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RELIABLE SOLUTIONS OF PROBLEMS IN THE DEFORMATION THEORY OF PLASTICITY WITH RESPECT TO UNCERTAIN MATERIAL FUNCTION

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Summary. Maximization problems are formulated for a class of quasistatic problems in the deformation theory of plasticity with respect to an uncertainty in the material function. Approximate problems are introduced on the basis of cubic Hermite splines and finite elements. The solvability of both continuous and approximate problems is proved and some convergence analysis presented.

Keywords: deformation theory of plasticity, physically nonlinear elasticity, uncertain data

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INTRODUCTION

One of the simplest models of elasto-plastic bodies is represented by the deformation theory of plasticity (see [3], [4], [5]). It is nothing else than an elastic model with a nonlinear stress-strain relations. From the mathematical point of view, the model is advantageous, being formulated by means of potential and strongly monotonous operators. The crucial role is played by a material function, which has to satisfy some differential inequalities.

Sometimes, however, the material function cannot be given uniquely, but only in some set of admissible functions. If maximal values of a functional are the main goal of all computations (e.g., some mean values of displacements, intensity of shear stresses or principal stresses, respectively), we can follow an approach used in Optimal Design and formulate a maximization problem, which expresses the requirement to remain "on the safe side".

In Section 1 of the present paper we recall the deformation theory of plasticity, setting a mixed boundary value problem. Conditions guaranteeing the existence and uniqueness of a weak solution are given in Section 2. A continuous dependence of the solution on the material function is derived in Section 3 and a general maximization problem is formulated in Section 4. Here we present three examples of functionals, which may be of practical interest. The short Section 5 contains the proof of solvability of the maximization problem.

In Section 6 we introduce approximations of the material function, using cubic Hermite splines and Ritz-Galerkin method is applied to define approximate displacement functions. In this way we introduce an approximate maximization problem and prove its solvability. Section 7 is devoted to a convergence analysis. We can show that having a sequence of approximate solutions (with the mesh-sizes of both the material function and the displacement function discretization tending to zero), one can choose a subsequence, which converges to a solution of the continuous maximization problem.

The results are valid for both three-and two-dimensional problems. The Kačanov (secant modules) method (see [3], [5], [6]) is proposed for computation of approximate displacement functions.

1. Setting of the state problem

Let us recall the basic relations of quasistatic problems in the deformation theory of plasticity (see Kačanov [3], Langenbach [4], Nečas and Hlaváček [5, 6]).

We consider a body occupying a bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary $\partial\Omega$ and assume that

$$\partial\Omega = \Gamma_u \cup \Gamma_\tau \cup \Gamma_M,$$

where Γ_u, Γ_τ are relatively open in $\partial\Omega$, $\Gamma_u \neq \emptyset$ and the surface measure of Γ_M vanishes.

Let the material of the body be governed by the following Hencky-Mises stress-strain relations

$$\tau_{ij} = \Big(k - \frac{2}{3}\mu(\gamma)\Big)\delta_{ij}\vartheta + 2\mu(\gamma)e_{ij},$$

where k is a (constant) bulk modulus,

$$\begin{aligned} \vartheta = \vartheta(\mathrm{u}) = e_{ii}(\mathrm{u}) = \operatorname{div}\mathrm{u},\ e_{ij} = e_{ij}(\mathrm{u}) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i), \\ \gamma = \gamma(\mathrm{u}) = \Gamma(\mathrm{u},\, \mathrm{u}), \\ \Gamma(\mathrm{u},\, \mathrm{v}) = -\frac{2}{3}\vartheta(\mathrm{u})\vartheta(\mathrm{v}) + 2e_{ij}(\mathrm{u})e_{ij}(\mathrm{v}) \end{aligned}$$

and a repeated index implies summation over $\{1,2,3\}$.

The function $\mu\colon [0,+\infty) \to \mathbb{R}$ belongs to a certain set of admissible functions, which will be specified in the following section.

Let body forces $\mathrm{f} \in [L^2(\Omega)]^3$, surface loads $\mathrm{g} \in [L^2(\Gamma_\tau)]^3$ and a displacement function $\mathrm{u}^0 \in [H^1(\Omega)]^3$ be given.

We are looking for a solution of the following non-linear boundary value problem

$$
\begin{aligned}
-\partial\tau_{ij}(\mathrm{u})/\partial x_j + f_i &= 0 \quad \text{in } \Omega,\ i = 1,2,3,\\
\mathrm{u} &= \mathrm{u}^0 \quad \text{on } \Gamma_u,\\
\nu_j\tau_{ij}(\mathrm{u}) &= g_i \quad \text{on } \Gamma_\tau,
\end{aligned}
\tag{1.1}
$$

where ν denotes the unit outward normal to $\partial\Omega$.

The solution of the problem (1.1) leads to the minimization of the functional of potential energy (see [5 - chapt. 8])

$$
\Phi(\mathrm{u}) = \frac{1}{2}\int_\Omega \left[k\vartheta^2(\mathrm{u}) + \int_0^{\gamma(\mathrm{u})} \mu(t)\,\mathrm{d}t\right] \mathrm{d}x - \int_\Omega f_i u_i \,\mathrm{d}x - \int_{\Gamma_\tau} g_i u_i \,\mathrm{d}s
\tag{1.2}
$$

over the affine set $\mathrm{u}^0 + V$, where

$$
V = \{\mathrm{u} \in [H^1(\Omega)]^3 : \mathrm{u} = 0 \quad \text{on } \Gamma_u\}.
$$

The minimization problem can be replaced by the following equivalent problem: find $\mathrm{u} \in \mathrm{u}^0 + V$ such that

$$
D\Phi(\mathrm{u},\mathrm{v}) \equiv \int_\Omega [k\vartheta(\mathrm{u})\vartheta(\mathrm{v}) + \mu(\gamma(\mathrm{u}))\Gamma(\mathrm{u},\mathrm{v})]\,\mathrm{d}x - \int_\Omega f_i v_i \,\mathrm{d}x - \int_{\Gamma_\tau} g_i v_i \,\mathrm{d}x = 0
\tag{1.3}
$$

for all $\mathrm{v} \in V$.

Here $D\Phi(\mathrm{u},\mathrm{v})$ denotes the Gâteaux differential at the point u.

2. The set of admissible material functions

To guarantee a unique solvability of the problem (1.3), we assume that the function $\mu \in C^1([0,+\infty))$ and there exist positive constants μ_0, κ, such that $3k/2 > \mu_0 \geqslant \kappa$,

$$
\mu_0 \leqslant \mu(t) \leqslant \frac{3}{2}k,\ \mathrm{d}\mu/\,\mathrm{d}t \leqslant 0,
\tag{2.1}
$$

$$
\kappa \leqslant \mu(t) + 2t\,\mathrm{d}\mu/\,\mathrm{d}t
\tag{2.2}
$$

holds for all $t \geqslant 0$.

Let us introduce the space $U = C^{(1)}([0,1])$ and the following sets of admissible functions:

$$\begin{aligned}
&U_{ad} = \{\varphi \in C^{(1),1}([0,1])\colon \mu_0 \leqslant \varphi(\xi) \leqslant \tfrac{3}{2}k,\ -C_1 \leqslant \mathrm{d}\varphi/\mathrm{d}\xi \leqslant 0,\\
&\quad \kappa \leqslant \varphi(\xi) + \xi(1-\xi)\,\mathrm{d}\varphi/\mathrm{d}\xi\\
&\text{for all } \xi \in [0,1] \text{ and } |\mathrm{d}^2\varphi/\mathrm{d}\xi^2| \leqslant C_2 \text{ for a.a. } \xi \in [0,1]\},\\
&\widetilde{U}_{ad} = \{\varphi \in C^{(1),1}([0,1]) : \mu_0/2 \leqslant \varphi(\xi) \leqslant 2k, -2C_1 \leqslant \mathrm{d}\varphi/\mathrm{d}\xi \leqslant 0,\\
&\quad \kappa/2 \leqslant \varphi(\xi) + \xi(1-\xi)\,\mathrm{d}\varphi/\mathrm{d}\xi\\
&\text{for all } \xi \in [0,1] \text{ and } |\mathrm{d}^2\varphi/\mathrm{d}\xi^2| \leqslant C_2 \text{ for a.a. } \xi \in [0,1]\},
\end{aligned}$$

where C_1 and C_2 are given positive parameters, $C^{(1),1}$ denotes the set of Lipschitz continuous functions with Lipschitz continuous derivatives.

For any $t \in [0,+\infty)$ we define the material function

$$\mu(t) = \varphi\Big(\frac{t^{1/2}}{1+t^{1/2}}\Big). \tag{2.3}$$

It is easy to verify that the *material function μ satisfies the conditions* (2.1), (2.2), *provided $\varphi \in U_{ad}$*. We have

$$\frac{\mathrm{d}\mu(t)}{\mathrm{d}t} = \frac{(1-\xi)^3}{2\xi}\frac{\mathrm{d}\varphi(\xi)}{\mathrm{d}\xi},\ \xi = \frac{t^{1/2}}{1+t^{1/2}},\ t = \frac{\xi^2}{(1-\xi)^2}. \tag{2.4}$$

Lemma 2.1. *The sets U_{ad} and $\widetilde{U}_{ad}$ are compact in U.*

Proof follows from a repeated use of Arzelà-Ascoli Theorem and a classic result for the derivatives of a uniformly convergent sequence. □

Remark 2.1. The function $t^{1/2}(1+t^{1/2})^{-1}$ can be replaced by $2\pi^{-1}\arctan t^{1/2}$.

3. Well-posedness of the state problem

Proposition 3.1. *There exists a unique solution $u(\varphi)$ of the state problem* (1.3) *for any $\varphi \in \widetilde{U}_{ad}$ and μ defined by the relation* (2.3).

Proof. See [5 – §8.2]. □

Proposition 3.2. *Let $\varphi_n \in U_{ad}$, $\varphi_n \to \varphi$ in $C([0,1])$, as $n \to \infty$. Then*

$$u(\varphi_n) \to u(\varphi) \quad \text{in } [H^1(\Omega)]^3. \tag{3.1}$$

Proof. Let us follow some ideas of Langenbach [4 – IV. §1.1]. Denote $W\colon = [H^1(\Omega)]^3$, W^* its dual, $[\cdot,\cdot]$ the dual pairing, $\|\cdot\|_1$ the norm in W, and $\|\cdot\|_*$ the norm in W^*, $u_n = u(\varphi_n)$, $u = u(\varphi)$. Introduce an operator $A(\varphi)\colon W \to W^*$ as follows:

$$[A(\varphi)u, v] = \int_\Omega \Big[\Big(k - \frac{2}{3}\mu(\gamma(u))\Big)\vartheta(u)\vartheta(v) + 2\mu(\gamma(u))e_{ij}(u)e_{ij}(v)\Big]\,\mathrm{d}x,$$

where $\mu(t) = \varphi(t^{1/2}(1+t^{1/2})^{-1})$.

Then for any $\varphi \in U_{ad}$ the operator $A(\varphi)$ is uniformly strongly monotone on the set $u^0 + V$, i.e.,

$$[A(\varphi)u - A(\varphi)v, u - v] \geqslant C\|u - v\|_1^2 \tag{3.2}$$

holds for any $\varphi \in U_{ad}$, $u, v \in u^0 + V$, where the constant C is independent of φ.

In fact,

$$[A(\varphi)u - A(\varphi)v, u - v] \geqslant 2\kappa \int_\Omega e_{ij}(u-v)e_{ij}(u-v)\,\mathrm{d}x$$

follows from the condition (2.2) (see [5 – §8.2, Lemma 2.1]). Combining this result with the Korn's inequality, we obtain (3.2).

Second,

$$\|A(\varphi_n)v - A(\varphi)v\|_* \to 0 \quad \text{as } n \to \infty \tag{3.3}$$

holds for any $v \in W$.

In fact, we may write

$$\begin{aligned}|[A(\varphi_n)v - A(\varphi)v, w]| &\leqslant C\int_\Omega \|\mu_n - \mu\|_{0,\infty}(|\vartheta(v)|\,|\vartheta(w)| + |e_{ij}(v)|\,|e_{ij}(w)|)\,\mathrm{d}x\\ &\leqslant \widetilde{C}\|\mu_n - \mu\|_{0,\infty}\|v\|_1\|w\|_1,\end{aligned}$$

where

$$\|\mu_n - \mu\|_{0,\infty} = \sup_{t\in[0,\infty)} |\mu_n(t) - \mu(t)| = \sup_{x\in[0,1]} |\varphi_n(x) - \varphi(x)| \to 0.$$

Using (3.2), we have

$$\begin{aligned}&[A(\varphi_n)\mathrm{u}_n - A(\varphi)\mathrm{u}, \mathrm{u}_n - \mathrm{u}]\\ &= [A(\varphi_n)\mathrm{u}_n - A(\varphi_n)\mathrm{u}, \mathrm{u}_n - \mathrm{u}] + [A(\varphi_n)\mathrm{u} - A(\varphi)\mathrm{u}, \mathrm{u}_n - \mathrm{u}]\\ &\geqslant C\|\mathrm{u}_n - \mathrm{u}\|_1^2 - \|A(\varphi_n)\mathrm{u} - A(\varphi)\mathrm{u}\|_*\,\|\mathrm{u}_n - \mathrm{u}\|_1.\end{aligned} \tag{3.4}$$

Since

$$[A(\varphi_n)\mathrm{u}_n, \mathrm{v}] = [A(\varphi)\mathrm{u}, \mathrm{v}] = \int_\Omega f_i v_i \,\mathrm{d}x + \int_{\Gamma_\tau} g_i v_i \,\mathrm{d}s$$

for all $\mathrm{v} \in V$ follows from the definition (1.3), the left-hand side of (3.4) vanishes. Thus we obtain

$$\|\mathrm{u}_n - \mathrm{u}\|_1 \leqslant C^{-1}\|A(\varphi_n)\mathrm{u} - A(\varphi)\mathrm{u}\|_*.$$

Using (3.3), we arrive at the strong convergence (3.1). □

4. Setting of a maximization problem

Let a functional $\Psi\colon U_{ad} \times W \to \mathbb{R}$ be given, such that if $\varphi_n \in U_{ad}, \varphi_n \to \varphi$ in U and $\mathrm{u}_n \to \mathrm{u}$ in W, then

$$\lim_{n\to\infty} \Psi(\varphi_n, \mathrm{u}_n) = \Psi(\varphi, \mathrm{u}). \tag{4.1}$$

We want to solve the following *Maximization Problem:* find

$$\varphi^0 = \arg \max_{\varphi\in U_{ad}} \Psi(\varphi, \mathrm{u}(\varphi)). \tag{4.2}$$

Example 4.1. Let G_j, $1 \leqslant j \leqslant N$, be given subsets of Γ_τ, with positive surface measure. Define

$$\psi_j(\mathrm{u}) = (\text{meas } G_j)^{-1} \int_{G_j} u_i \nu_i \,\mathrm{d}s$$

and

$$\Psi(\mathrm{u}) = \max_{1\leqslant j\leqslant N} \psi_j(u).$$

Using the Trace Theorem, it is easy to see that the assumption (4.1) is satisfied.

Example 4.2. Let G_j, $1 \leqslant j \leqslant N$, be given subdomains of Ω, meas $G_j > 0$. Let us define

$$\psi_j(\varphi, \mathrm{u}) = (\text{meas } G_j)^{-1} \int_{G_j} \mu(\gamma(\mathrm{u}))(\gamma(\mathrm{u}))^{1/2} \,\mathrm{d}x.$$

The integrand represents the square root of the intensity of shear stress (i.e., an invariant of the stress tensor)—see [5 – §3.3]. We define

$$\Psi(\varphi, u) = \max_{1\leqslant j\leqslant N} \psi_j(\varphi, u)$$

and show that the condition (4.1) is satisfied. Obviously, it suffices to consider a single functional $\psi_j(\varphi, u)$.

First, we prove the following

Lemma 4.1. *If $u_n \to u$ in W, then*

$$(\gamma(u_n))^{1/2} \to (\gamma(u))^{1/2} \quad \text{in } L^2(G_j).$$

Proof. Using the definition of $\gamma(u)$ and the inequality

$$|\Gamma(u,v)| \leqslant \gamma^{1/2}(u)\gamma^{1/2}(v), \tag{4.3}$$

we derive that

$$\begin{aligned}(\gamma^{1/2}(u_n) - \gamma^{1/2}(u))^2 &= \gamma(u_n) + \gamma(u) - 2\gamma^{1/2}(u_n)\gamma^{1/2}(u) \\ &\leqslant \Gamma(u_n - u, u_n - u) = \gamma(u_n - u).\end{aligned}$$

Hence using also the estimate

$$\int_{G_j} \gamma(w)\,\mathrm{d}x \leqslant C\|w\|_1^2 \quad \forall w \in W, \tag{4.4}$$

we obtain

$$\int_{G_j} (\gamma^{1/2}(u_n) - \gamma^{1/2}(u))^2\,\mathrm{d}x \leqslant \int_{G_j} \gamma(u_n - u)\,\mathrm{d}x \leqslant C\|u_n - u\|_1^2 \to 0. \tag{4.5}$$

□

Lemma 4.2. *If $\varphi \in U_{ad}$ and $u_n \to u$ in W, then*

$$\mu(\gamma(u_n)) \to \mu(\gamma(u)) \quad \text{in } L^2(G_j).$$

Proof. Let us introduce an auxiliary function

$$\widetilde{\mu}(s) = \varphi(s(1+s)^{-1}), \quad s \in [0, +\infty).$$

Then

$$\mu(t) = \widetilde{\mu}(t^{1/2}), \quad |\,\mathrm{d}\widetilde{\mu}/\,\mathrm{d}s| = (1+s)^{-2}|\,\mathrm{d}\varphi/\,\mathrm{d}x| \leqslant C_1.$$

Then we may write

$$\begin{aligned}\int_{G_j} [\mu(\gamma(u_n) - \mu(\gamma(u))]^2\,\mathrm{d}x &= \int_{G_j} [\widetilde{\mu}(\gamma^{1/2}(u_n)) - \widetilde{\mu}(\gamma^{1/2}(u))]^2\,\mathrm{d}x \\ &\leqslant C_1^2\|\gamma^{1/2}(u_n) - \gamma^{1/2}(u)\|_0^2 \leqslant C_1^2 C\|u_n - u\|_1^2,\end{aligned} \tag{4.6}$$

using (4.5).

□

Combining Lemma 4.1 and Lemma 4.2, we obtain

$$(\mu(\gamma(u_n)), \gamma^{1/2}(u_n))_0 \to (\mu(\gamma(u)), \gamma^{1/2}(u))_0 \quad \text{as } n \to \infty.$$

Since the definition of μ, μ_n and the estimate (4.4) yield that

$$\begin{aligned}
&|(\mu_n(\gamma(u_n)) - \mu(\gamma(u_n)), \gamma^{1/2}(u_n))_0| \\
&\leqslant C\|\mu_n - \mu\|_{0,\infty}\Big(\int_{G_j} \gamma(u_n)\,\mathrm{d}x\Big)^{1/2} \leqslant \widetilde{C}\|\varphi_n - \varphi\|_{0,\infty}\|u_n\|_1,
\end{aligned}$$

we obtain that

$$\begin{aligned}
&|(\mu_n(\gamma(u_n)), \gamma^{1/2}(u_n))_0 - (\mu(\gamma(u)), \gamma^{1/2}(u))_0| \\
&\qquad \leqslant |(\mu_n(\gamma(u_n)) - \mu(\gamma(u_n)), \gamma^{1/2}(u_n))_0| \\
&\qquad\quad + |(\mu(\gamma(u_n), \gamma^{1/2}(u_n))_0 - (\mu(\gamma(u)), \gamma^{1/2}(u))_0| \to 0,
\end{aligned}$$

as $\varphi_n \to \varphi$ in $C([0,1])$ and $u_n \to u$ in W.

As a consequence,

$$\psi_j(\varphi_n, u_n) \to \psi_j(\varphi, u),\ 1 \leqslant j \leqslant N, \quad \text{as } n \to \infty$$

and the same holds true for $\Psi = \max\limits_j \psi_j$.

Example 4.3. Let us consider a corresponding two-dimensional *plane stress* problem. Then the coefficient $(-2/3)$ in the formulae for τ_{ij}, $\Gamma(u,v)$ has to be replaced by (-1) and $(3k/2)$ in the condition (2.1) and in the definition of U_{ad} by (k).

Let the *principal stresses* be denoted by τ_1, τ_2, $\tau_1 \geqslant \tau_2$. We define

$$\psi_j(\varphi, u) = (\text{meas } G_j)^{-1} \int_{G_j} \tau_1(\varphi, u)\,\mathrm{d}x,\ 1 \leqslant j \leqslant N,$$

where G_j is a given subdomain of $\Omega \subset \mathbb{R}^2$, meas $G_j > 0$. Let us introduce

$$\Psi(\varphi, u) = \max_{1 \leqslant j \leqslant N} \psi_j(\varphi, u).$$

It is easy to derive that

$$\int_{G_j} \tau_1(\varphi, u)\,\mathrm{d}x = \int_{G_j} [k\vartheta(u) + \mu(\gamma(u))(\beta(u))^{1/2}]\,\mathrm{d}x$$

where

$$\beta(u) \equiv B(u,u),$$
$$B(u,v) = (e_{11}(u) - e_{22}(u))(e_{11}(v) - e_{22}(v)) + 4e_{12}(u)e_{12}(v).$$

For β we prove an analogue of (4.3), (4.4) and of Lemma 4.1, using a parallel argument. Consequently, the condition (4.1) follows, provided $\varphi_n \to \varphi$ in U and $u_n \to u$ in W.

Remark 4.1. Note that Propositions 3.1 and 3.2 hold true also for the plane stress problem. Their proofs are completely analogous.

Remark 4.2. In the proof of (4.1) for the three examples 4.1–4.3 we have not needed the convergence in $U \equiv C^{(1)}$ but only in $C([0,1])$.

5. Existence of a solution to the Maximization Problem

There exists at least one solution of the Maximization Problem (4.2). Indeed, we can define the following functional

$$J(\varphi) = \Psi(\varphi, u(\varphi)), \quad \varphi \in U_{ad},$$

on the basis of Proposition 3.1. Let $\{\varphi_n\}$ be a sequence of functions $\varphi_n \in U_{ad}$, such that

$$\lim_{n\to\infty} J(\varphi_n) = \sup_{\varphi \in U_{ad}} J(\varphi).$$

Using Lemma 2.1, we can choose a subsequence $\{\varphi_m\}$ such that $\varphi_m \to \varphi^0$ in U and $\varphi^0 \in U_{ad}$. Proposition 3.2 implies that

$$u(\varphi_m) \to u(\varphi^0) \quad \text{in } W, \quad \text{as } m \to \infty.$$

By virtue of the assumption (4.1), we may write

$$\sup_{\varphi \in U_{ad}} J(\varphi) = \lim_{m\to\infty} \Psi(\varphi_m, \mathrm{u}(\varphi_m)) = \Psi(\varphi^0, \mathrm{u}(\varphi^0)),$$

so that φ^0 is a solution of the problem (4.2).

Let us assume that

$$C_1 < 4(3k/2 - \mu_0). \tag{6.1}$$

Let M be an integer such that

$$M > \frac{1}{2}C_2/C_1. \tag{6.2}$$

Denote $\Delta_j = [(j-1)/M, j/M]$, $j = 1, \dots, M$, $\varphi'_M = \mathrm{d}\varphi_M/\,\mathrm{d}x$ and introduce the following approximation of the set U_{ad}:

$$\begin{aligned} U_{ad}^M = \{\varphi_M \in C^{(1)}([0,1])\colon & \varphi_{M|\Delta j} \in P_3(\Delta_j),\ j = 1, \dots, M; \\ & \mu_0 \leqslant \varphi_M(j/M) \leqslant 3k/2; \\ & -C_1 \leqslant \varphi'_M(j/M) \leqslant -\tfrac{1}{2}C_2/M; \\ & \varphi_M(j/M) + j/M(1 - j/M)\varphi'_M(j/M) \geqslant \kappa; \\ & |\varphi''_M(j/M\pm)| \leqslant C_2, \quad j = 0, 1, \dots, M\}. \end{aligned} \tag{6.3}$$

Here $P_3(\Delta_j)$ is the space of cubic polynomials on the interval Δ_j and

$$\varphi''_M(j/M\pm) = \lim \varphi''_M(x) \quad \text{for } x \to j/M \pm .$$

Functions $\varphi_M \in U_{ad}^M$ are Hermite cubic splines. It is easy to verify that

$$U_{ad}^M \not\subset U_{ad},\ U_{ad}^M \subset \widetilde{U}_{ad}, \text{ for } M \text{ great enough.} \tag{6.4}$$

In fact,

$$|\varphi_M(\xi) - \varphi_M(j/M)| \leqslant \frac{1}{2}C_1/M + \frac{1}{8}C_2/M^2, \tag{6.5}$$

$$|\varphi'_M(\xi) - \varphi'_M(j/M)| \leqslant \frac{1}{2}C_2/M \tag{6.6}$$

holds for all $\xi \in [0,1]$ and the closest nodal point $x_j = j/M$.

The condition (6.2) guarantees that the interval in (6.3) has a positive length.

Lemma 6.1. *The set U_{ad}^M is compact in $\mathbb{R}^{2(M+1)}$.*

Proof. Every $\varphi_M \in U_{ad}^M$ can be identified with the vector of nodal values $\varphi_M(j/M)$, $\varphi'_M\,(j/M)$, $j = 0, 1, \dots, M$. It is readily seen that a bounded and closed set $\mathcal{A} \subset \mathbb{R}^{2(M+1)}$ corresponds to U_{ad}^M. As a consequence, the set $\mathcal{A}$ is compact in $\mathbb{R}^{2(M+1)}$. $\square$

Let $W_h \subset W$ be a finite-dimensional subspace (e.g. a finite element space) and $V_h = V \cap W_h$.

We define the *Galerkin approximation* $u_h \equiv u_h(\varphi)$ for $\varphi \in \widetilde{U}_{ad}$ as follows:

$$
\begin{aligned}
&u_h \in u^0 + V_h,\\
(6.7)\qquad &a_\varphi(u_h; u_h, v_h) = L(v_h) \quad \forall v_h \in V_h.
\end{aligned}
$$

where

$$(6.8)\quad a_\varphi(u; w, v) = \int_\Omega \Big[\Big(k - \frac{2}{3}\mu(\gamma(u))\Big)\vartheta(w)\vartheta(v) + 2\mu(\gamma(u))e_{ij}(w)e_{ij}(v)\Big]\,\mathrm{d}x,$$

$$(6.9)\qquad L(v) = \int_\Omega f_i v_i \,\mathrm{d}x + \int_{\Gamma_\tau} g_i v_i \,\mathrm{d}s$$

and

$$\mu(t) = \varphi(t^{1/2}(1 + t^{1/2})^{-1}).$$

Lemma 6.2. *For any $\varphi \in U_{ad}$ there exists a unique Galerkin approximation $u_h(\varphi)$.*

Proof. The condition (6.7) is equivalent with the following minimization problem

$$(6.10)\qquad u_h = \arg \min_{v \in u^0 + V_h} \Phi(v)$$

(cf. (1.2) and (1.3)). The functional

$$J(y_h) = \Phi(u^0 + y_h)$$

is coercive and lower semicontinuous on the space V_h (see the analogous proof of Theorem 2.1 in [5 – §8.2]). As a consequence a minimizer $u^0 + w_h \equiv u_h$ exists.

Let u_h and $\bar{u}_h$ be two Galerkin approximations. On the basis of (3.2) we obtain that

$$a_\varphi(u_h; u_h, u_h - \bar{u}_h) - a_\varphi(\bar{u}_h, \bar{u}_h, u_h - \bar{u}_h) \geqslant C\|u_h - \bar{u}_h\|_1^2.$$

Since the left-hand side vanishes, $u_h - \bar{u}_h = 0$ follows. □

The Galerkin approximation $u_h(\varphi)$ can be calculated by means of the *secant modules (Kačanov)* method as follows:

Let $y^0 \in V_h$ be arbitrary. If $y^k \in V_h$ is known, let $y^{k+1} \in V_h$ be defined by the relation

$$(6.11)\qquad a_\varphi(u^0 + y^k; u^0 + y^{k+1}, v) = L(v) \quad \forall v \in V_h.$$

Then

$$\|u_h - (u^0 + y^k)\|_1 \to 0 \quad \text{as } k \to \infty. \tag{6.12}$$

For the proof—see [5 – §11.5]. We can modify the argument there slightly, replacing the Hilbert space H by the affine set $u^0 + V_h \subset W_h$.

Remark 6.1. Note that only now the assumption $\mathrm{d}\varphi/\mathrm{d}x \leqslant 0$ is employed, for the proof of (6.12).

Proposition 6.1. *If $\{\varphi_n\}, \varphi_n \in \tilde{U}_{ad}$ and $\varphi_n \to \varphi$ in $C([0,1])$ as $n \to \infty$, then*

$$u_h(\varphi_n) \to u_h(\varphi) \quad \text{as } n \to \infty.$$

Proof. is analogous to that of Proposition 3.2. □

Let us introduce the following *Approximate Maximization Problem:*

$$\varphi_M^0(h) = \arg \max_{\varphi_M \in U_{ad}^M} \Psi(\varphi_M, u_h(\varphi_M)). \tag{6.13}$$

Lemma 6.3. *Let the functional Ψ satisfy the condition* (4.1). *Then the Approximate Maximization Problem* (6.13) *has at least one solution.*

Proof. Let $\{\varphi_M^n\}, n \to \infty, \varphi_M^n \in U_{ad}^M$, be a sequence such that

$$\lim_{n\to\infty} \Psi(\varphi_M^n, u_h(\varphi_M^n)) = \sup_{\varphi_M \in U_{ad}^M} \Psi(\varphi_M, u_h(\varphi_M)). \tag{6.14}$$

By Lemma 6.1 the set U_{ad}^M is compact and therefore a subsequence $\{\varphi_M^m\} \subset \{\varphi_M^n\}$ and $\varphi_M \in U_{ad}^M$ exist such that

$$\varphi_M^m \to \varphi_M \quad \text{in} \quad \mathbb{R}^{2(M+1)} \quad \text{as } m \to \infty.$$

Proposition 6.1 yields that $u_h(\varphi_M^m) \to u_h(\varphi_M)$. Using (4.1), we obtain

$$\Psi(\varphi_M^m, u_h(\varphi_M^m)) \to \Psi(\varphi_M, u_h(\varphi_M)) \quad \text{as } m \to \infty. \tag{6.15}$$

From (6.14) and (6.15), we deduce

$$\Psi(\varphi_M, u_h(\varphi_M)) = \sup_{\varphi_M \in U_{ad}^M} \Psi(\varphi_M, u_h(\varphi_M)),$$

so that $\varphi_M = \varphi_M^0(h)$ is a solution of the problem (6.13). □

7. Some convergence analysis

We will study the behaviour of the solutions $\varphi_M^0(h)$ and approximations $u_h(\varphi_M^0(h))$, when M tends to infinity and h (the mesh-size of finite element discretization) tends to zero. To this end, we shall need the following.

Lemma 7.1. *For any $\varphi \in U_{ad}$ there exists a sequence $\{\varphi_M\}$, $M = M_0, M_0+1, \dots$ such that $\varphi_M \in U_{ad}^M$ and $\varphi_M \to \varphi$ in $U \equiv C^{(1)}([0,1])$, as $M \to \infty$.*

Proof. Denote (cf. the definition of U_{ad}^M)

$$b(M) = \frac{1}{2} C_2/M.$$

For $\nu \in (0,1)$ define

$$\begin{aligned} &\varphi_\nu(s) = \varphi^0((1-\nu)s), \text{ where} \\ &\varphi^0(s) = \varphi(s+1/2) \text{ and } s \in I_\nu = [-(1-\nu)^{-1}, (1-\nu)^{-1}]. \end{aligned}$$

Then we have for $s \in I = [-\frac{1}{2}, \frac{1}{2}]$

$$\mu_0 \leqslant \varphi_\nu(s) \leqslant \frac{3}{2}k, \ -C_1 \leqslant d\varphi_\nu(s)/\,\mathrm{d}s \leqslant 0,$$

since

$$\frac{\mathrm{d}\varphi_\nu(s)}{\mathrm{d}s} = \frac{\mathrm{d}\varphi}{\mathrm{d}x}(1-\nu), \quad \|\varphi_\nu - \varphi\|_{0,\infty,I} \leqslant C_1\nu/2, \tag{7.1}$$
$$\|\varphi_\nu' - \varphi'\|_{0,\infty,I} \leqslant (C_1 + \frac{1}{2}C_2)\nu.$$

Let us apply the regularization

$$R_H\varphi_\nu(t) = (\kappa_0 H)^{-1} \int_{-\infty}^{\infty} \omega(t-s)\varphi_\nu(s)\,\mathrm{d}s$$

where $H = \text{const} > 0$,

$$\omega(z,H) = \begin{cases} \exp \dfrac{z^2}{z^2 - H^2}, & \text{if } |z| < H, \\ 0 & \text{if } |z| \geqslant H \end{cases}$$

and

$$\kappa_0 = H^{-1} \int_{-H}^{H} \omega(z,H)\,\mathrm{d}z.$$

Thus we obtain

$$R_H\varphi_\nu \in C^\infty(I), \quad \mu_0 \leqslant R_H\varphi_\nu(s) \leqslant \frac{3}{2}k, \tag{7.2}$$

$$-C_1 \leqslant (R_H\varphi_\nu)'(s) \leqslant 0 \quad \text{for all } s \in I. \tag{7.3}$$

Let us denote

$$\begin{aligned} E_1 &= \|\varphi - \varphi_\nu\|_{1,\infty,I}, \quad E_2 = \|R_H\varphi_\nu - \varphi_\nu\|_{1,\infty,I}, \\ \varepsilon(\nu, H) &= E_1 + E_2. \end{aligned}$$

Introduce a function

$$\begin{aligned} \eta(s) &= (1-\lambda)R_H\varphi_\nu(s) + S_0\lambda - b\Big(s + \frac{1}{2}\Big) + 4b\Delta/C_1 \\ \text{where} \quad S_0 &= \frac{1}{2}\Big(\frac{3}{2}k + \mu_0\Big), \ \Delta = \frac{1}{2}\Big(\frac{3}{2}k - \mu_0\Big), \\ \lambda &= \lambda(M,\nu,H) = \varepsilon/\Delta + 4b/C_1. \end{aligned}$$

Next we show that if M is great enough and ν, H small enough, $0 < \lambda < 1$ and η satisfies the following conditions for all $s \in I$:

$$\mu_0 \leqslant \eta(s) \leqslant \frac{3}{2}k \tag{7.4}$$

$$-C_1 \leqslant \eta'(s) \leqslant -b(M) \tag{7.5}$$

$$\eta(s) + \Big(\frac{1}{4} - s^2\Big)\eta'(s) \geqslant \kappa. \tag{7.6}$$

In fact, using (7.2), (7.3) we may write

$$\min_{s\in I} \eta(s) \geqslant (1-\lambda)\mu_0 + S_0\lambda - b + 4b\Delta/C_1 = \mu_0 + \varepsilon + 8b\Delta/C_1 - b > \mu_0,$$

since $b(8\Delta/C_1 - 1) > 0$ (by virtue of (6.1));

$$\begin{aligned} \max_{s\in I} \eta(s) &\leqslant (1-\lambda)\frac{3}{2}k + S_0\lambda + 4b\frac{\Delta}{C_1} \leqslant \frac{3}{2}k - \lambda\Delta + 4b\Delta/C_1 = \frac{3}{2}k - \varepsilon \leqslant \frac{3}{2}k; \\ \eta'(s) &= (1-\lambda)(R_H\varphi_\nu)' - b, \end{aligned}$$

so that

$$\begin{aligned} -b \geqslant \eta'(s) &\geqslant (1-\lambda)(-C_1) - b = -C_1 + \lambda C_1 - b \\ &= -C_1 + 3b + \varepsilon C_1/\Delta \geqslant -C_1. \end{aligned}$$

Since

$$\begin{aligned}
&\Big|R_H\varphi_\nu + \Big(\frac{1}{4}-s^2\Big)(R_H\varphi_\nu)' - \Big(\varphi + \Big(\frac{1}{4}-s^2\Big)\varphi'\Big)\Big| \\
&\quad \leqslant |R_H\varphi_\nu - \varphi| + \frac{1}{4}|R_H\varphi_\nu' - \varphi'| \leqslant \|R_H\varphi_\nu - \varphi\|_{1,\infty,I} \\
&\quad \leqslant \|R_H\varphi_\nu - \varphi_\nu\|_{1,\infty,I} + \|\varphi_\nu - \varphi\|_{1,\infty,I} = \varepsilon(\nu, H),
\end{aligned}$$

we conclude that

$$R_H\varphi_\nu + \Big(\frac{1}{4}-s^2\Big)R_H\varphi_\nu' \geqslant \kappa - \varepsilon(\nu, H).$$

Then we may write

$$\begin{aligned}
\eta + \Big(\frac{1}{4}-s^2\Big)\eta' &= (1-\lambda)R_H\varphi_\nu + S_0\lambda - b\Big(\frac{1}{2}+s\Big) + 4b\Delta/C_1 \\
&\quad + \Big(\frac{1}{4}-s^2\Big)[(1-\lambda)(R_H\varphi_\nu)' - b] \\
&\geqslant (1-\lambda)(R_H\varphi_\nu + \Big(\frac{1}{4}-s^2\Big)R_H\varphi_\nu') + S_0\lambda + 4b\Delta/C_1 - b \\
&\geqslant (1-\lambda)(\kappa-\varepsilon) + (\mu_0+\Delta)\lambda + 4b\Delta/C_1 - b \\
&= \kappa - \varepsilon + \lambda\varepsilon + \lambda(\mu_0-\kappa) + \varepsilon + 8b\Delta/C_1 - b \geqslant \kappa + \lambda\varepsilon \geqslant \kappa,
\end{aligned}$$

using (6.1) and $\mu_0 \geqslant \kappa$. Altogether, the function η satisfies the conditions (7.4)–(7.6).

Let $I_M\eta$ denote the Hermite cubic interpolate of η with the nodes $x_j = j/M$, $j = 0, 1, \ldots, M$. Then we have

$$\|\varphi - I_M\eta\|_{1,\infty,I} \leqslant \|\varphi-\varphi_\nu\| + \|\varphi_\nu - R_H\varphi_\nu\| + \|R_H\varphi_\nu - \eta\| + \|\eta - I_M\eta\| = \sum_{i=1}^{4} E_i \tag{7.7}$$

where all the norms are in the space $C^{(1)}(I)$. From (7.1) we obtain

$$E_1 \leqslant \frac{1}{2}(3C_1 + C_2)\nu. \tag{7.8}$$

It is well-known that (see e.g. [7])

$$E_2 \leqslant C_0\|\varphi_\nu - R_H\varphi_\nu\|_{2,2,I} \to 0 \quad \text{as } H \to 0+. \tag{7.9}$$

It is readily seen that

$$\begin{aligned}
E_3 &= \Big\|\lambda(S_0 - R_H\varphi_\nu) + 4b\Delta/C_1 - b\Big(\frac{1}{2}+s\Big)\Big\|_{1,\infty,I} \\
&\leqslant \widetilde{C}(\lambda + b) = \widehat{C}(E_1 + E_2 + b(M))
\end{aligned} \tag{7.10}$$

where $\widetilde{C}$, $\widehat{C}$ are independent of ν, H, M.

Finally, we have

$$E_4 \leqslant C_0\|\eta - I_M\eta\|_{2,2,I} \leqslant CM^{-2}|\eta|_{4,2,I} \tag{7.11}$$

where $|\eta|_{4,2,I}$ stands for the seminorm of fourth derivatives and

$$|\eta|_{4,2,I} \leqslant \|(R_H\varphi_\nu)^{(4)}\|_{0,2,I} \tag{7.12}$$

follows from the definition of η.

Combining (7.7) till (7.12), we can prove that

$$\|\varphi - I_M\eta\|_{1,\infty,I} \to 0 \quad \text{as } M \to \infty.$$

(Indeed, we can choose ν_0 such that

$$E_1 < \varepsilon/(16\widehat{C}),$$

and H_0 such that $E_2 < \varepsilon/(16\widehat{C})$, where $\varepsilon > 0$ is arbitrary. We choose M_3 such that

$$\widehat{C}b(M) < \varepsilon/8 \quad \text{for } M > M_3$$

and M_4 such that

$$CM^{-2}\|(R_{H_0}\varphi_{\nu_0})^{(4)}\|_{0,2,I} < \varepsilon/4 \quad \text{for } M > M_4.$$

Then

$$\sum_{i=1}^{4} E_i < \varepsilon \quad \text{for } M > \max\{M_3, M_4\}$$

follows from (7.10)–(7.12), as $\varepsilon/(16\widehat{C}) \leqslant \varepsilon/4$ can be supposed without any loss of generality.)

It remains to prove that

$$\|(I_M\eta)''\|_{0,\infty,I} \leqslant C_2. \tag{7.13}$$

In fact, we can derive this bound for $M \geqslant 3g(H)(2\nu - \nu^2)^{-1}C_2^{-1}$, where $g(H)$ is some function, such that $g(H) \to +\infty$ as $H \to 0+$.

It is easy to derive that

$$\|(I_M\eta)''\|_{0,\infty,I} \leqslant \|\eta''\|_{0,\infty,I} + 3M^{-1}\|\eta'''\|_{0,\infty,I}.$$

On the other hand,

$$\|\eta''\|_{0,\infty,I} \leqslant \|(R_H\varphi_\nu)''\|_{0,\infty,I} \leqslant C_2(1-\nu)^2$$

follows from the definitions of η and φ_ν. Moreover, we have

$$|\eta'''| \leqslant |(R_H\varphi_\nu)'''| \leqslant (\kappa_0 H)^{-1} C_3(H) \int_{-H}^{H} \varphi_\nu(t)\,\mathrm{d}t \leqslant 3kC_3(H)/\kappa_0 \equiv g(H),$$

where $\lim g(H) = +\infty$ as $H \to 0+$, since the kernel $\omega(z,H)$ has continuous third derivatives. Thus we obtain

$$|(I_M\eta)''(s_j\pm)| \leqslant C_2(1-\nu)^2 + 3M^{-1}g(H)$$

and if M satisfies the inequality

$$3M^{-1}g(H) \leqslant C_2(1-(1-\nu)^2) = C_2(2\nu-\nu^2),$$

then (7.13) holds true. □

Lemma 7.2. *Assume that:*
(i) *the set $V \cap [C^\infty(\overline{\Omega})]^3$ is dense in V and*
(ii) *for any $v \in V \cap [C^\infty(\overline{\Omega})]^3$ there exists a sequence $\{v_h\}$, $h \to 0+$, such that $v_h \in V_h$ and $v_h \to v$ in W as $h \to 0+$.*
Then

$$u_h(\varphi) \to u(\varphi) \quad \text{in } W \quad \text{as } h \to 0+$$

holds for any $\varphi \in \widetilde{U}_{ad}$, where $u_h(\varphi)$ is the Galerkin approximation and $u(\varphi)$ is the solution of the problem (1.3).

Proof. follows from the general theorem on the convergence of Ritz-Galerkin approximations [1 – Chapter 4, Theorem 0.6]. We employ the inequalities

$$D^2\Phi(u;v,w) \leqslant C\|v\|_1\|w\|_1 \quad \forall u,v,w \in W,$$
$$D^2\Phi(u,v,v) \geqslant \kappa \int_\Omega e_{ij}(v)e_{ij}(v)\,\mathrm{d}x \geqslant C_0\|v\|_1^2 \quad \forall u \in W, v \in V,$$

(see [5 – §8.2, Lemma 2.1]). □

Remark 7.1. Both assumptions (i) and (ii) are easy to satisfy. We can use standard finite element spaces.

Proposition 7.1. *Let the assumptions of Lemma 7.2 be satisfied. Let $\{\varphi_M\}$, $M \to \infty$, be a sequence of $\varphi_M \in U_{ad}^M$, such that $\varphi_M \to \varphi$ in U, as $M \to \infty$.*

Then there is a subsequence $\{\varphi_{M_n}\}$ and a function

$$\lambda\colon (0, +\infty) \to (0, +\infty) \text{ such that } \lim \lambda(h) = +\infty \text{ as } h \to 0+ \text{ and}$$
$$u_h(\varphi_{M_n}) \to u(\varphi) \text{ in } W \text{ as } h \to 0+ \text{ and } M_n \geqslant \lambda(h).$$

Proof. Consider a fixed subspace W_h. By (6.4) and Lemma 2.1, $\varphi \in \widetilde{U}_{ad}$ follows. Lemma 6.2 and Proposition 6.1 imply that

$$u_h(\varphi_M) \to u_h(\varphi) \text{ in } W \text{ as } M \to \infty.$$

Using Lemma 7.2, we obtain

$$u_h(\varphi) \to u(\varphi) \text{ in } W \text{ as } h \to 0+.$$

As a consequence, we have

$$\|u_h(\varphi_M) - u(\varphi)\|_1 \leqslant \|u_h(\varphi_M) - u_h(\varphi)\|_1 + \|u_h(\varphi) - u(\varphi)\|_1 \to 0$$

as $h \to 0+$ and $M \to +\infty$, $M \geqslant \lambda(h)$ for some function λ, which grows to infinity as $h \to 0+$. □

Theorem 7.1. *Let $\{\varphi_M^0(h)\}$, $h \to 0+$, $M \geqslant \lambda(h)$, be a sequence of solutions of the Approximate Maximization Problems* (6.13), *(where λ is the function from Proposition* 7.1*). Let the assumptions of Lemma* 7.2 *be satisfied.*

Then there exists a subsequence $\{\varphi_{M_n}^0(h_n)\}$ and $\varphi^0 \in U_{ad}$ such that

$$\varphi_{M_n}^0(h_n) \to \varphi^0 \quad \text{in } U, \tag{7.14}$$
$$u_{h_n}(\varphi_{M_n}^0(h_n)) \to u(\varphi^0) \quad \text{in } W, \tag{7.15}$$
$$\Psi(\varphi_{M_n}^0(h_n), u_{h_n}(\varphi_{M_n}^0(h_n))) \to \Psi(\varphi^0, u(\varphi^0)) \tag{7.16}$$

as $M_n \to +\infty$, $h_n \to 0+$, where φ^0 is a solution of the Maximization Problem (4.2).

Proof. Let $\varphi \in U_{ad}$ be arbitrary. Using Lemma 7.1 we find a sequence $\{\varphi_M\}$ such that $\varphi_M \in U_{ad}^M, \varphi_M \to \varphi$ in U as $M \to \infty$.

By definition, we have

$$\Psi(\varphi_M^0(h), u_h(\varphi_M^0(h))) \geqslant \Psi(\varphi_M, u_h(\varphi_M)) \tag{7.17}$$

for all couples (h, M) under consideration. Let us apply (6.4), Lemma 2.1 and Proposition 7.1 to both sides of (7.17). On the left-hand side we can choose a subsequence $\{\varphi^0_{M_n}(h_n)\}$ such that

$$\begin{aligned} \varphi^0_{M_n}(h_n) &\to \varphi^0 \quad \text{in } U,\ \varphi^0 \in \widetilde{U}_{ad}, \\ u_{h_n}(\varphi^0_{M_n}(h_n)) &\to u(\varphi^0) \quad \text{in } W. \end{aligned}$$

Moreover, we can prove that $\varphi^0 \in U_{ad}$, using (6.5) and (6.6).

By virtue of (4.1) we have

$$\Psi(\varphi^0_{M_n}(h_n), u_{h_n}(\varphi^0_{M_n}(h_n))) \to \Psi(\varphi^0, u(\varphi^0)).$$

On the right-hand side of (7.17) we obtain that

$$\begin{aligned} u_{h_n}(\varphi_{M_n}) &\to u(\varphi) \quad \text{in } W, \\ \Psi(\varphi_{M_n}, u_{h_n}(\varphi_{M_n})) &\to \Psi(\varphi, u(\varphi)). \end{aligned}$$

Thus we are led to (7.14), (7.15), (7.16) and to the inequality

$$\Psi(\varphi^0, u(\varphi^0)) \geqslant \Psi(\varphi, u(\varphi)),$$

so that φ^0 is a solution of the problem (4.2). □

Remark 7.1. In practice, (7.16) is the most important result. Indeed, the maximizing data φ^0 are usually not needed, whereas the "safest" value of the functional Ψ is required.

Remark 7.2. In *two-dimensional problems*, we have to modify the conditions (6.1) and the proof of Lemma 7.1 replacing everywhere $(3k/2)$ by (k). In the definition (6.8), the coefficient $(-2/3)$ has to be replaced by (-1). The assumptions (i), (ii) of Lemma 7.2 are prescribed for $V \cap [C^\infty(\overline{\Omega})]^2$.

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