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BIFURCATION OF PERIODIC SOLUTIONS IN DIFFERENTIAL INCLUSIONS

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Abstract. Ordinary differential inclusions depending on small parameters are considered such that the unperturbed inclusions are ordinary differential equations possessing manifolds of periodic solutions. Sufficient conditions are determined for the persistence of some of these periodic solutions after multivalued perturbations. Applications are given to dry friction problems.

Keywords: multivalued mappings, differential inclusions, periodic solutions

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1. Introduction

Consider a mass attached to a spring and putting horizontally on a moving ribbon with a speed $v_0 \sin \omega t$. The resulting differential equation [2] has the form

$$\ddot{x} + q(x) + \mu \operatorname{sgn}(\dot{x} + v_0 \sin \omega t) = 0, \tag{1.1}$$

where $\operatorname{sgn} r = \frac{r}{|r|}$, $r \neq 0$ corresponds to the dry friction between the mass and ribbon, $q \in C^2(\mathbb{R}, \mathbb{R})$ represents the force of the spring and $\mu > 0$, $v_0 > 0$, $\omega > 0$ are constants. Since $\operatorname{sgn} r$ is discontinuous in $r = 0$, by taking the multivalued mapping

$$\operatorname{Sgn} r = \begin{cases} \operatorname{sgn} r & \text{for } r \neq 0 \\ [-1, 1] & \text{for } r = 0, \end{cases}$$

(1.1) is considered as a perturbed differential inclusion of the form

$$\dot{x} = y, \quad \dot{y} \in -q(x) - \mu \operatorname{Sgn}(y + v_0 \sin \omega t). \tag{1.2}$$

By assuming the existence of a $2\pi/\omega$-periodic solution γ of $\ddot{x} + q(x) = 0$, we are interested in the persistence of a $2\pi/\omega$-periodic solution near γ of (1.2) for $\mu > 0$ small.

Periodic and almost periodic solutions to dry friction problems are also investigated in [4–7]. The numerical analysis is given in the papers [14–16] for a mechanical model of a friction oscillator with simultaneous self- and external excitation. These papers [14–16] present a nice introduction to the phenomenon of dry friction as well. Finally let us note that equations similar to (1.1) also appear in electrical engineering (see [1, Chap. III]), related problems are studied in control systems (see [21]) as well, and dry friction problems were investigated already in [19], [20].

To deal with differential inclusions much more general than (1.2), in Section 3 we consider systems of perturbed differential inclusions which take the form

$$\dot{x}(t) \in f(x(t)) + \sum_{j=1}^{k} \mu_j f_j(x(t), \mu, t) \quad \text{a.e. on} \quad \mathbb{R} \tag{1.3}$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$, $\mu = (\mu_1, \ldots, \mu_k)$. By a solution of any differential inclusion we mean in this paper a function which is absolute continuous and satisfies the differential inclusion almost everywhere. The inner product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. $\mathbb{N}$, $\mathbb{Z}$ are the sets of natural or integer numbers, respectively.

We make the following assumptions about (1.3):

(i) $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $f_j\colon \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to 2^{\mathbb{R}^n} \setminus \emptyset$, $j = 1, \ldots, k$ are all upper-semicontinuous [4, 17] with compact and convex values.

(ii) The unperturbed equation $\dot{x} = f(x)$ has a manifold of 1-periodic solutions, i.e. there is an open subset $\mathcal{O} \subset \mathbb{R}^{d-1}$, $d \geqslant 1$ and a C^2-mapping $\gamma\colon \mathcal{O} \times \mathbb{R} \to \mathbb{R}^n$ such that $\gamma(\theta, t+1) = \gamma(\theta, t)$ and $\gamma(\theta, \cdot)$ is a solution of $\dot{x} = f(x)$.

(iii) $f_j(x, \mu, t+1) = f_j(x, \mu, t)$ for $j = 1, \ldots, k$.

We ask if any of these periodic solutions persists after perturbation (1.3). The case when f_j are all singlevalued and smooth is a classical problem and we refer the reader to the paper [3] for more details. The purpose of this paper is to extend some results of [3] to the multivalued case (1.3).

In Section 4, we consider the autonomous case of (1.3) when f_j are all independent of t. We also consider both a saddle-node and the Poincaré-Andronov bifurcations of periodic solutions (see [11]) for the multivalued case of (1.3). Section 5 is devoted to applications to dry friction perturbations and a multivalued van der Pol oscillator is studied as well. A discussion about modelling dry frictions is given in Section 6 together with a hint at an extension of results of this paper to singularly perturbed differential inclusions studied in [10]. In addition, a piecewise smoothly perturbed problem is presented.

Proofs of results of this paper are similar to [9, 10]. So we use the method of Lyapunov-Schmidt decomposition, which is presented in Section 2 together with the theory of generalized Leray-Schauder degree for multivalued mappings (see [4, p. 181], [17, p. 21] and [18, p. 962]).

2. Preliminary results

Consider the non-homogeneous variational equation

$$
\begin{aligned}
&\dot{x} = A(\theta,t)x + h(t) \quad \text{a.e. on} \quad [0,1],\\
&A(\theta,t) = D_x f(\gamma(\theta,t)), \quad h \in L^2 = L^2([0,1],\mathbb{R}^n),\\
&x \in C^p = \{y \in C([0,1],\mathbb{R}^n) \colon\ y(0) = y(1)\}
\end{aligned}
\tag{2.1}
$$

along with the homogeneous one

$$
\dot{x} = A(\theta,t)x, \quad x \in C^p. \tag{2.2}
$$

Differentiation with respect to θ and t of $\dot{\gamma}(\theta,t) = f(\gamma(\theta,t))$ yields respectively

$$
\begin{aligned}
&\frac{\partial}{\partial\theta_i}\dot{\gamma}(\theta,t) = A(\theta,t)\frac{\partial}{\partial\theta_i}\gamma(\theta,t), \quad \theta = (\theta_1,\dots,\theta_{d-1}),\\
&\ddot{\gamma}(\theta,t) = A(\theta,t)\dot{\gamma}(\theta,t).
\end{aligned}
$$

Consequently, (2.2) has solutions $\frac{\partial}{\partial\theta_i}\gamma(\theta,t)$, $\dot{\gamma}(\theta,t)$, $i = 1,\dots,d-1$. We assume that the following condition is satisfied:

- (iv) The only 1-periodic solutions of $\dot{x} = A(\theta,\cdot)x$ are linear combinations of $\frac{\partial}{\partial\theta_i}\gamma(\theta,t)$, $\dot{\gamma}(\theta,t)$, $i = 1,\dots,d-1$. Moreover, $\frac{\partial}{\partial\theta_i}\gamma(\theta,t)$, $\dot{\gamma}(\theta,t)$, $i = 1,\dots,d-1$ are linearly independent.

Let $U(\theta,t)$ be the fundamental solution of $\dot{x} = A(\theta,t)x$. Then of course $\ker(I - U(\theta,1))$ corresponds to the 1-periodic solutions of $\dot{x} = A(\theta,t)x$, hence

$$
\ker(I - U(\theta,1)) = \operatorname{span}\left\{\frac{\partial}{\partial\theta_i}\gamma(\theta,0),\ \dot{\gamma}(\theta,0),\ i = 1,\dots,d-1\right\}, \tag{2.3}
$$

and $\operatorname{im}(I - U(\theta,1))$ is a continuous vector bundle over $\mathcal{O}$.

The adjoint equation to (2.2) has the form

$$
\dot{x} = -A(\theta,t)^* x, \quad x \in C^p. \tag{2.4}
$$

The fundamental solution of (2.4) is $(U^{-1}(\theta,t))^*$. A 1-periodic solution of (2.4) has the form $(U^{-1}(\theta,t))^* y$, $y \in \mathbb{R}^n$ and $(U^{-1}(\theta,1))^* y = y$. We have

$$(U^{-1}(\theta,1))^* y = y \Longleftrightarrow U(\theta,1)^* y = y.$$

Consequently,

$$\ker(I - (U^{-1}(\theta,1))^*) = \ker(I - U(\theta,1)^*) = \big(\operatorname{im}(I - U(\theta,1))\big)^{\perp}.$$

Since by (2.3) $\operatorname{im}(I-U(\theta,1))$ is a continuous vector bundle over $\mathcal{O}$, $\big(\operatorname{im}(I-U(\theta,1))\big)^{\perp}$ is also a continuous vector bundle over $\mathcal{O}$. So there are linearly independent continuous mappings $v_i(\theta,t)$, $i = 1,\dots,d$, $v_i\colon\ \mathcal{O}\times\mathbb{R}\to\mathbb{R}^n$, $v_i(\theta,t+1) = v_i(\theta,t)$ and all v_i are solutions of (2.4). Then it is well-known [12] that (2.1) has a solution if and only if

$$\int_0^1 \langle h(s), v_i(\theta,s)\rangle\,\mathrm{d}s = 0, \qquad \forall\, i = 1,\dots,d,$$

and moreover, this solution x is unique provided that it satisfies

$$\int_0^1 \langle x(s), \frac{\partial}{\partial\theta_i}\gamma(\theta,s)\rangle\,\mathrm{d}s = 0, \qquad \forall\, i = 1,\dots,d-1, \qquad \int_0^1 \langle x(s), \dot\gamma(\theta,s)\rangle\,\mathrm{d}s = 0.$$

It is not hard to construct a projection $\Pi(\theta)\colon\ L^2 \to L^2$ such that Π depends continuously on θ, $\|\Pi(\theta)\|$ is bounded on any bounded closed subset of $\mathcal{O}$ and

$$\operatorname{im}\Pi(\theta) = \Big\{h \in L^2\colon \int_0^1 \langle h(s), v_i(\theta,s)\rangle\,\mathrm{d}s = 0, \qquad \forall\, i = 1,\dots,d\Big\}.$$

Let $K(\theta)\colon\ L^2 \to C^p$ be the linear operator defined so that $K(\theta)h$ is the unique solution of $\dot x = A(\theta,t)x + \Pi(\theta)h$, $x \in C^p$ satisfying $\int_0^1 \langle x(s), \frac{\partial}{\partial\theta_i}\gamma(\theta,s)\rangle\,\mathrm{d}s = 0, \quad \forall\, i = 1,\dots,d-1, \quad \int_0^1 \langle x(s), \dot\gamma(\theta,s)\rangle\,\mathrm{d}s = 0$. Of course, $K(\theta)$ depends continuously on θ, and $K(\theta)$ is a compact operator.

3. Periodic solutions for non-autonomous case

Let us scale $\mu \to s\mu$ for $s \in \mathbb{R} \setminus \{0\}$ and let us take the change of variables

$$x(t+\alpha) = \gamma(\theta, t) + sz(t), \quad \alpha \in \mathbb{R}, \quad s \in \mathbb{R} \setminus \{0\}. \tag{3.1}$$

Then (1.3) has the form

$$\dot{z}(t) - A(\theta, t)z(t) \in g(z(t), \theta, \alpha, \mu, s, t) \quad \text{a.e. on} \quad [0,1], \tag{3.2}$$

where

$$g(x, \theta, \alpha, \mu, s, t) = \Big\{ v \in \mathbb{R}^n \colon\ v \in \frac{1}{s}\Big[f(sx + \gamma(\theta, t)) - f(\gamma(\theta, t)) - A(\theta, t)sx\Big] \\ + \sum_{j=1}^{k} \mu_j f_j(sx + \gamma(\theta, t), s\mu, t + \alpha) \Big\}.$$

We note that g is defined only for $s \neq 0$ in the above form, but it is naturally extended to $s = 0$ by putting the term in the square brackets equal to zero. It is clear that $g \colon \mathbb{R}^n \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}^n} \setminus \emptyset$ is upper-semicontinuous with compact and convex values.

Using g we define a multivalued mapping

$$G \colon C^p \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to 2^{L^2}$$

by the formula

$$G(z, \theta, \alpha, \mu, s) = \Big\{ h \in L^2 \colon\ h(t) \in g(z(t), \theta, \alpha, \mu, s, t) \quad \text{a.e. on} \quad [0,1] \Big\},$$

so (3.2) is a multivalued equation for z of the form

$$\dot{z} - A(\theta, \cdot)z \in G(z, \theta, \alpha, \mu, s). \tag{3.3}$$

According to [4, p. 29, Problem 10] and [17, p. 17, Propositions 3.3–3.4], each of these sets $G(z, \theta, \alpha, \mu, s)$ is non-empty. Moreover, they are all closed, convex and bounded in L^2. Hence they are all weakly compact in L^2 (see [22]).

Now we put (3.3) in the homotopy

$$\dot{z} - A(\theta, \cdot)z \in F(z, \theta, \alpha, \mu, s, \lambda), \quad \lambda \in [0,1], \tag{3.4}$$

where

$$F(z,\theta,\alpha,\mu,s,\lambda) = \Big\{h \in L^2\colon\ h(t) \in p(z(t),\theta,\alpha,\mu,s,\lambda,t) \text{ a.e. on } [0,1]\Big\},$$

and

$$p(x,\theta,\alpha,\mu,s,\lambda,t) = \Big\{v \in \mathbb{R}^n:\ v \in \frac{\lambda}{s}\Big[f(sx+\gamma(\theta,t)) - f(\gamma(\theta,t)) - A(\theta,t)sx\Big]$$
$$+ \lambda \sum_{j=1}^{k} \mu_j f_j(sx+\gamma(\theta,t), s\mu, t+\alpha) + (1-\lambda)\sum_{j=1}^{k} \mu_j f_j(\gamma(\theta,t),0,t+\alpha)\Big\}.$$

Again p is extended to $s=0$ like g above. We put

$$L(\theta)\colon\ L^2 \to \mathbb{R}^d, \quad L(\theta)h = \left(\int_0^1 \langle h(s), v_1(\theta,s)\rangle\,\mathrm{d}s, \dots, \int_0^1 \langle h(s), v_d(\theta,s)\rangle\,\mathrm{d}s\right).$$

To solve (3.4), we decompose it by using the results of Section 2 in the following way:

$$(3.5) \qquad \begin{cases} 0 \in H(z,\theta,\alpha,\mu,s,\lambda) \\ H(z,\theta,\alpha,\mu,s,\lambda) = \Big\{\big(z - \lambda K(\theta)h, L(\theta)h\big)\colon\ h \in F(z,\theta,\alpha,\mu,s,\lambda)\Big\}. \end{cases}$$

We view H as a multivalued mapping

$$H\colon\ C^p \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times [0,1] \to 2^{C^p \times \mathbb{R}^d} \setminus \emptyset.$$

It is again clear that the multivalued mapping

$$p\colon\ \mathbb{R}^n \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times [0,1] \times \mathbb{R} \to 2^{\mathbb{R}^n} \setminus \emptyset$$

is upper-semicontinuous with compact and convex values. Furthermore, ranges of H are bounded provided that z, μ are bounded, s is small, θ is from compact subsets of $\mathcal{O}$ and $\alpha \in \mathbb{R}$, $\lambda \in [0,1]$ are arbitrary.

By using standard arguments (see [4, Remarks 5.5.1], [17, p. 18, Proposition 3.6] and [18, Proposition 1.7]), the mapping H is upper-semicontinuous with compact convex values and maps bounded sets into relatively compact ones. Hence topological degree methods like in [4, p. 154] and [17, p. 20] can be applied to (3.5).

Now we introduce a multivalued mapping

$$(3.6) \qquad M_\mu\colon\ \mathcal{O} \times \mathbb{R} \to 2^{\mathbb{R}^d} \setminus \emptyset, \quad M_\mu(\theta,\alpha) = \Big\{L(\theta)h\colon\ h \in L^2, \quad \text{satisfying}$$
$$\text{the relation} \quad h(t) \in \sum_{j=1}^{k} \mu_j f_j(\gamma(\theta,t),0,t+\alpha) \quad \text{a.e. on} \quad [0,1]\Big\}.$$

The mapping M_μ is again upper-semicontinuous with compact convex values and maps bounded sets into bounded ones. The boundedness is clear, so we show the upper-semicontinuity. Let

$$\begin{aligned}
&\mu_p \to \mu_0, \quad (\theta_p, \alpha_p) \to (\theta_0, \alpha_0), \quad h_p \in L^2, \quad p \in \mathbb{N}, \\
&h_p(t) \in \sum_{j=1}^{k} \mu_{pj} f_j(\gamma(\theta_p, t), 0, t + \alpha_p) \quad \text{a.e. on} \quad [0,1], \\
&L(\theta_p) h_p \to \overline{M}_0.
\end{aligned}$$

Since $\{h_p\}_1^p$ is bounded in L^2, we can assume that h_p tends weakly to some $h_0 \in L^2$. Then by applying the standard arguments (see the proof of [18, Proposition 1.4]), we obtain

$$\begin{aligned}
&h_0(t) \in \sum_{j=1}^{k} \mu_{0j} f_j(\gamma(\theta_0, t), 0, t + \alpha_0) \quad \text{a.e. on} \quad [0,1], \\
&L(\theta_0) h_0 = \overline{M}_0.
\end{aligned}$$

The upper-semicontinuity of M_μ is proved by [4, Proposition 1.2.(b)].

Let $S^{k-1} = \{b \in \mathbb{R}^k \colon |b| = 1\}$ be the $(k-1)$-dimensional sphere. Now we are ready to prove the main results of this section.

Theorem 3.1. *Let $d > 1$. Let there exist a non-empty open bounded set $\mathcal{B} \subset \mathcal{O} \times \mathbb{R}$ and $\mu_0 \in S^{k-1}$ such that*

(i) $0 \notin M_{\mu_0}(\partial \mathcal{B})$,

(ii) $\deg(M_{\mu_0}, \mathcal{B}, 0) \neq 0$,

where deg *is the topological degree in the sense of* [4, pp. 154–155] *and* [17, p. 20].

Then there is a constant $K > 0$ and a region in $\mathbb{R}^k$ for μ of the form

$$\mathcal{R} = \Big\{ s\tilde{\mu} \colon s \text{ and } \tilde{\mu} \text{ are from open small connected neighborhoods } U_1 \text{ and } U_2 \subset S^{k-1} \text{ of } 0 \in \mathbb{R} \text{ and of } \mu_0, \text{ respectively} \Big\}$$

such that for any $\mu \in \mathcal{R}$ of the form $\mu = s\tilde{\mu}$, $s \in U_1$, $\tilde{\mu} \in U_2$, the differential inclusion (1.3) *possesses a* 1*-periodic solution x_μ satisfying, according to* (3.1),

$$\sup_{0 \leqslant t \leqslant 1} \big| x_\mu(t) - \gamma(\theta_\mu, t - \alpha_\mu) \big| \leqslant Ks,$$

where $\alpha_\mu \in \mathbb{R}$ and $\theta_\mu \in \mathcal{O}$.

P r o o f. We need the following results.

Claim A. *The above condition* (i) *implies for any $A > 0$ the existence of constants $\delta > 0$, $1 > s_0 > 0$ such that*

$$|L(\theta)h| \geqslant \delta, \quad \forall\, h \in F(z, \theta, \alpha, \mu, s, \lambda)$$

for any $|s| < s_0$, $\lambda \in [0, 1]$, $|z| \leqslant A$, $(\theta, \alpha) \in \partial\mathcal{B}$, $|\mu - \mu_0| \leqslant \delta$.

P r o o f o f C l a i m A. Assume the contrary. So there is an $A > 0$ and

$$\begin{aligned} & s_p \to 0, \quad \lambda_p \to \lambda_0, \quad |z_p| \leqslant A, \quad \mu_p \to \mu_0, \quad p \in \mathbb{N}, \\ & \partial\mathcal{B} \ni (\theta_p, \alpha_p) \to (\theta_0, \alpha_0) \in \partial\mathcal{B}, \quad h_p \in F(z_p, \theta_p, \alpha_p, \mu_p, s_p, \lambda_p) \end{aligned}$$

such that

$$L(\theta_p)h_p \to 0 \quad \text{as} \quad p \to \infty.$$

Since $\{h_p\}_1^p$ is bounded in L^2, we can assume that h_p tends weakly to some $h_0 \in L^2$. Then by applying the standard arguments (see the proof of [18, Proposition 1.4]), we obtain

$$\begin{aligned} & h_0(t) \in \sum_{j=1}^{k} \mu_{0j} f_j(\gamma(\theta_0, t), 0, t + \alpha_0) \quad \text{a.e. on} \quad [0, 1], \\ & L(\theta_0)h_0 = 0. \end{aligned}$$

This contradicts (i) of this theorem. □

Claim B. *There are open small connected neighborhoods $U_1 \subset \mathbb{R}$, $U_2 \subset S^{k-1}$ of 0 and μ_0, respectively, and a constant $K_1 > 0$ such that*

$$0 \notin H(\partial\Omega, \mu, s, \lambda)$$

for any $s \in U_1,\ \mu \in U_2,\ \lambda \in [0, 1]$, where

$$\Omega = \Big\{(z, \theta, \alpha) \in C^p \times \mathbb{R}^d \colon\ |z| < K_1,\ (\theta, \alpha) \in \mathcal{B}\Big\}.$$

P r o o f o f C l a i m B. By applying Claim A, the result follows from the construction of H. □

By Claim B for any $s \in U_1$, $\mu \in U_2$ we obtain

$$\deg\big(H(\cdot, \mu, s, 1), \Omega, 0\big) = \deg\big(H(\cdot, \mu_0, 0, 0), \Omega, 0\big) = \deg(M_{\mu_0}, \mathcal{B}, 0) \neq 0.$$

Hence (3.5) has a solution in Ω for any $s \in U_1$, $\mu \in U_2$ and $\lambda = 1$. This solution gives a solution of (3.3) according to the definition of (3.5). $\square$

Corollary 3.2. *Let $d = 1$. Then M_μ depends only on α. If there are constants $a < b$ and $\mu_0 \in S^{k-1}$ such that*

$M_{\mu_0}(a)$ contains only positive (only negative) numbers
and $M_{\mu_0}(b)$ contains only negative (only positive) ones,

then the conclusion of Theorem 3.1 holds.

P r o o f. We apply Theorem 3.1 with $\mathcal{B} = (a, b)$. The assumption (i) of Theorem 3.1 is clearly satisfied. To prove (ii), it is enough to consider the case (the other one is similar) that $M_{\mu_0}(a)$ contains positive and $M_{\mu_0}(b)$ negative numbers, and then to take the homotopy

$$\lambda M_{\mu_0}(\alpha) + (1-\lambda)\Big(\frac{a+b}{2} - \alpha\Big).$$

The proof now follows directly from Theorem 3.1. $\square$

R e m a r k 3.3. The restriction $|\mu_0| = 1$ is not essential, since M_μ in (3.6) is homogeneous with respect to the variable μ.

R e m a r k 3.4. Let $f(x) = Bx$ for a matrix B and let $\gamma_1, \dots, \gamma_d$ be the maximum number of linearly independent 1-periodic solutions of $\dot{x} = Bx$. Then we take $\gamma(\theta, t) = \theta_1\gamma_1 + \dots + \theta_d\gamma_d$ and $\frac{\partial\gamma}{\partial\theta_i}$, $i = 1, \dots, d$ represent the maximum number of linearly independent 1-periodic solutions of $\dot{x} = Bx$. Consequently (see the assumption (iv)), we take now (3.1) with $\alpha = 0$ and we also put $\alpha = 0$ in (3.6). We note that now $\mathcal{O} = \mathbb{R}^d$ and $M_\mu\colon\ \mathbb{R}^d \to 2^{\mathbb{R}^d} \setminus \emptyset$. Then Theorem 3.1 is valid with such M_μ.

In this section we study (1.3) when f_j are all independent of t. So we consider the differential inclusion

$$\dot{x}(t) \in f(x(t)) + \sum_{j=1}^{k} \mu_j f_j(x(t), \mu) \quad \text{a.e. on} \quad \mathbb{R} \tag{4.1}$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^k$, $\mu = (\mu_1, \ldots, \mu_k)$ and f, f_j satisfying the assumptions (i), (ii) and (iv). We scale $\mu \to s\mu$ for $s \in \mathbb{R} \setminus \{0\}$ and take now, instead of (3.1), the change of variables

$$x((1 + s\alpha)t) = \gamma(\theta, t) + sz(t), \quad \alpha \in \mathbb{R}, \quad s \in \mathbb{R} \setminus \{0\}. \tag{4.2}$$

Then (4.1) has the form

$$\dot{z}(t) - A(\theta, t)z(t) \in \tilde{g}(z(t), \theta, \alpha, \mu, s, t) \quad \text{a.e. on} \quad [0, 1], \tag{4.3}$$

where

$$\begin{aligned}\tilde{g}(x, \theta, \alpha, \mu, s, t) = \bigg\{ v \in \mathbb{R}^n \colon\ v \in \frac{1}{s}\Big[f(sx + \gamma(\theta, t)) - f(\gamma(\theta, t)) - A(\theta, t)sx\Big] \\ + \alpha f(\gamma(\theta, t) + sx) + (1 + s\alpha)\sum_{j=1}^{k} \mu_j f_j(sx + \gamma(\theta, t), s\mu)\bigg\}.\end{aligned}$$

We see that (4.3) has a similar form like (3.2). Consequently, we can repeat the approach of Section 3 to (4.3) and we arrive at the multivalued mapping (like (3.6)) of the form

$$\begin{aligned}N_\mu \colon \mathcal{O} \times \mathbb{R} \to 2^{\mathbb{R}^d} \setminus \emptyset, \quad N_\mu(\theta, \alpha) = \Big\{ L(\theta)h \colon\ h \in L^2, \quad \text{satisfying} \\ \text{the relation} \quad h(t) \in \alpha\dot{\gamma}(\theta, t) + \sum_{j=1}^{k} \mu_j f_j(\gamma(\theta, t), 0) \quad \text{a.e. on} \quad [0, 1] \Big\}.\end{aligned} \tag{4.4}$$

The mapping N_μ is again upper-semicontinuous with compact convex values and maps bounded sets into bounded ones.

The proofs of the following results are very similar to the proofs of Theorem 3.1 and Corollary 3.2, so we omit them.

Theorem 4.1. *Let $d > 1$. Let there exist a non-empty open bounded set $\mathcal{B} \subset \mathcal{O} \times \mathbb{R}$ and $\mu_0 \in S^{k-1}$ such that*

(i) $0 \notin N_{\mu_0}(\partial\mathcal{B})$,
(ii) $\deg(N_{\mu_0}, \mathcal{B}, 0) \neq 0$.

Then there is a constant $K > 0$ and a region in $\mathbb{R}^k$ for μ of the form

$$\mathcal{R} = \bigl\{ s\tilde{\mu}\colon\ s \text{ and } \tilde{\mu} \text{ are from open small connected neighborhoods } U_1 \text{ and } U_2 \subset S^{k-1} \text{ of } 0 \in \mathbb{R} \text{ and of } \mu_0, \text{ respectively} \bigr\}$$

such that for any $\mu \in \mathcal{R}$ of the form $\mu = s\tilde{\mu}$, $s \in U_1$, $\tilde{\mu} \in U_2$, the differential inclusion (4.1) *possesses a* $(1 + s\alpha_\mu)$*-periodic solution x_μ satisfying, according to* (4.2),

$$\sup_{0 \leqslant t \leqslant 1+s\alpha_\mu} \bigl| x_\mu(t) - \gamma(\theta_\mu, t/(1 + s\alpha_\mu)) \bigr| \leqslant Ks,$$

where $\alpha_\mu \in \mathbb{R}$, $|\alpha_\mu| \leqslant K$ and $\theta_\mu \in \mathcal{O}$.

Corollary 4.2. *Let $d = 1$. Then $\gamma(\theta, t) = \gamma(t)$ and the adjoint variational equation* (2.4) *has a unique (up to scalar multiples)* 1*-periodic solution $v(t)$. If*

$$\int_0^1 \langle \dot{\gamma}(s), v(s) \rangle \, \mathrm{d}s \neq 0,$$

then there is a constant $K > 0$ such that for any sufficiently small μ, the differential inclusion (4.1) *possesses a periodic solution x_μ with the properties of Theorem* 4.1.

Remark 4.3. Corollary 4.2 is an extension of [12, p. 416, Theorem 2.4] for the multivalued case (4.1).

To complete the subject of Corollary 4.2, we consider the case $d = 1$ and

$$\int_0^1 \langle \dot{\gamma}(s), v(s) \rangle \, \mathrm{d}s = 0,$$

where γ, v are from Corollary 4.2. Then we also put $A(t) = D_x f(\gamma(t))$. Now we deal with a higher-order singularity of (4.1). To this end, we scale $\mu \to s^2\mu$ for $s > 0$ and make the change of variables

$$x((1 + s\alpha)t) = \gamma(t) + s\alpha u(t) + s^2 z(t), \quad \alpha \in \mathbb{R}, \quad s > 0, \tag{4.5}$$

where $u \in C^p$ is the unique solution of $\dot{u} - A(t)u = \dot{\gamma}$, $\int_0^1 \langle u(s), \dot{\gamma}(s) \rangle \, \mathrm{d}s = 0$. Then (4.1) has the form

$$\dot{z}(t) - A(t)z(t) \in \bar{g}(z(t), \alpha, \mu, s, t) \quad \text{a.e. on} \quad [0, 1], \tag{4.6}$$

where

$$\bar{g}(x,\alpha,\mu,s,t)=\Big\{v\in\mathbb{R}^n:\ v\in\frac{1}{s^2}\Big[f(s\alpha u(t)+s^2x+\gamma(t))-f(\gamma(t))$$

$$-A(t)(s\alpha u(t)+s^2x)\Big]+\frac{\alpha}{s}\Big[f(\gamma(t)+s\alpha u(t)+s^2x)-f(\gamma(t))\Big]$$

$$+(1+s\alpha)\sum_{j=1}^{k}\mu_j f_j(s\alpha u(t)+s^2x+\gamma(t),s^2\mu)\Big\}.$$

Again we can repeat the procedure of Section 3 for (4.6). The resulting multivalued function is (like (3.6)) as follows:

(4.7)

$$P_\mu(\alpha)=\Big\{\int_0^1\langle h(s),v(s)\rangle\,\mathrm{d}s:\ h\in L^2,\quad\text{satisfying a.e. on}\quad[0,1]\quad\text{the relation}$$

$$h(t)\in\alpha^2\Big[\frac{1}{2}D_x^2f(\gamma(t))(u(t),u(t))+D_xf(\gamma(t))u(t)\Big]+\sum_{j=1}^{k}\mu_jf_j(\gamma(t),0)\Big\}.$$

The proof of the next result is very similar to the proof of Corollary 3.2, so we omit it.

Theorem 4.4. *Let*

$$\int_0^1\Big\langle\frac{1}{2}D_x^2f(\gamma(s))(u(s),u(s))+D_xf(\gamma(s))u(s),v(s)\Big\rangle\,\mathrm{d}s>0$$

and let there exist $\mu_0\in S^{k-1}$ *such that* $\int_0^1\langle h(s),v(s)\rangle\,\mathrm{d}s<0$ *for any* $h\in L^2$ *satisfying*

$$h(t)\in\sum_{j=1}^{k}\mu_{0j}f_j(\gamma(t),0)\quad\text{a.e. on}\quad[0,1].$$

Then there is a constant $K>0$ *and a wedge-shaped region in* $\mathbb{R}^k$ *for* μ *of the form*

$$\mathcal{R}=\ \big\{s^2\tilde{\mu}:\ s>0\ \text{and}\ \tilde{\mu}\ \text{are from open small connected neighborhoods}$$
$$U_1\ \text{and}\ U_2\subset S^{k-1}\ \text{of}\ 0\in\mathbb{R}\ \text{and of}\ \mu_0,\ \text{respectively}\ \big\}$$

such that for any $\mu\in\mathcal{R}$ *of the form* $\mu=s^2\tilde{\mu}$, $0<s\in U_1$, $\tilde{\mu}\in U_2$, *the differential inclusion* (4.1) *possesses two* $(1+s\alpha_{\pm,\mu})$*-periodic solutions* $x_{\pm,\mu}$ *satisfying, according to* (4.5),

$$\sup_{0\leqslant t\leqslant 1+s\alpha_{\pm,\mu}}\big|x_{\pm,\mu}(t)-\gamma(t/(1+s\alpha_{\pm,\mu}))-s\alpha_{\pm,\mu}u(t/(1+s\alpha_{\pm,\mu}))\big|\leqslant Ks^2,$$

where $\alpha_{+,\mu} > 0$, $\alpha_{-,\mu} < 0$ *and* $|\alpha_{\pm,\mu}| \leqslant K$.

Remark 4.5. Theorem 4.4 is an extension of a saddle-node bifurcation of periodic solutions [11] to the multivalued case (4.1).

To conclude this section, we deal with the case $d = 2$ and $n = 2$. Hence (4.1) is a planar differential inclusion satisfying the condition (i), and (ii), (iv) are replaced by

(v) There are numbers $0 < c < e$ and a C^2-mapping $\gamma\colon (c,e)\times\mathbb{R} \to \mathbb{R}^2$ such that $\gamma(\theta,t)$ has the minimum period θ in t and $\gamma(\theta,\cdot)$ is a solution of $\dot{x} = f(x)$.

Since $\gamma(\theta,t+\theta) = \gamma(\theta,t)$ implies $\frac{\partial}{\partial\theta}\gamma(\theta,t+\theta)+\dot{\gamma}(\theta,t) = \frac{\partial}{\partial\theta}\gamma(\theta,t)$, we see that the linear equation $\dot{u} = D_x f(\gamma(\theta,t))u$ has the single (up to scalar multiples) θ-periodic solution $\dot{\gamma}(\theta,t)$.

We scale $\mu \to s\mu$ for $s \in \mathbb{R}\setminus\{0\}$ and make the change of variables

$$x(\theta t) = \gamma(\theta,\theta t) + sz(t), \quad s \in \mathbb{R}\setminus\{0\}. \tag{4.8}$$

Then (4.1) has the form

$$\dot{z}(t) - \theta A(\theta,\theta t)z(t) \in \bar{p}(z(t),\theta,\mu,s,t) \quad \text{a.e. on} \quad [0,1], \tag{4.9}$$

where

$$\bar{p}(x,\theta,\mu,s,t) = \Big\{v \in \mathbb{R}^2\colon\ v \in \frac{\theta}{s}\Big[f(sx+\gamma(\theta,\theta t)) - f(\gamma(\theta,\theta t)) - A(\theta,\theta t)sx\Big] + \theta\sum_{j=1}^{k}\mu_j f_j(sx+\gamma(\theta,\theta t),s\mu)\Big\}.$$

According to the considerations of Section 2, the condition (v) implies the existence of a non-zero, continuous mapping $v\colon (c,e) \to \mathbb{R}^2$ such that $v(\theta,t)$ is θ-periodic in t and satisfies the adjoint equation $\dot{v} = -A(\theta,t)^* v$. Then $\dot{w} = -\theta A(\theta,\theta t)^* w$ has the unique (up to scalar multiples) 1-periodic solution $w(\theta,t) = v(\theta,t/\theta)$.

Consequently, we solve (4.9) in $z \in C^p$ by regarding $\theta \in (c,e)$ as a parameter. So for any fixed $\theta \in (c,e)$, we apply the procedure of Section 3 with $d = 1$, $\alpha = 0$ and the corresponding multivalued mapping (3.6) has now the form

$$Q_\mu(\theta) = \Big\{\int_0^1 \langle h(s), w(\theta,s)\rangle\,\mathrm{d}s\colon\ h \in L^2,\ h(t) \in \theta\sum_{j=1}^{k}\mu_j f_j(\gamma(\theta,\theta t),0) \quad \text{a.e. on} \quad [0,1]\Big\}.$$

We note that

$$Q_\mu(\theta) = \left\{ \int_0^\theta \langle h(s), v(\theta,s)\rangle \,\mathrm{d}s : \ h \in L^2([0,\theta],\mathbb{R}^2), \right. \tag{4.10}$$

$$\left. h(t) \in \sum_{j=1}^k \mu_j f_j(\gamma(\theta,t),0) \quad \text{a.e. on} \quad [0,\theta] \right\}.$$

R e m a r k 4.6. If $\dot{x} = f(x)$ is a Hamiltonian system then (see [11])

$$v(\theta,t) = (\dot{\gamma}_2(\theta,t), -\dot{\gamma}_1(\theta,t)),$$

where $\gamma(\theta,t) = (\gamma_1(\theta,t),\gamma_2(\theta,t))$. Hence

$$\int_0^\theta \langle h(s), v(\theta,s)\rangle \,\mathrm{d}s = \int_0^\theta h(s)\wedge\dot{\gamma}(\theta,s)\,\mathrm{d}s = \int_0^\theta h(s)\wedge f(\gamma(\theta,s))\,\mathrm{d}s,$$

where $\wedge$ is the wedge product [11, p. 187].

Summarizing, we obtain the following result.

Theorem 4.7. *Let* (i) *and* (v) *be satisfied. If there are constants* $c < a < b < e$ *and* $\mu_0 \in S^{k-1}$ *such that*

$Q_{\mu_0}(a)$ *contains only positive (only negative) numbers*
and $Q_{\mu_0}(b)$ *contains only negative (only positive) ones,*

then there is a constant $K > 0$ *and a region in* $\mathbb{R}^k$ *for* μ *of the form*

$$\mathcal{R} = \{s\tilde{\mu}\colon\ s \text{ and } \tilde{\mu} \text{ are from open small connected neighborhoods } U_1 \text{ and } U_2 \subset S^{k-1} \text{ of } 0 \in \mathbb{R} \text{ and of } \mu_0, \text{ respectively}\}$$

such that for any $\mu \in \mathcal{R}$ *of the form* $\mu = s\tilde{\mu}$, $s \in U_1$, $\tilde{\mu} \in U_2$, *the differential inclusion* (4.1) *possesses a* θ_μ*-periodic solution* x_μ *satisfying, according to* (4.8),

$$\sup_{0\leqslant t\leqslant\theta_\mu} \left|x_\mu(t) - \gamma(\theta_\mu,t)\right| \leqslant Ks,$$

where $\theta_\mu \in (c,e)$.

R e m a r k 4.8. $Q_\mu(\theta)$ in (4.10) is an extension of a classical Melnikov function [3] to the multivalued case (4.1). Hence Theorem 4.7 is a generalization of the Poincaré-Andronov bifurcation theorem.

5. Applications to dry friction problems

We begin by considering the following simple version of (1.1):

$$\ddot{x} + \mu_1 \tau(x) + \mu_2 \operatorname{sgn}(\dot{x} + v_0 \beta(t)) = 0, \tag{5.1}$$

where $\tau \in C(\mathbb{R}, \mathbb{R})$, $\beta \in C(\mathbb{R}, \mathbb{R})$ is 1-periodic and $\mu_{1,2} > 0$, $v_0 > 0$ are constants. We rewrite (5.1) in the form of (1.3)

$$\dot{z} = y, \quad \dot{y} \in -\mu_1 \tau(z) - \mu_2 \operatorname{Sgn}(y + v_0 \beta(t))\,. \tag{5.2}$$

We see that $f = 0$, so we apply Remark 3.4. The 1-periodic solutions of the variational and adjoint equations

$$\dot{z} = y, \quad \dot{y} = 0 \quad \text{and} \quad \dot{u} = 0, \quad \dot{v} = -u, \quad \text{respectively}$$

are constant ones $(\theta, 0)$ and $(0, \theta)$, $\theta \in \mathbb{R}$, respectively. So $\gamma(\theta, t) = (\theta, 0)$ and $\Pi(\theta)(h_1, h_2) = \Bigl(h_1, h_2 - \int_0^1 h_2(s)\,\mathrm{d}s\Bigr)$. Hence the corresponding multivalued function (3.6), according to Remark 3.4, has now the form

$$M_\mu(\theta) = \Biggl\{ \int_0^1 h(s)\,\mathrm{d}s \colon\ h \in L^2,$$
$$h(t) \in -\mu_1 \tau(\theta) - \mu_2 \operatorname{Sgn}(v_0 \beta(t)) \quad \text{a.e. on} \quad [0,1] \Biggr\}. \tag{5.3}$$

Theorem 5.1. *Let $0 \leqslant t_1 < t_2 < \ldots < t_{2j} < 1$, $j \geqslant 1$ be the only zero points of β. Let $\inf \tau = \Gamma_1 < 0 < \Gamma_2 = \sup \tau$. If $\mu_{1,2} > 0$ are sufficiently small satisfying*

$$\mu_1/\mu_2 > \max\Bigl\{ -\frac{\Gamma_3}{\Gamma_1}, -\frac{\Gamma_3}{\Gamma_2} \Bigr\}, \quad \textit{where} \tag{5.4}$$
$$\Gamma_3 = \Bigl(-1 + 2\sum_{i=1}^{2j} (-1)^i t_i \Bigr) \operatorname{sgn} \beta\Bigl(\frac{t_1 + t_2}{2}\Bigr),$$

then (5.1) *has a* 1*-periodic solution.*

P r o o f. By the assumptions of this theorem, (5.3) has the form

$$M_\mu(\theta) = -\mu_1 \tau(\theta) - \mu_2 \Bigl(-1 + 2\sum_{i=1}^{2j} (-1)^i t_i \Bigr) \operatorname{sgn} \beta\Bigl(\frac{t_1 + t_2}{2}\Bigr).$$

Now it is clear that there are $a < b$ such that $M_\mu(a) M_\mu(b) < 0$. The proof is completed by Corollary 3.2 and Remark 3.4. □

R e m a r k 5.2. The condition (5.4) is important when either $\Gamma_3 > 0$, $\Gamma_1 > -\infty$ or $\Gamma_3 < 0$, $\Gamma_2 < \infty$; then we see that $\mu_1 > 0$ has to be relatively large with respect to $\mu_2 > 0$. If $j = 1$ and $t_1 = 0$, the equation $\Gamma_3 \neq 0$ means $t_2 \neq 1/2$, so then β changes sign asymmetrically with respect to the middle point $1/2$.

Similarly we study the problem

$$\ddot{x} + \mu_1 \tau_1(x,t) + \mu_2 \operatorname{sgn} \dot{x} = 0, \tag{5.5}$$

where $\tau_1 \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is 1-periodic in t and $\mu_{1,2} > 0$ are constants. Then we obtain in the same way as above that

$$M_\mu(\theta) = \left[-\mu_1 \int_0^1 \tau_1(\theta, s)\,\mathrm{d}s - \mu_2, -\mu_1 \int_0^1 \tau_1(\theta, s)\,\mathrm{d}s + \mu_2 \right].$$

Consequently, we have the following result.

Theorem 5.3. *Assume*

$$\inf_\theta \int_0^1 \tau_1(\theta, s)\,\mathrm{d}s < 0 < \sup_\theta \int_0^1 \tau_1(\theta, s)\,\mathrm{d}s.$$

If $\mu_{1,2} > 0$ are sufficiently small satisfying

$$\mu_2/\mu_1 < \min \left\{ \sup_\theta \int_0^1 \tau_1(\theta, s)\,\mathrm{d}s, -\inf_\theta \int_0^1 \tau_1(\theta, s)\,\mathrm{d}s \right\},$$

then (5.5) *has a* 1*-periodic solution.*

R e m a r k 5.4. We see that again $\mu_1 > 0$ has to be relatively large with respect to $\mu_2 > 0$ for the validity of the inequality of Theorem 5.3 provided that the right-hand side of this inequality is a finite constant.

Now we study the general form of (1.2) by assuming:

(vi) There are numbers $0 < c < e$ and a C^2-mapping $\gamma\colon (c,e) \times \mathbb{R} \to \mathbb{R}$ such that $\gamma(\theta, t)$ has the minimum period θ in t, $\dot{\gamma}(\theta, 0) = 0$ and $\gamma(\theta, \cdot)$ is a solution of $\ddot{x} + q(x) = 0$.

Hence (iv) holds with $d = 1$ and $\theta = 2\pi/\omega$, provided that $c < 2\pi/\omega < e$. We apply Corollary 3.2. Like in Remark 4.6, we have $v_1(t) = (\ddot{\gamma}(2\pi/\omega, t), -\dot{\gamma}(2\pi/\omega, t))$.

Lemma 5.5. *$\dot{\gamma}(2\pi/\omega, \pi/\omega) = 0$ and $\dot{\gamma}(2\pi/\omega, t) \neq 0$ for any $\pi/\omega \neq t \in (0, 2\pi/\omega)$.*

Proof. Since $\dot{\gamma}(2\pi/\omega, t)$ is periodic, there is a smallest t_1, $0 < t_1 \leqslant 2\pi/\omega$ such that $\dot{\gamma}(2\pi/\omega, t_1) = 0$. Then $y(t) = \gamma(2\pi/\omega, 2t_1 - t)$ is also a solution of $\ddot{x} + q(x) = 0$ such that $y(t_1) = \gamma(2\pi/\omega, t_1)$, $\dot{y}(t_1) = \dot{\gamma}(2\pi/\omega, t_1)$. Hence $\gamma(2\pi/\omega, t) = \gamma(2\pi/\omega, 2t_1 - t)$. Similarly we have $\gamma(2\pi/\omega, -t) = \gamma(2\pi/\omega, t)$. So $\gamma(2\pi/\omega, t)$ is $2t_1$ periodic, and this gives $2t_1 = 2\pi/\omega$. □

If $v_0 > 0$ is sufficiently large then

$$\dot{\gamma}(2\pi/\omega, t) + v_0 \sin \omega t = 0 \iff t = \pi j/\omega, \quad j \in \mathbb{Z}.$$

Moreover, for $v_0 > 0$ sufficiently large the equation

$$\dot{\gamma}(2\pi/\omega, \alpha) + v_0 \sin \omega(t + \alpha) = 0, \quad 0 \leqslant \alpha \leqslant 2\pi/\omega$$

has precisely the solutions $t_1(\alpha) + 2\pi j/\omega$, $t_2(\alpha) + 2\pi j/\omega$, $j \in \mathbb{Z}$, where t_1, t_2 are continuous functions such that $t_1(\alpha) < t_2(\alpha)$, $t_1(0) = \pi/\omega$, $t_2(0) = 2\pi/\omega$, $t_1(\pi/\omega) = 0$, $t_2(\pi/\omega) = \pi/\omega$, $t_1(\alpha)$ is near $\pi/\omega - \alpha$ and $t_2(\alpha)$ is near $2\pi/\omega - \alpha$. Now (3.6) has the form

$$M_1(\alpha)\colon \mathbb{R} \to 2^{\mathbb{R}}, \quad M_1(\alpha) = \Bigg\{ -\int_0^1 \langle h(s), \dot{\gamma}(2\pi/\omega, s)\rangle \,\mathrm{d}s \colon$$
$$h \in L^2, \quad h(t) \in -\operatorname{Sgn}\big(\dot{\gamma}(2\pi/\omega, t) + v_0 \sin \omega(t + \alpha)\big) \quad \text{a.e. on} \quad [0, 2\pi/\omega] \Bigg\}.$$

According to the above results, for $v_0 > 0$ sufficiently large,

$$\dot{\gamma}(2\pi/\omega, t) + v_0 \sin \omega(t + \alpha)$$

is positive on $\big(t_2(\alpha), t_1(\alpha) + 2\pi/\omega\big)$ and negative on $\big(t_1(\alpha), t_2(\alpha)\big)$.

Hence

$$\begin{aligned} M_1(\alpha) &= \int_{t_2(\alpha)}^{t_1(\alpha)+2\pi/\omega} \dot{\gamma}(2\pi/\omega, s)\,\mathrm{d}s - \int_{t_1(\alpha)}^{t_2(\alpha)} \dot{\gamma}(2\pi/\omega, s)\,\mathrm{d}s \\ &= 2\big(\gamma(2\pi/\omega, t_1(\alpha)) - \gamma(2\pi/\omega, t_2(\alpha))\big). \end{aligned}$$

We have

$$
\begin{aligned}
M_1(0) &= 2\big(\gamma(2\pi/\omega, t_1(0)) - \gamma(2\pi/\omega, t_2(0))\big) = 2\big(\gamma(2\pi/\omega, \pi/\omega) - \gamma(2\pi/\omega, 2\pi/\omega)\big)\\
M_1(\pi/\omega) &= 2\big(\gamma(2\pi/\omega, t_1(\pi/\omega)) - \gamma(2\pi/\omega, t_2(\pi/\omega))\big)\\
&= 2\big(\gamma(2\pi/\omega, 0) - \gamma(2\pi/\omega, \pi/\omega)\big) = -M_1(0).
\end{aligned}
$$

Since $\dot\gamma(2\pi/\omega, \pi/\omega) = \dot\gamma(2\pi/\omega, 0) = 0$ and $\gamma(2\pi/\omega, t)$ has the minimum period $2\pi/\omega$, then necessarily $M_1(\pi/\omega) \neq 0$. This gives that $M_1(\pi/\omega)M_1(0) < 0$. Consequently, by taking $a = 0$, $b = \pi/\omega$ in Corollary 3.2, we obtain the following result.

Theorem 5.6. *Let $v_0 > 0$ be sufficiently large. If (vi) holds and $c < 2\pi/\omega < e$, then for any sufficiently small $\mu > 0$, (1.2) has a $2\pi/\omega$-periodic solution near the family $\gamma(\theta, t)$, $\theta \in (c, e)$ from (vi).*

The following example is motivated by a clock-pendulum [13]. Consider the equation

$$
\ddot x + q(x) + \mu \operatorname{sgn} x \cdot \operatorname{sgn} \dot x = 0, \tag{5.6}
$$

where (vi) holds. According to the proof of Lemma 5.5, $\dot\gamma(\theta, t) = 0$, $0 < t < \theta$ if and only if $t = \theta/2$. It is clear that $\gamma(\theta, 0) \neq \gamma(\theta, \theta/2)$. We assume in addition that

(vii) $\gamma(\theta, 0) > 0$ and $\gamma(\theta, \theta/2) < 0$.

We apply Theorem 4.7 to (5.6). (vii) implies the existence of continuous mappings $\tilde t_1(\cdot)$, $\tilde t_2(\cdot)\colon (c, e) \to (0, \infty)$ such that $0 < \tilde t_1(\theta) < \theta/2 < \tilde t_2(\theta) < \theta$ and $\gamma(\theta, t)$, $\dot\gamma(\theta, t)$ have the signs $(+,-)$, $(-,-)$, $(-,+)$, $(+,+)$ on the intervals

$$
(0, \tilde t_1(\theta)),\ (\tilde t_1(\theta), \theta/2),\ (\theta/2, \tilde t_2(\theta)),\ (\tilde t_2(\theta), \theta), \quad \text{respectively.}
$$

We note that $\gamma(\theta, \tilde t_1(\theta)) = \gamma(\theta, \tilde t_2(\theta)) = 0$. Now it is not hard to see that $v(\theta, t) = \big(\ddot\gamma(\theta, t), -\dot\gamma(\theta, t)\big)$ and (4.10) for (5.6) has the form

$$
\begin{aligned}
Q_1(\theta) = &-\int_0^{\tilde t_1(\theta)} \dot\gamma(\theta, s)\,\mathrm{d}s + \int_{\tilde t_1(\theta)}^{\theta/2} \dot\gamma(\theta, s)\,\mathrm{d}s - \int_{\theta/2}^{\tilde t_2(\theta)} \dot\gamma(\theta, s)\,\mathrm{d}s + \int_{\tilde t_2(\theta)}^{\theta} \dot\gamma(\theta, s)\,\mathrm{d}s\\
= &-\gamma(\theta, \tilde t_1(\theta)) + \gamma(\theta, 0) + \gamma(\theta, \theta/2) - \gamma(\theta, \tilde t_1(\theta)) - \gamma(\theta, \tilde t_2(\theta))\\
&+ \gamma(\theta, \theta/2) + \gamma(\theta, \theta) - \gamma(\theta, \tilde t_2(\theta)) = 2\big(\gamma(\theta, 0) + \gamma(\theta, \theta/2)\big).
\end{aligned}
$$

Theorem 5.7. *Let* (vi) *and* (vii) *be valid. If there are constants $c < a < b < e$ such that*

$$
\big(\gamma(a, 0) + \gamma(a, a/2)\big)\big(\gamma(b, 0) + \gamma(b, b/2)\big) < 0,
$$

then (5.6) *has a periodic solution for any sufficiently small* $\mu > 0$.

P r o o f. Since the assumptions of this theorem imply $Q_1(a)Q_1(b) < 0$, the proof is completed by Theorem 4.7. $\square$

R e m a r k 5.8. We note that $\max\limits_{[0,\theta]} \gamma(\theta,t) = \gamma(\theta,0)$ and $\min\limits_{[0,\theta]} \gamma(\theta,t) = \gamma(\theta,\theta/2)$. Moreover, $\bar{q}(\gamma(\theta,0)) = \bar{q}(\gamma(\theta,\theta/2))$ for $\bar{q}(x) = \int\limits_0^x q(s)\,\mathrm{d}s$, so the inequality of Theorem 5.7 can be verified from the graph of $\bar{q}$.

The next example can be regarded as a multivalued van der Pol oscillator (see [3]) of the form

$$\dot{x} = y, \quad \dot{y} \in -x + \mu\varphi(x)y, \tag{5.7}$$

where $\varphi(x) = \bigl[\varphi_1(x), \varphi_2(x)\bigr]$, $\mu > 0$ and $\varphi_{1,2} \in C(\mathbb{R},\mathbb{R})$, $\varphi_1(x) \leqslant \varphi_2(x)$. We apply Theorem 4.1. Now we have

$$\begin{aligned} &\gamma(\theta,t) = \theta(\cos t, -\sin t), \quad \dot{\gamma}(\theta,t) = -\theta(\sin t, \cos t), \quad \theta > 0\\ &v_1(\theta,t) = (\cos t, -\sin t), \quad v_2(\theta,t) = (\sin t, \cos t). \end{aligned}$$

Consequently, (4.4) now has the form

$$\begin{aligned} N_1(\theta,\alpha) = \Biggl\{ \Biggl(-\int\limits_0^{2\pi} h(s)\sin s\,\mathrm{d}s, -2\theta\alpha\pi + \int\limits_0^{2\pi} h(s)\cos s\,\mathrm{d}s \Biggr) :\\ h \in L^2([0,2\pi],\mathbb{R}), \quad h(t) \in \bigl[\varphi_1(\theta\cos t), \varphi_2(\theta\cos t)\bigr](-\theta\sin t) \quad \text{a.e. on} \quad [0,1] \Biggr\}. \end{aligned}$$

If $|\alpha|$ is sufficiently large and $\theta > 0$ is from a compact interval, then $0 \notin N_1(\theta,\alpha)$. For this reason, it is enough to study the multivalued function

$$\psi(\theta) = \Biggl[\int\limits_0^{2\pi} \varphi_1(\theta\cos s)\sin^2 s\,\mathrm{d}s, \int\limits_0^{2\pi} \varphi_2(\theta\cos t)\sin^2 s\,\mathrm{d}s \Biggr].$$

Theorem 4.1 gives the following result.

Theorem 5.9. *If there are constants* $0 < \theta_1$, $0 < \theta_2$ *such that*

$$0 < \int\limits_0^{2\pi} \varphi_1(\theta_1\cos s)\sin^2 s\,\mathrm{d}s, \quad \int\limits_0^{2\pi} \varphi_2(\theta_2\cos t)\sin^2 s\,\mathrm{d}s < 0,$$

then (5.7) *has a periodic solution for any sufficiently small* $\mu > 0$.

We conclude this section by considering coupled oscillators (see [14–16])

$$\begin{aligned} \ddot{x}_1 + q_1(x_1) + \mu_1 \operatorname{sgn}(\dot{x}_1 - \dot{x}_2) &= 0, \\ \ddot{x}_2 + q_2(x_2) + \mu_2 \operatorname{sgn}(\dot{x}_2 - \dot{x}_1) &= \mu_3 \operatorname{sw} t\,, \end{aligned} \tag{5.8}$$

where $q_{1,2} \in C^2(\mathbb{R}, \mathbb{R})$, $\mu_{1,2,3} > 0$ are parameters and

$$\operatorname{sw} t = \begin{cases} 1 & \text{for } [2t] \text{ even} \\ 0 & \text{for } [2t] \text{ odd.} \end{cases}$$

Here $[t]$ is the integer part of t. (5.8) may be related to (1.1), since the second equation of (5.8) may represent the movement of the ribbon coupled with an interference of the mass given by the relative velocity $\dot{x}_1 - \dot{x}_2$. The term $\operatorname{sw} t$ is a switching. In this interpretation of (5.8), the coupling is given by the dry friction. We assume that there are constants $0 < c < 1 < e$ and mappings $\varrho_{1,2}\colon (c,e) \times \mathbb{R} \to \mathbb{R}$ such that ϱ_i and $\ddot{x}_i + q_i(x_i) = 0$, $i = 1, 2$ satisfy the conditions (vi) and (vii).

Define a multivalued mapping by

$$\operatorname{Sw} t = \begin{cases} 1 & \text{for } [2t] \text{ even and } 2t \notin \mathbb{Z} \\ [0,1] & \text{for } 2t \in \mathbb{Z} \\ 0 & \text{for } [2t] \text{ odd and } 2t \notin \mathbb{Z}. \end{cases}$$

By rewriting (5.8) in the form

$$\begin{aligned} \dot{x}_1 &= y_1, \quad \dot{y}_1 \in -q(x_1) - \mu_1 \operatorname{Sgn}(y_1 - y_2), \\ \dot{x}_2 &= y_2, \quad \dot{y}_2 \in -q(x_2) - \mu_2 \operatorname{Sgn}(y_2 - y_1) + \mu_3 \operatorname{Sw} t, \end{aligned} \tag{5.9}$$

(5.9) has the form of (1.3) and according to Remark 4.6, (ii) (iv) hold with

$$\begin{aligned} &\gamma(\theta, t) = \big(\varrho_1(1,t), \dot{\varrho}_1(1,t), \varrho_2(1, t+\theta), \dot{\varrho}_2(1, t+\theta)\big), \\ &v_1(\theta, t) = \big(\ddot{\varrho}_1(1,t), -\dot{\varrho}_1(1,t), 0, 0\big), \quad v_2(\theta, t) = \big(0, 0, \ddot{\varrho}_2(1, t+\theta), -\dot{\varrho}_2(1, t+\theta)\big). \end{aligned}$$

We apply Theorem 3.1 to (5.9). In this case, (3.6) reads

$$\begin{aligned} M_\mu(\theta, \alpha) = \bigg\{ \bigg(&- \int_0^1 h_1(s) \dot{\varrho}_1(1,s)\,\mathrm{d}s, - \int_0^1 h_2(s) \dot{\varrho}_2(1, s+\theta)\,\mathrm{d}s \bigg) \colon h_1,\ h_2 \in L^2, \\ &h_1(t) \in -\mu_1 \operatorname{Sgn}\big(\dot{\varrho}_1(1,t) - \dot{\varrho}_2(1, t+\theta)\big) \quad \text{a.e. on} \quad [0,1], \\ &h_2(t) \in -\mu_2 \operatorname{Sgn}\big(\dot{\varrho}_2(1, t+\theta) - \dot{\varrho}_1(1,t)\big) + \mu_3 \operatorname{Sw}(t+\alpha) \quad \text{a.e. on} \quad [0,1] \bigg\}. \end{aligned}$$

We assume

(viii) There are continuous functions $\bar{t}_1,\ \bar{t}_2\colon [0,1] \to \mathbb{R}$ such that $\bar{t}_1(\theta) < \bar{t}_2(\theta)$, $\bar{t}_1(0) = 1/2$, $\bar{t}_2(0) = 1$, $\bar{t}_1(1/2) = 0$, $\bar{t}_2(1/2) = 1/2$ and $\dot{\varrho}_1(1,t) - \dot{\varrho}_2(1,t+\theta)$ is positive or negative on $(\bar{t}_1(\theta), \bar{t}_2(\theta))$, $(\bar{t}_2(\theta), \bar{t}_1(\theta)+1)$, respectively.

Now we can simplify $M_\mu = (M_{1\mu}, M_{2\mu})$ by using (viii) as follows:

$$\begin{aligned}
M_{1\mu}(\theta) &= \mu_1\Biggl(\int_{\bar{t}_1(\theta)}^{\bar{t}_2(\theta)} \dot{\varrho}_1(1,s)\,\mathrm{d}s - \int_{\bar{t}_2(\theta)}^{\bar{t}_1(\theta)+1} \dot{\varrho}_1(1,s)\,\mathrm{d}s\Biggr)\\
&= 2\mu_1\bigl(\varrho_1(1,\bar{t}_2(\theta)) - \varrho_1(1,\bar{t}_1(\theta))\bigr),\\
M_{2\mu}(\theta,\alpha) &= \mu_2\Biggl(\int_{\bar{t}_2(\theta)}^{\bar{t}_1(\theta)+1} \dot{\varrho}_2(1,s+\theta)\,\mathrm{d}s - \int_{\bar{t}_1(\theta)}^{\bar{t}_2(\theta)} \dot{\varrho}_2(1,s+\theta)\,\mathrm{d}s\Biggr)\\
&\quad - \mu_3\Bigl(\varrho_2(1,\frac{1}{2}+\theta-\alpha\Bigr) - \varrho_2(1,\theta-\alpha)\Bigr)\\
&= 2\mu_2\bigl(\varrho_2(1,\bar{t}_1(\theta)+\theta) - \varrho_2(1,\bar{t}_2(\theta)+\theta)\bigr) - \mu_3\Bigl(\varrho_2\Bigl(1,\frac{1}{2}+\theta-\alpha\Bigr) - \varrho_2(1,\theta-\alpha)\Bigr).
\end{aligned}$$

We have

$$M_{1\mu}(0) = 2\mu_1\bigl(\varrho_1(1,1) - \varrho_1(1,1/2)\bigr) = -M_{1\mu}(1/2).$$

Since $\dot{\varrho}_1(1,0) = \dot{\varrho}_1(1,1/2) = 0$ and ϱ_1 has the minimum period 1, we have $\varrho_1(1,0) \neq \varrho_1(1,1/2)$. Hence $M_{1\mu}(0) \neq 0$ and $M_{1\mu}(0)M_{1\mu}(1/2) < 0$ provided that $\mu_1 > 0$.

Theorem 5.10. *Let* (viii) *hold. If* $\mu_{1,2,3} > 0$ *are sufficiently small such that* $\mu_3 > 2\mu_2$, *then* (5.8) *has a* 1*-periodic solution.*

Proof. We apply Theorem 3.1 with the above M_μ and

$$\mathcal{B} = \Bigl\{(\theta,\alpha)\colon\ \theta \in (0,1/2),\quad \theta < \alpha < \frac{1}{2}+\theta\Bigr\}.$$

We put M_μ in the homotopy $M_{\mu,\lambda} = \bigl(M_{1\mu,\lambda}, M_{2\mu,\lambda}\bigr)$ given by

$$\begin{aligned}
M_{1\mu,\lambda}(\theta) &= \lambda M_{1\mu}(\theta) + (1-\lambda)\Bigl(\frac{1}{4}-\theta\Bigr),\\
M_{2\mu,\lambda} &= \lambda M_{2\mu}(\theta,\alpha) + (1-\lambda)\Bigl(\theta-\alpha+\frac{1}{4}\Bigr).
\end{aligned}$$

Since $\max\limits_{[0,1]} \varrho_2(1,t) = \varrho_2(1,0)$ and $\min\limits_{[0,1]} \varrho_2(1,t) = \varrho_2(1,1/2)$, we see that $\mu_1 > 0$, $\mu_3 > 2\mu_2 > 0$ implies $0 \notin M_{\mu,\lambda}(\partial\mathcal{B})$, $\lambda \in [0,1]$. Hence

$$\deg(M_\mu,\mathcal{B},0) = \deg(M_{\mu,0},\mathcal{B},0) = 1.$$

Now the result follows from Theorem 3.1. □

6. Concluding remarks

R e m a r k 6.1. In this paper, the dry friction is modelled by a Coulomb law [8], [13] which includes a statistic coefficient of friction μ_s and a dynamic coefficient of friction μ_d. If $\mu_s = \mu_d = \mu$, then the friction law may be written as $\dot{x} \to \mu \operatorname{sgn} \dot{x}$. On the other hand, since usually $\mu_s > \mu_d$, the smooth approximation of $\operatorname{sgn} r$ given, for instance, by

$$\Phi(r) = \frac{1}{\pi}\bigl(7 \arctan 8sr - 5 \arctan 4sr\bigr), \quad s \gg 1$$

seems to be physically more relevant than the mathematically convenient approximation of the form

$$r \to \frac{2}{\pi} \arctan sr, \quad s \gg 1.$$

The function Φ has two symmetric spikes at $r = \pm\frac{\sqrt{6}}{8s}$ of the values

$$\pm\frac{1}{\pi}\Bigl(7 \arctan \sqrt{6} - 5 \arctan \frac{\sqrt{6}}{2}\Bigr) \doteq \pm 1.226, 134, 4.$$

Moreover, $\Phi(r)$ is quickly near 1 or -1 when $r > 0$ or $r < 0$, respectively, tends off 0. Summarizing, we can take for any $\eta \geqslant 0$, $\zeta \geqslant 1$, $0 < \kappa \leqslant 1$ the multivalued function $\operatorname{Sgn}_{\eta,\zeta,\kappa} r$ defined by

$$\operatorname{Sgn}_{\eta,\zeta,\kappa} r = \begin{cases} -1 & \text{for } r < -\eta, \\ [-\zeta, -\kappa] & \text{for } -\eta \leqslant r < 0, \\ [-\zeta, \zeta] & \text{for } r = 0, \\ [\kappa, \zeta] & \text{for } 0 < r \leqslant \eta, \\ 1 & \text{for } r > \eta. \end{cases}$$

The term $\operatorname{Sgn}_{\eta,\zeta,\kappa} \dot{x}$ can be viewed as an extension for modelling dry friction including static and dynamic frictions as well.

R e m a r k 6.2. The aim of this paper is to deal with multivalued perturbation problems, but our method is clearly applied to piecewise smoothly perturbed problems. For instance, let us consider the problem

$$\ddot{x} + \mu_1 (x^+)^2 + \mu_2 x^- = \mu_3 \beta(t), \tag{6.1}$$

where $\mu_{1,2,3} \in \mathbb{R}$ are small parameters, $\beta \in C(\mathbb{R}, \mathbb{R})$ is 1-periodic and $z^+ = \max\{0, z\}$, $z^- = \min\{0, z\}$. Then we have similarly as for (5.1) that (3.6) now assumes the form

$$M_\mu(\theta) = -\mu_1 (\theta^+)^2 - \mu_2 \theta^- + \mu_3 \int_0^1 \beta(s)\, \mathrm{d}s.$$

By estimating the number of simple roots of M_μ, we obtain that if $\int\limits_0^1 \beta(s)\,\mathrm{d}s \neq 0$ then (6.1) has a 1-periodic solution for any sufficiently small $\mu_{1,2,3}$ satisfying one of the following conditions:

a) $\mu_3 \int\limits_0^1 \beta(s)\,\mathrm{d}s < 0$ and either $\mu_1 < 0$ or $\mu_2 > 0$;

b) $\mu_3 \int\limits_0^1 \beta(s)\,\mathrm{d}s > 0$ and either $\mu_1 > 0$ or $\mu_2 < 0$.

Moreover, if $\int\limits_0^1 \beta(s)\,\mathrm{d}s \neq 0$ then (6.1) has at least two 1-periodic solutions for any sufficiently small $\mu_{1,2,3}$ satisfying one of the following conditions:

c) $\mu_3 \int\limits_0^1 \beta(s)\,\mathrm{d}s < 0$ and $\mu_1 < 0$, $\mu_2 > 0$;

d) $\mu_3 \int\limits_0^1 \beta(s)\,\mathrm{d}s > 0$ and $\mu_1 > 0$, $\mu_2 < 0$.

Remark 6.3. Finally we note that by combining the method of this paper with that of [10], we can straightforwardly extend the results of this paper to singularly perturbed differential inclusions of the form

$$\begin{aligned} &\dot{x}(t) \in f(x(t), y(t)) + \varepsilon h_1(x(t), y(t), t) \quad \text{a.e. on} \quad \mathbb{R}, \\ &\varepsilon \dot{y}(t) \in g(x(t), y(t)) + \varepsilon h_2(x(t), y(t), t) \quad \text{a.e. on} \quad \mathbb{R} \end{aligned} \tag{6.2}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$, $\varepsilon > 0$ is small and the following assumptions are satisfied:

(I) $f \in C^2(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^n)$, $g \in C^2(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R}^k)$ and $h_1\colon \mathbb{R}^n \times \mathbb{R}^k \to 2^{\mathbb{R}^n} \setminus \emptyset$, $h_2\colon \mathbb{R}^n \times \mathbb{R}^k \to 2^{\mathbb{R}^k} \setminus \emptyset$ are upper-semicontinuous with compact and convex values.

(II) $g(\cdot, 0) = 0$, $g(x, y) = A(x)y + o(|y|_k)$ for $A(x) \in \mathcal{L}(\mathbb{R}^k)$ satisfying

$$B(x)A(x)B^{-1}(x) = (D_1(x), D_2(x)) \quad \forall\, x \in \mathbb{R}^n,$$

where $B\colon \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^k)$, $D_1\colon \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^{k_1})$, $D_2\colon \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^{k_2})$ are C^1-smooth mappings, $k = k_1 + k_2$ and

$$\begin{gathered} \langle D_1(x)v, v\rangle_{k_1} > a|v|_{k_1}^2, \quad \langle D_2(x)w, w\rangle_{k_2} < -a|w|_{k_2}^2 \\ \forall\, x \in \mathbb{R}^n, \forall\, v \in \mathbb{R}^{k_1}, \forall\, w \in \mathbb{R}^{k_2}, \end{gathered}$$

where $a > 0$ is a constant. Here $\langle \cdot, \cdot \rangle_i$ is an inner product on $\mathbb{R}^i$, $i \in \mathbb{N}$ with the corresponding norm $|\cdot|_i$.

(III) The reduced equation of (6.2) of the form $\dot{x} = f(x, 0)$ has a manifold of 1-periodic solutions, i.e. there is an open subset $\mathcal{O} \subset \mathbb{R}^{d-1}$, $d \geqslant 1$ and a C^2-mapping $\gamma\colon \mathcal{O} \times \mathbb{R} \to \mathbb{R}^n$ such that $\gamma(\theta, t+1) = \gamma(\theta, t)$ and $\gamma(\theta, \cdot)$ is a solution of $\dot{x} = f(x, 0)$.

(IV) The only 1-periodic solutions of $\dot{x} = D_x f(\gamma(\theta, t), 0)x$ are linear combinations of $\frac{\partial}{\partial \theta_i}\gamma(\theta, t)$, $\dot{\gamma}(\theta, t)$, $i = 1, \ldots, d-1$. Moreover, $\frac{\partial}{\partial \theta_i}\gamma(\theta, t)$, $\dot{\gamma}(\theta, t)$, $i = 1, \ldots, d-1$ are linearly independent.

(V) $h_i(x, y, t+1) = h_i(x, y, t)$ for $t \in \mathbb{R}$ and $i = 1, 2$.

By assuming in addition the validity of a condition similar to **(H)** of [10], multivalued mappings like (3.6), (4.4) and (4.10) can be derived for both the non-autonomous and autonomous versions of (6.2). For instance, the multivalued mapping corresponding to (3.6) has the form

$$
\begin{aligned}
&M\colon \mathbb{R}^d \to 2^{\mathbb{R}^d} \setminus \emptyset, \quad M(\theta, \alpha) = \bigl\{L(\theta)h \,\mathrm{d}s\colon\ h \in L^2 \text{ satisfying a.e. on } [0,1] \\
&\text{the relation} \quad h(t) \in D_y f(\gamma(\theta, t), 0)(C(\theta, t, \alpha)) + h_1(\gamma(\theta, t), 0, t+\alpha)\bigr\},
\end{aligned}
$$

where $C\colon \mathcal{O} \times [0,1] \times \mathbb{R} \to 2^{\mathbb{R}^k} \setminus \emptyset$ is an upper-semicontinuous mapping with compact convex values such that $C(\mathcal{O} \times [0,1] \times \mathbb{R})$ is bounded, $C(\theta, t, \alpha+1) = C(\theta, t, \alpha)$ and it satisfies the above mentioned condition similar to **(H)** of [10].

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