langle \sqrt{2}\alpha - \alpha; \infty\right) \cup \langle 1 - 2\alpha; \infty\rangle,$$

then the implication

$$\boxed{k < \min\{\alpha^2, \beta\} \quad \Longrightarrow \quad k < \beta^2 + 2\alpha\beta}$$

holds and the result of Alonso and Ortega is stronger than (28).

(ii) On the other hand, if the condition

$$\alpha \in \left\langle \beta + \sqrt{2}\beta; \frac{1-\beta}{2} \right\rangle$$

is satisfied, which again means

$$\beta \in \left(0; \sqrt{2}\alpha - \alpha\right\rangle \cap (0; 1 - 2\alpha\rangle\,,$$

then the implication

$$\boxed{k < \beta^2 + 2\alpha\beta \quad \Longrightarrow \quad k < \min\{\alpha^2, \beta\}}$$

holds and our result (28) is stronger than that of Alonso and Ortega.

Remark 3.2.

1. For physical reasons we take into account only positive values of the parameters α and β.
2. In particular, the previous discussion means that for sufficiently small α and β in a certain relation, our result is stronger than the result published in Alonso, Ortega [1].

3.2. The second case—the coupling of a beam and a string

In this part we complete the previous model by a movable main cable, which holds the nonlinear cable stays and is represented by a vibrating string. Our model is described by a coupled system of partial differential equations (see A. C. Lazer, P. J. McKenna [6])

$$\begin{aligned}
&m_1 v_{tt} - T v_{xx} + b_1 v_t - \kappa(u-v)^+ = h_1(x,t),\\
&m_2 u_{tt} + EI u_{xxxx} + b_2 u_t + \kappa(u-v)^+ = h_2(x,t),\\
&u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = v(0,t) = v(L,t) = 0,\\
&-\infty < t < \infty, x \in (0,L),
\end{aligned} \tag{29}$$

where $v(x,t)$ means the displacement of the vibrating string and $u(x,t)$ represents the deflection of the beam. Both functions are considered to be periodic in time.

We can transform both equations into a simpler form in the same way as in Section 2.1.1. It means that we divide by the mass m_1, and m_2 respectively, and change the scale of the space variable x. Then we obtain

$$
\begin{aligned}
&v_{tt} - \alpha_1^2 v_{xx} + \beta_1 v_t - k_1(u-v)^+ = h_1(x,t),\\
&u_{tt} + \alpha_2^2 u_{xxxx} + \beta_2 u_t + k_2(u-v)^+ = h_2(x,t),\\
&u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = v(0,t) = v(\pi,t) = 0,\\
&-\infty < t < \infty, x \in (0,\pi),
\end{aligned}
\tag{30}
$$

where $\alpha_1^2 = \frac{T}{m_1}\left(\frac{\pi}{L}\right)^2$, $\alpha_2^2 = \frac{EI}{m_2}\left(\frac{\pi}{L}\right)^4$, $k_1 = \frac{\kappa}{m_1}$, $k_2 = \frac{\kappa}{m_2}$, $\beta_1 = \frac{b_1}{m_1}$ a $\beta_2 = \frac{b_2}{m_2}$. We use the same symbols as in the previous equation for the other transformed parameters.

3.2.1. Preliminaries

If we introduce a new vector function

$$
\mathbf{w} = \begin{bmatrix} v \\ u \end{bmatrix},
\tag{31}
$$

we can rewrite the system (30) into the matrix form

$$
\begin{aligned}
&\underbrace{\begin{bmatrix} 10 \\ 01 \end{bmatrix}}_{\mathbf{I}} \mathbf{w}_{tt} + \underbrace{\begin{bmatrix} 00 \\ 0\alpha_2^2 \end{bmatrix}}_{\mathbf{A}_2} \mathbf{w}_{xxxx} + \underbrace{\begin{bmatrix} -\alpha_1^2 0 \\ 00 \end{bmatrix}}_{\mathbf{A}_1} \mathbf{w}_{xx}\\
&+ \underbrace{\begin{bmatrix} \beta_1 0 \\ 0\beta_2 \end{bmatrix}}_{\mathbf{B}} \mathbf{w}_t + \mathbf{F}(\mathbf{w}) = \underbrace{\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}_{\mathbf{h}}
\end{aligned}
\tag{32}
$$

and thus

$$
\mathbf{w}_{tt} + \mathbf{A}_2 \mathbf{w}_{xxxx} + \mathbf{A}_1 \mathbf{w}_{xx} + \mathbf{B}\mathbf{w}_t + \mathbf{F}(\mathbf{w}) = \mathbf{h},
\tag{33}
$$

where $\mathbf{F}(\mathbf{w})$ is a nonlinear vector function

$$
\mathbf{F}(\mathbf{w}) = \begin{bmatrix} -k_1(u-v)^+ \\ k_2(u-v)^+ \end{bmatrix}.
$$

Moreover, we require the unknown function $\mathbf{w}(x,t)$ to be time-periodic and to satisfy the boundary conditions prescribed for a vibrating string in its first component, and the boundary conditions prescribed for a supported beam in its second component (see (30)).

If we again denote our domain by $\Omega = (0, \pi) \times (0, 2\pi)$, we have

$$\mathbf{w} \in \mathbf{H} = H \times H = L_2(\Omega) \times L_2(\Omega).$$

The corresponding norm can be introduced e.g. in this way:

$$\|\mathbf{w}\| = \sum_{i=1}^{2} \|w_i\|_{L_2} = \left[\int_\Omega v^2 \,\mathrm{d}x\right]^{\frac{1}{2}} + \left[\int_\Omega u^2 \,\mathrm{d}x\right]^{\frac{1}{2}}.$$

We can again define the notion of a generalized solution.

Definition 3.2. A vector function $\mathbf{w} = [v, u]^{\mathrm{T}} \in \mathbf{H}$ is called *a generalized solution* of the boundary value problem (30) with the right hand side $\mathbf{h}(x, t) = [h_1, h_2]^{\mathrm{T}}$ if and only if the integral identities

$$\int_\Omega v(\tilde{v}_{tt} - \alpha_1^2 \tilde{v}_{xx} - \beta_1 \tilde{v}_t) \,\mathrm{d}x \,\mathrm{d}t = \int_\Omega [h_1 + k_1(u - v)^+] \tilde{v} \,\mathrm{d}x \,\mathrm{d}t,$$
$$\int_\Omega u(\tilde{u}_{tt} + \alpha_2^2 \tilde{u}_{xxxx} - \beta_2 \tilde{u}_t) \,\mathrm{d}x \,\mathrm{d}t = \int_\Omega [h_2 - k_2(u - v)^+] \tilde{u} \,\mathrm{d}x \,\mathrm{d}t$$

hold for all $\tilde{\mathbf{w}} = [\tilde{v}, \tilde{u}]^{\mathrm{T}} \in \mathscr{D}$, where $\mathscr{D}$ is the set of all smooth vector functions satisfying the prescribed boundary conditions.

As the system

$$\{\mathrm{e}^{\mathrm{i}nt} \sin mx ;\ n \in \mathbb{Z}, m \in \mathbb{N}\}$$

forms a complete orthogonal system in the subspace $\tilde{H} = H + \mathrm{i}H$, we can write

$$\mathbf{w} = \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} v_{nm} \mathrm{e}^{\mathrm{i}nt} \sin mx \\ \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} u_{nm} \mathrm{e}^{\mathrm{i}nt} \sin mx \end{bmatrix}. \tag{34}$$

3.2.2. The operator formulation

Let us denote

$$\mathbf{L}(\mathbf{w}) = \mathbf{w}_{tt} + \mathbf{A}_2 \mathbf{w}_{xxxx} + \mathbf{A}_1 \mathbf{w}_{xx} + \mathbf{B} \mathbf{w}_t.$$

Then $\mathbf{L}$ is a linear operator and the equation (33) can be written in the form

$$\mathbf{L}(\mathbf{w}) = -\mathbf{F}(\mathbf{w}) + \mathbf{h}. \tag{35}$$

First of all, we should determine the real eigenvalues of the operator $\mathbf{L}$. This problem means to find all $\lambda \in \mathbb{R}$ for which the equation

$$\mathbf{L}(\mathbf{w}) = \lambda \mathbf{w}$$

has a nontrivial solution. If we use the description of $\mathbf{w}$ by the Fourier series (34), we obtain an equivalent expression

$$\begin{aligned} (-n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n - \lambda) v_{nm} &= 0, \\ (-n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n - \lambda) u_{nm} &= 0, \end{aligned} \tag{36}$$

where $n \in \mathbb{Z}$ a $m \in \mathbb{N}$. Since the parameter λ is real, the only possible n for which the system (36) has a nontrivial solution is

$$n = 0.$$

If we express the previous system in the matrix form, we get

$$\begin{bmatrix} \alpha_1^2 m^2 - \lambda & 0 \\ 0 & \alpha_2^2 m^4 - \lambda \end{bmatrix} \begin{bmatrix} v_{m0} \\ u_{m0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and the corresponding characteristic equation is

$$(\alpha_1^2 m^2 - \lambda)(\alpha_2^2 m^4 - \lambda) = 0.$$

Then we obtain the eigenvalues and the corresponding eigenvectors:

$$\begin{aligned} \lambda_1 &= \alpha_1^2 m^2 \ldots \mathbf{w}_1 = \begin{bmatrix} v_{m0} \sin mx \\ 0 \end{bmatrix} \\ \lambda_2 &= \alpha_2^2 m^4 \ldots \mathbf{w}_2 = \begin{bmatrix} 0 \\ u_{m0} \sin mx \end{bmatrix} \qquad \forall m \in \mathbb{N}. \end{aligned}$$

The set of all real eigenvalues of the operator $\mathbf{L}$ will be denoted by

$$\sigma = \{\lambda \in \mathbb{R};\ \lambda = \alpha_1^2 m^2 \vee \lambda = \alpha_2^2 m^4, \forall m \in \mathbb{N}\}.$$

3.2.3. The linear "string-beam" model

As a starting point, we will consider the solvability of the linear equation

$$\mathbf{L}(\mathbf{w}) - \lambda \mathbf{w} = \mathbf{h}, \tag{37}$$

which can be written in an equivalent form

$$
\begin{aligned}
&v_{tt} - \alpha_1^2 v_{xx} + \beta_1 v_t - \lambda v = h_1, \\
&u_{tt} + \alpha_2^2 u_{xxxx} + \beta_2 u_t - \lambda u = h_2.
\end{aligned}
\tag{38}
$$

The following theorem can be proved for this system.

Theorem 3.4. *Let $\lambda \in \mathbb{R}$. Then for an arbitrary $\mathbf{h} \in \mathbf{H}$ the equation (37) has a unique solution $\mathbf{w} \in \mathbf{H}$ if and only if*

$$\lambda \notin \sigma.$$

If $\lambda \notin \sigma$ then there exists a mapping

$$\mathbf{T}_\lambda \colon \mathbf{H} \to \mathbf{H}, \quad \mathbf{T}_\lambda \colon \mathbf{h} \to \mathbf{w}$$

with the following properties:

(i) $\mathbf{T}_\lambda$ *is linear and* $\operatorname{Im}\mathbf{T}_\lambda \subset C(\overline{\Omega}) \times C(\overline{\Omega})$*;*

(ii) $\mathbf{T}_\lambda$ *is compact from* $\mathbf{H}$ *into* $C(\overline{\Omega}) \times C(\overline{\Omega})$ *(and thus from* $\mathbf{H}$ *into* $\mathbf{H}$*), and for its norm we have an estimate*

$$\|\mathbf{T}_\lambda\| \leqslant \max\left\{ \max_{m,n} \frac{1}{|A_{nm}^\lambda|}\,;\ \max_{m,n} \frac{1}{|B_{nm}^\lambda|} \right\},$$

where

$$
\begin{aligned}
A_{nm}^\lambda &= -n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n - \lambda, \\
B_{nm}^\lambda &= -n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n - \lambda.
\end{aligned}
$$

Proof. I. ($\Rightarrow$) Let there exist a generalized solution $\mathbf{w}$ of the equation (37) for an arbitrary right hand side $\mathbf{h} \in \mathbf{H}$. Then the system

$$
\begin{aligned}
(-n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n - \lambda) v_{nm} &= (h_1)_{nm}, \\
(-n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n - \lambda) u_{nm} &= (h_2)_{nm}
\end{aligned}
$$

must be solvable for all $m \in \mathbb{N}$, $n \in \mathbb{Z}$. Hence it follows that

$$\lambda \neq \alpha_1^2 m^2 \quad \wedge \quad \lambda \neq \alpha_2^2 m^4,$$

which means

$$\lambda \notin \sigma.$$

II. ($\Leftarrow$) Let $\lambda \notin \sigma$ and $\mathbf{h} \in \mathbf{H}$. Let us define auxiliary functions $k_{\lambda 1}$, $k_{\lambda 2}$ by the expressions

$$\begin{aligned} k_{\lambda 1} \colon\ (x,t) &\to \sum_{m,n} (-n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n - \lambda)^{-1} \mathrm{e}^{\mathrm{i}nt} \cos mx, \\ k_{\lambda 2} \colon\ (x,t) &\to \sum_{m,n} (-n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n - \lambda)^{-1} \mathrm{e}^{\mathrm{i}nt} \cos mx. \end{aligned}$$

If we make an odd extension of the functions h_1, h_2 (or v, u) in the space variable x to the interval $\langle -\pi, \pi\rangle$, we can put

$$\begin{aligned} v(x,t) &= \frac{1}{2\pi^2}(k_{\lambda 1} * h_1), \\ u(x,t) &= \frac{1}{2\pi^2}(k_{\lambda 2} * h_2). \end{aligned}$$

Thanks to the orthogonality of the basis functions on $\langle -\pi, \pi\rangle \times \langle 0, 2\pi\rangle$, we have

$$\begin{aligned} v(x,t) &= \sum_{m,n} (-n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n - \lambda)^{-1} (h_1)_{nm} \mathrm{e}^{\mathrm{i}nt} \sin mx, \\ u(x,t) &= \sum_{m,n} (-n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n - \lambda)^{-1} (h_2)_{nm} \mathrm{e}^{\mathrm{i}nt} \sin mx, \end{aligned}$$

and $v_{nm} = (k_{\lambda 1})_{nm}(h_1)_{nm}$, $u_{nm} = (k_{\lambda 2})_{nm}(h_2)_{nm}$ for $m \in \mathbb{N}$, $n \in \mathbb{Z}$. Thus $\mathbf{w} = [v,u]^{\mathrm{T}}$ is a unique solution of the system (38) and so a unique solution of the operator equation (37).

An operator $\mathbf{T}_\lambda$ will be defined in the following way:

$$\mathbf{T}_\lambda(\mathbf{h}) = \frac{1}{2\pi^2} \begin{bmatrix} k_{\lambda 1} * h_1 \\ k_{\lambda 2} * h_2 \end{bmatrix}. \tag{39}$$

It is clear that this operator is linear, and if we again define both functions u, v equal to zero outside the set $\langle -\pi, \pi\rangle \times \langle 0, 2\pi\rangle$, we have the following estimate for $\delta_1, \delta_2 \in \mathbb{R}$:

$$
\begin{aligned}
&\|\mathbf{w}(x+\delta_1, t+\delta_2) - \mathbf{w}(x,t)\| = \|\mathbf{T}_\lambda(\mathbf{h}(x+\delta_1, t+\delta_2)) - \mathbf{T}_\lambda(\mathbf{h}(x,t))\| \\
&= \frac{1}{2\pi^2}\|(k_{\lambda 1} * h_1)(x+\delta_1, t+\delta_2) - (k_{\lambda 1} * h_1)(x,t)\| \\
&\quad + \frac{1}{2\pi^2}\|(k_{\lambda 2} * h_2)(x+\delta_1, t+\delta_2) - (k_{\lambda 2} * h_2)(x,t)\| \\
&= \frac{1}{2\pi^2}\left\|\int_0^{2\pi}\int_{-\pi}^{\pi}\left[k_{\lambda 1}(x+\delta_1-\xi, t+\delta_2-\tau) - k_{\lambda 1}(x-\xi, t-\tau)\right] h_1(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau\right\| \\
&\quad + \frac{1}{2\pi^2}\left\|\int_0^{2\pi}\int_{-\pi}^{\pi}\left[k_{\lambda 2}(x+\delta_1-\xi, t+\delta_2-\tau) - k_{\lambda 2}(x-\xi, t-\tau)\right] h_2(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau\right\| \\
&\leqslant \frac{1}{\sqrt{2}\pi^2}\|h_1\| \\
&\quad \times \left\|\int_0^{2\pi}\int_{-\pi}^{\pi}\left|k_{\lambda 1}(x+\delta_1-\xi, t+\delta_2-\tau) - k_{\lambda 1}(x-\xi, t-\tau)\right|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t\right\|^{\frac{1}{2}} \\
&\quad + \frac{1}{\sqrt{2}\pi^2}\|h_2\| \\
&\quad \times \left\|\int_0^{2\pi}\int_{-\pi}^{}\pi\left|k_{\lambda 2}(x+\delta_1-\xi, t+\delta_2-\tau) - k_{\lambda 2}(x-\xi, t-\tau)\right|^2\,\mathrm{d}\xi\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}t\right\|^{\frac{1}{2}}.
\end{aligned}
$$

As $k_{\lambda 1}, k_{\lambda 2} \in L_2(\Omega)$, the function v, u are continuous and thus $\mathscr{R}(\mathbf{T}_\lambda) \subset C(\overline{\Omega}) \times C(\overline{\Omega})$. Moreover, $\mathbf{T}_\lambda$ maps any bounded subset of the space $\mathbf{H}$ onto a bounded subset of uniformly continuous functions in the space $C(\overline{\Omega}) \times C(\overline{\Omega})$. The Arzelà-Ascoli theorem implies that the mapping $\mathbf{T}_\lambda\colon \mathbf{H} \to C(\overline{\Omega}) \times C(\overline{\Omega})$ is compact. As $C(\overline{\Omega}) \subset H$, the operator $\mathbf{T}_\lambda\colon \mathbf{H} \to \mathbf{H}$ is compact as well.

If we denote

$$
\begin{aligned}
A^\lambda_{nm} &= -n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n - \lambda, \\
B^\lambda_{nm} &= -n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n - \lambda,
\end{aligned}
$$

we obtain by an easy calculation estimate of the norm of the operator $\mathbf{T}_\lambda$:

$$
\begin{aligned}
\|\mathbf{T}_\lambda(\mathbf{h})\| = \|\mathbf{w}\| &= \|v\| + \|u\| \\
&= \left[\int_\Omega \left|\sum_{m,n}(A^\lambda_{nm})^{-1}(h_1)_{nm}\mathrm{e}^{\mathrm{i}nt}\sin mx\right|^2\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}} \\
&\quad + \left[\int_\Omega \left|\sum_{m,n}(B^\lambda_{nm})^{-1}(h_2)_{nm}\mathrm{e}^{\mathrm{i}nt}\sin mx\right|^2\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{2}}
\end{aligned}
$$

$$\leqslant \max_{m,n} \frac{1}{|A^{\lambda}_{nm}|} \underbrace{\left[\int_{\Omega} \left|\sum_{m,n} (h_1)_{nm} \mathrm{e}^{\mathrm{i}nt} \sin mx\right|^2 \mathrm{d}x \, \mathrm{d}t\right]^{\frac{1}{2}}}_{\|h_1\|}$$

$$+ \max_{m,n} \frac{1}{|B^{\lambda}_{nm}|} \underbrace{\left[\int_{\Omega} \left|\sum_{m,n} (h_2)_{nm} \mathrm{e}^{\mathrm{i}nt} \sin mx\right|^2 \mathrm{d}x \, \mathrm{d}t\right]^{\frac{1}{2}}}_{\|h_2\|}$$

$$\leqslant \max \left\{ \max_{m,n} \frac{1}{|A^{\lambda}_{nm}|} ; \; \max_{m,n} \frac{1}{|B^{\lambda}_{nm}|} \right\} \|\mathbf{h}\|$$

and hence

$$\|\mathbf{T}_{\lambda}\| \leqslant \max \left\{ \max_{m,n} \frac{1}{|A^{\lambda}_{nm}|} ; \; \max_{m,n} \frac{1}{|B^{\lambda}_{nm}|} \right\}.$$

□

3.2.4. Banach contraction theorem

Now we turn our attention to the original equation $\mathbf{L}(\mathbf{w}) = -\mathbf{F}(\mathbf{w}) + \mathbf{h}$ and its solvability.

As zero is not an eigenvalue of the operator $\mathbf{L}$, we can—in accordance with the previous paragraph—define the operator $\mathbf{T}_0$ and estimate its norm as follows:

$$\|\mathbf{T}_{0}\| \leqslant \max \left\{ \max_{m,n} \frac{1}{|A^{0}_{nm}|} ; \; \max_{m,n} \frac{1}{|B^{0}_{nm}|} \right\}.$$

Further,

$$\begin{aligned}
\max_{m,n} \frac{1}{|A^{0}_{nm}|} &= \max_{m,n} \frac{1}{|-n^2 + \alpha_1^2 m^2 + \mathrm{i}\beta_1 n|} = \max_{m,n} \frac{1}{\sqrt{\beta_1^2 n^2 + (\alpha_1^2 m^2 - n^2)^2}} \\
&\leqslant \frac{1}{\min\{\alpha_1^2, \sqrt{\beta_1^2 + (\alpha_1^2 m_{01}^2 - 1)^2}\}}, \\
\max_{m,n} \frac{1}{|B^{0}_{nm}|} &= \max_{m,n} \frac{1}{|-n^2 + \alpha_2^2 m^4 + \mathrm{i}\beta_2 n|} = \max_{m,n} \frac{1}{\sqrt{\beta_2^2 n^2 + (\alpha_2^2 m^4 - n^2)^2}} \\
&\leqslant \frac{1}{\min\{\alpha_2^2, \sqrt{\beta_2^2 + (\alpha_2^2 m_{02}^4 - 1)^2}\}},
\end{aligned}$$

where m_{01}, m_{02} solve the problems

$$\min_{m \in \mathbb{N}} |\alpha_1^2 m^2 - 1|, \qquad \min_{m \in \mathbb{N}} |\alpha_2^2 m^4 - 1|, \quad \text{respectively.}$$

Hence we finally obtain

(40)

$$\|\mathbf{T}_0\| \leqslant \max\left\{\frac{1}{\min\{\alpha_1^2, \sqrt{\beta_1^2+(\alpha_1^2 m_{01}^2-1)^2}\}}; \frac{1}{\min\{\alpha_2^2, \sqrt{\beta_2^2+(\alpha_2^2 m_{02}^4-1)^2}\}}\right\}$$
$$= \frac{1}{\min\left\{\alpha_1^2, \alpha_2^2, \sqrt{\beta_1^2+(\alpha_1^2 m_{01}^2-1)^2}, \sqrt{\beta_2^2+(\alpha_2^2 m_{02}^4-1)^2}\right\}} = \overline{K}_0.$$

If we use this operator $\mathbf{T}_0$, we can rewrite our equation (35) to the equivalent form

$$\mathbf{w} = \mathbf{T}_0(\mathbf{h} - \mathbf{F}(\mathbf{w})). \tag{41}$$

Since we want to prove its unique solvability, it is again suitable to apply the Banach contraction theorem.

In our case $\mathbf{G}(\mathbf{w}) = \mathbf{T}_0(\mathbf{h} - \mathbf{F}(\mathbf{w}))$. We have to verify whether this operator is a contraction:

$$\begin{aligned}
\|\mathbf{G}(\mathbf{w}_1) - \mathbf{G}(\mathbf{w}_2)\| &= \|\mathbf{T}_0(\mathbf{h} - \mathbf{F}(\mathbf{w}_1)) - \mathbf{T}_0(\mathbf{h} - \mathbf{F}(\mathbf{w}_2))\| \\
&= \|\mathbf{T}_0\|\|\mathbf{F}(\mathbf{w}_2) - \mathbf{F}(\mathbf{w}_1)\| \\
&\leqslant \|\mathbf{T}_0\|(k_1+k_2)\|(u_2-v_2)^+ - (u_1-v_1)^+\| \\
&\leqslant \|\mathbf{T}_0\|(k_1+k_2)\|(u_2-v_2) - (u_1-v_1)\| \\
&= \|\mathbf{T}_0\|(k_1+k_2)\|(u_2-u_1) - (v_2-v_1)\| \\
&\leqslant \|\mathbf{T}_0\|(k_1+k_2)\,[\|u_2-u_1\| + |v_2-v_1\|] \\
&\leqslant (k_1+k_2)\overline{K}_0\|\mathbf{w}_2 - \mathbf{w}_1\|.
\end{aligned}$$

Hence it follows that the operator $\mathbf{G}$ is a contraction if the condition

$$0 < (k_1+k_2)\overline{K}_0 < 1$$

holds. Equivalently,

$$k_1 + k_2 < \min\left\{\alpha_1^2, \alpha_2^2, \sqrt{\beta_1^2+(\alpha_1^2 m_{01}^2-1)^2}, \sqrt{\beta_2^2+(\alpha_2^2 m_{02}^4-1)^2}\right\}.$$

As we have $k_1 = \frac{\kappa}{m_1}$, $k_2 = \frac{\kappa}{m_2}$, we obtain a condition of existence of a unique solution of the operator equation (35) in the form

$$\kappa < \frac{m_1 m_2}{m_1+m_2}\min\left\{\alpha_1^2, \alpha_2^2, \sqrt{\beta_1^2+(\alpha_1^2 m_{01}^2-1)^2}, \sqrt{\beta_2^2+(\alpha_2^2 m_{02}^4-1)^2}\right\}, \tag{42}$$

or (if we use the roughest estimate)

$$\boxed{\kappa < \frac{m_1 m_2}{m_1 + m_2} \min\left\{\alpha_1^2, \alpha_2^2, \beta_1, \beta_2\right\}.} \tag{43}$$

4. Discussion

The question left is whether the condition (42) is stronger or weaker than the condition

$$\kappa = m_2 k < m_2 \min\{\alpha_2^2, \sqrt{\beta_2^2 + (\alpha_2^2 m_{02}^4 - 1)^2}\},$$

obtained in the same way for the bridge modelled only as a supported beam (i.e. by a scalar equation)—see (27).

We can expect that the mass of the main cable will be considerably less than the mass of the beam, and thus $\frac{m_1 m_2}{m_1+m_2} \simeq m_1$. The damping coefficients β_1 a β_2 can be considered almost the same.

The relation between α_1^2 and α_2^2 is still an open problem.

As for the real parameters of particular suspension bridges, we have found in the preprint [3] by A. Fonda, Z. Schneider and F. Zanolin the following values.

	Tacoma	Golden Gate	Bronx-Whitestone
m	$8.5 \times 10^3\,\mathrm{kg\,m^{-1}}$	$3.1 \times 10^4\,\mathrm{kg\,m^{-1}}$	$1.6 \times 10^4\,\mathrm{kg\,m^{-1}}$
I	$0.2\,\mathrm{m}^4$	$5.3\,\mathrm{m}^4$	$0.4\,\mathrm{m}^4$
L	$855\,\mathrm{m}$	$1280\,\mathrm{m}$	$700\,\mathrm{m}$

However, we still have not found anything concerning the real values of the stiffness of the cable stays k, of the inner tension T and the mass m_1 of the main cable.

5. Conclusion

All models described in Chapter 2 are very simplified (some of them more, some of them less) and do not correspond to the real behaviour of a suspension bridge exactly; but the more complicated and realistic the model is, the worse we can deal with it. On the other hand, if even the simplest models exhibit some anomalies (e.g. nonunique solutions, large-scale oscillations), it is reasonable to assume that the more accurate and complicated model will do so.

All cited authors came to following conclusions:

The jumping nonlinearity in the suspension bridge models causes that under some values of the particular parameters there exist not only the expected low-amplitude oscillations about the equilibrium, but also other solutions, which correspond to large scale oscillations and probably lead to the collapse of the bridge. This phenomenon occurs paradoxically in the case when the cable stays are sufficiently stiff (and the nonlinearity is more considerable).

A. C. Lazer and P. J. McKenna suggest in their paper [6] to solve this problem by a "linearization of the bridge", which means to equip the bridge construction with cables above and below the roadbed, thus eliminating the asymmetry.

All cited authors supported their theoretical statements by numerical results, which exactly correspond to the established fact, and moreover, to the conclusions made after the observation of the real behaviour of suspension bridges.

We have tried to deal with the models that have not been studied in detail yet, and to bring some new pieces of knowledge into this field.

By using the Banach contraction theorem, we have found a sufficient condition under which the existence of a unique solution of a beam equation, as well as of the "beam-string" system, is guaranteed.

Unfortunately, we are afraid that the conditions obtained are too restrictive and are not satisfied by the real values of the bridge parametres.

However, we hope that our results are contributive, and in future we would like to study these problems in more detail and to include other, so far not considered elements.

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