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ON EQUILIBRIUM FINITE ELEMENTS IN THREE-DIMENSIONAL CASE

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Abstract. The space of divergence-free functions with vanishing normal flux on the boundary is approximated by subspaces of finite elements that have the same property. The easiest way of generating basis functions in these subspaces is considered.

Keywords: divergence-free functions, finite elements, internal approximation, stream function.

MSC 2000: 65N30

1. INTRODUCTION

The goal of the paper is to construct finite element subspaces of the spaces of divergence-free functions. Such a problem is frequently met when we treat numerically some phenomena in continuum mechanics, electromagnetism, heat and fluid flow problems, etc.

In this paper we shall describe an internal finite element approximation of the following space which appears in variational formulations of a considerable number of problems, see, e.g., [1], [2], [4], [5], [6]:

$$H_0(\overset{0}{\operatorname{div}};\Omega) = \left\{\vec{q} \in [L^2(\Omega)]^d \mid (\vec{q}, \nabla z)_0 = 0 \ \forall z \in H^1(\Omega)\right\}, \quad d = 2,3. \tag{1.1}$$

We will deal only with the three-dimensional case: $\Omega \subset \mathbb{R}^3$ is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$, $(\cdot\,,\cdot)_0$ is the inner product in $[L_2(\Omega)]^l$, $l = 1,2,3$, $H^k(\Omega)$ is the standard Sobolev space with the norm $\|\cdot\|_k$ and $\vec{l}\cdot\vec{w}$ is the standard inner product of vectors $\vec{l}$ and $\vec{w}$ in $\mathbb{R}^3$.

In this work we will generalize the results which were obtained in [3] for wider class of domains.

2. Auxiliary results

First we recall some known important facts.

Introduce a space of vector-functions the divergence of which exists in the sense of distributions (see, for example, [1])

$$H(\mathrm{div};\Omega) = \big\{\vec{q} \in [L^2(\Omega)]^3 \mid \exists \varphi \in L^2(\Omega)\colon\ (\vec{q}, \nabla z)_0 + (\varphi, z)_0 = 0\ \forall z \in H_0^1(\Omega)\big\} \tag{2.1}$$

and its subspace of divergence-free (so-called solenoidal) functions

$$H(\overset{0}{\mathrm{div}};\Omega) = \big\{\vec{q} \in [L^2(\Omega)]^3 \mid (\vec{q}, \nabla z)_0 = 0\ \forall z \in H_0^1(\Omega)\big\}. \tag{2.2}$$

Note that for both spaces the test-functions z vanish on the boundary $\partial\Omega$, so there are no conditions upon the normal flux $\vec{n} \cdot \vec{q}$ on $\partial\Omega$, where $\vec{n}$ is the outward normal to Ω.

Let $\vec{w} = (w_1, w_2, w_3) \in [H^1(\Omega)]^3$ and $z \in C_0^\infty(\Omega)$ be arbitrary functions. Then $(\mathrm{curl}\,\vec{w}, \nabla z)_0 = (\vec{w}, \mathrm{curl}\,\nabla z)_0 = 0$ due to the Green formula, where

$$\mathrm{curl}\,\vec{w} = (\partial_2 w_3 - \partial_3 w_2, \partial_3 w_1 - \partial_1 w_3, \partial_1 w_2 - \partial_2 w_1). \tag{2.3}$$

Hence, the density $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ implies

$$\mathrm{curl}\,\vec{w} \in H(\overset{0}{\mathrm{div}};\Omega) \quad \forall \vec{w} \in [H^1(\Omega)]^3. \tag{2.4}$$

Recall (see [1, p. 16]) that the functional $\vec{q} \to \vec{n} \cdot \vec{q}\ |_{\partial\Omega}$ defined on $[C^\infty(\overline{\Omega})]^3$ can be extended by continuity to a linear continuous mapping from the space $H(\mathrm{div};\Omega)$ into $H^{-1/2}(\partial\Omega)$, where the latter is the dual space to the space of traces $H^{1/2}(\partial\Omega)$ of functions from $H^1(\Omega)$. In this case, the Green formula takes the form

$$(\vec{q}, \nabla z)_0 + (\mathrm{div}\,\vec{q}, z)_0 = \langle \vec{n} \cdot \vec{q}, z\rangle_{\partial\Omega} \quad \forall \vec{q} \in H(\mathrm{div};\Omega) \quad \forall z \in H^1(\Omega), \tag{2.5}$$

where $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$.

Now we will formulate and prove Theorem 2.1.

Theorem 2.1. *Let $\vec{l} = (l_1, l_2, l_3)$ be a constant vector in $\mathbb{R}^3$ and $\Omega \subset \mathbb{R}^3$ a bounded domain with a Lipschitz continuous boundary. Then*

$$H_0(\overset{0}{\mathrm{div}};\Omega) = \mathrm{curl}\,W, \tag{2.6}$$

where

(2.7) $\quad W = \big\{\vec{w} = (w_1, w_2, w_3) \in [H^1(\Omega)]^3 \mid \vec{n}\cdot \operatorname{curl}\vec{w} = 0 \ \textit{on}\ \partial\Omega,\ \vec{l}\cdot\vec{w} = 0\ \textit{in}\ \Omega\big\}.$

P r o o f. To prove this theorem we use an idea similar to that which was used to prove Theorem 3.2 in [1].

Let $\vec{w} \in W$ be given. Then

$$(\operatorname{curl}\vec{w}, \nabla z)_0 = (-\operatorname{div}\operatorname{curl}\vec{w}, z)_0 + \langle \vec{n}\cdot\operatorname{curl}\vec{w}, z\rangle_{\partial\Omega} = 0 \quad \forall z \in H^1(\Omega)$$

(see formulae (2.4), (2.5)). Hence, it follows from $\operatorname{curl}\vec{w} \in H_0(\operatorname{div}^0;\Omega)$ that

$$H_0(\overset{0}{\operatorname{div}};\Omega) \supset \operatorname{curl} W. \tag{2.8}$$

Conversely, let $\vec{q} = (q_1, q_2, q_3) \in H_0(\operatorname{div}^0;\Omega)$, i.e.,

$$\begin{aligned} \operatorname{div}\vec{q} &= 0 \quad \text{in } \Omega,\\ \langle \vec{q}\cdot\vec{n}, 1\rangle_{\partial\Omega} &= 0. \end{aligned}$$

We can extend $\vec{q}$ (according to [1, pp. 27–28]) to the whole space so that the extended function $\vec{\tilde{q}} \in [L^2(\mathbb{R}^3)]^3$ would be still divergence-free and have a compact support. Let $\hat{q}_j$ be the Fourier transform of q_j, $j = 1, 2, 3$,

$$\hat{q}_j(\xi) = \int_{\mathbb{R}^3} e^{-2\mathrm{i}\pi x\cdot\xi}\tilde{q}_j(x)\,\mathrm{d}x, \quad \xi \in \mathbb{R}^3. \tag{2.9}$$

Here i is the imaginary unit, i.e., $\mathrm{i}^2 = -1$. In what follows we will write $\mathbb{R}^3_\xi$ for the three-dimensional space with coordinates (ξ_1, ξ_2, ξ_3). The condition $\operatorname{div}\vec{\tilde{q}} = 0$ implies that

$$\sum_{i=1}^{3} \xi_i\hat{q}_i = 0. \tag{2.10}$$

We seek a function $\vec{\tilde{\varphi}}$ in $[L^2(\mathbb{R}^3_\xi)]^3$ such that $\operatorname{curl}\vec{\varphi} = \vec{\tilde{q}}$, i.e.,

$$\begin{cases} \hat{q}_1 = 2\mathrm{i}\pi(\xi_2\hat{\varphi}_3 - \xi_3\hat{\varphi}_2),\\ \hat{q}_2 = 2\mathrm{i}\pi(\xi_3\hat{\varphi}_1 - \xi_1\hat{\varphi}_3),\\ \hat{q}_3 = 2\mathrm{i}\pi(\xi_1\hat{\varphi}_2 - \xi_2\hat{\varphi}_1). \end{cases} \tag{2.11}$$

Obviously, the third equation of (2.11) is a consequence of the first two and equation (2.10), hence, in fact, we have only two equations to define three unknown functions $\hat{\varphi}_1$, $\hat{\varphi}_2$, $\hat{\varphi}_3$.

Further, we add the following third condition which is suitable for our purposes:

$$\sum_{i=1}^{3} l_i \varphi_i = 0 \tag{2.12}$$

which, after the Fourier transform, takes the form

$$\sum_{i=1}^{3} l_i \hat{\varphi}_i = 0, \tag{2.13}$$

due to the fact that $\vec{l}$ is a constant vector. Equation (2.13) is the third relation connecting the functions $\hat{\varphi}_1$, $\hat{\varphi}_2$, $\hat{\varphi}_3$.

Hence, taking the first two equations from system (2.11) and equation (2.13) we obtain the system

$$\begin{cases} \xi_2\hat{\varphi}_3 - \xi_3\hat{\varphi}_2 = \dfrac{\hat{q}_1}{2\pi\mathrm{i}}, \\ \xi_3\hat{\varphi}_1 - \xi_1\hat{\varphi}_3 = \dfrac{\hat{q}_2}{2\pi\mathrm{i}}, \\ l_1\hat{\varphi}_1 + l_2\hat{\varphi}_2 + l_3\hat{\varphi}_3 = 0. \end{cases} \tag{2.14}$$

In the matrix form it may be rewritten as follows:

$$\begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ l_1 & l_2 & l_3 \end{bmatrix} \begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \\ \hat{\varphi}_3 \end{bmatrix} = \begin{bmatrix} \hat{q}_1/2\pi\mathrm{i} \\ \hat{q}_2/2\pi\mathrm{i} \\ 0 \end{bmatrix}. \tag{2.15}$$

The solution is

$$\begin{bmatrix} \hat{\varphi}_1 \\ \hat{\varphi}_2 \\ \hat{\varphi}_3 \end{bmatrix} = \frac{1}{2\pi\mathrm{i}\xi_3 \vec{l}\cdot\vec{\xi}} \begin{bmatrix} \xi_1 l_2 \hat{q}_1 + \xi_3 l_3 \hat{q}_2 + \xi_2 l_2 \hat{q}_2 \\ -\xi_3 l_3 \hat{q}_1 - \xi_1 l_1 \hat{q}_1 - \xi_2 l_1 \hat{q}_2 \\ \xi_3 l_2 \hat{q}_1 - \xi_3 l_1 \hat{q}_2 \end{bmatrix}. \tag{2.16}$$

The function defined by (2.16) represents the unique solution of system (2.14), because the determinant of the matrix in (2.15) is not zero.

Now, we have the following facts:

1) $\hat{q}_j$ are holomorphic in $\mathbb{R}^3_\xi$, since the supports of $\tilde{q}_j$ are compact (see [1, p. 27]).
2) The Fourier transform is a linear continuous operator from $L_2(\mathbb{R}^3)$ to $L_2(\mathbb{R}^3_\xi)$, hence $\hat{q}_j \in L_2(\mathbb{R}^3_\xi), j = 1, 2, 3$.

We recall the following theorem:

Theorem 2.2. *Let k and d be any integers. Then*

$$u(x) \in H^k(\mathbb{R}^d) \iff \xi^\alpha \hat{u}(\xi) \in L_2(\mathbb{R}^d_\xi) \quad \forall \alpha \text{ such that } |\alpha| \leqslant k$$

(see, for example, [7]), where the sign "ˆ" means the Fourier transform.

According to Theorem 2.2, in order to get $\varphi_j \in H^1(\Omega)$, $j = 1, 2, 3$, we shall prove the following theorem.

Theorem 2.3. *The statements*
(a) $\xi_j \hat{\varphi}_i(\xi) \in L_2(\mathbb{R}^3_\xi)$, $i, j = 1, 2, 3$,
(b) $\hat{\varphi}_i(\xi) \in L_2(\mathbb{R}^3_\xi)$, $i = 1, 2, 3$
are valid, where $\hat{\varphi}_i(\xi)$ and $\mathbb{R}^3_\xi$ are described above.

P r o o f. Condition (a) can be proved immediately from formula (2.10). We also have

$$|\hat{\varphi}_i| \leqslant \frac{C(|\hat{q}_j| + |\hat{q}_k|)}{\|\xi\|},$$

where $C > 0$ is a constant. Hence, we must check only the boundedness of $\hat{\varphi}_i$ in the neighbourhood of zero.

Condition (2.10) implies

$$\hat{q}_i(0) = 0. \tag{2.17}$$

From 1) we have

$$\hat{q}_i(\xi) = \sum_{j=1}^{3} \xi_j \frac{\partial \hat{q}_i}{\partial \xi_j}(0) + \mathcal{O}(\|\xi\|^2)$$

in a neighbourhood of 0. Here $\|\xi\|$ means the usual Euclidean norm of the vector $\vec{\xi} = (\xi_1, \xi_2, \xi_3)$. Hence, $\vec{\hat{\varphi}}$ is bounded as $\xi \to 0$. □

By restricting the inverse transform $\vec{\varphi}$ of $\vec{\hat{\varphi}}$ to Ω, we get a function $\vec{\varphi} \in [H^1(\Omega)]^3$ such that

$$\operatorname{curl} \vec{\varphi} = \vec{q}$$

and, moreover, the important identity $\vec{l} \cdot \vec{\varphi} = 0$ is valid. Note that in [1] and [3] the vector $\vec{l}$ is, in fact, equal to $(0, 0, 1)$. □

3. Equilibrium finite elements

Let W_h be an arbitrary finite element space of W whose functions are continuous and piecewise polynomial on some partition of $\overline{\Omega}$. We define the space of equilibrium finite elements as

$$Q_h = \operatorname{curl} W_h.$$

Due to Theorem 2.1, Q_h is a subspace of $H_0(\operatorname{div}^0;\Omega)$. Recall (see [3], Corollary of Theorem 1) that, if $\{W_h\}$ is a system of finite element subspaces of W such that the union $\bigcup_h W_h$ is dense in W with respect to the $\|\cdot\|_1$ norm, then $\bigcup_h Q_h$ is dense in $H_0(\operatorname{div}^0;\Omega)$ in the $\|\cdot\|_0$ norm.

Definition 3.1. *A domain $\Omega \subset \mathbb{R}^3$ is said to belong to the class $\mathcal{L}^*$ if it can be transformed by a rotation in $\mathbb{R}^3$ to the domain Ω' from the class $\mathcal{L}$ (see [3]), i.e.,*

(i) *Ω' is a bounded domain with a Lipschitz boundary,*

(ii) *there exists a simply connected domain $\omega \subset \mathbb{R}^2$ and a positive function $F\colon \omega \to \mathbb{R}^1$ (in general discontinuous) such that*

$$\Omega' = \big\{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid (x_1,x_2) \in \omega,\ 0 < x_3 < F(x_1,x_2)\big\}.$$

Remark 3.1. Denote by $\partial\Omega_0$ the base of the domain Ω, i.e., ω is the image of $\partial\Omega_0$ under the above rotation. Then there exists a constant vector $\vec{l} \in \mathbb{R}^3$ which is perpendicular to the base of such a domain.

Further we shall require the following property of finite element subspaces ($\Omega \in \mathcal{L}^*$ with the vector $\vec{l}$) to be valid:

$$\vec{w} \in W_h \Longrightarrow \vec{\mathring{w}} \in W_h, \tag{3.1}$$

where

$$\vec{\mathring{w}}(x_1,x_2,x_3) = \vec{w}(x_1^0,x_2^0,x_3^0) \tag{3.2}$$

and the points

$$x^0 = (x_1^0,x_2^0,x_3^0) \in \partial\Omega_0, \quad x = (x_1,x_2,x_3) \in \Omega$$

are connected by the following relation:

$$x_i - x_i^0 = \alpha \cdot l_i, \quad i = 1,2,3 \quad (\alpha \text{ is a constant}), \tag{3.3}$$

i.e., the point x^0 is the projection of the point x onto the base of the domain along the vector $\vec{l}$.

For simplicity we choose the vector $\vec{l}$ to be of the unit length, i.e.,

$$\|\vec{l}\| = (l_1^2 + l_2^2 + l_3^2)^{1/2} = 1. \tag{3.4}$$

Note that the operator curl: $W_h \to Q_h = \operatorname{curl} W_h$ is not bijective in general, so we need to define $V_h \subset W_h$ such that curl: $V_h \to Q_h$ is bijective.

The next theorem generalizes Theorem 2 from [3].

Theorem 3.1. *Let $\Omega \in \mathcal{L}^*$, let the vector $\vec{l}$ correspond to this domain (see Remark* 3.1*), let $W_h \subset W$ satisfy* (3.2) *and $Q_h = \operatorname{curl} W_h$. Then for the space $V_h \subset W_h$ such that*

$$V_h = \left\{\vec{v} \in W_h \mid \vec{v} = 0 \text{ on } \partial\Omega_0\right\}$$

the mapping

$$\operatorname{curl}\colon V_h \to Q_h \subset H_0(\overset{0}{\operatorname{div}}; \Omega)$$

is bijective.

P r o o f. *Injectivity.* If $\operatorname{curl} \vec{v} = 0$ for some $\vec{v} \in W_h$ then there exists $s \in H^1(\Omega)$ (note that Ω is simply connected) such that

$$\vec{v} = \operatorname{grad} s.$$

Moreover, $s \in H^2(\Omega) \subset C(\overline{\Omega})$. Hence, s is continuous in $\overline{\Omega}$ and, of course, s is a piecewise polynomial function. Due to these facts the following formula makes sense:

$$s(x_1, x_2, x_3) = s(x_1^0, x_2^0, x_3^0) + \int_{x_0}^{x} \frac{\partial s}{\partial \vec{l}} \, \mathrm{d}\xi,$$

where the point $(x_1^0, x_2^0, x_3^0) \in \partial\Omega_0$ is the projection of the point (x_1, x_2, x_3) to the base of Ω along the vector $\vec{l}$. It is obvious that

$$\frac{\partial s}{\partial \vec{l}} = \vec{l} \cdot \nabla s = \vec{l} \cdot \vec{v} = 0,$$

which implies that $s(x) = s(x^0)$.

Since $\vec{v} = 0$ on $\partial\Omega_0$, we get that s is constant on $\partial\Omega_0$ and, then, in the whole domain Ω. This means that $\vec{v} \equiv 0$ in $\overline{\Omega}$.

Surjectivity. Let $\vec{q} \in Q_h$ be an arbitrary vector function. According to Theorem 2.1, there exists a continuous piecewise polynomial function $\vec{w} = (w_1, w_2, w_3)$ such that $\vec{w} \in W_h$, $\vec{l} \cdot \vec{w} = 0$ and

$$\vec{q} = \operatorname{curl} \vec{w}.$$

Let $\vec{v} = \vec{w} - \vec{\overset{\circ}{w}}$, where $\vec{\overset{\circ}{w}} = (\overset{\circ}{w}_1, \overset{\circ}{w}_2, \overset{\circ}{w}_3)$ is defined by (3.2) and (3.3). Then $\vec{v} = 0$ on $\partial\Omega_0$ and $\vec{v} \in V_h \subset W_h$.

Now we check whether the relation

$$\vec{q} = \operatorname{curl} \vec{v}$$

holds.

In fact, we must show that

$$\operatorname{curl} \vec{\overset{\circ}{w}} = 0 \quad \text{in } \Omega.$$

Let us introduce the following convenient notation:

$$\begin{cases} \dfrac{\partial \overset{\circ}{w}_3}{\partial x_2} - \dfrac{\partial \overset{\circ}{w}_2}{\partial x_3} = \Delta_1, \\ \dfrac{\partial \overset{\circ}{w}_1}{\partial x_3} - \dfrac{\partial \overset{\circ}{w}_3}{\partial x_1} = \Delta_2, \\ \dfrac{\partial \overset{\circ}{w}_2}{\partial x_1} - \dfrac{\partial \overset{\circ}{w}_1}{\partial x_2} = \Delta_3. \end{cases} \tag{3.5}$$

Since $\vec{\overset{\circ}{w}}$ is, in fact, the trace of $\vec{w}$ on $\partial\Omega_0$ along the vector $\vec{l}$ we have the following obvious conditions:

$$\sum_{i=1}^{3} l_i \frac{\partial \overset{\circ}{w}_j}{\partial x_i} = 0, \;\; j = 1, 2, 3 \quad \text{in } \Omega. \tag{3.6}$$

As $\vec{l} \cdot \vec{\overset{\circ}{w}} = 0$, we also have

$$\sum_{i=1}^{3} l_i \overset{\circ}{w}_i = 0 \quad \text{in } \Omega. \tag{3.7}$$

And, of course, the following condition will be taken into account:

$$\vec{n} \cdot \operatorname{curl} \vec{\overset{\circ}{w}}\Big|_{\partial\Omega_0} = 0. \tag{3.8}$$

For simplicity, we suppose that $l_3 \neq 0$. Then (3.7) yields

$$\overset{\circ}{w}_3 = -\frac{l_2}{l_3}\overset{\circ}{w}_2 - \frac{l_1}{l_3}\overset{\circ}{w}_1.$$

Hence,

$$
\begin{aligned}
\Delta_1 &= \frac{\partial \mathring{w}_3}{\partial x_2} - \frac{\partial \mathring{w}_2}{\partial x_3} \\
&= \frac{\partial}{\partial x_2}\left(-\frac{l_2}{l_3}\mathring{w}_2 - \frac{l_1}{l_3}\mathring{w}_1 \right) - \frac{\partial}{\partial x_3}\mathring{w}_2 \\
&= -\frac{1}{l_3}\left(l_2 \frac{\partial \mathring{w}_2}{\partial x_2} + l_1 \frac{\partial \mathring{w}_1}{\partial x_2} + l_3 \frac{\partial \mathring{w}_2}{\partial x_3} \right) \\
&= -\frac{1}{l_3}\left(l_1 \frac{\partial \mathring{w}_2}{\partial x_1} + l_2 \frac{\partial \mathring{w}_2}{\partial x_2} + l_3 \frac{\partial \mathring{w}_2}{\partial x_3} + l_1 \frac{\partial \mathring{w}_1}{\partial x_2} - l_1 \frac{\partial \mathring{w}_2}{\partial x_1} \right) \\
&= \frac{l_1}{l_3}\Delta_3.
\end{aligned}
$$

and consequently,

$$l_3\Delta_1 = l_1\Delta_3.$$

It is easy to check that if l_3 is zero then the above equality also holds. Similar argument leads to the equalities $l_2\Delta_1 = l_1\Delta_2$ and $l_3\Delta_2 = l_2\Delta_3$.

These equalities constitute the system

$$
\begin{cases}
l_3\Delta_1 = l_1\Delta_3, \\
l_2\Delta_1 = l_1\Delta_2, \\
l_3\Delta_2 = l_2\Delta_3.
\end{cases}
\tag{3.9}
$$

Obviously, only two equalities from system (3.9) are independent. Condition (3.8) implies

$$l_1\Delta_1 + l_2\Delta_2 + l_3\Delta_3 = 0$$

(since $\|\vec{l}\| = 1$ and $\vec{l} = -\vec{n}$ on $\partial\Omega_0$, if $\Omega \in \mathcal{L}^*$).

Taking the system

$$
\begin{cases}
\hfill l_3\Delta_1 - l_1\Delta_3 = 0, \\
\hfill l_2\Delta_1 - l_1\Delta_2 = 0, \\
l_1\Delta_1 + l_2\Delta_2 + l_3\Delta_3 = 0,
\end{cases}
$$

with zero right-hand side, we see that, if

$$
\det \begin{bmatrix} l_3 & 0 & -l_1 \\ l_2 & -l_1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} = -l_1 \cdot \|\vec{l}\| = -l_1 \neq 0,
$$

then the only solution is $\Delta_1 = \Delta_2 = \Delta_3 = 0$. Obviously, if $l_1 = 0$, then we take other two equations from (3.9). $\square$

Note that we have to form finite elements according to the position of the base of such domains in the space, so conditions (3.2) and (3.3) are quite natural and can be easily satisfied when employing prismatic or rectangular C^0-elements.

Also, the restriction $\vec{l} \cdot \vec{w} = 0$ is not very difficult, because $\vec{l}$ is a constant vector. Namely, the basis in V_h can be easily obtained from the finite element basis of finite element subspaces of $H^1(\Omega)$.

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