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HOMOGENIZATION OF PARABOLIC EQUATIONS AN ALTERNATIVE APPROACH AND SOME CORRECTOR-TYPE RESULTS

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Abstract. We extend and complete some quite recent results by Nguetseng [Ngu1] and Allaire [All3] concerning two-scale convergence. In particular, a compactness result for a certain class of parameterdependent functions is proved and applied to perform an alternative homogenization procedure for linear parabolic equations with coefficients oscillating in both their space and time variables. For different speeds of oscillation in the time variable, this results in three cases. Further, we prove some corrector-type results and benefit from some interpolation properties of Sobolev spaces to identify regularity assumptions strong enough for such results to hold.

Keywords: partial differential equations, homogenization, two-scale convergence, linear parabolic equations, oscillating coefficients in space and time variable, dissimilar speeds of oscillation, admissible test functions, corrector results, compactness result, interpolation

MSC 2000: 35B27, 35K99, 73B27, 73K20

1. Introduction

Homogenization is a mathematical technique for the study of effective properties and microvariations in heterogeneous media through convergence analysis applied to the classical equations of mechanics. Various concepts of convergence, such as G-convergence, Γ-convergence and a number of related approaches (see [Att], [Bens], [DM], [Defr], [Per] and [SaPa]) have been developed for this purpose.

In this paper we apply and extend a quite recent method, two-scale convergence, to homogenize a class of linear parabolic equations and to prove some corrector-type

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results. Two-scale convergence was first introduced by Nguetseng in [Ngu1] and further improved by Allaire in [All3]. To be more precise, we will study the limit behaviour as ε passes to zero of sequences of solutions to equations of the type (see next page for the notation)

$$
\begin{aligned}
\partial_t u^\varepsilon(x,t) - \partial_{x_j}\Big(a_{ij}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\partial_{x_i}u^\varepsilon(x,t)\Big) &= f(x,t) \text{ in } \Omega\times I,\\
u^\varepsilon(x,0) &= u_0(x) \text{ in } \Omega \text{ and}\\
u^\varepsilon(x,t) &= 0 \text{ on } \partial\Omega,
\end{aligned}
\tag{1}
$$

where $r>0$. For $f\in L^2(I;W^{-1,2}(\Omega))$, $u_0\in L^2(\Omega)$, and $a_{ij}\in L^\infty_\sharp(Y\times J)$ positively definite, the operator equation (1) possesses a unique solution u^ε that belongs to $W^1_2(I;W^{1,2}_0(\Omega),L^2(\Omega))$. Further, (1) is equivalent to the corresponding weak formulation

$$
\begin{aligned}
\int_0^T\!\!\int_\Omega -u^\varepsilon(x,t)v(x)\partial_t c(t) + a_{ij}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\partial_{x_i}u^\varepsilon(x,t)\partial_{x_j}v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t\\
= \int_0^T\!\!\int_\Omega f(x,t)v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{2}
$$

for all $v\in W^{1,2}_0(\Omega)$ and $c\in D(I)$. See [Zei, Ch. 23.7].

This paper is organized as follows: In Section 2, we discuss and slightly amend some previous results concerning two-scale convergence. In Section 3, we prove an essential compactness result (Theorem 3.1). In particular, we make use of this result to perform a quick and convenient homogenization procedure in Section 4. Section 5 is devoted to proving some stronger convergence results (corrector results); see our Theorems 5.1 and 5.2. Finally, Section 6 is reserved for further results and concluding remarks. Especially, we prove a theorem (Theorem 6.1) which points out sufficient regularity assumptions on (1) to guarantee that a certain corrector result holds.

Throughout this report, we adopt the Einstein tensor summation convention. However, where beneficial to the brevity or transparency of the text, we may use standard operator symbols such as gradient (∇) or Laplacian (Δ). All the notation for Sobolev spaces is standard and can be found in e.g. [Ada] and in [Zei, Chapter 23], and all limits with respect to ε mean that ε passes to zero. The spaces H^m, H^s, and $H^{r,s}$ described below can also be found in Section 4.2.1 in [LiMa].

We introduce some more specific notation used in this report.

I: The intervall $]0,T[$.

J: The intervall $]0,1[$.

$K(\Omega)$: The space of all continuous functions with compact support in Ω.

$H^m(I;L^2(\Omega)) = \big\{u\colon\ u,\partial_t u,\ldots,\frac{\partial^m}{\partial^m t}u\in L^2(\Omega\times I)\big\}$, m integer.

$H^s(I;L^2(\Omega)) = [H^m(I;L^2(\Omega)), L^2(\Omega\times I)]_\theta$, $s=(1-\theta)m$, $0\leqslant\theta\leqslant 1$, where the brackets mean interpolation between the spaces $H^m(I;L^2(\Omega))$ and $L^2(\Omega\times I)$, s is not necessarily an integer.

$H^{r,s}(\Omega\times I) = L^2(I;W^{r,2}(\Omega))\cap H^s(I;L^2(\Omega))$.

$A(\Omega;B_\sharp(Y))$: A mapping of type $A(\Omega)$ into a space of type $B_\sharp(Y)$. $\sharp$ means that the functions in $B_\sharp(Y)$ are periodic with respect to the unit cube Y.

$B_\sharp(Y;A(\Omega))$: A Y-periodic mapping of type $B_\sharp(Y)$ into a space of type $A(\Omega)$.

$\mathrm{d}Z = \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$.

2. Two-scale convergence

In this section we study the notion of two-scale convergence that was originally invented by Nguetseng (see [Ngu1]) and further improved by Allaire (see [All1] and [All3]). We define two-scale convergence in the shape it was first introduced by Nguetseng in [Ngu1] and state the corresponding compactness result.

Definition 2.1. A sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ is said to two-scale converge to $u_0\in L^2(\Omega\times Y)$ if

$$\lim_\varepsilon \int_\Omega u^\varepsilon(x)a\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega\int_Y u_0(x,y)a(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{3}$$

for all $a\in D(\Omega;C^\infty_\sharp(Y)$. Sometimes, we will write this as

$$u^\varepsilon \rightharpoonup\!\!\rightharpoonup u_0 \text{ in } L^2(\Omega\times Y).$$

Theorem 2.2. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then $\{u^\varepsilon\}$ contains a subsequence satisfying* (3) *for some (unique) $u_0\in L^2(\Omega\times Y)$, all $a\in K(\Omega;C_\sharp(Y))$ and all $a=a_1\cdot a_2$, $a_1\in K(\Omega)$, $a_2\in L^2_\sharp(Y)$.*

In [All1] and [All3], Allaire enlarges the class of test functions for which (3) holds to all functions in $L^2(\Omega;C_\sharp(Y))$ and, for Ω bounded, in $L^2_\sharp(Y,C(\overline{\Omega}))$ and $C(\overline{\Omega};C_\sharp(Y))$, and provides an independent proof of Theorem 2.2. We prove the L^p-version of this result and characterize the corresponding spaces of admissible test functions. Further, we demonstrate that two-scale convergence works in an unaltered fashion even if we allow different variables to possess dissimilar speed of oscillation.

Theorem 2.3. *Assume that $\{u^\varepsilon\}$ is a bounded sequence in $L^p(\Omega\times I)$, $p\in]1,\infty]$, and that X is a separable subspace of $L^q(\Omega\times I\times Y\times J)$ such that, for any $a\in X$ and all $r>0$,*

$$\lim_\varepsilon \Big\|a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\Big\|_{L^q(\Omega\times I)} = \|a(x,t,y,s)\|_{L^q(\Omega\times I\times Y\times J)} \tag{4}$$

and

$$\left\| a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\right\|_{L^q(\Omega\times I)} \leqslant C\, \|a(x,t,y,s)\|_X\,. \tag{5}$$

Then, at least for a subsequence and for any $r>0$, *there exists* $u_0\in L^p(\Omega\times I\times Y\times J)$ *such that*

$$\lim_{\varepsilon}\int_0^T\!\!\int_\Omega u^\varepsilon(x,t)a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\!\!\int_\Omega\!\int_0^1\!\!\int_Y u_0(x,t,y,s)a(x,t,y,s)\,\mathrm{d}Z$$

for all $a\in X$.

Proof. We first note that the Hölder inequality, the boundedness of $\{u^\varepsilon\}$ and property (5) yield that

$$\int_0^T\!\!\int_\Omega u^\varepsilon(x,t)a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\leqslant C\left\| a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\right\|_{L^q(\Omega\times I)}\leqslant C\,\|a(x,t,y,s)\|_X\,.$$

Obviously, $\{u^\varepsilon\}$ represents a bounded sequence of bounded linear functionals F^ε on X defined through

$$(F^\varepsilon,a)_{X',X}=\int_0^T\!\!\int_\Omega u^\varepsilon(x,t)a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t.$$

By assumption, X is a separable Banach space and thus weakly$*$ sequentially compact. Hence, up to a subsequence, there exists $F\in X'$ such that

$$F^\varepsilon\rightharpoonup F \text{ weakly} * \text{ in } X'.$$

Moreover, by the Hölder inequality, the boundedness of $\{u^\varepsilon\}$ in $L^p(\Omega\times I)$, and assumption (4) on a, we achieve

$$\begin{aligned}(F,a)_{X',X}&=\lim_{\varepsilon}\int_0^T\!\!\int_\Omega u^\varepsilon(x,t)a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C\lim_{\varepsilon}\left\| a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\right\|_{L^q(\Omega\times I)}=C\,\|a(x,t,y,s)\|_{L^q(\Omega\times I\times Y\times J)}\,.\end{aligned}$$

This means that F remains a bounded linear functional on X also if we replace the X-norm by the $L^q(\Omega\times I\times Y\times J)$-norm. This space, which consists of the same elements as X but is normed by the $L^q(\Omega\times I\times Y\times J)$-norm instead of the X-norm, we denote by X_1. The Hahn-Banach theorem for linear functionals (see [Alt, p. 97, Satz 4.2]) yields the existence of a functional $G \in (L^q(\Omega\times I\times Y\times J))'$, that extends F from X_1 to $L^q(\Omega\times I\times Y\times J)$ and satisfies

$$\|G\|_{(L^q(\Omega\times I\times Y\times J))'} = \|F\|_{X_1'}\,.$$

By the Riesz representation theorem ([Kuf, p. 79, 2.9.5]), there exists a unique $u_0 \in L^p(\Omega\times I\times Y\times J)$ such that

$$(G,a)_{(L^q(\Omega\times I\times Y\times J))',L^q(\Omega\times I\times Y\times J)} = \int_0^T\int_\Omega\int_Y\int_0^1 u_0(x,t,y,s)a(x,t,y,s)\,\mathrm{d}Z$$

for all $a \in L^q(\Omega\times I\times Y\times J)$ and thus

$$(F,a)_{X',X} = (F,a)_{X_1',X_1} = \int_0^T\int_\Omega\int_Y\int_0^1 u_0(x,t,y,s)a(x,t,y,s)\,\mathrm{d}Z$$

for any $a \in X$. We have proved that, for any bounded sequence $\{u^\varepsilon\}$ in $L^p(\Omega\times I)$, $p \in]1,\infty]$, we can extract a subsequence, also denoted $\{u^\varepsilon\}$, such that

$$\lim_\varepsilon \int_0^T\int_\Omega u^\varepsilon(x,t)a\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega\int_Y\int_0^1 u_0(x,t,y,s)a(x,t,y,s)\,\mathrm{d}Z$$

for some $u_0 \in L^p(\Omega\times I\times Y\times J)$ and any a that meets (4) and (5). The theorem is proved $\square$

R e m a r k 1. The careful reader may have noticed that the extended map G in the proof of Theorem 2.3 is not necessarily unique, unless X is dense in $L^q(\Omega\times I\times Y\times J)$. However, the important point is that we can use the same u_0 to characterize the limit for any $a \in X$.

Definition 2.4. We say that $a \in L^q(\Omega\times I\times Y\times J), q \in [1,\infty[$, is an admissible test function if it complies with (4) and (5).

R e m a r k 2. Important examples of admissible test functions are those in $L^q(\Omega\times I; C_\sharp(Y\times J))$ and, for Ω bounded, in $L^q_\sharp(Y\times J; C(\overline{\Omega}\times\overline{I}))$ (see Section 5 in [All3]). For the sake of simplicity, in the sequel we will assume that Ω is bounded.

R e m a r k 3. The corresponding result for traditional two-scale convergence is obtained if we just remove the dependence of t and s from all the functions involved.

Clearly, the usual kind of two-scale convergence exhibited in (3) is immediately generalized to the version shown in (6) and, therefore, we formulate the rest of the results in this section in the standard setting, where only one speed of oscillation appears. For $p = 2$, the following slightly different version of Theorem 2.3 holds.

Theorem 2.5. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$ with a two-scale limit u_0. Further, assume that $\{a^\varepsilon\}$ is a sequence in $L^2(\Omega)$ with a two-scale limit a and that*

$$\lim_\varepsilon \|a^\varepsilon(x)\|_{L^2(\Omega)} = \|a(x,y)\|_{L^2(\Omega\times Y)}\,. \tag{6}$$

Then

$$\lim_\varepsilon \int_\Omega u^\varepsilon(x)a^\varepsilon(x)\,\mathrm{d}x = \int_\Omega\int_Y u_0(x,y)a(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{7}$$

and if, in addition,

$$\lim_\varepsilon \Big\|a\Big(x,\frac{x}{\varepsilon}\Big)\Big\|_{L^2(\Omega)} = \|a(x,y)\|_{L^2(\Omega\times Y)}\,,$$

then

$$\lim_\varepsilon \Big\|a^\varepsilon(x) - a\Big(x,\frac{x}{\varepsilon}\Big)\Big\|_{L^2(\Omega)} = 0. \tag{8}$$

Proof. See Theorem 2.4 in [All1] and Theorem 1.8 in [All3]. □

Next we claim that any function in $L^2(\Omega\times Y)$ will appear as the two-scale limit of some bounded sequence in $L^2(\Omega)$.

Proposition 2.6. *Let u be any function in $L^2(\Omega\times Y)$. There then exists a bounded sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ that two-scale converges to u.*

Proof. See Lemma 1.13 in [All3]. □

Proposition 2.7. *Assume that $u\in L^2(\Omega\times Y)$ is an admissible test function. Then $\{u(x,\frac{x}{\varepsilon})\}$ two-scale converges to u.*

Proof. Proposition 2.6 says that there exists a bounded sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ that two-scale converges to u. By construction (see [All3] or [HolWel]),

$$\|u^\varepsilon\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega\times Y)}$$

and hence, by (8) and the Hölder inequality,

$$\int_\Omega \Big(u^\varepsilon(x) - u\Big(x,\frac{x}{\varepsilon}\Big)\Big)a\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \leqslant \Big\|u^\varepsilon(x) - u\Big(x,\frac{x}{\varepsilon}\Big)\Big\|_{L^2(\Omega)}\Big\|a\Big(x,\frac{x}{\varepsilon}\Big)\Big\|_{L^2(\Omega)} \to 0.$$

We have proved that, for all admissible a,

$$\lim_\varepsilon \int_\Omega u\Big(x,\frac{x}{\varepsilon}\Big)a\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \lim_\varepsilon \int_\Omega u^\varepsilon(x)a\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega\int_Y u(x,y)a(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

The proof is complete. □

R e m a r k 4. In the light of Theorem 2.5, the close relationship is worth noticing between on the one hand the strong and weak convergences in the usual L^2-meaning and on the other the corresponding notions in the sense of two-scale convergence. Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then, up to a subsequence and for some $u \in L^2(\Omega)$,

$$u^\varepsilon \rightharpoonup u \text{ weakly in } L^2(\Omega).$$

If, in addition,

$$\|u^\varepsilon\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)}\,, \tag{9}$$

then

$$u^\varepsilon \to u \text{ strongly in } L^2(\Omega).$$

We compare this with the corresponding cases of two-scale convergence and find that, still up to a subsequence,

$$u^\varepsilon \rightharpoonup\!\!\rightharpoonup u_0 \ \text{ in } L^2(\Omega \times Y),$$

i.e. $\{u^\varepsilon\}$ passes to u_0 in the sense of usual (weak) two-scale convergence.

Under the supplementary assumptions that

$$\|u^\varepsilon\|_{L^2(\Omega)} \to \|u_0\|_{L^2(\Omega\times Y)} \tag{10}$$

and

$$\left\|u_0\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \to \|u_0\|_{L^2(\Omega\times Y)} \tag{11}$$

Theorem 2.5 yields that

$$\left\|u^\varepsilon(x) - u_0\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \to 0$$

or, in other words, $\{u^\varepsilon\}$ two-scale converges strongly to u_0. Clearly, usual two-scale convergence plays the role of weak convergence and the assumptions (10) and (11) strengthen the weak (usual) two-scale convergence in a similar way as assumption (9) turns the weak $L^2(\Omega)$-convergence of $\{u^\varepsilon\}$ into the corresponding strong convergence.

Proposition 2.8. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$ and let v belong to $L^\infty_\sharp(Y)$. Then, up to a subsequence, $\{u^\varepsilon(x)v(\frac{x}{\varepsilon})\}$ two-scale converges to $u_0(x,y)v(y)$, where u_0 is the two-scale limit to $\{u^\varepsilon\}$.*

P r o o f. Obviously, $\{u^\varepsilon(x)v(\frac{x}{\varepsilon})\}$ is a bounded sequence of functions in $L^2(\Omega)$ and thus possesses a unique two-scale limit $w_0 \in L^2(\Omega\times Y)$. Moreover, for e.g. $a = a_1\cdot a_2$, $a_1 \in D(\Omega), a_2 \in C^\infty_\sharp(Y)$, $a\cdot v$ is an admissible test function and hence

$$\int_\Omega u^\varepsilon(x)a_1(x)a_2\Big(\frac{x}{\varepsilon}\Big)v\Big(\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \to \int_\Omega\int_Y u_0(x,y)a_1(x)a_2(y)v(y)\,\mathrm{d}x\,\mathrm{d}y.$$

This, together with the fact that the class of test functions a_1, a_2 used here is large enough to provide uniqueness, means exactly that the two-scale limit w_0 coincides with $u_0\cdot v$ and the proof is complete. □

Proposition 2.9. *Assume that $\{u^\varepsilon\}$ is a sequence in $L^2(\Omega)$ that two-scale converges to $u \in L^2(\Omega\times Y)$. Then $\{u^\varepsilon\}$ converges weakly to $\int_Y u(x,y)\,\mathrm{d}y$ in $L^2(\Omega)$.*

P r o o f. See Proposition 1.16 in [All3]. □

Proposition 2.10. *Assume that $\{u^\varepsilon\}$ converges strongly to u in $L^2(\Omega)$. Then $\{u^\varepsilon\}$ two-scale converges to u.*

P r o o f. Let a be any function in $L^2(\Omega; C_\sharp(Y))$. By assumption, $u \in L^2(\Omega)$ and thus $u\cdot a \in L^1(\Omega; C_\sharp(Y))$. This means (see (5.8) in the proof of Lemma 5.3 in [All3]) that

$$\int_\Omega u(x)a\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \to \int_\Omega\int_Y u(x)a(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Further, by Hölder's inequality and the fact that a obeys (4), we arrive at

$$\left|\int_\Omega(u^\varepsilon(x)-u(x))a\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x\right| \leqslant \|u^\varepsilon-u\|_{L^2(\Omega)}\cdot\left\|a\Big(x,\frac{x}{\varepsilon}\Big)\right\|_{L^2(\Omega)} \to 0$$

and the proof is complete. □

R e m a r k 5. In [HolWel], it is proved that the results in Theorem 2.5 and Propositions 2.6–2.10 hold also in $L^p(\Omega)$, where p may be different from two.

R e m a r k 6. Both Nguetseng and Allaire characterize limits of bounded sequences in $W^{1,2}(\Omega)$ and gradients of such sequences. In Chapter 3, we will prove such results in a more general setting that is particularily well suited for homogenization procedures for a quite large class of linear and nonlinear parabolic equation.

Below we prove an extension of Theorem 2.3 that is a generalization in a certain evolution sense of the corresponding compactness results for gradients mentioned in Remark 6.

Theorem 3.1. *Assume that $\{u^\varepsilon\}$ is a bounded sequence in $W_p^1(I; W_0^{1,p}(\Omega), L^2(\Omega))$ and that $p \in [2, \infty[$.*

Then, up to a subsequence,

$$\lim_\varepsilon \int_0^T \int_\Omega u^\varepsilon(x,t) a\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \int_0^1 \int_Y u(x,t) a(x,t,y,s)\,\mathrm{d}Z$$

and

$$\begin{aligned}\lim_\varepsilon \int_0^T \int_\Omega \partial_{x_i} u^\varepsilon(x,t) a_i\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T \int_\Omega \int_0^1 \int_Y (\partial_{x_i} u(x,t) + \partial_{y_i} u_1(x,t,y,s)) a_i(x,t,y,s)\,\mathrm{d}Z\end{aligned}$$

for all admissible $a\colon \Omega \times I \times Y \times J \to R^N$, where u is the weak $L^p(I; W^{1,p}(\Omega))$-limit to $\{u^\varepsilon\}$ and $u_1 \in L^p(\Omega \times I; L^p(J; W_\sharp^{1,p}(Y)/R))$.

The L^2-version of the result below is found in Lemma 4 in [Ngu1] (see also Remark 1.9 in Chapter 1 in [Tem]) and is essential for the proof of Theorem 3.1. The generalization to the L^p-case follows immediately from [Ziem] Theorem 2.1.4.

Lemma 3.2. *Let $f \in [L_\sharp^p(Y)]^N$, $p \geqslant 2$, and assume that $\int_Y f(y) \cdot a(y)\,\mathrm{d}y = 0$ for all $a \in [C_\sharp^\infty(Y)]^N$ with zero divergence. Then there exists a unique $u \in W_\sharp^{1,p}(Y)/R$ such that $\nabla u = f$.*

Proof of Theorem 3.1. $\{u^\varepsilon\}$ is bounded in $L^p(I; W^{1,p}(\Omega))$ and thus $\{\partial_{x_i} u^\varepsilon\}$ is bounded in $L^p(\Omega \times I)$. Consequently, there exists two-scale limits u_0 in $L^p(\Omega \times I \times Y \times J)$ and v_0 in $[L^p(\Omega \times I \times Y \times J)]^N$ such that, up to a subsequence,

$$\int_0^T \int_\Omega u^\varepsilon(x,t) a\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \int_0^1 \int_Y u_0(x,t,y,s) a(x,t,y,s)\,\mathrm{d}Z$$

and

$$\int_0^T \int_\Omega \partial_{x_i} u^\varepsilon(x,t) a_i\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \int_0^1 \int_Y v_{0,i}(x,t,y,s) a_i(x,t,y,s)\,\mathrm{d}Z \tag{12}$$

for all admissible a and a_i. We have established the existence of the respective two-scale limits. It remains to force as much regularity as possible on them. By the boundedness of $\{u^\varepsilon\}$ in $L^p(I; W^{1,p}(\Omega))$ and the weak sequential compactness of unit balls in reflexive Banach spaces it follows that, up to a subsequence,

$$u^\varepsilon \rightharpoonup u \text{ weakly in } L^p(I; W^{1,p}(\Omega)) \tag{13}$$

and hence, of course, in $L^p(\Omega \times I)$. Moreover, in e.g. Lemmas 8.2 and 8.4 in [CoFo] it is proven that for $\{u^\varepsilon\}$ a bounded sequence in $W_p^1(I; W_0^{1,p}(\Omega), L^2(\Omega))$

$$u^\varepsilon \to u \text{ strongly in } L^2(\Omega \times I) \tag{14}$$

up to a subsequence. Strong L^2-convergence to a certain limit u implies two-scale convergence to this same limit (see Proposition 2.10) and hence we have proved that $\{u^\varepsilon\}$ two-scale converges to its weak $L^2(I; W^{1,2}(\Omega))$-limit u. Further, again by (13), it is clear that $u \in L^p(I; W^{1,p}(\Omega))$.

Next we characterize v_0. For this purpose, it will prove sufficient to use a smaller class of test functions. Therefore, let

$$a(x,t,y,s) = a_1(x) \cdot a_2(t) \cdot a_3(y) \cdot a_4(s),$$

where $a_1 \in D(\Omega)$, $a_2 \in D(I)$, $a_3 \in [C_\sharp^\infty(Y)]^N$, $a_4 \in C_\sharp^\infty(J)$, and a_3 has zero divergence. Obviously, a is an admissible test function and so

$$b(x,t,y,s) = \partial_{x_i} a_1(x) a_2(t) a_{3,i}(y) a_4(s)$$

and thus, integrating by parts, once before and once after the passage to the limit, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_{x_i} u^\varepsilon(x,t) a_1(x) a_2(t) a_{3,i}\left(\frac{x}{\varepsilon}\right) a_4\left(\frac{t}{\varepsilon^r}\right) \mathrm{d}x\, \mathrm{d}t \\
&\qquad \to \int_0^T \int_\Omega \int_0^1 \int_Y \partial_{x_i} u(x,t) a_1(x) a_2(t) a_{3,i}(y) a_4(s)\, \mathrm{d}Z.
\end{aligned}$$

We have proved that, by (12), for these test functions,

$$\begin{aligned}
&\int_0^T \int_\Omega \int_0^1 \int_Y v_{0,i}(x,t,y,s) a_1(x) a_2(t) a_{3,i}(y) a_4(s)\, \mathrm{d}Z \\
&\qquad = \int_0^T \int_\Omega \int_0^1 \int_Y \partial_{x_i} u(x,t) a_1(x) a_2(t) a_{3,i}(y) a_4(s)\, \mathrm{d}Z.
\end{aligned}$$

This implies that v_0 and ∇u differ only up to a certain term that will vanish during the above integration process. Formally, this means that for

$$U_{1,i}(x,t,y,s) = v_{0,i}(x,t,y,s) - \partial_{x_i} u(x,t)$$

we have

$$\int_0^T \int_\Omega \int_0^1 \int_Y U_{1,i}(x,t,y,s) a_1(x) a_2(t) a_{3,i}(y) a_4(s) \, \mathrm{d}Z = 0$$

and, consequently,

$$\int_Y U_{1,i}(x,t,y,s) a_{3,i}(y) \, \mathrm{d}y = 0$$

a.e. in $\Omega \times I \times J$ and for all divergence-free $a_3 \in [C_\sharp^\infty(Y)]^N$. Lemma 3.2 now yields that, for a.e. fixed $(x,t,s) \in \Omega \times I \times J$, there exists $u_1(x,t,\cdot,s) \in W_\sharp^{1,p}(Y)/R$ such that

$$\partial_{y_i} u_1(x,t,y,s) = U_{1,i}(x,t,y,s) = v_{0,i}(x,t,y,s) - \partial_{x_i} u(x,t).$$

It remains to prove that u_1 provides a measurable function

$$u_1 \colon \Omega \times I \to L^p(J; W_\sharp^{1,p}(Y)/R)$$

and is bounded with respect to the $L^p(\Omega \times I; L^p(J; W_\sharp^{1,p}(Y)/R))$-norm. We first prove measurability. Lusin characterization and Petti's theorem say that u_1 is measurable iff it is continuous up to small sets (see Remark 7). We know that $L^p(J; L^p(Y))$ is separable and that $v_{0,i}$ and $\partial_{x_i} u$ belong to $L^p(\Omega \times I; L^p(J; L^p(Y))$ and hence, for a compact K with $\mu(A - K) < \delta$, $v_{0,i}$ and $\partial_{x_i} u$ are continuous on K.

For $(x_j, t_j) \to (x,t)$ in K, we then have

$$\begin{aligned}
&\|u_1(x,t,y,s) - u_1(x_j,t_j,y,s)\|_{L^p(J; W_\sharp^{1,p}(Y)/R))} \\
&= \|\nabla_y u_1(x,t,y,s) - \nabla_y u_1(x_j,t_j,y,s)\|_{[L^p(J;L^p(Y))]^N} \\
&\leqslant \|\nabla u(x,t) - \nabla u(x_j,t_j)\|_{[L^p(J;L^p(Y))]^N} \\
&\quad + \|v_0(x,t,y,s) - v_0(x_j,t_j,y,s)\|_{[L^p(J;L^p(Y))]^N} \to 0.
\end{aligned}$$

We have proved that u_1 is continuous on K and the measurability of u_1 follows.

Finally, we prove that u_1 is bounded in $L^p(\Omega \times I; L^p(J; W_\sharp^{1,p}(Y)/R))$. This follows directly by Minkowski's inequality through

$$\begin{aligned}
&\|u_1(x,t,y,s)\|_{L^p(\Omega \times I; L^p(J; W_\sharp^{1,p}(Y)/R))} = \|\nabla_y u_1(x,t,y,s)\|_{[L^p(\Omega \times I; L^p(J; L^p(Y)))]^N} \\
&\leqslant \|\nabla u(x,t)\|_{[L^p(\Omega \times I; L^p(J; L^p(Y)))]^N} + \|v_0(x,t,y,s)\|_{[L^p(\Omega \times I; L^p(J; L^p(Y)))]^N} < \infty.
\end{aligned}$$

We have proved that $u_1 \in L^p(\Omega \times I; L^p(J; W_\sharp^{1,p}(Y)/R)$. The proof is complete. $\square$

The corollary below is essential, especially for the case $r = 2$, in the homogenization procedures executed in Section 4.

Corollary 3.3. *Assume that* $\{u^\varepsilon\}$ *is a bounded sequence in* $W_p^1(I; W_0^{1,p}(\Omega), L^2(\Omega))$ *and that* $p \in [2, \infty[$.

Then

$$\lim_\varepsilon \int_0^T \int_\Omega (u^\varepsilon(x,t) - u(x,t))(1/\varepsilon) b\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\Big)\Big) \,\mathrm{d}x \\ = \int_\Omega \int_Y u_1(x,t,y,s) b(x,t,y,s) \,\mathrm{d}Z$$

for any $b\colon \Omega \times I \to Y \times J$ *such that* $b = b_1 \cdot b_2 \cdot c_1 \cdot c_2$, *where* $b_1 \in D(\Omega), b_2 \in L^2_\sharp(Y)/R$, $c_1 \in D(I)$, *and* $c_2 \in L^2_\sharp(J)$.

Proof. For some $a \in [W_\sharp^{1,2}(Y)]^N$, we achieve any $b_2 \in L^2_\sharp(Y)/R$ as the divergence of a (see Lemma 2.4 in Chapter 1 in [Tem]). Theorem 3.1 yields that

$$\int_0^T \int_\Omega (\partial_{x_i} u^\varepsilon(x,t) - \partial_{x_i} u(x,t)) b_1(x) a_i\Big(\frac{x}{\varepsilon}\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \\ \to \int_0^T \int_\Omega \int_0^1 \int_Y \partial_{y_i} u_1(x,t,y,s) b_1(x) a_i(y) c_1(t) c_2(s) \,\mathrm{d}Z.$$

Integrating by parts, we find that this means

$$\int_0^T \int_\Omega (u^\varepsilon(x,t) - u(x,t))(\partial_{x_i} b_1(x) a_i(x/\varepsilon) + (1/\varepsilon) b_1(x) \partial_{y_i} a_i\Big(\frac{x}{\varepsilon}\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \\ \to \int_0^T \int_\Omega \int_0^1 \int_Y u_1(x,t,y,s) b_1(x) \partial_{y_i} a_i(y) c_1(t) c_2(s) \,\mathrm{d}Z.$$

The boundedness assumptions on $\{u^\varepsilon\}$ and (14) in the proof of Theorem 3.1 suffices to conclude that $\{u^\varepsilon\}$ converges strongly to u in $L^2(\Omega)$. Hence, by Proposition 2.10 and the Hölder inequality

$$\int_0^T \int_\Omega (u^\varepsilon(x,t) - u(x,t))(\partial_{x_i} b_1(x)) a_i\Big(\frac{x}{\varepsilon}\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \to 0.$$

Consequently

$$\int_\Omega (u^\varepsilon(x,t) - u(x,t))(1/\varepsilon) b_1(x) \partial_{y_i} a_i\Big(\frac{x}{\varepsilon}\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t \\ \to \int_\Omega \int_Y u_1(x,t,y,s) b_1(x) \partial_{y_i} a_i(y) c_1(t) c_2(s) \,\mathrm{d}Z$$

and thus, for any b of the type specified in the theorem, the assertion follows and the proof is complete. $\square$

R e m a r k 7. Lusin's theorem, as stated in e.g. [Edw, Corollary 4.8.5, Remark 4.8.6], treats only functions from a measurable set A to R. Below we make the investigations necessary to justify the use of Lusin characterization also for functions from A into a separable Banach space Y. We say that $f\colon A \to Y$ is continuous up to small sets if, for any $\delta > 0$, there exists a compact K with $\mu(A - K) < \delta$ such that f is continuous on $A - K$. The following four statements are equivalent:

1) $f\colon A \to Y$ is measurable,
2) $p \circ f\colon A(\to Y) \to R$ is measurable for all $p \in Y'$,
3) $p \circ f\colon A \to R$ is continuous up to small sets,
4) $f\colon A \to Y$ is continuous up to small sets.

Clearly, Petti's theorem means that 1) is equivalent to 2) and 2) is equivalent to 3) by Lusin's theorem. Moreover, 4) implies 3) because

$$|p(f(x_j) - f(x))| \leqslant \|p\|_{Y'} \, \|f(x_j) - f(x)\|_Y \to 0$$

if $x_j \to x$ in $K \subset A$ and $f\colon A \to Y$ is continuous. Finally, in [Alt, A 4.11 pp. 131] it is proved that 1) implies 4).

R e m a r k 8. Two-scale limits for sequences of the type $\{\varepsilon \partial_{x_i} u^\varepsilon\}$, when bounded in $L^2(\Omega \times I)$, appear with much less effort as a fairly direct consequence of Theorem 2.2 and repeated integration by parts. See [All2] or [Nand].

4. A homogenization procedure

In this section we apply the results from the preceding sections to perform a quick and convenient homogenization procedure for (1). However, let us first make a brief comparison with classical homogenization techniques. Homogenization of linear parabolic equations may also be carried out using a method attributed to Luc Tartar, that is usually but not quite adequately, called the energy method.

A more appropriate name for this approach would be the "crosswise test function method". Rather heuristic methods, such as asymptotic expansion (see [Bens]), are utilized to infer suitable homogenized and local equations. The solution to the local problem is exploited to construct test functions that are introduced in the weak formulation of the original sequence of equations, and vice versa to prove certain convergence results.

A striking advantage of two-scale convergence in the light of the above is that the homogenized and local problems appear directly as strict convergence results and do not have to be derived by tedious and, from a theoretical point of view, more or less dubious calculations whose relevance has to be verified afterwards.

We present an alternative homogenization procedure that is based solely on our developments of Nguetseng's fundamental convergence results (Theorem 3.1) and standard functional analysis. First, we list some well known a priori estimates.

Lemma 4.1. *The solutions* $\{u^\varepsilon\}$ *of* (1) *are bounded in* $L^\infty(I;L^2(\Omega))$, $W_2^1(I;W_0^{1,2}(\Omega),L^2(\Omega))$, *and, consequently, in* $L^2(I;W_0^{1,2}(\Omega))$.

Theorem 4.2. *Any sequence* $\{u^\varepsilon\}$ *of solutions to* (1) *converges weakly in* $L^2(I;W_0^{1,2}(\Omega))$ *to a limit* $u\in W_2^1(I;W_0^{1,2}(\Omega),L^2(\Omega))$, *the unique solution to the homogenized problem*

$$\partial_t u(x,t)-\partial_{x_j}(\overline{a}_{ij}\partial_{x_i}u(x,t))=f(x,t)\text{ in }\Omega\times I,\qquad u(x,0)=u_0(x)\text{ in }\Omega. \tag{15}$$

For $r<2$ *we have*

$$\overline{a}_{jk}=\int_0^1\int_Y a_{ij}(y,s)(\partial_{ik}+\partial_{y_i}z^k(y,s))\,\mathrm{d}y\,\mathrm{d}s, \tag{16}$$

where $z^k\in L^2_\sharp(J;W^{1,2}_\sharp(Y)/R)$, $k=1,2,\ldots,N$ *is the unique solution to the local problem*

$$\partial_{y_j}(a_{ij}(y,s)(\delta_{ik}+\partial_{y_i}z^k(y,s)))=0. \tag{17}$$

For $r=2$, $\overline{a}_{jk}$ *is again computed by* (16) *but* z^k *is the solution to*

$$\partial_s z^k(y,s)-\partial_{y_j}(a_{ij}(y,s)(\delta_{ik}+\partial_{y_i}z^k(y,s)))=0. \tag{18}$$

For $r>2$, *finally,*

$$\overline{a}_{jk}=\int_Y\left(\int_0^1 a_{ij}(y,s)\,\mathrm{d}s\right)(\delta_{ik}+\partial_{y_i}z^k(y))\,\mathrm{d}y \tag{19}$$

and z^k *solves*

$$\partial_{y_j}\left(\left(\int_0^1 a_{ij}(y,s)\,\mathrm{d}s\right)(\delta_{ik}+\partial_{y_i}z^k(y))\right)=0. \tag{20}$$

P r o o f. We introduce $v\in W_0^{1,2}(\Omega)$ and $c\in D(I)$ in (2), pass to the two-scale limit, and obtain through Theorem 3.1

$$\begin{aligned}\int_0^T\int_\Omega &-u(x,t)v(x)\partial_t c(t)\\ &+\left[\int_0^1\int_Y a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\,\mathrm{d}y\,\mathrm{d}s\right]\partial_{x_j}v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t\\ =&\int_0^T\int_\Omega f(x,t)v(x)c(t)\,\mathrm{d}Z.\end{aligned} \tag{21}$$

Our approach is to study the limit behaviour of the difference between (2) and (21) for

$$v(x)=\varepsilon^{r-1}v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big),\ c(t)=c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big),$$

where $v_1\in D(\Omega)$, $v_2\in W^{1,2}_\sharp(Y)/R$, $c_1\in D(I)$ and $c_2\in C^\infty_\sharp(J)$. We obtain

(22)
$$\begin{aligned}&\int_0^T\int_\Omega((u^\varepsilon(x,t)-u(x,t))/\varepsilon)v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)\Big(\varepsilon^r(\partial_t c_1(t))c_2\Big(\frac{t}{\varepsilon^r}\Big)+c_1(t)\partial_s c_2\Big(\frac{t}{\varepsilon^r}\Big)\Big)\\&\qquad+\Big(\Big[\int_0^1\int_Y a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\,\mathrm{d}y\,\mathrm{d}s\Big]\\&\qquad-a_{ij}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\partial_{x_i}u^\varepsilon(x,t)\Big)\varepsilon^{r-2}(\varepsilon(\partial_{x_j}v_1(x))v_2\Big(\frac{x}{\varepsilon}\Big)\\&\qquad+v_1(x)\partial_{y_j}v_2\Big(\frac{x}{\varepsilon}\Big))c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

Further, multiplication by ε^{2-r} transforms this equation into a very useful shape. This version of (22) is exhibited below.

(23)
$$\begin{aligned}&\int_0^T\int_\Omega((u^\varepsilon(x,t)-u(x,t))/\varepsilon)v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)\Big(\varepsilon^2(\partial_t c_1(t))c_2\Big(\frac{t}{\varepsilon^r}\Big)+\varepsilon^{2-r}c_1(t)\partial_s c_2\Big(\frac{t}{\varepsilon^r}\Big)\Big)\\&\qquad+\Big(\Big[\int_0^1\int_Y a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\,\mathrm{d}y\,\mathrm{d}s\Big]\\&\qquad-a_{ij}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\partial_{x_i}u^\varepsilon(x,t)\Big)\Big(\varepsilon(\partial_{x_j}v_1(x))v_2\Big(\frac{x}{\varepsilon}\Big)\\&\qquad+v_1(x)\partial_{y_j}v_2\Big(\frac{x}{\varepsilon}\Big)\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

Clearly, $\partial_{x_j}v_1(x)v_2(y)c_1(t)c_2(s)$ and $v_1(x)\partial_{y_j}v_2(y)c_1(t)c_2(s)$ are test functions of e.g. the type $L^2_\sharp(Y\times J;C(\overline{\Omega}\times\overline{I}))$. In view of Theorem 3.1, Corollary 3.3, and Proposition 2.8, we study the limit processes for the three different cases singled out in the theorem.

For $0<r<2$, we pass to the two-scale limit in (23) arriving at

$$\begin{aligned}&\int_0^T\int_\Omega\int_0^1\int_Y(a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\\&-\Big[\int_0^1\int_Y a_{ij}(y,s)\Big(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s)\,\mathrm{d}y\,\mathrm{d}s\Big]\Big)v_1(x)\partial_{y_j}v_2(y)c_1(t)c_2(s)\,\mathrm{d}Z=0.\end{aligned}$$

Obviously,

$$\begin{aligned}\int_0^T\int_\Omega\int_0^1\int_Y\Big[\int_0^1\int_Y a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\,\mathrm{d}y\,\mathrm{d}s\Big]&\\\times v_1(x)\partial_{y_j}v_2(y)c_1(t)c_2(s)\,\mathrm{d}Z=0&\end{aligned}\tag{24}$$

(the expression within the brackets is independent of y and s) and hence

$$\int_Y a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\partial_{y_j}v_2(y)\,\mathrm{d}y=0$$

for all $v_2 \in W_\sharp^{1,2}(Y)/R$ and a.e. in $\Omega \times I \times J$. Separating variables, we get

$$u_1(x,t,y,s)=\partial_{x_k}u(x,t)z^k(y,s), \tag{25}$$

where $z \in L_\sharp^2(I;[W_\sharp^{1,2}(Y)/R]^N)$ is the unique solution to the decoupled system

$$\int_Y a_{ij}(y,s)(\delta_{ik}+\partial_{y_i}z^k(y,s))\partial_{y_j}v_2(y)\,\mathrm{d}y=0, \tag{26}$$

which holds for all $v_2 \in W_\sharp^{1,2}(Y)/R$. A similar separation of variables turns (21) into the weak form of (15).

For $r=2$, we face a different situation. In this case, Corollary 3.3 is essential. We pass to the two-scale limit in (22) and arrive at

$$\int_0^T\int_\Omega\int_0^1\int_Y u_1(x,t,y,s)v_1(x)v_2(y)c_1(t)\partial_s c_2(s)-a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))$$
$$+\left[\int_0^1\int_Y a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s)\,\mathrm{d}y\,\mathrm{d}s\right]\cdot v_1(x)\partial_{y_j}v_2(y)c_1(t)c_2(s)\,\mathrm{d}Z=0$$

which through (24) and (25) immediately yields the local problem

$$\int_0^1\int_Y -z^k(y,s)v_2(y)\partial_s c_2(s)+a_{ij}(y,s)(\delta_{ik}+\partial_{y_i}z^k(y,s))\partial_{y_j}v_2(y)c_2(s)\,\mathrm{d}y\,\mathrm{d}s=0. \tag{27}$$

The solution z^k to the local problem is utilized in exactly the same way as for $0<r<2$ to compute the homogenized coefficients $\overline{a}_{jk}$ by means of (16).

Now, for $r>2$, we let ε pass to zero in (22). This means that (22) approaches

$$\int_0^T\int_\Omega\int_0^1\int_Y u_1(x,t,y,s)v_1(x)v_2(y)c_1(t)\partial_s c_2(s)\,\mathrm{d}Z=0.$$

Here the distributional derivative of u_1 with respect to s is zero and we conclude that u_1 is in fact independent of s. Finally, we choose c_2 as a constant equal to one in (23), pass to the two-scale limit and obtain from (24) that

$$\int_0^T\int_\Omega\int_Y\left(\int_0^1 a_{ij}(y,s)\,\mathrm{d}s\right)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y))\partial_{y_j}v_2(y)v_1(x)c_1(t)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y=0.$$

A separation of variables by (25) yields that

$$\int_Y\left(\int_0^1 a_{ij}(y,s)\,\mathrm{d}s\right)(\delta_{ik}+\partial_{y_i}z^k(y))\partial_{y_j}v_2(y)\,\mathrm{d}y=0. \tag{28}$$

The homogenized coefficients are computed by (19) and the proof is complete. □

5. Correctors

In this section, we benefit from the properties of u_1 to prove some stronger convergence results (corrector results), which include also a characterization of strong convergence for the gradients to sequences of solutions to (1). Clearly, if $\partial_{x_i} u$ is regular enough (most naturally $\partial_{x_i} u \in C(\overline{\Omega} \times \overline{I})$, then $\partial_{y_i} u_1$ will be admissible (e.g. $\partial_{y_i} u_1 \in L^2_\sharp(Y \times J; C(\overline{\Omega} \times \overline{I}))$. Depending on whether $\partial_{y_i} u_1$ is admissible or not, we get the two different types of corrector results that are found in Theorems 5.1 and 5.2.

Theorem 5.1. *Assume that* $\{u^\varepsilon\}$ *is a sequence of solutions to* (1) *and that* $\{\partial_{x_i} u^\varepsilon\}$ *two-scale converges to* $\partial_{x_i} u + \partial_{y_i} u_1$, *where* u *is the unique solution to* (15), u_1 *is obtained through* (17), (18), (20), *and* (25), *and* $\partial_{y_i} u_1$ *are admissible test functions.*

Then

$$\lim_{\varepsilon} \|u^\varepsilon(x,t) - u(x,t)\|_{L^2(\Omega \times I)} = 0 \tag{29}$$

and

$$\lim_{\varepsilon} \left\| \nabla u^\varepsilon(x,t) - \left(\nabla u(x,t) + \nabla_y u_1\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right)\right)\right\|_{[L^2(\Omega \times I)]^N} = 0. \tag{30}$$

If $\partial_{y_i} u_1$ are not admissible test functions, the results below still hold.

Theorem 5.2. *Assume that* $\{u^\varepsilon\}$ *is a sequence of solutions to* (1) *and that* $\{\partial_{x_i} u^\varepsilon\}$ *two-scale converges to* $\partial_{x_i} u + \partial_{y_i} u_1$, *where* u *is the unique solution to* (15) *and* u_1 *is obtained through* (17), (18), (20) *and* (25). *Moreover, let* $\{s_{n,i}\}$ *be a sequence of elements in* $C(\overline{\Omega} \times \overline{I})$ *which converges strongly to* $\partial_{x_i} u$ *in* $L^2(\Omega \times I)$.

Then

$$\lim_{\varepsilon} \left\| \nabla u^\varepsilon(x,t) - \left(\nabla u(x,t) + \nabla_y u_1\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right)\right)\right\|_{[L^1(\Omega \times I)]^N} = 0 \tag{31}$$

and

$$\lim_{\varepsilon} \left\| \nabla u^\varepsilon(x,t) - \left(\nabla u(x,t) + s_{n,k}(x,t) \nabla_y z^k\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}\right)\right)\right\|_{[L^2(\Omega \times I)]^N} = a_n, \tag{32}$$

where $a_n \to 0$ *for* $n \to \infty$.

P r o o f o f T h e o r e m 5.1. The positive definiteness of a_{ij} yields that

$$C\left\|\nabla u^\varepsilon(x,t)-\Big(\nabla u(x,t)+\nabla_y u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\Big)\right\|_{[L^2(\Omega\times I)]^N} \tag{33}$$
$$\leqslant \int_0^T\int_\Omega a_{ij}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\Big(\partial_{x_i}u^\varepsilon(x,t)-\Big(\partial_{x_i}u(x,t)+\partial_{y_i}u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\Big)\Big)$$
$$\times\,\partial_{x_j}u^\varepsilon(x,t)-\Big(\partial_{x_j}u(x,t)+\partial_{y_j}u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t$$

holds for some constant C. We now let the operator equation (1) act on u^ε as a test function and, for the right-hand side of (33) written in full, we may replace

$$\int_0^T\int_\Omega a_{ij}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r})\partial_{x_i}u^\varepsilon(x,t)\partial_{x_j}u^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t$$

with

$$\int_0^T\int_\Omega -\partial_t u^\varepsilon(x,t)u^\varepsilon(x,t)+f(x,t)u^\varepsilon(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

We note that, for this rewritten version of (33), Theorem 3.1, Proposition 2.8, and the admissibility of $\partial_{y_i}u_1$ allow us to pass to the limit. We obtain

$$C\lim_\varepsilon\left\|\nabla u^\varepsilon(x,t)-(\nabla u(x,t)+\nabla_y u_1(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}))\right\|^2_{[L^2(\Omega\times I)]^N}$$
$$=\int_0^T\int_\Omega\int_0^1\int_Y -\partial_t u(x,t)u(x,t)+f(x,t)u(x,t)$$
$$-\,a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\cdot(\partial_{x_j}u(x,t)+\partial_{y_j}u_1(x,t,y,s))\,\mathrm{d}Z.$$

We first observe that

$$\int_0^T\int_\Omega\int_0^1\int_Y -\partial_t u(x,t)u(x,t)+f(x,t)u(x,t)$$
$$-\,a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\partial_{x_j}u(x,t)\,\mathrm{d}Z=0$$

means nothing but the homogenized operator equations acting on u as a test function.

For $1<r<2$ and $r>2$, a separation of variables as in (25) yields that

$$\int_0^T\int_\Omega\int_0^1\int_Y a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\partial_{y_j}u_1(x,t,y,s)\,\mathrm{d}Z=0$$

easily reduces to the local problems (26) and (28), respectively.

For $r=2$ we obtain

$$\int_0^T\int_\Omega\int_0^1\int_Y a_{ij}(y,s)(\partial_{x_i}u(x,t)+\partial_{y_i}u_1(x,t,y,s))\partial_{y_j}u_1(x,t,y,s)\,\mathrm{d}Z$$
$$= -\int_0^T\int_\Omega\int_0^1\int_Y u_1(x,t,y,s)\partial_s u_1(x,t,y,s)\,\mathrm{d}Z.$$

Integrating by parts we find that

$$\int_0^T\int_\Omega\int_0^1\int_Y u_1(x,t,y,s)\partial_s u_1(x,t,y,s)\,\mathrm{d}Z$$
$$= \int_0^T\int_\Omega\int_0^1\int_Y \partial_s(u_1(x,t,y,s))^2-\partial_s u_1(x,t,y,s)u_1(x,t,y,s)\,\mathrm{d}Z$$

and, by the periodicity of u_1 with respect to s, we conclude that

$$\int_0^T\int_\Omega\int_0^1\int_Y \partial_s(u_1(x,t,y,s))^2\,\mathrm{d}Z=0.$$

Consequently,

$$\int_0^T\int_\Omega\int_0^1\int_Y u_1(x,t,y,s)\partial_s u_1(x,t,y,s)\,\mathrm{d}Z$$
$$= -\int_0^T\int_\Omega\int_0^1\int_Y \partial_s u_1(x,t,y,s)u_1(x,t,y,s)\,\mathrm{d}Z=0.$$

We have proved that

$$\lim_\varepsilon\left\|\nabla u^\varepsilon(x,t)-\Bigl(\nabla u(x,t)+\nabla_y u_1\Bigl(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Bigr)\Bigr)\right\|_{[L^2(\Omega\times I)]^N}=0$$

and the proof is complete. □

R e m a r k 9. Note that (25) characterizes the corrector $\partial_{y_i}u_1$ explicitly in terms of the solutions to the local problems (17), (18), and (20) and the homogenized problem (15). Further, the proof of Theorem 3.1 contains also a proof of strong convergence for $\{u^\varepsilon\}$ to u in $L^2(\Omega\times I)$ which will appear independently of whether $\partial_{y_i}u_1$ is admissible or not.

P r o o f o f T h e o r e m 5.2. We first note that the existence of the approximating sequence $\{s_n\}$ follows immediately from the density of $C(\overline{\Omega}\times\overline{I})$ in $L^2(\Omega\times I)$. Obviously, $s_{n,k}(x,t)\partial_{y_i}z^k(y,s)\in L^2_\sharp(Y\times J;C(\overline{\Omega}\times\overline{I}))$ and thus they are admissible test functions. The proof for (32) is then exactly the same as for (30) in Theorem 5.1, if we let $s_{n,i}$ go strongly to $\partial_{x_i} u$ in $L^2(\Omega \times I)$ after the passage to the limit zero for ε. We prove (31).

$$\begin{aligned}
&\lim_{\varepsilon} \left\| \nabla u^\varepsilon(x,t) - \Big(\nabla u(x,t) + \nabla_y u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\Big)\right\|_{[L^1(\Omega\times I)]^N} \\
&\leqslant \lim_{\varepsilon} \left\| \nabla u^\varepsilon(x,t) - \Big(\nabla u(x,t) + s_{n,k}(x,t)\nabla_y z^k\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\Big)\right\|_{[L^1(\Omega\times I)]^N} \\
&\quad + \lim_{\varepsilon} \left\| s_{n,k}(x,t)\nabla_y z^k\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big) - \nabla_y u_1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\right\|_{[L^1(\Omega\times I)]^N} \\
&\leqslant \lim_{\varepsilon} \left\| \nabla u^\varepsilon(x,t) - (\nabla u(x,t) + s_{n,k}(x,t)\nabla_y z^k\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big))\right\|_{[L^1(\Omega\times I)]^N} \\
&\quad + \lim_{\varepsilon} \left\| (s_{n,k}(x,t) - \partial_{x_k} u(x,t))\nabla_y z^k\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\right\|_{[L^1(\Omega\times I)]^N} \\
&\leqslant \lim_{\varepsilon} C(\left\| \nabla u^\varepsilon(x,t) - (\nabla u(x,t) + s_{n,k}(x,t)\nabla_y z^k\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big))\right\|_{[L^2(\Omega\times I)]^N} \\
&\quad + \|s_n(x,t) - \nabla u(x,t)\|_{[L^2(\Omega\times I))]^N} \cdot \lim_{\varepsilon} \left\| \nabla_y z^k\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\right\|_{[L^2(\Omega\times I)]^N}) \\
&= 0 + 0 \cdot \left\|\nabla_y z^k(y,s)\right\|_{[L^2(\Omega\times I\times Y\times J)]^N} = 0
\end{aligned}$$

as a consequence of (32) and the admissibility of $\partial_{y_i} z^k$ if we let $n \to \infty$. □

The p r o o f of (31) is complete. □

6. Further results and concluding remarks

First we note that $\partial_{y_i} u_1$ may be admissible under the regularity assumptions made in (1). However, these assumptions absolutely do not *guarantee* enough regularity for this to hold. In Theorem 6.1 below we give examples of regularity assumptions strong enough for this aim.

Theorem 6.1. *Assume that $\Omega \subset \mathbb{R}^N$, $N = 1, 2$, and 3 is bounded with a C^∞ boundary, that $f \in H^{2,1}(\Omega \times I)$ and $u_0 \in W^{3,2}(\Omega)$. Then, after a possible modification on a negligible subset of Ω, $\partial_{y_i} u_1$ is an admissible test function. For $N = 1$, it suffices to require that $f \in H^{1,1}(\Omega \times I)$ and $u_0 \in W^{2,2}(\Omega)$.*

For the proof of this, we state a number of lemmas.

Lemma 6.2. *Assume that $f \in H^{k,\frac{k}{2}}(\Omega \times I)$, $u_0 \in W^{k+1,2}(\Omega)$, $k > 0$ integer and that Ω is bounded with a C^∞ boundary. Then* (15) *possesses a unique solution $u \in H^{k+2,k/3}(\Omega \times I)$.*

Proof. The lemma follows directly from [LiMa, Chapter 4, Theorem 5.3] for $g_0 = 0$, $m = 1$, and B_0 the identity boundary operator on $\partial\Omega$. $\square$

Lemma 6.3. *Assume that $\Omega \subset R^N$ is strongly locally Lipschitz (e.g. bounded and with locally Lipschitz boundary). Then $W^{j+m,p}(\Omega)$ is continuously embedded into $C^j(\overline{\Omega})$ if $mp > N > (m-1)p$.*

Proof. See [Ada, Theorem 5.4 C^I]. $\square$

Lemma 6.4. *Let Ω be bounded with a C^∞ boundary and assume that $0 < \theta < 1$. Moreover, let s, s_0, s_1, p, p_0, p_1, $s_0 \neq s_1$, $1 < p_0$, $p_1 < \infty$ be real numbers. Furthermore, assume that $s = (1-\theta)s_0 + \theta s_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then, the interpolation space $[W^{s_0,p_0}(\Omega), W^{s_1,p_1}(\Omega)]_\theta$ coincides with $W^{s,p}(\Omega)$.*

Proof. See [BeLö, Theorem 6.4.5 (7)]. $\square$

The result in the next lemma is found on p. 1111 in [Zei].

Lemma 6.5. *Let $V \subset H \subset V^*$ be an evolution triple and assume that $u \in L^2(I;V)$ and $\frac{\partial^m}{\partial^m t}u \in L^2(I;H)$. Then $\frac{\partial^j}{\partial^j t}u \in C(\overline{I};[V,H]_{(j+\frac{1}{2})/m})$ for $j = 0,1,\dots,m-1$.*

Proof of Theorem 6.1. For $k = 2$ it follows immediately from Lemma 6.2 that u belongs to $H^{4,\frac{2}{3}}(\Omega \times I) \subset L^2(I;W^{4,2}(\Omega))$ and thus a simple reformulation of the homogenized problems implies that $\partial_t u$ lies in $L^2(I;W^{2,2}(\Omega))$.

Further, we note that the embeddings $W^{4,2}(\Omega) \subset W^{2,2}(\Omega) \subset (W^{4,2}(\Omega))^*$ are dense and continuous and thus represent an evolution triple. Hence, by interpolation (see Lemmas 6.4 and 6.5) we find that u belongs to the interpolation space $C(\overline{I};[W^{4,2}(\Omega), W^{2,2}(\Omega)]_{1/2})$, which coincides with $C(\overline{I};W^{3,2}(\Omega))$.

For $N = 1,2$, and 3 and $\partial\Omega$ Lipschitz it follows directly from Lemma 6.3 that $W^{3,2}(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$ and hence, changing u on at most a set of measure zero, $u \in C(\overline{I};C^1(\overline{\Omega}))$. Clearly, $\partial_{x_i}u \in C(\overline{\Omega} \times \overline{I})$.

We have proved that $\partial_{y_i}u_1 \in L^2_\sharp(Y \times J; C(\overline{\Omega} \times \overline{I}))$ and thus it is an admissible test function. For $N = 1$, it suffices to assume that $f \in H^{1,1}(\Omega \times I)$ and that $u_0 \in W^{2,2}(\Omega)$, because, in this case, $W^{2,2}(\Omega)$ is continuously embedded in $C^1(\overline{\Omega})$.

The proof is complete. $\square$

Remark 10. Let us remark that the sacrifice necessary to ensure the admissibility of $\partial_{y_i}u_1$ is solely to require more regularity of the right-hand side of (1), but not of the from the point of view of physical relevance more important coefficients a_{ij}. This is exactly the reason why we avoid the second possibility, namely to increase the regularity of a_{ij} enough to obtain $\partial_{y_i}z^k \in C_\sharp(Y \times J)$, making $\partial_{y_i}u_1$ an admissible test function of the type $L^2(\Omega \times I; C_\sharp(Y \times J))$. For some texts on function spaces, interpolation theory and regularity that contain an essential background to the above discussion we refer to [Ada], [BeLö, Ch 6.4], [Kuf, Ch 5], [Alt, Ch 5], and [Zei, part IIB, p. 1101–1110 and part IIA, Ch 23].

Remark 11. In [BraOts] Brahim-Otsmane et al. apply classical homogenization methods to obtain corrector results for linear parabolic equations where a_{ij} oscillates only in the space variable. Further, in [Bens], Bensoussan et al. study homogenization and correctors for parabolic problems and obtain the three cases exhibited in Theorem 4.2 by means of asymptotic expansions.

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