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A RECOVERED GRADIENT METHOD APPLIED TO SMOOTH OPTIMAL SHAPE PROBLEMS

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Summary. A new postprocessing technique suitable for nonuniform triangulations is employed in the sensitivity analysis of some model optimal shape design problems.

Keywords: shape optimization, sensitivity analysis, superconvergence, recovered gradient.

AMS classification: 65N30, 65K10, 49D07

INTRODUCTION

This article intends to contribute to the sensitivity analysis in optimal shape design. A model problem of a single elliptic equation in $\mathbb{R}^2$ with Dirichlet or mixed boundary conditions is considered.

It is well-known (see [9—§2.4]) that the simplest formulae for the gradient of standard cost functionals with respect to the design variables involve integrals of the boundary flux. Thus one needs an efficient numerical method for computing both the solution and the boundary flux. The direct differentiation of the finite element solution, however, does not yield satisfactory results, as it suffers from a significant loss of accuracy. Consequently, a postprocessing has been proposed by many authors (see [16] or [22] for a recent survey of such methods), which recovers the gradient of the simplest finite element approximations.

In the present paper we employ a new technique suitable in case of quasiuniform triangulations [12, 13]. Note that a related method of Lazarov at al. [17, 18] would lead essentially to the second type of sensitivity formulae, called "method of domains" [9—§2.4], where the gradient of the cost functional is represented by an integral over the whole domain. Let us mention two other methods, which can be used in the first type of the sensitivity formulae, as they claim to be more accurate in the boundary flux calculation. The primal hybrid method, based on the theory of Raviart and Thomas [20], has been used by Chleboun and Mäkinen [5,6]. The penalty method with extrapolation of King and Serbin [15] has been employed in [14] by Hlaváček.

The paper is organized as follows. In Section 1 we introduce a class of admissible domains, several elliptic state problems and cost functionals. Sensitivity formulae are derived in Section 2 for three standard cost functionals. Section 3 is devoted to the discretization by linear finite elements and to the definition of approximate optimization problems. In the final section we recall the technique of recovered gradients on quasiuniform triangulations and propose its application in the sensitivity formulae.

1. Optimal shape problems

An admissible domain (Fig. 1)

$$\Omega(\theta) = \{(x_1, x_2) \in \mathbb{R}^2;\ 0 < x_2 < \theta(x_1),\ 0 < x_1 < 1\}$$

is controlled by the design function θ which belongs to a set $\mathcal{U}_{ad}$ of *admissible design functions* (see (1.2)).

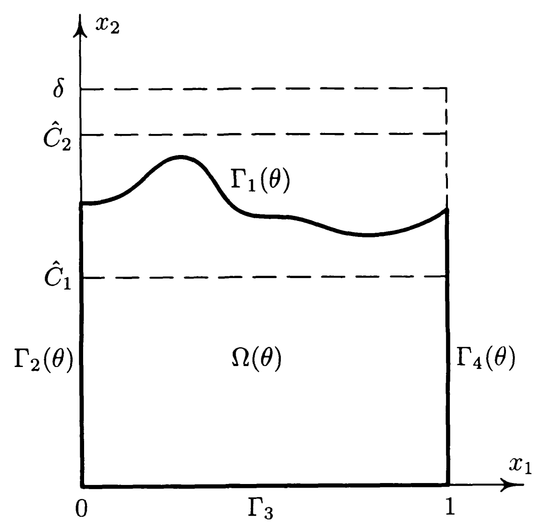

Fig. 1: An admissible domain $\Omega(\theta)$.

Let $n > 1$ be a fixed integer. Like in the paper [6], we define the following set of *admissible design variables*

$$(1.1) \qquad Q = \Big\{ \alpha \in \mathbb{R}^{n+1}\,;\ \hat{C}_1 \leqslant \alpha_i \leqslant \hat{C}_2, i = 0, \dots, n,$$
$$|\alpha_{i+1} - \alpha_i| \leqslant \hat{C}_3/n,\ i = 0, \dots, n-1,\ \sum_{i=0}^{n} \alpha_i = \hat{C}_4(n+1) \Big\},$$

where $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$ and $\hat{C}_4$ are given positive parameters. Then the design functions will be defined as Bézier curves [1]

$$(1.2) \qquad \theta(s) = F(\alpha)(s) = \sum_{i=0}^{n} \alpha_i \beta_i^n(s), \quad \beta_i^n(s) = \binom{n}{i} s^i (1-s)^{n-i}.$$

The design variables are the x_2-coordinates of the control points

$$\{B_i\}_{i=0}^{n}, \quad B_i = (\tfrac{i}{n}, \alpha_i) \in \mathbb{R}^2.$$

We set

$$\mathcal{U}_{ad} = F(Q)$$

where the mapping $F\colon \mathbb{R}^{n+1} \to C([0,1])$ is determined by means of the formula (1.2).

It is easy to verify, that for any integer $N \geqslant n$ and any $\theta \in \mathcal{U}_{ad}$ a *unique* $\alpha_\theta \in Q$ exists such that

$$F(\alpha_\theta)(j/N) = \theta(j/N), \quad j = 0, 1, \dots, N.$$

Moreover, we can prove that

$$(1.3) \qquad \mathcal{U}_{ad} \subset \hat{\mathcal{U}}_{ad} \equiv \Big\{ \theta \in C^{(0),1}([0,1])\,;\ \hat{C}_1 \leqslant \theta(x_1) \leqslant \hat{C}_2,$$
$$|\theta'(x_1)| \leqslant \hat{C}_3,\ \int_0^1 \theta(x_1)\,\mathrm{d}x_1 = \hat{C}_4 \Big\},$$

where $C^{(0),1}([0,1])$ denotes the space of Lipschitz-continuous functions and the prime denotes $(\mathrm{d}/\,\mathrm{d}x_1)$.

Indeed, since any Bézier curve is contained in the convex hull of its control points it follows that

$$\hat{C}_1 \leqslant \min_{0 \leqslant i \leqslant n} \{\alpha_i\} \leqslant F(\alpha)(s) \leqslant \max_{0 \leqslant i \leqslant n} \{\alpha_i\} \leqslant \hat{C}_2, \quad \forall s \in [0,1].$$

Differentiating $F(\alpha)$ with respect to s, we find that

$$F(\alpha)'(s) = \sum_{i=0}^{n-1} \gamma_i \beta_i^{n-1}(s), \quad \gamma_i = n(\alpha_{i+1} - \alpha_i).$$

Thus $F(\alpha)'$ is also a Bézier curve with control points $(i/(n-1), \gamma_i)$, $i = 0, \dots, n-1$ and

$$|F(\alpha)'(s)| \leqslant \max_{0 \leqslant i \leqslant n-1} \{n|\alpha_{i+1} - \alpha_i|\} \leqslant \hat{C}_3.$$

Integrating the basis functions, we obtain for all $i = 0, \dots, n$

$$\int_0^1 \beta_i^n(s)\,\mathrm{d}s = 1/(n+1)$$

and finally

$$\int_0^1 F(\alpha)(s)\,\mathrm{d}s = \sum_{i=0}^{n} \alpha_i \int_0^1 \beta_i^n(s)\,\mathrm{d}s = (n+1)^{-1} \sum_{i=0}^{n} \alpha_i = \hat{C}_4.$$

Remark 1.1. Instead of Bézier curves we could use B-splines as well. For details see [1] and [3]. □

We shall consider a few *state problems* based on elliptic boundary value problems with a symmetric matrix $A = (a_{ij})_{i,j=1}^2$, $a_{ij} \in W_2^2(\Omega_\delta)$ and the right-hand side $f \in L^2(\Omega_\delta)$. Henceforth we denote $W_p^k(\Omega)$ the standard Sobolev space of functions the derivatives of which in the sense of distributions up to the order k belong to the space $L^p(\Omega)$; $\Omega_\delta = (0,1) \times (0,\delta)$ with $\delta > \hat{C}_2$.

We suppose that a constant $a_0 > 0$ exists such that

$$A(x)t \cdot t \geqslant a_0 \|t\|^2 \quad \forall x \in \Omega_\delta,\ \forall t \in \mathbb{R}^2. \tag{1.4}$$

We subdivide the boundary $\partial\Omega(\theta)$ into four parts corresponding to the four sides of the "generalized rectangle" $\partial\Omega(\theta)$, i.e., $\partial\Omega(\theta) = \Gamma_1(\theta) \cup \Gamma_2(\theta) \cup \Gamma_3 \cup \Gamma_4(\theta)$ (see Fig. 1).

Let us introduce the bilinear form

$$a(\theta; y, w) = \int_{\Omega(\theta)} A\nabla y \cdot \nabla w\,\mathrm{d}x, \quad y, w \in W_2^1(\Omega(\theta)),$$

and the subspace

$$H_0^1(\Omega(\theta)) = \{w \in W_2^1(\Omega(\theta)); w = 0 \text{ on } \partial\Omega(\theta)\}.$$

The *first state problem*: Find a function $y(\theta) \in H_0^1(\Omega(\theta))$ such that

$$a(\theta; y(\theta), w) = \int_{\Omega(\theta)} fw\,\mathrm{d}x \quad \forall w \in H_0^1(\Omega(\theta)). \tag{1.5$_1$}$$

Let us introduce both another bilinear form

$$b(\theta; y, w) = a(\theta; y, w) + \int_{\Gamma_2(\theta)\cup\Gamma_4(\theta)} \sigma y w \,\mathrm{d}\gamma$$

where $\sigma \in L^\infty(\hat{\Gamma}_2 \cup \hat{\Gamma}_4)$ is a given function, $\hat{\Gamma}_2 = \{(x_1, x_2);\, x_1 = 0, x_2 \in (0, \delta)\}$, $\hat{\Gamma}_4 = \{(x_1, x_2);\, x_1 = 1, x_2 \in (0, \delta)\}$, $\sigma \geqslant 0$, and the following subspace

$$V(\theta) = \{w \in W_2^1(\Omega(\theta));\, w = 0 \text{ on } \Gamma_1(\theta) \cup \Gamma_3\}.$$

Then we may define the *second state problem*: Find $y(\theta) \in V(\theta)$ such that

$$(1.5_2) \qquad a(\theta; y(\theta), w) = \int_{\Omega(\theta)} f w \,\mathrm{d}x + \int_{\Gamma_2(\theta)\cup\Gamma_4(\theta)} g w \,\mathrm{d}\gamma \quad \forall w \in V(\theta),$$

where $g \in L^2(\hat{\Gamma}_2 \cup \hat{\Gamma}_4)$ is a given function, and the *third state problem*: Find $y(\theta) \in V(\theta)$ such that

$$(1.5_3) \qquad b(\theta; y(\theta), w) = \int_{\Omega(\theta)} f w \,\mathrm{d}x + \int_{\Gamma_2(\theta)\cup\Gamma_4(\theta)} g w \,\mathrm{d}\gamma \quad \forall w \in V(\theta).$$

Remark 1.2. The problems (1.5_i), $i = 1, 2, 3$, express a weak formulation of the equation $-\operatorname{div}(A\nabla y(\theta)) = f$ with homogeneous Dirichlet boundary conditions, mixed Dirichlet (on $\Gamma_1(\theta) \cup \Gamma_3$) and Neumann (on $\Gamma_2(\theta) \cup \Gamma_4(\theta)$) or mixed Dirichlet (on $\Gamma_1(\theta) \cup \Gamma_3$) and Newton (on $\Gamma_2(\theta) \cup \Gamma_4(\theta)$) boundary conditions, respectively.

Since the bilinear forms a and b are $V(\theta)$-elliptic, the above mentioned problems are uniquely solvable. □

We choose the following cost functionals

$$(1.6) \qquad J_1(\theta) = \int_{\Omega(\theta)} (y(\theta) - y_p)^2 \,\mathrm{d}x,$$

$$(1.7) \qquad J_2(\theta) = \int_{\Omega(\theta)} f y(\theta) \,\mathrm{d}x,$$

$$(1.8) \qquad J_3(\theta) = \int_{\Omega(\theta)} |\nabla y(\theta)|^2 \,\mathrm{d}x,$$

where $y(\theta)$ is the solution of some of the problems (1.5_i) and $y_p \in L^2(\Omega_\delta)$ is a given function.

Finally, we define *the optimization problems* P_{ij}, $i, j \in \{1, 2, 3\}$: Find $\alpha^{ij}_{\mathrm{opt}} \in Q$ such that

$$(1.9) \qquad \alpha^{ij}_{\mathrm{opt}} = \operatorname*{argmin}_{\alpha \in Q} J_j(F(\alpha)),$$

where $y(F(\alpha))$ solves the state problem (1.5_i) for $\theta = F(\alpha)$.

In what follows, we use extensions of functions $y \in V(\theta)$ by zero in $\Omega(\delta) - \Omega(\theta)$, so that the extensions belong to $V(\delta)$. For brevity, we shall not denote the extended functions by new symbols.

Proposition 1.3. *The optimization problem P_{ij}, $i, j \in \{1,2,3\}$, has at least one solution.*

Proof. It is readily seen that the set Q is compact in $\mathbb{R}^{n+1}$. If $\alpha_k \to \alpha$ as $k \to \infty$, then $\theta_k \equiv F(\alpha_k) \to F(\alpha) \equiv \theta$ in $C([0,1])$. Following step by step the ideas of e.g. the proof of Lemma 2.1 in [10], we can show that the corresponding solutions of (1.5_i) (extensions)

$$y(\theta_k) \to y(\theta) \quad \text{in } W_2^1(\Omega_\delta). \tag{1.10}$$

Any of the cost functionals J_j, $j \in \{1,2,3\}$, is continuous as a mapping from $W_2^1(\Omega_\delta)$ into $\mathbb{R}$, so that

$$J_j(F(\alpha_k)) \to J_j(F(\alpha)), \quad \text{as } k \to \infty,$$

follows from (1.10). We conclude that $J_j(F(\cdot))$ attains its minimum on Q, being continuous on a compact subset of $\mathbb{R}^{n+1}$. □

2. Sensitivity analysis

To give the reader an idea of sensitivity formulae we compute the Gâteaux differential of cost functionals. For simplicity, let us assume that the coefficients a_{ij} are constants. We use the method of material derivative [9], [21] and suppose $v \in \mathbb{R}^2$ is a material velocity field, $v = (0, v_2)$, $v_2 = 0$ on Γ_3 and

$$v_2(x_1, x_2) = x_2 \frac{\hat\theta(x_1)}{\theta(x_1)}, \quad (x_1, x_2) \in \Omega(\theta),$$

where $\hat\theta \in C^{(0),1}([0,1])$. For brevity, θ is mostly omitted, e.g. $y \equiv y(\theta)$, $\Omega \equiv \Omega(\theta)$, $V = V(\theta)$.

Let us consider the *first state problem* (1.5_1).

The adjoint equation to (1.6) is (see [9—(3.3.13)])

$$\int_\Omega A\nabla z \cdot \nabla w \,\mathrm{d}x = \int_\Omega 2(y - y_p) w \,\mathrm{d}x \quad \forall w \in H_0^1(\Omega),$$

where $z \in H_0^1(\Omega)$. According to [9–(3.3.17)],

$$(2.1)\quad DJ_1(\theta,\hat\theta) = \lim_{t\to 0}\frac{J_1(\theta+t\hat\theta)-J_1(\theta)}{t}$$
$$= \int_\Omega [A\nabla y\cdot\nabla(\nabla z\cdot v) - f\nabla z\cdot v + A\nabla(\nabla y\cdot v)\cdot\nabla z - 2(y-y_p)\nabla y\cdot v]\,\mathrm{d}x$$
$$+ \int_{\partial\Omega}[(y-y_p)^2 + fz - A\nabla y\cdot\nabla z]v\cdot\nu\,\mathrm{d}\gamma,$$

where ν denotes the unit outward normal to the boundary. Some more regularity for the functions y, z, f and y_p is required to justify the formula (2.1) and next calculations.

We know that

$$(2.2)\quad \int_{\partial\Omega}\frac{\partial y}{\partial\nu_A}\nabla z\cdot v\,\mathrm{d}\gamma = \int_\Omega A\nabla y\cdot\nabla(\nabla z\cdot v)\,\mathrm{d}x - \int_\Omega f\nabla z\cdot v\,\mathrm{d}x,$$
$$(2.3)\quad \int_{\partial\Omega}\frac{\partial z}{\partial\nu_A}\nabla y\cdot v\,\mathrm{d}\gamma = \int_\Omega A\nabla z\cdot\nabla(\nabla y\cdot v)\,\mathrm{d}x - \int_\Omega 2(y-y_p)(\nabla y\cdot v)\,\mathrm{d}x,$$

where $\partial/\partial\nu_A \equiv \sum_{i,j=1}^2 a_{ij}\nu_i\partial/\partial x_j$. Substituting (2.2) and (2.3) into (2.1), we obtain

$$DJ_1(\theta,\hat\theta) = \int_{\partial\Omega}\Big[\frac{\partial y}{\partial\nu_A}\nabla z\cdot v + \frac{\partial z}{\partial\nu_A}\nabla y\cdot v + \big(fz - A\nabla y\cdot\nabla z + (y-y_p)^2\big)v\cdot\nu\Big]\,\mathrm{d}\gamma.$$

The vanishing of y and z on $\partial\Omega$ implies a direct simplification of DJ_1 and

$$(2.4)\qquad \nabla y\,|_{\partial\Omega} = \frac{\partial y}{\partial\nu}\nu,\quad \nabla z\,|_{\partial\Omega} = \frac{\partial z}{\partial\nu}\nu.$$

Then, since A is symmetric, we arrive at

$$(2.5)\qquad DJ_1(\theta,\hat\theta) = \int_{\partial\Omega}\Big[\frac{\partial y}{\partial\nu}\frac{\partial z}{\partial\nu_A} + y_p^2\Big]v\cdot\nu\,\mathrm{d}\gamma = \int_0^1\Big(\frac{\partial y}{\partial\nu}\frac{\partial z}{\partial\nu_A} + y_p^2\Big)\hat\theta\,\mathrm{d}x_1.$$

The adjoint equation to (1.7) is

$$\int_\Omega A\nabla z\cdot\nabla w\,\mathrm{d}x = \int_\Omega fw\,\mathrm{d}x \quad \forall w\in H_0^1(\Omega),$$

where $z \in H_0^1(\Omega)$. Obviously, $z = y$. According to [9—(3.3.17)],

$$DJ_2(\theta,\hat\theta) = \int_\Omega[2A\nabla y\cdot\nabla(\nabla y\cdot v) - 2f\nabla y\cdot v]\,\mathrm{d}x + \int_{\partial\Omega}[2fy - A\nabla y\cdot\nabla y]v\cdot\nu\,\mathrm{d}\gamma.$$

Using the formula (2.2) for $z = y$, the boundary condition and (2.4), we arrive at

$$DJ_2(\theta, \hat{\theta}) = \int_{\partial\Omega} (A\nu \cdot \nu)\Big(\frac{\partial y}{\partial \nu}\Big)^2 v \cdot \nu \, d\gamma = \int_0^1 (A\nu \cdot \nu)\Big(\frac{\partial y}{\partial \nu}\Big)^2 \hat{\theta} \, dx_1. \tag{2.6}$$

The adjoint equation to (1.8) is

$$\int_\Omega A\nabla z \cdot \nabla w \, dx = 2 \int_\Omega \nabla y \cdot \nabla w \, dx \quad \forall w \in H_0^1(\Omega),$$

where $z \in H_0^1(\Omega)$. Obviously,

$$\nabla z = 2A^{-1}\nabla y. \tag{2.7}$$

According to [9—(3.3.17)],

$$\begin{aligned} DJ_3(\theta, \hat{\theta}) = \int_\Omega [&A\nabla y \cdot \nabla(\nabla z \cdot v) - f\nabla z \cdot v + A\nabla(\nabla y \cdot v) \cdot \nabla z \\ &- 2\nabla y \cdot \nabla(\nabla y \cdot v)] \, dx + \int_{\partial\Omega} [|\nabla y|^2 + fz - A\nabla y \cdot \nabla z] v \cdot \nu \, d\gamma. \end{aligned}$$

From (2.2), (2.7), symmetry of A and (2.4) we obtain

$$DJ_3(\theta, \hat{\theta}) = \int_0^1 \Big(\frac{\partial y}{\partial \nu}\Big)^2 \hat{\theta} \, dx_1. \tag{2.8}$$

The link with control variables is given by the formulae

$$\frac{\partial J_j}{\partial \alpha_k}\big(F(\alpha)\big) = DJ_j\big(F(\alpha), F'(\alpha, \varepsilon_k)\big), \quad k = 0, 1, \ldots, n, \ j = 1, 2, 3, \tag{2.9}$$

where $F'(\alpha, \varepsilon_k)$ is the derivative of the mapping F at the point α in the direction of the unit $\mathbb{R}^{n+1}$—coordinate vector ε_k, i.e.

$$F'(\alpha, \varepsilon_k) = \beta_k^n.$$

Next let us consider the *second state* problem (1.5_2).

The adjoint equation to (1.6) is

$$\int_\Omega A\nabla z \cdot \nabla w \, dx = \int_\Omega 2(y - y_p) w \, dx \quad \forall w \in V, \tag{2.10}$$

where $z \in V$. Following the derivation of (3.3.17) in [9], we arrive at

$$DJ_1(\theta, \hat{\theta}) = (2.1) + B(\hat{\theta}, y, z),$$

where (2.1) denotes the right-hand side of the equation (2.1) and

$$\text{(2.11)} \qquad B(\hat\theta, y, z) = \frac{\hat\theta(0)}{\theta(0)} \int_0^{\theta(0)} \Big(\frac{\mathrm{d}g}{\mathrm{d}x_2} x_2 + g\Big) z \,|_{x_1=0}\, \mathrm{d}x_2 + \frac{\hat\theta(1)}{\theta(1)} \int_0^{\theta(1)} \Big(\frac{\mathrm{d}g}{\mathrm{d}x_2} x_2 + g\Big) z \,|_{x_1=1}\, \mathrm{d}x_2.$$

Since (2.2) and (2.3) hold, inserting the homogeneous boundary conditions on $\Gamma_1 \cup \Gamma_3$, we obtain

$$DJ_1(\theta, \hat\theta) = \int_0^1 \Big(\frac{\partial z}{\partial \nu_A}\frac{\partial y}{\partial \nu} + y_p^2\Big)\hat\theta \,\mathrm{d}x_1 + C(\hat\theta, y, z),$$

where

$$\text{(2.12)} \quad C(\hat\theta, y, z) = \frac{\hat\theta(0)}{\theta(0)} \int_0^{\theta(0)} \Big[\Big(g - \frac{\partial y}{\partial \nu_A}\Big) + x_2 \frac{\mathrm{d}}{\mathrm{d}x_2}\Big(g - \frac{\partial y}{\partial \nu_A}\Big)\Big] z \,|_{x_1=0}\, \mathrm{d}x_2 + \frac{\hat\theta(1)}{\theta(1)} \int_0^{\theta(1)} [\ldots] z \,|_{x_1=1}\, \mathrm{d}x_2.$$

Assuming a sufficient regularity of the solution y, z and of the data, we conclude that $C(\hat\theta, y, z) = 0$, since

$$\frac{\partial y}{\partial \nu_A} = g \quad \text{and} \quad \frac{\partial z}{\partial \nu_A} = 0 \quad \text{on } \Gamma_2 \cup \Gamma_4.$$

Consequently, we have again

$$\text{(2.13)} \qquad DJ_1(\theta, \hat\theta) = \int_0^1 \Big(\frac{\partial y}{\partial \nu}\frac{\partial z}{\partial \nu_A} + y_p^2\Big)\hat\theta \,\mathrm{d}x_1,$$

like in (2.5) for the first state problem.

The adjoint equation to (1.7) is

$$\text{(2.14)} \qquad \int_\Omega A\nabla z \cdot \nabla w \,\mathrm{d}x = \int_\Omega f w \,\mathrm{d}x \quad \forall w \in V,$$

where $z \in V$. Arguing as before, we arrive at

$$\text{(2.15)} \qquad DJ_2(\theta, \hat\theta) = \int_0^1 \frac{\partial y}{\partial \nu}\frac{\partial z}{\partial \nu_A}\hat\theta \,\mathrm{d}x_1.$$

The adjoint equation to (1.8) is

$$\text{(2.16)} \qquad \int_\Omega A\nabla z \cdot \nabla w \,\mathrm{d}x = 2\int_\Omega \nabla y \cdot \nabla w \,\mathrm{d}x \quad \forall w \in V,$$

where $z \in V$. Obviously, (2.7) holds again. Following the previous argument, we arrive at the formula (2.8).

Finally, let us consider the *third state problem* (1.5_3).

The adjoint equation to (1.6) is

$$\int_\Omega A\nabla z \cdot \nabla w \, \mathrm{d}x + \int_{\Gamma_2 \cup \Gamma_4} \sigma z w \, \mathrm{d}\gamma = 2\int_\Omega (y - y_p) w \, \mathrm{d}x \quad \forall w \in V,$$

where $z \in V$.

In the same way, as before, we derive

$$DJ_1(\theta, \hat{\theta}) = \int_0^1 \Big(\frac{\partial y}{\partial \nu}\frac{\partial z}{\partial \nu_A} + y_p^2\Big)\hat{\theta} \, \mathrm{d}x_1 + E(\hat{\theta}, y, z),$$

where

$$\begin{aligned} E(\hat{\theta}, y, z) = \frac{\hat{\theta}(0)}{\theta(0)} \int_0^{\theta(0)} \Big[& \frac{\partial y}{\partial \nu_A}\frac{\partial z}{\partial x_2} x_2 + \frac{\partial z}{\partial \nu_A}\frac{\partial y}{\partial x_2} x_2 \\ & - yz\frac{\mathrm{d}}{\mathrm{d}x_2}(x_2\sigma) + z\frac{\mathrm{d}}{\mathrm{d}x_2}(x_2 g)\Big]_{x_1=0} \mathrm{d}x_2 \\ + \frac{\hat{\theta}(1)}{\theta(1)} \int_0^{\theta(1)} & [\ldots]_{x_1=1} \, \mathrm{d}x_2. \end{aligned}$$

Assuming a sufficient regularity of y, z, σ and g, we conclude that $E(\hat{\theta}, y, z)$ vanishes, by virtue of the conditions

$$\frac{\partial y}{\partial \nu_A} + \sigma y = g, \quad \frac{\partial z}{\partial \nu_A} + \sigma z = 0 \quad \text{on } \Gamma_2 \cup \Gamma_4.$$

Consequently, we arrive at (2.13) again.

The adjoint equation to (1.7) is

$$\int_\Omega A\nabla z \cdot \nabla w \, \mathrm{d}x + \int_{\Gamma_2 \cup \Gamma_4} \sigma z w \, \mathrm{d}\gamma = \int_\Omega f w \, \mathrm{d}x \quad \forall w \in V,$$

where $z \in V$. The same calculations as in the previous case lead to the formula (2.15).

The adjoint equation to (1.8) is

$$\int_\Omega A\nabla z \cdot \nabla w \, \mathrm{d}x + \int_{\Gamma_2 \cup \Gamma_4} \sigma z w \, \mathrm{d}\gamma = 2\int_\Omega \nabla y \cdot \nabla w \, \mathrm{d}x \quad \forall w \in V,$$

where $z \in V$. Then

$$- \operatorname{div}(A\nabla z) = -2\Delta y \quad \text{in } \Omega,$$
$$\frac{\partial z}{\partial \nu_A} + \sigma z = 2\frac{\partial y}{\partial \nu} \quad \text{on } \Gamma_2 \cup \Gamma_4,$$

follows, provided the solutions z and y are sufficiently regular. Using this and the previous approach, we arrive at the formula

$$DJ_3(\theta, \hat{\theta}) = \int_0^1 \Big[\frac{\partial y}{\partial \nu_A}\frac{\partial z}{\partial \nu} - \Big(\frac{\partial y}{\partial \nu}\Big)^2\Big]\hat{\theta}\,\mathrm{d}x_1. \tag{2.17}$$

3. Discretization

Passing to numerical analysis of the optimal design problems, we introduce triangulations of admissible domains. Let N be a positive integer and $h = 1/N$. Denoting by e_j the subintervals $[(j-1)h, jh]$, $j = 1, \ldots, N$, we consider the set

$$U^h = \{\theta_h \in C^{(0),1}([0,1]);\, \theta_h|_{e_j} \in P_1(e_j),\ j = 1, \ldots, N\},$$

where $P_1(e_j)$ is the set of linear functions on e_j. Let us define the following mapping $F_h \colon Q \to U^h$ by means of the relations

$$F_h(\alpha)(jh) = F(\alpha)(jh), \quad j = 0, 1, \ldots, N.$$

We approximate the set U_{ad} of admissible design functions by the set

$$\mathcal{U}_{ad}^h = \{\theta_h \in U^h;\, \exists \alpha \in Q,\ F_h(\alpha) = \theta_h\}.$$

For any given $\theta_h \in \mathcal{U}_{ad}^h$, there exists a unique control variable α, such that $F_h(\alpha) = \theta_h$, if h is sufficiently small $(h \leqslant 1/n)$.

Having the discretization of a control function, we subdivide the domain $\Omega(\theta_h)$ making two demands on a triangulation.

(3.1) Any domain $\Omega(\theta_h)$, $\theta_h \in \mathcal{U}_{ad}^h$, is unambiguously related to a triangulation $\mathcal{T}_h(\theta_h)$ belonging to a family $\mathcal{T} = \{\mathcal{T}_h(\theta_h);\, \theta_h \in \mathcal{U}_{ad}^h,\ h \to 0_+\}$.

(3.2) The family $\mathcal{T}$ of triangulations is strongly regular, i.e., there exist constants $C_1 > 0$, $C_2 > 0$ independent of h and θ_h, such that for any triangle $K \in \mathcal{T}_h(\theta_h)$

$$C_1 h^2 \leqslant \operatorname{meas}(K), \quad \operatorname{diam}(K) \leqslant C_2 h.$$

A triangular mesh satisfying (3.1), (3.2) can be constructed as follows (see Fig. 2). The x_1-coordinates of nodes are uniquely determined by the end points of the segments e_j. Let $M > 1$ be the number of nodes in the x_2-direction. If we denote by $x_2^i(s)$ the x_2-coordinate of the i-th mesh node which has s for x_1-coordinate, we define

$$x_2^i(s) = \frac{i-1}{M-1} F_h(\alpha)(s), \quad i = 1, \dots, M. \tag{3.3}$$

So the position of each node depends on $F_h(\alpha)(s)$, $s \in [0,1]$. For more details we refer to [10], [11], where strongly regular "uniform slope" triangulations are constructed.

However, a uniform slope mesh can contain "badly shaped" triangles, i.e., triangles with obtuse inner angle. To avoid this source of numerical inaccuracy, we shall use "chevron meshes" like on Fig. 2, see [19].

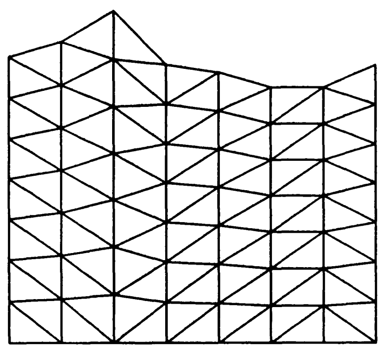

Fig. 2: A chevron mesh.

These meshes can be created by means of the following simple algorithm:

a) for fixed h and θ_h positions of mesh nodes are computed (see (3.3)) and the corresponding quadrilateral mesh is constructed;

b) let a strip of quadrangles be denoted by I_j,

$$I_j = \{(x_1, x_2) \in \mathbb{R}^2;\ (j-1)h \leqslant x_1 \leqslant jh,\ 0 \leqslant x_2 \leqslant \theta_h(x_1)\},\ j = 1, \dots, N.$$

For any quadrangle in I_j, there are two possible diagonals subdividing it into two triangles. The respective slopes of the two diagonals are always different, i.e., one (denoted by "+" in what follows) is greater than the other ("−"). To subdivide a quadrangle, the diagonal with

ba) the "+" slope is chosen iff $\theta_h\big((j-1)h\big) > \theta_h(jh)$,

bb) the "−" slope is chosen iff $\theta_h\big((j-1)h\big) < \theta_h(jh)$, bc) the slope sign taken over from the strip I_{j-1} is chosen iff $\theta_h((j-1)h) = \theta_h(jh)$ for $j > 1$;

c) if $\theta_h(0) = \theta_h(h)$, then diagonals with "+" slopes are used in I_1.

To discretize the state problems (1.5_i), $i = 1, 2, 3$, we apply the finite element method. To this end we introduce the spaces

$$\begin{aligned} X_h(\Omega(\theta_h)) &= \{w_h \in C(\overline{\Omega(\theta_h)})\,;\, w_h|_K \in P_1(K)\ \forall K \in \mathcal{T}_h(\theta_h)\} \subset W_2^1(\Omega(\theta_h)), \\ V_h(\Omega(\theta_h)) &= \{w_h \in X_h(\Omega(\theta_h))\,;\, w_h = 0 \text{ on } \Gamma_1(\theta_h) \cup \Gamma_3\}, \\ H_{0h}^1(\Omega(\theta_h)) &= \{w_h \in X_h(\Omega(\theta_h))\,;\, w_h = 0 \text{ on } \partial\Omega(\theta_h)\}. \end{aligned}$$

The approximate state problems are derived directly from the equations (1.5_i). We only substitute a finite dimensional subspace H_{0h}^1 or V_h for the space $H_0^1(\Omega(\theta_h))$ or $V(\theta_h)$.

The *first approximate state problem*: find $y_h(\theta_h) \in H_{0h}^1(\Omega(\theta_h))$ such that

$$(3.4_1) \qquad a(\theta_h; y_h(\theta_h), w_h) = \int_{\Omega(\theta_h)} f w_h \,\mathrm{d}x \quad \forall w_h \in H_{0h}^1(\Omega(\theta_h)).$$

The *second approximate state problem*: find $y_h(\theta_h) \in V_h(\Omega(\theta_h))$ such that

$$(3.4_2) \quad a(\theta_h; y_h(\theta_h), w_h) = \int_{\Omega(\theta_h)} f w_h \,\mathrm{d}x + \int_{\Gamma_2(\theta_h)\cup\Gamma_4(\theta_h)} g w_h \,\mathrm{d}\gamma \ \forall w_h \in V_h(\Omega(\theta_h)).$$

The *third approximate state problem*: find $y_h(\theta_h) \in V_h(\Omega(\theta_h))$ such that

$$(3.4_3) \quad b(\theta_h; y_h(\theta_h), w_h) = \int_{\Omega(\theta_h)} f w_h \,\mathrm{d}x + \int_{\Gamma_2(\theta_h)\cup\Gamma_4(\theta_h)} g w_h \,\mathrm{d}\gamma \ \forall w_h \in V_h(\Omega(\theta_h)).$$

The approximate domain optimization problem $\mathcal{P}_h^{ij}$ for fixed h and $i, j \in \{1, 2, 3\}$: find $\alpha_{\mathrm{opt},h}^{ij} \in Q$ such that

$$\alpha_{\mathrm{opt},h}^{ij} = \operatorname*{argmin}_{\alpha\in Q} J_{hj}(F_h(\alpha)),$$

where the solution of (3.4_i) is used to compute $J_{hj}(F_h(\alpha))$ (i.e., we substitute $y_h(\theta_h)$ for $y(\theta_h)$ in (1.6)–(1.8), $\theta_h \equiv F_h(\alpha)$).

The function $y_h(\theta_h)$ can be identified with a vector which solves a linear algebraic system related to (3.4_i). It can be shown that both the matrix of the system and $y_h(\theta_h)$ depend continuously on design variables α. Thus the problem $\mathcal{P}_h^{ij}$ can be converted to a problem of the minimization of a continuous function on a compact set.

Lemma 3.1. *Let a sequence $\{\theta_h\}$, $h \to 0_+$, $\theta_h \in \mathcal{U}_{ad}^h$, converge in $C([0,1])$ to a function θ. Then $\theta \in \mathcal{U}_{ad}$ and*

$$\lim_{h\to 0_+} \|y_h(\theta_h) - y(\theta)\|_{1,\Omega_\delta} = 0,$$

where $y_h(\theta_h)$ and $y(\theta)$ solve (3.4_i) and (1.5_i), respectively.

Proof is standard—see e.g. [2]. □

As a consequence we deduce that under the assumptions of Lemma 3.1

$$\lim_{h\to 0_+} J_{hj}(\theta_h) = J_j(\theta), \quad i,j = 1,2,3.$$

Theorem 3.2. *Let $\{\alpha^{ij}_{\mathrm{opt},h}\}$, $h \to 0_+$, be a sequence of solutions of the approximate domain optimization problems $\mathcal{P}^{ij}_h$, $i,j \in \{1,2,3\}$. Then a subsequence $\{\alpha^{ij}_{\mathrm{opt},\hat{h}}\}$ exists such that*

$$\begin{gathered} \alpha^{ij}_{\mathrm{opt},\hat{h}} \to \alpha^{ij}_{\mathrm{opt}} \quad \textit{in } \mathbb{R}^{n+1}, \\ \theta^{ij}_{\mathrm{opt},\hat{h}} \equiv F_{\hat{h}}(\alpha^{ij}_{\mathrm{opt},\hat{h}}) \to \theta^{ij}_{\mathrm{opt}} \equiv F(\alpha^{ij}_{\mathrm{opt}}) \quad \textit{in } C([0,1]), \\ y_{\hat{h}}(\theta^{ij}_{\mathrm{opt},\hat{h}}) \to y(\theta^{ij}_{\mathrm{opt}}) \quad \textit{in } W_2^1(\Omega_\delta) \end{gathered}$$

hold as $\hat{h} \to 0_+$, where $y_{\hat{h}}(\theta^{ij}_{\mathrm{opt},\hat{h}})$ solves (3.4_i) on $\Omega(\theta^{ij}_{\mathrm{opt},\hat{h}})$, $y(\theta^{ij}_{\mathrm{opt}})$ solves (1.5_i) on $\Omega(\theta^{ij}_{\mathrm{opt}})$ and $\alpha^{ij}_{\mathrm{opt}}$ is a solution of the problem $\mathcal{P}_{ij}$ (1.9).

Proof. The theorem can be proved by a slight modification of the proof of [2—Theorem 7.1]. □

4. Recovered gradient

In this section we shall propose how a superconvergent postprocessing technique can be incorporated into a shape optimization process. Our approach is based on some results of Levine [19], Hlaváček, Křížek and Pištora [12], [13].

Above all, we should use triangulations which can be obtained from a uniform reference mesh via W^2_∞-diffeomorphism. The reference mesh is either a chevron mesh or a uniform slope one on the unit square $S = [0,1]^2$, generated by triangulation of a uniform square grid.

Meshes described in Section 3 comply with this requirement. Indeed, if we consider a Bézier curve $\theta \in \mathcal{U}_{ad}$ and if we put

$$\Phi(X_1, X_2) = \big(X_1, X_2\theta(X_1)\big), \tag{4.1}$$

then $\Phi(S) = \overline{\Omega(\theta)}$ and the nodes of the mesh $\mathcal{T}_h(\theta_h)$ are the Φ-images of the reference mesh-nodes on S. The mapping Φ can be extended (e.g. via the formula (4.1)) to a mapping $\hat{\Phi}$ defined on a domain $\hat{\Omega}$, $S \subset \hat{\Omega}$. A proper extension $\hat{\Phi}$ is a W^2_∞-diffeomorphism, that is, $\hat{\Phi}$ is injective, $\hat{\Phi}(\hat{\Omega})$ is an open set, $\hat{\Phi}$ and $\hat{\Phi}^{-1}$ belong to W^2_∞ on $\hat{\Omega}$ and $\hat{\Phi}(\hat{\Omega})$, respectively.

The recovered gradient method enables us to compute a better approximation of $\nabla y(\theta_h)$ than a direct differentiation of the finite element solution $y_h(\theta_h)$ can give. Since only the derivatives $\partial y/\partial \nu$, $\partial y/\partial \nu_A$, $\partial z/\partial \nu$ and $\partial z/\partial \nu_A$ occur in the sensitivity formulae (2.5), (2.6), (2.8), (2.13), (2.15) and (2.17), we are mainly interested in approximations of the boundary flux by means of the recovered gradient method.

We define the second component of the recovered gradient $G_h y_h(\theta_h)$ at a grid point Z (mesh node) as follows [12], [13]:

$$[G_h y_h]_2 = \alpha y_h(A) - (\alpha + \beta) y_h(Z) + \beta y_h(B) \tag{4.2}$$

where A, B are the two nearest grid points in the x_2-direction and

$$\alpha = \frac{b}{a(b-a)}, \quad \beta = \frac{a}{b(a-b)},$$
$$a = x_2(A) - x_2(Z), \quad b = x_2(B) - x_2(Z).$$

Then $[G_h y_h]_2$ is extended to the whole domain $\Omega \equiv \Omega(\theta_h)$ by piecewise linear interpolation.

Let us employ a *local error estimate up to the boundary* [13—Theorem 6.4]. The standard norm in the space $W^k_q(\Omega)$ will be denoted by $\|.\|_{k,q,\Omega}$ and if $q = 2$, by $\|.\|_{k,\Omega}$.

Assume that domains $\mathcal{G}_1 \subset\subset \mathcal{G}_2$ exist (which are not contained in Ω) such that the solution of the first state problem (1.5_1)

$$y \equiv y(\theta_h) \in W^3_q(\Omega_1) \cap W^3_2(\Omega_2),$$

where $\Omega_j = \mathcal{G}_j \cap \Omega$, $j = 1, 2$ and $q > 2$. Moreover, assume that a $\gamma \in (0,1]$ exists such that $y \in W^{1+\gamma}_2(\Omega)$ for any $f \in L^2(\Omega)$ and

$$\|y\|_{1+\gamma,\Omega} \leqslant C \|f\|_{0,\Omega}. \tag{4.3}$$

Then

$$\left\| \frac{\partial y}{\partial x_2} - [G_h y_h]_2 \right\|_{0,\Omega_0} \leqslant C(\Omega_0, \Omega_1) h^{2\gamma} (\|y\|_{3,\Omega_2} + \|y\|_{3,q,\Omega_1} + \|f\|_{0,\Omega})$$

holds for any subdomain

$$\Omega_0 = \mathcal{G}_0 \cap \Omega, \quad \text{where } \mathcal{G}_0 \subset\subset \mathcal{G}_1$$

and sufficiently small h.

Remark 4.1. Recall that the standard finite element theory [7] yields only the estimate

$$\left\| \frac{\partial y}{\partial x_2} - \frac{\partial y_h}{\partial x_2} \right\|_{0,\Omega_0} \leqslant |y - y_h|_{1,\Omega} \leqslant \tilde{C} h^{\gamma} \|y\|_{1+\gamma,\Omega} \leqslant C h^{\gamma} \|f\|_{0,\Omega},$$

provided the solution of the state problem has the regularity considered in (4.3). □

Using (4.2), we thus arrive at a piecewise linear approximation

$$q_{2h} = [G_h y_h]_2 \in C\big(\Gamma_1(\theta_h)\big),$$

which is close to $\partial y(\theta_h)/\partial x_2$ on $\Gamma_1(\theta_h)$.

To approximate the first component of $\nabla y(\theta_h)$ on $\Gamma_1(\theta_h)$, we employ the homogeneous boundary condition. We have

$$\frac{\partial y_h(\theta_h)}{\partial t_j} = 0, \quad j = 1, \ldots, N, \tag{4.4}$$

where

$$t_j = (t_{j1}, t_{j2}) = \big(h, \theta(jh) - \theta((j-1)h)\big)$$

is the tangent vector to the segment $\Gamma_{1j}(\theta_h)$ of the optimized boundary $\Gamma_1(\theta_h)$. In accordance with (4.4), we prescribe

$$q_{1h} = -q_{2h} t_{j2}/t_{j1} = -q_{2h}[\theta(jh) - \theta((j-1)h)]/h \tag{4.5}$$

on $\Gamma_{1j}(\theta_h)$, $j = 1, \ldots, N$.

The vector $q_h = (q_{1h}, q_{2h})$ can be used to approximate $\partial y/\partial \nu$, $\partial y/\partial \nu_A$ in the sensitivity formulae of Section 2.

The same approach can be employed to obtain approximations of $\partial z/\partial \nu$ and $\partial z/\partial \nu_A$.

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