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STABILITY OF INVARIANT LINEARLY SUFFICIENT STATISTICS IN THE GENERAL GAUSS-MARKOV MODEL

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Abstract. Necessary and sufficient conditions are derived for the inclusions $J_0 \subset J$ and $J_0^* \subset J^*$ to be fulfilled where J_0, J_0^* and J, J^* are some classes of invariant linearly sufficient statistics (Oktaba, Kornacki, Wawrzosek (1988)) corresponding to the Gauss-Markov models $GM_0 = (y, X_0\beta_0, \sigma_0^2 V_0)$ and $GM = (y, X\beta, \sigma^2 V)$, respectively.

Keywords: Gauss-Markov model, linearly sufficient statistics, invariant linearly sufficient statistics

MSC 2000: 62B05, 62F10

1. Introduction

Let us consider a fixed linear Gauss-Markov model GM of the form

$$(y, X\beta, \sigma^2 V) \tag{1.1}$$

where the vector y of n observations has the expected value $X\beta$ and dispersion matrix σ^2V. Matrix X is known, parameters β, σ^2 unknown, while matrix V is known and nonnegative definite. Following Rao (1971) let us introduce the matrix

$$T = V + XUX', \tag{1.2}$$

where U is an arbitrary symmetric matrix such that

$$R(T) = R(V \vdots X). \tag{1.3}$$

The symbol $R(T)$ in 1.3 denotes the vector space generated by the columns of the matrix T.

Let us assume now that instead of the model GM we have an alternative model GM_0 of the form

$$(y, X_0\beta_0, \sigma_0^2 V_0). \tag{1.4}$$

Analogously to the matrix T in 1.2 we introduce the matrix

$$T_0 = V_0 + X_0 U_0 X_0^{'} \tag{1.5}$$

with an arbitrary matrix U_0 such that

$$R(T_0) = R(V_0 \vdots X_0). \tag{1.6}$$

Further, let $\mathscr{F}_0$ be the class of all linear statistics with a certain property in the model GM_0, and let $\mathscr{F}$ be the class of all linear statistics with the same property but in the model GM. The main problem of this paper consists in determining conditions under which the class $\mathscr{F}_0$ remains valid under GM in the sense that $\mathscr{F}_0 \subseteq \mathscr{F}$.

The stability problem so defined has been discussed in literature very often especially in the context of best linear unbiased estimators; see e.g. Rao(1971), Rao and Mitra (1971 chapter 8), Kala (1981), Mathew and Bhimasankaram (1983) and others. Baksalary and Mathew (1986) studied this topic for classes S of linearly sufficient statistics, classes $S^*$of linearly minimal sufficient statistics and classes C of linearly complete statistics. The authors obtained conditions for inclusions $S_0 \subset S$, $S_0^* \subset S^*$, $C_0 \subset C$ and equality in the above classes.

The aim of this paper is to continue investigations of similiar nature by considering the stability problem with reference to the classes J_0, J and J_0^*, J^* of invariant linearly sufficient statistics introduced by Oktaba, Kornacki, Wawrzosek (1988). The main results of this paper are presented in section 4 in Theorems 4.1–4.3.

2. Lemmas and auxiliary facts

In this section we give some necessary notation, lemmas and facts that will be useful for the presentation of the main results of the paper. The transpose of a matrix A, the rank of A and the vector space generated by columns of the matrix A are denoted by A', $r(A)$ and $R(A)$, respectively. The symbol A^- is reserved for the g-inverse of a matrix A satisfying the property

$$AA^-A = A. \tag{2.1}$$

Further, A^+ will stand for the Moore-Penrose inverse of A, i.e. a matrix satisfying the following four conditions:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (AA^+)' = AA^+, \quad (A^+A)' = A^+A. \tag{2.2}$$

The lemma given below plays the fundamental role in the proofs of the main results of this paper (cf. Lemma 3 Baksalary, Mathew 1986).

Lemma 1. *Let matrices A, B, C, D with dimensions $m \times n$, $p \times n$, $p \times q$ and $m \times q$ respectively be given. Moreover, let $R(C) \subseteq R(B)$ and $R(D) \subset R(A)$. Then the inclusion*

$$R(D) \subset R(AB^-C) \tag{2.3}$$

is satisfied regardless of the choice of a g-inverse B^- if and only if:

$$R(D) \subset R(AB^+C) \tag{2.4}$$

holds along with

$$R(A') \subset R(B'). \tag{2.5}$$

3. Invariant linearly sufficient statistics

The main object of investigations in this paper is the class of invariant linearly sufficient statistics. This notion was introduced by Oktaba, Kornacki and Wawrzosek (1988). Let us recall appropriate definitions and properties. Let us consider a fixed general linear Gauss-Markov model 1.1. Then for an arbitrary linear transformation $u = Py$ instead of the model GM we have the model PGM

$$(Py, PX\beta, \sigma^2 PVP'), \tag{3.1}$$

where P is a $k \times n$ matrix. Drygas (1983) introduced the notion of a linearly sufficient statistic.

Definition 3.1. A statistic $u = Py$ is said to be linearly sufficient (LS) (cf. Drygas 1983, p. 91) if there is a linear transformation $Gu = GPy$ which is BLUE of $X\beta$ in GM 1.1.

Figuratively one can define a linearly sufficient statistic as a transformation of the Gauss-Markov model preserving information needed for the estimation $X\beta$. Oktaba, Kornacki, Wawrzosek (1988) extended this notion considering the transformations which preserve information needed for both the estimation and the testing of hypotheses. The conception of invariant linearly sufficient statistics originated on the basis of what has just been presented.

Definition 3.2. A statistic $u = Py$ is said to be invariant linearly sufficient (ILS) if

(1) Py is linearly sufficient (LS),

(2) the parametric function $\lambda'\beta$ is estimable in GM if and only if it is estimable in PGM,

(3) the unbiased estimators

$$\hat{\sigma}^2 = \frac{(y - X\hat{\beta})T^-(y - X\hat{\beta})}{r(V \vdots X) - r(X)} \tag{3.2}$$

and

$$\tilde{\sigma}^2 = \frac{(Py - PX\tilde{\beta})'(\mathrm{PTP}')^-(Py - PX\tilde{\beta})}{r(\mathrm{PVP}' \vdots PX) - r(PX)} \tag{3.3}$$

of σ^2 in GM and PGM are identical, where $\hat{\beta}$ and $\tilde{\beta}$ are solutions of the normal equations in GM and PGM, respectively;

(4) the null hypothesis: $L\beta = \varphi_0$ is consistent in GM if and only if it is consistent in PGM;

(5) with the normality assumption of the vector y the statistics

$$F_{GM} = \frac{v'[D(v)]^- v}{r[D(v)] \cdot \hat{\sigma}^2}, \quad v = L\hat{\beta} - \varphi_0 \tag{3.4}$$

and

$$F_{PGM} = \frac{w'[D(w)]^- w}{r[D(w)] \cdot \tilde{\sigma}^2}, \quad w = L\tilde{\beta} - \varphi_0 \tag{3.5}$$

for testing the hypothesis $L\beta = \varphi_0$ have F distributions and are the same.

The theorem given below gives sufficient conditions for a statistic to be invariant linearly sufficient (ILS).

Theorem 3.1. (Oktaba, Kornacki, Wawrzosek (1988)) *If any one of the conditions*

$$R(T) = R(TP') \quad \text{and} \quad R(X) \subset R(P') \tag{3.6}$$

$$r(T) = r(TP') \quad \text{and} \quad R(X) \subset R(P') \tag{3.7}$$

$$R(T) \subset R(P') \tag{3.8}$$

is satisfied then the statistic $u = Py$ is ILS.

Let us note that conditions 3.6 and 3.7 are equivalent while 3.8 imply 3.6.

4. Main results

In this section the basic results of this paper will be presented. Let J_0 denote the class of statistics ILS satisfying equivalent conditions 3.6 and 3.7 in the model GM_0 and let J denote the analogous class in the model GM. Similarly let J_0^* indicate the class of statistics ILS satifying condition (3.8) in the model GM_0 and J^* the analogous class of statistics in the model GM. Now we will try to find conditions under which the above mentioned classes of statistics in the model GM_0 remain valid under passing over to the model GM. In other words we look for conditions under which the following relations are fulfilled:

$$J_0 \subset J \tag{4.1}$$

$$J_0^* \subset J^*. \tag{4.2}$$

The following three basic theorems contain answers to the questions given above.

Theorem 4.1. *Let a linear Gauss-Markov model* 1.1 GM *and* 1.4 GM_0 *be given. Then relation* 4.2 *is fulfilled if and only if*

$$R(T) \subset R(T_0). \tag{4.3}$$

Proof. *Sufficiency.* Let 4.3 be fulfilled. Then for every statistics $Py \in J_0^*$ we have $R(T_0) \subset R(P')$. To connect it with 4.3 we obtain: $R(T) \subset R(T_0) \subset R(P')$, and this means $Py \in J^*$, hence $J_0^* \subset J^*$.

Necessity. Let 4.2 be fulfilled. Taking $P' = T_0$ we get $Py \in J_0^*$. Due to 4.2 we have also $Py \in J^*$,which means that $R(T) \subset R(T_0)$. □

The following theorems are connected with condition 4.1.

Theorem 4.2. *Let two classes*

$$K_0 = \{Py : R(T_0) \subset R(T_0P')\},$$
$$K = \{Py : R(T) \subset R(TP')\}$$

in the models GM_0 *and* GM *be given. Then*

$$K_0 \subset K \tag{4.4}$$

if and only if

$$R(T) \subset R(T_0). \tag{4.5}$$

P r o o f. *Necessity.* Let 4.4 be fulfilled. It is easy to see that for $P' = T_0^- T_0$, $Py \in K_0$ regardless of the choice of T_0^-. According to 4.4 likewise $Py \in K$. Therefore

$$R(T) \subset R(TT_0^- T_0) \tag{4.6}$$

for every g-inverse matrix T_0^-. Hence by virtue of Lemma 1 we get

$$R(T) \subset R(TT_0^+ T_0) \text{ and } R(T) \subset R(T_0). \tag{4.7}$$

4.5 results naturally from 4.7.

Sufficiency. If 4.5 is satisfied then of course 4.7 is fulfilled. Let now $Py \in K_0$. We have

$$\forall_{T_0^-}\ R(T) \subset R(TT_0^- T_0 P') = R(TP') \tag{4.8}$$

from Lemma 1, $Py \in K_0$ and from 4.5. The formula 4.8 yields that $Py \in K$ and the assumption $Py \in K_0$ implies $K_0 \subset K$. □

Theorem 4.3. *Let the models GM_0 and GM be given. Then a sufficient condition for the inclusion*

$$J_0 \subset J \tag{4.9}$$

to be fulfilled is

$$R(T) \subset R(T_0) \quad \textit{and} \quad R(X) \subset R(X_0). \tag{4.10}$$

P r o o f. It is evident that the condition $R(T) = R(TP')$ which appears in the definition of classes J(and analogously J_0) is equivalent to the inclusion $R(T) \subset R(TP')$. Therefore the first part of the proof results from Theorem 4.2. Let now the condition $R(X_0) \subset R(P')$ be satisfied. Due to 4.10 we have

$$R(X) \subset R(X_0) \subset R(P'), \tag{4.11}$$

that is $J_0 \subset J$. □

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