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HOW TO RECOVER THE GRADIENT OF LINEAR ELEMENTS ON NONUNIFORM TRIANGULATIONS

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Summary. We propose and examine a simple averaging formula for the gradient of linear finite elements in R^d whose interpolation order in the L^q-norm is $\mathcal{O}(h^2)$ for $d < 2q$ and nonuniform triangulations. For elliptic problems in R^2 we derive an interior superconvergence for the averaged gradient over quasiuniform triangulations. A numerical example is presented.

Keywords: weighted averaged gradient, linear elements, nonuniform triangulations, superapproximation, superconvergence

AMS classification: 65N30

1. MOTIVATION

Finite element or difference schemes of higher order are developed to reach quickly highly accurate approximate solutions. This fact can be rigorously proved usually when the true solution is sufficiently smooth. Nevertheless, in practice the use of higher order schemes yields also quite good results even when the true solution has not theoretically required smoothness (see e.g. [18, p. 232]).

The theoretical optimal rate of approximation of linear finite elements in the L^2-norm is of order $\mathcal{O}(h^2)$ whereas for their gradient it is only $\mathcal{O}(h)$ for regular families of triangulations. This paper can be regarded as a continuation of the article [17], where Křížek and Neittaanmäki analyzed a simple postprocessing which yields locally the $\mathcal{O}(h^2)$-accuracy in the L^2-norm of the so-called averaged gradient applied to solving the Poisson equation in special plane domains. The averaged gradient is defined over uniform triangulations as a piecewise linear continuous vector field, the value of which at any nodal point is the average of gradients of linear elements surrounding the nodal point. Similar postprocessing techniques were later generalized by many authors in many directions, in particular, to tetrahedral elements [15], to quadratic elements [10], to global estimates [18], to elliptic systems [9, 12] and nonlinear elliptic problems [14], to parabolic problems [25], to L^∞-norm error estimates [18], to locally symmetric triangulations with respect to a point [27], to piecewise uniform [22] and quasiuniform triangulations [20].

Note that some superconvergence results on nonuniform rectangular meshes were obtained by [21, 29]. Also in [2, 23], the authors introduced higher order approximations of the gradient by integral smoothing operators over some nonuniform meshes. However, their use for practical calculations is complicated. We introduce a simpler averaging operator (see (2.2)). Our concern in Section 3 is with interpolation (superapproximation) properties of a weighted averaged gradient of linear elements on nonuniform meshes, especially in two- and three-dimensional space. The piecewise constant gradient of linear elements will be replaced by a continuous piecewise linear recovered gradient defined via appropriate weights. In contrast to piecewise constant gradients, we prove the $\mathcal{O}(h^2)$-accuracy of the weighted averaged gradient in the L^q-norm (see Theorem 3.8). In Theorem 4.1, we prove the $\mathcal{O}(h)$-accuracy of the averaged gradient in the W_q^1-norm provided a family of triangulations is strongly regular. Thus we can approximate the second derivatives of a smooth function the values of which are given only in a finite number of nodes.

In Section 5, we apply the results of previous sections to the finite element solution of an elliptic boundary value problem. Here, however, more severe restrictions on the triangulation are required. Recall that finite element schemes often produce some superconvergence phenomena on uniform meshes (see e.g. [6, 19, 26, 27]). We derive an interior superconvergence error estimate for the proposed averaged gradient over quasiuniform triangulations. Note that our averaging technique differs from that presented in [20], where also different norms (discrete L^2-norms) were used. Section 6 is devoted to numerical tests.

The proposed technique enables us to obtain not only good a priori error estimates but also efficient a posteriori error estimates using recovery based estimators like in [1, 8, 30]. Moreover, the knowledge of the recovered gradient is a useful tool in magnetic field computations, in sensitivity analysis of optimization problems (see [11]), in adaptive mesh refinements, in calculation of the boundary flux and many other problems. These problems need not have any connection with the Galerkin method. For instance, suppose that some data (measurement of a potential, etc.) were obtained in nodes of a given nonuniform triangulation. The gradient of their piecewise linear interpolation can be recovered by the averaging method of Sections 2 and 3.

2. Weighted averaged gradient

Throughout the paper $\Omega \subset R^d$ ($d = 1, 2, 3$) will be a bounded domain with a polyhedral Lipschitz boundary. Let $\mathscr{F} = \{T_h\}_{h\to 0}$ be a family of decompositions (triangulations) of $\overline{\Omega}$ into closed simplexes in the standard sense (cf. [7]). As usual, the discretization parameter h is the maximum diameter of all elements $K \in T_h$. We define

$$V_h = \{v_h \in C(\overline{\Omega}) \mid v_h\big|_K \in P_1(K) \quad \forall K \in T_h\},$$

where $P_1(K)$ is the space of linear polynomials over K. Let N_h be the set of all nodal points associated to T_h, i.e., the set of all vertices of all $K \in T_h$. For $Z \in N_h$ denote by $\ell_i = \ell_i(Z)$ that straight line passing through Z which is parallel to the axis x_i, $i \in \{1, \dots, d\}$. Set

$$U = U(Z) = \bigcup_{\substack{K\in T_h\\ K\cap Z\neq\emptyset}} K.$$

The dependence of $U, \ell, \dots$ upon Z will be usually not explicitly indicated in what follows. Let $Z \in N_h \cap \Omega$ be a fixed interior node (the case $Z \in N_h \cap \partial\Omega$ is treated in Section 3). Let A_iB_i be the line segment $\ell_i \cap U$. Then we have $A_i, B_i \in \partial U$ (see Figure 1).

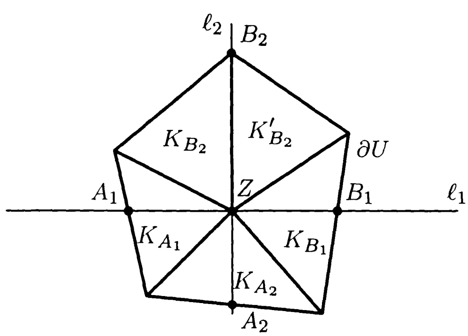

Figure 1

Put

$$a_i = (A_i - Z)_i, \quad b_i = (B_i - Z)_i, \quad i = 1, \dots, d, \tag{2.1}$$

where $(.)_i$ stands for the i-th component, i.e., we have $\mathrm{dist}(A_i, Z) = |a_i|$ and $\mathrm{dist}(B_i, Z) = |b_i|$. For $v_h \in V_h$ we define *the weighted averaged gradient* $G_h v_h$ at interior nodal points as follows

$$(G_h v_h(Z))_i = \frac{b_i}{b_i - a_i}\partial_i v_h\big|_{K_{A_i}} + \frac{a_i}{a_i - b_i}\partial_i v_h\big|_{K_{B_i}} \tag{2.2}$$

for $i = 1, \ldots, d$, where $K_{A_i}, K_{B_i} \subset U$ are such simplexes from T_h that $A_i \in K_{A_i}$ and $B_i \in K_{B_i}$. The position of K_{A_i} and K_{B_i} is indicated in Figure 1. It is clear that $a_i \neq b_i$ (since they have converse signs) and that both the weights $b_i/(b_i - a_i)$ and $a_i/(a_i - b_i)$ are positive and their sum is 1. Notice that the choice of the simplexes K_{A_i} or K_{B_i} need not be uniquely determined, since A_i or B_i can be contained in more than one simplex $K \subset U$. Nevertheless, the value of $G_h v_h(Z)$ is still uniquely defined. For instance, in Figure 1 there are two triangles K_{B_2} and K'_{B_2} containing the point B_2. In this case, however,

$$\partial_2 v_h \big|_{K_{B_2}} = \partial_2 v_h \big|_{K'_{B_2}}.$$

Remark 2.1. Each component of the weighted averaged gradient $G_h v_h$ is defined by another weights, in general. But for uniform triangulations (where any two adjacent triangles of T_h form a parallelogram) we have

$$(G_h v_h(Z))_i = \tfrac{1}{2} \partial_i v_h \big|_{K_{A_i}} + \tfrac{1}{2} \partial_i v_h \big|_{K_{B_i}},$$

in other words, the weights are independent of i (cf. [12, p. 147]).

3. Global superapproximation properties of the weighted averaged gradient

Throughout the paper the Sobolev space $W_p^k(\Omega)$ and the product space $(W_p^k(\Omega))^d$ are equipped with the standard norm $\|.\|_{k,p} = \|.\|_{k,p,\Omega}$ and seminorm $|.|_{k,p} = |.|_{k,p,\Omega}$. The symbol $\|.\|$ stands for the Euclidean norm. As usual, we denote by $C, C', \ldots$ the so-called generic positive constants which are not necessarily the same at each occurrence and which do not depend on relating functions and the discretization parameter h.

Let $\pi_h \colon C(\overline{\Omega}) \to V_h$ be the usual linear interpolation operator such that

$$\pi_h v(Z) = v(Z) \quad \forall Z \in N_h.$$

Moreover, let

$$\pi_K v = \pi_h v \big|_K \quad \forall K \in T_h.$$

We will extend the definition (2.2) to boundary nodes (see Remark 3.1). Let $N_h^0 \subset N_h$ be the set of those nodes Z of $\partial\Omega$ for which there exists $i \in \{1, \ldots, d\}$ such that $\ell_i(Z) \cap U(Z) = Z$. Consider for instance a triangulation T_h of the unit square $\overline{\Omega}$. If its sides are parallel to the coordinate axes then $N_h^0 = \emptyset$ otherwise N_h^0 contains all four vertices of $\overline{\Omega}$.

For $Z \in \partial\Omega \cap (N_h \setminus N_h^0)$ the set $\ell_i \cap U$ is a straight line segment for any $i \in \{1, \ldots, d\}$. If Z is not the endpoint of this segment (see Figure 2 for $i = 1$) then the points A_i and B_i are defined as in Section 2. If Z is one of the endpoints, we denote by A_i the opposite endpoint and take $B_i \in \ell_i \cap \overline{\Omega}$ so that A_i is in the interior of ZB_i (see Figure 2 for $i = 2$) and

$$C|a_i| \leqslant |b_i| \leqslant \overline{C}|a_i|, \tag{3.1}$$

where a_i, b_i are defined by (2.1) and $C > 1$ and $\overline{C} > 1$ are independent of h. Note that the points B_i can be chosen on element boundaries as well as inside of elements.

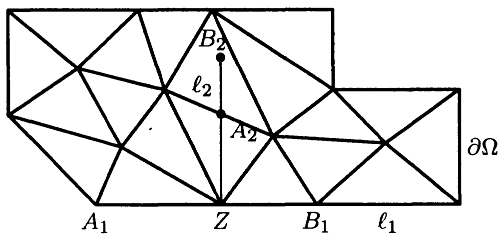

Figure 2

Now if $v \in C(\overline{\Omega})$ and $i \in \{1, \ldots, d\}$ we set

$$(G_h v(Z))_i = \alpha_i v(A_i) - (\alpha_i + \beta_i) v(Z) + \beta_i v(B_i) \quad \text{for } Z \in N_h \setminus N_h^0, \tag{3.2}$$

where

$$\alpha_i = \frac{b_i}{a_i(b_i - a_i)}, \qquad \beta_i = \frac{a_i}{b_i(a_i - b_i)}, \tag{3.3}$$

with a_i and b_i given in (2.1).

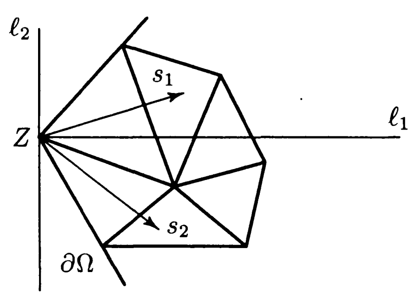

Figure 3

Next, let us consider $Z \in N_h^0$. Let $s_1, \ldots, s_d$ be unit (column) vectors independent of h, let the matrix $S = (s_1, \ldots, s_d)$ be nonsingular. Assume that the vectors s_i tend from the node Z into $\overline{\Omega}$, as illustrated in Figure 3 for $d = 2$. Let $i \in \{1, \ldots, d\}$ and v smooth be fixed. Then we choose a local Cartesian coordinate system $\mathscr{C}_i$ in which the axis x'_i is parallel to s_i. Denote by g_i an approximation of the directional derivative

$$\frac{\partial v}{\partial s_i}(Z) = (\operatorname{grad} v(Z))^T s_i$$

obtained by (3.2) in $\mathscr{C}_i$ such that (3.1) holds. Then it is natural to define

$$G_h v(Z) = (S^{-1})^T g_h v(Z), \tag{3.4}$$

where $g_h v(Z) = (g_1, \ldots, g_d)^T$.

The operator $G_h v$ is thus defined for all nodal points $Z \in N_h$. Thus we may introduce a continuous piecewise linear vector function (still denoted by $G_h v$) which is uniquely defined by the values at nodes, i.e., from now on

$$G_h v \in V_h \times \ldots \times V_h \quad (d\text{-times}).$$

Remark 3.1. The formula (3.2) for $v_h \in V_h$ can be interpreted as the equivalent definition of the weighted averaged gradient from (2.2). To see this, consider an interior node $Z \in N_h \cap \Omega$ and some $v_h \in V_h$. Using the fact that

$$\partial_i v_h\big|_{K_{A_i}} = \frac{v_h(A_i) - v_h(Z)}{a_i}, \quad \partial_i v_h\big|_{K_{B_i}} = \frac{v_h(B_i) - v_h(Z)}{b_i},$$

we find, by (3.3), that the definition (2.2) coincides with that in (3.2). That is why the operator $G_h v$ will still be called *the weighted averaged gradient.*

For $Z \in N_h$ we set

$$h(Z) = \max_{K \in T_h, K \subset U} h_K,$$

where $h_K = \operatorname{diam} K$. Thus we have $h(Z) \leqslant h$. By (2.1) and (3.1), obviously

$$\max(|a_i|, |b_i|) \leqslant \overline{C} h(Z). \tag{3.5}$$

Lemma 3.2. *There exists a constant $C > 0$ such that*

$$|\alpha_i| \leqslant \frac{C}{|a_i|}, \quad |\beta_i| \leqslant \frac{C}{|b_i|}, \tag{3.6}$$

where a_i, b_i and α_i, β_i are of the form (2.1) *and* (3.3), *respectively.*

Proof. Let $Z \in N_h \setminus N_h^0$ and $i \in \{1, \ldots, d\}$ be given. We shall distinguish two cases:

1) Z is in the interior of A_iB_i. Then, by (2.1), a_i and b_i have the converse signs and thus using (3.3), we obtain

$$|\alpha_i| = \frac{|b_i|}{|a_i|\,|b_i - a_i|} = \frac{|b_i|}{|a_i|(|b_i| + |a_i|)} < \frac{|b_i|}{|a_i|\,|b_i|} = \frac{1}{|a_i|},$$

$$|\beta_i| = \frac{|a_i|}{|b_i|\,|b_i - a_i|} < \frac{|a_i|}{|b_i|\,|a_i|} = \frac{1}{|b_i|}.$$

2) A_i is in the interior of ZB_i. Then, a_i and b_i have the same signs and, by (3.1) and (3.3),

$$|\alpha_i| = \frac{|b_i|}{|a_i|(|b_i| - |a_i|)} \leqslant \frac{|b_i|}{|a_i|(1 - 1/C)|b_i|} = \frac{C'}{|a_i|},$$

$$|\beta_i| = \frac{|a_i|}{|b_i|(|b_i| - |a_i|)} \leqslant \frac{|a_i|}{|b_i|(C|a_i| - |a_i|)} = \frac{C''}{|b_i|}.$$

If $Z \in N_h^0$ and $i \in \{1, \ldots, d\}$ then we use the same procedure as above in the local Cartesian coordinate system $\mathscr{C}_i$. □

Definition 3.3. A family $\mathscr{F} = \{T_h\}_{h \to 0}$ of decompositions of $\overline{\Omega}$ into simplexes is said to be regular (strongly regular) if there exists a constant $\varkappa > 0$ such that for any decomposition $T_h \in \mathscr{F}$ and any simplex $K \in T_h$ there exists a ball $\mathscr{B}_K$ with radius ϱ_K such that $\mathscr{B}_K \subset K$ and

$$\varkappa h_K \leqslant \varrho_K \qquad (\varkappa h \leqslant \varrho_K). \tag{3.7}$$

Remarks 3.4. Any strongly regular family is obviously regular. In the case $d = 1$, any family is regular. If $d > 1$ we have for the radius of the ball inscribed to K that

$$\varrho_K = \frac{d \operatorname{meas}_d K}{\operatorname{meas}_{d-1} \partial K}. \tag{3.8}$$

A constructive proof of the existence of a strongly regular family of decompositions of an arbitrary polyhedron into tetrahedra can be found in [16, p. 58].

Lemma 3.5. *Let $\mathscr{F}$ be regular and for $d = 1$ strongly regular. Then there exists a constant C such that for any $Z \in N_h$*

$$h(Z) \leqslant C h_K \quad \forall K \in T_h,\ K \subset U(Z). \tag{3.9}$$

P r o o f. The proof is evident for $d = 1$. So let $d \in \{2, 3\}$. First we prove that there exists an integer M (independent of h) such that for any $T_h \in \mathcal{F}$ there are around any nodal point at most M simplexes.

The case $d = 2$ immediately follows from the Zlámal condition which is equivalent with the regularity of $\mathcal{F}$ (see e.g. [7, p. 128]).

Finally let $d = 3$. We show that all angles between faces of any $K \in T_h \in \mathcal{F}$ and all angles at vertices of these faces satisfy the minimum angle condition.

1. Let ψ be the angle between two arbitrary faces T', T'' of K. Let S be the area of T' and let s be the length of the corresponding spatial altitude of K perpendicular to T'—see Figure 4. Then by (3.7) and (3.8)

$$\varkappa \leqslant \frac{\varrho_K}{h_K} = \frac{3 \text{ meas}_3 K}{h_K \text{ meas}_2 \partial K} < \frac{sS}{h_K S} = \frac{s}{h_K} \leqslant \sin \psi.$$

Hence, $\psi \in (\gamma, \pi - \gamma)$, where

$$\gamma = \arcsin \varkappa \in (0, \tfrac{\pi}{2}) \quad \text{for a given } \varkappa \in (0, 1).$$

This means that all angles between faces of K satisfy the minimum angle condition.

2. Let T be an arbitrary face of K and let φ be the angle between two arbitrary edges of T the lengths of which are e, f—see Figure 4. Then by (3.7) and (3.8),

$$\varkappa \leqslant \frac{\varrho_K}{h_K} < \frac{r_K}{h_K} = \frac{2 \text{ meas}_2 T}{h_K \text{ meas}_1 \partial T} < \frac{ef \sin \varphi}{h_K f} \leqslant \sin \varphi,$$

where r_K is the radius of the circle inscribed to T. We again see that all angles at vertices of faces of K are in the interval $(\gamma, \pi - \gamma)$, i.e., the minimum angle condition holds.

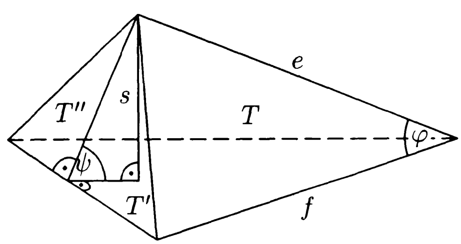

Figure 4

Let $Z \in N_h$ be arbitrary. Consider a sphere

$$\mathcal{S} = \{z \in R^3 \mid \|z - Z\| = r\},$$

where $r > 0$ is less than any edge of any $K \subset U(Z)$, $K \in T_h$. Then $K \cap \mathscr{S}$ is a spherical triangle. The radius ϱ of its inscribed circle is given by (see [3, Chap. 3.5.3])

$$\varrho = r \arctan\left(\sin \frac{\varphi_3 + \varphi_2 - \varphi_1}{2} \tan \frac{\psi_1}{2}\right), \tag{3.10}$$

where $\varphi_1 \leqslant \varphi_2$ is assumed, and $\varphi_1, \varphi_2, \varphi_3$ and ψ_1 are sketched in Figure 5. Note that ψ_1 is the angle between two faces of K. Since

$$\frac{\gamma}{2} \leqslant \frac{\varphi_3}{2} \leqslant \frac{\varphi_3 + \varphi_2 - \varphi_1}{2} < \frac{\varphi_3 + \varphi_2}{2} < \pi - \gamma < \pi - \frac{\gamma}{2},$$
$$\frac{\gamma}{2} \leqslant \frac{\psi_1}{2} \leqslant \frac{\pi}{2} - \frac{\gamma}{2},$$

we have by (3.10) that

$$\varrho > r \arctan\left(\sin \frac{\gamma}{2} \tan \frac{\gamma}{2}\right) \equiv r\hat{\varkappa} > 0$$

which implies that

$$M \leqslant \frac{\operatorname{meas}_2 \mathscr{S}}{\operatorname{meas}_2(K \cap \mathscr{S})} < \frac{4\pi r^2}{\pi \varrho^2} < \frac{4}{\hat{\varkappa}^2}.$$

Thus we have got the existence of the number M independent of h also in the case $d = 3$.

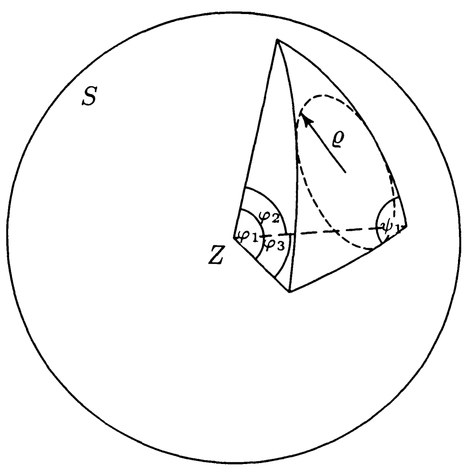

Figure 5

Since $2\varrho_K$ is less than the shortest spatial altitude which is not greater than any edge, we have by (3.7) that

$$h_K \leqslant \frac{1}{\varkappa} \varrho_K \leqslant \frac{1}{2\varkappa} e_K,$$

where e_K is the length of an arbitrary edge of K. The same relation holds clearly also for $d = 2$. Thus we obtain

$$h_K \leqslant \frac{1}{2\varkappa} h_{K'},$$

where K, K' are any adjacent triangles or tetrahedra form $T_h \in \mathscr{F}$. Consequently,

$$h_K \leqslant (2\varkappa)^{-M} h_{K''},$$

where K and K'' are arbitrary simplexes having the same vertex $Z \in N_h$. □

Lemma 3.6. *Let $q \in (\frac{d}{2}, \infty)$, $q \geqslant 1$, let $\mathscr{F}$ be regular and for $d = 1$ strongly regular. Then there exists a constant $C > 0$ such that for any decomposition $T_h \in \mathscr{F}$ and any $Z \in N_h$ there exists a closed neighbourhood $\mathscr{U} = \mathscr{U}(Z)$, $\operatorname{diam} \mathscr{U} \leqslant Ch(Z)$, such that*

$$\|\operatorname{grad} v(Z) - G_h v(Z)\| \leqslant C(h(Z))^{2-d/q} |v|_{3,q,\mathscr{U}} \quad \forall v \in W_q^3(\mathscr{U}).$$

Proof. Let $Z \in N_h \setminus N_h^0$ and $i \in \{1, \ldots, d\}$ be arbitrary but fixed. From (3.3), we easily find that

$$\alpha_i a_i + \beta_i b_i = 1, \tag{3.11}$$

and

$$\alpha_i a_i^2 + \beta_i b_i^2 = 0. \tag{3.12}$$

By the Taylor formula for any quadratic polynomial $p \in P_2(\overline{\Omega})$ we come to

$$p(A_i) - p(Z) = \partial_i p(Z) a_i + \tfrac{1}{2} \partial_i^2 p(Z) a_i^2, \tag{3.13}$$

$$p(B_i) - p(Z) = \partial_i p(Z) b_i + \tfrac{1}{2} \partial_i^2 p(Z) b_i^2. \tag{3.14}$$

Multiplying (3.13) by α_i, and (3.14) by β_i, and summing this, we get from (3.2), (3.11) and (3.12) that

$$(G_h p(Z))_i = \partial_i p(Z).$$

For $m_i = \min(|a_i|, |b_i|)$ define a linear functional

$$F_i(v) = m_i(\partial_i v(Z) - (G_h v(Z))_i), \quad v \in C^1(\overline{\Omega}). \tag{3.15}$$

Clearly,

$$(3.16) \qquad F_i(p) = 0 \quad \forall p \in P_2(\overline{\Omega}).$$

By (3.9) and (3.7) we have for $m_i = |a_i|$ that

$$(3.17) \qquad \frac{h(Z)}{m_i} = \frac{h(Z)}{|a_i|} \leqslant C\frac{h_K}{|a_i|} \leqslant C\frac{h_K}{2\varrho_K} \leqslant \frac{C}{2\varkappa},$$

where $K \subset U$ is that simplex for which $A_i \in K$. Since $|b_i| > |a_i|$ for boundary nodes, the proof of (3.17) for internal nodes in the case $m_i = |b_i|$ is analogous.

Let $\mathscr{U} = \mathscr{U}(Z)$ be a closed domain for which $\overline{ZA_i} \cup \overline{ZB_i} \subset \mathscr{U}$ for all i, $\operatorname{diam} \mathscr{U} \leqslant Ch(Z)$ and $U \subset \mathscr{U} \subset \overline{\Omega}$ (e.g. $\mathscr{U} = U$ for interior nodes and the existence of $\mathscr{U}$ for boundary nodes follows from (3.1)). Then by (3.15), (3.2) and (3.6), we have

$$\begin{aligned} |F_i(v)| &\leqslant m_i(\|\operatorname{grad} v\|_{0,\infty,\mathscr{U}} + 2(|\alpha_i| + |\beta_i|)\|v\|_{0,\infty,\mathscr{U}}) \\ &\leqslant m_i(\|\operatorname{grad} v\|_{0,\infty,\mathscr{U}} + C\Big(\frac{1}{|a_i|} + \frac{1}{|b_i|}\Big)\|v\|_{0,\infty,\mathscr{U}}) \\ &\leqslant \operatorname{diam} \mathscr{U} \|\operatorname{grad} v\|_{0,\infty,\mathscr{U}} + 2C\|v\|_{0,\infty,\mathscr{U}} \quad \forall v \in C^1(\overline{\Omega}). \end{aligned}$$

From here, (3.16) and the Bramble-Hilbert theorem [5, Theorem 3], we obtain

$$|F_i(v)| \leqslant C(h(Z))^{3-d/q}|v|_{3,q,\mathscr{U}} \quad \forall v \in W_q^3(\mathscr{U})$$

whenever $q \in (\frac{d}{2}, \infty)$ and $q \geqslant 1$. Therefore, from the definition (3.15) and (3.17)

$$(3.18) \qquad |\partial_i v(Z) - (G_h v(Z))_i| \leqslant C(h(Z))^{2-d/q}|v|_{3,q,\mathscr{U}},$$

which completes the proof for $q \in (\frac{d}{2}, \infty)$, $q \geqslant 1$ and $Z \in N_h \setminus N_h^0$.

The case $Z \in N_h^0$ follows similarly from the definition (3.4), where the matrix S is independent of h. We again use the local coordinate systems $\mathscr{C}_i$ for $i = 1, \ldots, d$. □

In the following we shall use the inequality for $q \in (\frac{1}{2}, \infty]$

$$(3.19) \qquad \|z\|_{0,q} \leqslant (\operatorname{meas}\Omega)^{1/q}\|z\|_{0,\infty} \leqslant \max(1, (\operatorname{meas}\Omega)^2)\|z\|_{0,\infty} \quad \forall z \in L^\infty(\Omega).$$

Corollary 3.7. *It is*

$$(3.20) \qquad G_h p(Z) = \operatorname{grad} p(Z) \quad \forall p \in P_2(\Omega)\, \forall Z \in N_h,$$

i.e., for any quadratic polynomial p *we have* $G_h p = \operatorname{grad} p$ *over all elements.*

Theorem 3.8. *Let $q \in (\frac{d}{2}, \infty)$, $q \geqslant 1$ and let $\mathscr{F}$ be strongly regular. Then there exists a constant $C > 0$ such that for any decomposition $T_h \in \mathscr{F}$ we have*

$$\|\operatorname{grad} v - G_h v\|_{0,q} \leqslant Ch^2 |v|_{3,q} \quad \forall v \in W_q^3(\Omega). \tag{3.21}$$

Proof. Let $q \in (\frac{d}{2}, \infty]$, $q \geqslant 1$ be arbitrary. Recall (see [7, p. 126]) that

$$\|v - \pi_K v\|_{0,q,K} \leqslant Ch_K^2 |v|_{2,q,K} \quad \forall K \in T_h \, \forall v \in W_q^2(K), \tag{3.22}$$

without any regularity assumptions upon the family $\mathscr{F}$. Let $L_h v$ be a continuous piecewise linear vector function over T_h such that

$$L_h v(Z) = \operatorname{grad} v(Z) \quad \forall Z \in N_h. \tag{3.23}$$

Then, by (3.22),

$$\|\operatorname{grad} v - L_h v\|_{0,q} \leqslant Ch^2 |v|_{3,q} \quad \forall v \in W_q^3(\Omega). \tag{3.24}$$

Let us introduce

$$\delta_i(Z) = \{K' \in T_h \mid \operatorname{meas}_1((\overline{ZA_i} \cup \overline{ZB_i}) \cap K') > 0\},$$
$$Q(K,i) = \bigcup_{Z \in K \cap N_h} \bigcup_{K' \in \delta_i(Z)} K'.$$

Note that simplexes, which have only one point (vertex) on the segment $\overline{ZA_i} \cup \overline{ZB_i}$, do not belong to $\delta_i(Z)$. The set $Q(K,i)$ is an "oriented garland" of simplexes around the simplex K.

Since $G_h v$ is a continuous piecewise linear vector function, for any $K \in T_h$ and $i \in \{1, \ldots, d\}$ we have by (3.23) and Lemma 3.6 (compare (3.18)) that

$$\|(L_h v - G_h v)_i\|_{0,\infty,K} = \max_{Z \in K \cap N_h} |\partial_i v(Z) - (G_h v(Z))_i| \leqslant Ch^{2-d/q} |v|_{3,q,Q(K,i)}. \tag{3.25}$$

Next, we may write

$$\|(L_h v - G_h v)_i\|_{0,q,K} \leqslant ch_K^{d/q} \|(L_h v - G_h v)_i\|_{0,\infty,K} \leqslant Ch^2 |v|_{3,q,Q(K,i)}.$$

The union $Q(K,i)$ contains at most M simplexes, where M is independent of h (see also the proof of Lemma 5.1 below) as follows from (3.1) and the strong regularity of $\mathscr{F}$. Consequently,

$$\|L_h v - G_h v\|_{0,q}^q \leqslant \sum_{K \in T_h} \|L_h v - G_h v\|_{0,q,K}^q \leqslant Ch^{2q} \sum_{K \in T_h} |v|_{3,q,Q(K,i)}^q \leqslant CMh^{2q} |v|_{3,q}^q$$

and

$$\|L_h v - G_h v\|_{0,q} \leqslant Ch^2 |v|_{3,q} \tag{3.26}$$

follows. Combining (3.24) and (3.26), we arrive at

$$\|\operatorname{grad} v - G_h v\|_{0,q} \leqslant \|\operatorname{grad} v - L_h v\|_{0,q} + \|L_h v - G_h v\|_{0,q} \leqslant Ch^2 |v|_{3,q}.$$

□

4. Approximation of the second derivatives

In this section we show that piecewise constant derivatives of the weighted averaged gradient can be used as a good approximation of the second derivatives.

Theorem 4.1. *Let $q \in (\frac{d}{2}, \infty)$, $q \geqslant 1$ and let $\mathscr{F}$ be a strongly regular family of decompositions. Then there exists a constant $C > 0$ such that for any decomposition $T_h \in \mathscr{F}$ we have*

$$\|\operatorname{grad} v - G_h v\|_{1,q} \leqslant Ch|v|_{3,q} \quad \forall v \in W_q^3(\Omega).$$

Proof. Since the family $\mathscr{F}$ of decompositions is regular, we have (see [7])

$$\|w - \pi_h w\|_{1,q} \leqslant Ch|w|_{2,q} \quad \forall w \in W_q^2(\Omega). \tag{4.1}$$

Thus for $L_h v$ defined by (3.23) we have by (4.1) that

$$\|\operatorname{grad} v - L_h v\|_{1,q} \leqslant Ch|v|_{3,q} \quad \forall v \in W_q^3(\Omega). \tag{4.2}$$

Since $\mathscr{F}$ is strongly regular, the standard inverse inequality (see e.g. [7, Theorem 17.2]) and (3.26) imply that

$$\|L_h v - G_h v\|_{1,q} \leqslant Ch^{-1}\|L_h v - G_h v\|_{0,q} \leqslant Ch|v|_{3,q}. \tag{4.3}$$

Using finally the triangle inequality, (4.2) and (4.3), we get

$$\|\operatorname{grad} v - G_h v\|_{1,q} \leqslant \|\operatorname{grad} v - L_h v\|_{1,q} + \|L_h v - G_h v\|_{1,q} \leqslant Ch|v|_{3,q}.$$

□

5. Application to the finite element method

Let us consider the following elliptic model problem

$$-\operatorname{div}(\lambda \operatorname{grad} u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{5.1}$$

where $\lambda = (\lambda_{ij})_{i,j=1}^{d}$ is a symmetric matrix, $\lambda_{ij} \in W_{2+\varepsilon}^{2}(\Omega)$ for some $\varepsilon > 0$, there exists a constant $C_0 \geqslant 0$ such that

$$\xi^{\top}\lambda(x)\xi \geqslant C_0\|\xi\|^2 \quad \forall x \in \Omega \quad \forall \xi \in R^d \tag{5.2}$$

and $f \in W_q^1(\Omega)$. Let us assume that the solution u of (5.1) belongs to $W_q^3(\Omega)$, where $q > d$.

We define the finite element approximation u_h in the standard way, i.e.,

$$u_h \in V_h^0 = \{v_h \in V_h \mid v_h = 0 \text{ on } \partial\Omega\},$$
$$a(u_h, v_h) = (f, v_h)_{0,2} \quad \forall v_h \in V_h^0,$$

where

$$a(u, v) = (\lambda \operatorname{grad} u, \operatorname{grad} v)_{0,2}$$

and $(.,.)_{0,2}$ is the L^2-inner product. It is well-known that

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h^0. \tag{5.3}$$

Now a natural question arises: having the finite element approximation u_h, is the weighted averaged gradient $G_h u_h$ superconvergent for some kind of meshes, i.e., does the estimate

$$\|\operatorname{grad} u - G_h u_h\|_{0,2} = \mathcal{O}(h^2)$$

hold? For simplicity we establish only a local estimate over a fixed subdomain $\Omega_0 \subset\subset \Omega$. Making use of the triangle inequality, we may write

$$\begin{aligned}\|\operatorname{grad} u - G_h u_h\|_{0,2,\Omega_0} &\leqslant \|\operatorname{grad} u - G_h u\|_{0,2,\Omega_0} \\ &+ \|G_h u - G_h(\pi_h u)\|_{0,2,\Omega_0} + \|G_h(\pi_h u) - G_h u_h\|_{0,2,\Omega_0}.\end{aligned} \tag{5.4}$$

The first term of the upper bound can be estimated on the basis of Theorem 3.8, namely

$$\|\operatorname{grad} u - G_h u\|_{0,2,\Omega_0} \leqslant Ch^2|u|_{3,q}, \tag{5.5}$$

provided the family of decompositions is regular. To estimate further terms we first prove the following lemmas.

Lemma 5.1. *Let $\mathscr{F} = \{T_h\}_{h\to 0}$ be a strongly regular family. Then there exists a constant C such that*

$$\|G_h v_h\|_{0,2} \leqslant C\|\operatorname{grad} v_h\|_{0,2} \quad \forall v_h \in V_h.$$

Proof. Let $K \in T_h$, $i \in \{1,\dots,d\}$ and $v_h \in V_h$ be arbitrary but fixed. Since $G_h v_h$ is a linear function on K, we have

$$\|(G_h v_h)_i\|^2_{0,2,K} \leqslant \operatorname{meas} K\,(G_h v_h(Z))^2_i, \tag{5.6}$$

where $Z = Z_K = (z_1,\dots,z_d)$ is an appropriate vertex of K. Obviously,

$$v_h(A_i) - v_h(Z) = \int_{z_i}^{z_i+a_i} \partial_i \overline{v}_h(t)\,\mathrm{d}t, \tag{5.7}$$

$$v_h(B_i) - v_h(Z) = \int_{z_i}^{z_i+b_i} \partial_i \overline{v}_h(t)\,\mathrm{d}t, \tag{5.8}$$

where $\overline{v}_h(t) = v_h(z_1,\dots,z_{i-1},t,z_{i+1},\dots,z_d)$ and where the local coordinate Cartesian system $\mathscr{C}_i$ (cf. (3.4)) is applied whenever $Z \in N_h^0$. Recall that N_h^0 contains only a finite number of points and this number is independent of h.

First, let $Z \in N_h \cap \Omega$. Multiplying (5.7) by α_i and (5.8) by β_i, summing this, and using the fact (see (3.3)) that $\alpha_i a_i = |\alpha_i|\,|a_i|$, $\beta_i b_i = |\beta_i|\,|b_i|$, we obtain by (3.2) and (3.11) that

$$\begin{aligned}|(G_h v_h(Z))_i| &\leqslant |\alpha_i|\Big|\int_{z_i}^{z_i+a_i} \partial_i\overline{v}_h(t)\,\mathrm{d}t\Big| + |\beta_i|\Big|\int_{z_i}^{z_i+b_i} \partial_i\overline{v}_h(t)\,\mathrm{d}t\Big| \\ &\leqslant (\alpha_i a_i + \beta_i b_i) \max_{K'\in\delta_i(Z)} |(\partial_i v_h|_{K'})| = |(\partial_i v_h|_E)|,\end{aligned} \tag{5.9}$$

where

$$\delta_i(Z) = \{K' \in T_h \mid \operatorname{meas}_1((\overline{ZA}_i \cup \overline{ZB}_i) \cap K') > 0\}$$

and $E = E_i(K)$ is that element from $\delta_i(Z)$, where $|\partial_i v_h|$ attains its maximum. Consequently, according to (5.9),

$$\operatorname{meas} K\,(G_h v_h(Z))^2_i \leqslant \operatorname{meas} K|(\partial_i v_h|_E)|^2 \leqslant \tilde{C}\operatorname{meas} E|(\partial_i v_h|_E)|^2,$$

and thus

$$\text{meas}\, K\, (G_h v_h(Z))_i^2 \leqslant \tilde{C}\, \text{meas}\, E \| \operatorname{grad} v_h|_E \|^2, \tag{5.10}$$

where $\tilde{C}$ does not depend on h due to the strong regularity of $\mathscr{F}$.

Secondly, assume that $Z \in N_h \cap \partial\Omega$. Then by (3.3) we have

$$\begin{aligned} |\alpha_i + \beta_i|\, |a_i| + |\beta_i|\, |b_i - a_i| &= \left|\frac{a_i^2 - b_i^2}{a_i b_i (a_i - b_i)}\right| |a_i| + \left|\frac{a_i}{b_i(a_i - b_i)}\right| |b_i - a_i| \\ &= \left|\frac{a_i + b_i}{b_i}\right| + \left|\frac{a_i}{b_i}\right| < 3, \end{aligned}$$

as $|a_i| < |b_i|$. From here we get analogously to (5.9) that

$$\begin{aligned} |(G_h v_h(Z))_i| &= \left|\alpha_i \int_{z_i}^{z_i + a_i} \partial_i \overline{v}_h(t)\, \mathrm{d}t + \beta_i \int_{z_i}^{z_i + b_i} \partial_i \overline{v}_h(t)\, \mathrm{d}t\right| \\ &= \left|(\alpha_i + \beta_i) \int_{z_i}^{z_i + a_i} \partial_i \overline{v}_h(t)\, dt + \beta_i \int_{z_i + a_i}^{z_i + b_i} \partial_i \overline{v}_h(t)\, \mathrm{d}t\right| \\ &\leqslant (|\alpha_i + \beta_i|\, |a_i| + |\beta_i|\, |b_i - a_i|) \max_{K' \in \delta_i(Z)} |(\partial_i v_h|_{K'})| \leqslant 3|(\partial_i v_h|_E)| \end{aligned}$$

for $Z \notin N_h^0$, and again the local coordinates of $\mathscr{C}_i$ are employed if $Z \in N_h^0$. This and (3.4) yield

$$\|G_h v_h(Z)\| \leqslant C \| \operatorname{grad} v_h|_E \|,$$

where C is independent of h and grad is in global Cartesian coordinates $(x_1, \ldots, x_d)$. Thus (5.10) holds also in the case $Z \in N_h \cap \partial\Omega$.

Let i be fixed. Since

$$\text{dist}(K, E) \leqslant \text{dist}(Z, E) \leqslant \overline{C} h(Z) \leqslant \overline{C} h$$

(see (3.5)) and since $\mathscr{F}$ is strongly regular, there exists an integer M independent of h such that E corresponds to at most M different simplexes from T_h, i.e., there exist $m \leqslant M$ and $K_1, \ldots, K_m \in T_h$ such that $E = E(K_1) = \ldots = E(K_m)$. In fact, a $\overline{C}h$-neighbourhood of the simplex K may contain at most M simplexes, where

$$M \leqslant \left(\frac{(1 + \overline{C}) h}{\varrho_{K_j}}\right)^d \leqslant (1 + \overline{C})^d \varkappa^{-d}$$

due to Definition 3.3. Here ϱ_{K_j} is the radius of the minimal ball among all balls inscribed in the simplexes of the $\overline{C}h$-neighbourhood. Setting

$$T_h^0 = \{E \in T_h \mid \exists K \in T_h \colon E = E(K)\},$$

we get by (5.6) and (5.10) that

$$\|(G_h v_h)_i\|_{0,2}^2 = \sum_{K\in T_h} \|(G_h v_h)_i\|_{0,2,K}^2 \leqslant \sum_{K\in T_h} \operatorname{meas} K\,(G_h v_h(Z_K))_i^2$$
$$\leqslant \tilde{C} \sum_{K\in T_h} \operatorname{meas} E(K) \|\operatorname{grad} v_h|_{E(K)}\|^2 \leqslant M\tilde{C} \sum_{E\in T_h^0} \operatorname{meas} E \|\operatorname{grad} v_h|_E\|^2$$
$$\leqslant M\tilde{C} \sum_{K\in T_h} \operatorname{meas} K \|\operatorname{grad} v_h|_K\|^2 = C \sum_{K\in T_h} \|\operatorname{grad} v_h|_K\|_{0,2,K}^2 = C\|\operatorname{grad} v_h\|_{0,2}^2,$$

which completes the proof. □

From now on we shall deal only with the case $d = 2$. The proof of the next lemma uses some relations and notions of Levine [20].

Lemma 5.2. *Assume that $d = 2$, $q \in (2, \infty)$ and that the family $\mathscr{F} = \{T_h\}_{h\to 0}$ is generated by smooth distortions (W_∞^2-diffeomorphism) of uniform triangulations of square grids of mesh-size h. Then*

$$|\alpha_i(v(A_i) - \pi_h v(A_i)) + \beta_i(v(B_i) - \pi_h v(B_i))| \leqslant Ch^{2-2/q}\|v\|_{3,q,U}$$

holds for $i = 1, 2$, $Z \in N_h \cap \Omega$, sufficiently small h and $v \in W_q^3(U)$ with $U = U(Z)$.

Proof. If $q > 2$ then the second derivatives of $v \in W_q^3(\Omega)$ are continuous by the Sobolev imbedding theorem. Without loss of generality, let us consider $i = 1$ and the hexagon U from Figure 6. We shall drop the index 1 in what follows and set $x_A = x(A), x_B = x(B)$, a.s.o. Obviously, we have

$$\pi_h v(A) = \tau_A v(G) + \tau_A' v(F), \tag{5.11}$$
$$\pi_h v(B) = \tau_B v(D) + \tau_B' v(E), \tag{5.12}$$

where

$$\tau_A = \frac{-y_F}{y_G - y_F}, \quad \tau_B = \frac{-y_E}{y_D - y_E}, \quad \tau_A, \tau_B \in [0,1],$$
$$\tau_A' = 1 - \tau_A, \quad \tau_B' = 1 - \tau_B.$$

Write

$$\xi = x - x_0, \quad \eta = y = y - y_0$$

and

$$f(v) = \alpha[\pi_h v(A) - v(A)] + \beta[\pi_h v(B) - v(B)].$$

We have by [20, Lemma 2.2] and Definition 3.3

$$C_1 h \leqslant h_K \leqslant C_2 h, \quad h_K/|a| \leqslant h_K/(2\varrho_K) \leqslant C, \quad h_{K'}/b \leqslant C \quad \forall K \in T_h. \tag{5.13}$$

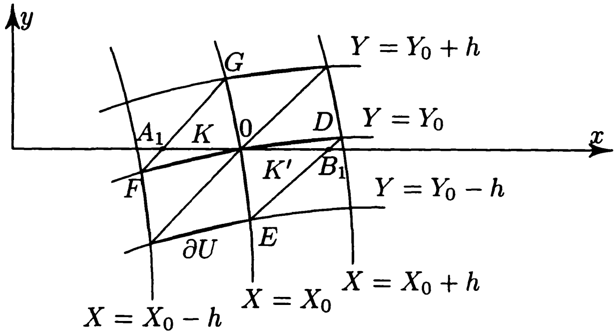

Figure 6

Let us find estimates for quadratic terms $v \in \{\xi^2, \eta^2, \xi\eta\}$. First, we realize that

$$\alpha v(A) + \beta v(B) = 0 \tag{5.14}$$

due to (3.12) (note that $\xi_A = a$ and $\xi_B = b$) and $\eta_A = \eta_B = 0$.

Using (5.11), (5.12) and (5.14), we may write for $v \in \{\xi^2, \eta^2, \xi\eta\}$

$$f(v) = \alpha(\tau_A v(G) + \tau'_A v(F)) + \beta(\tau_B v(D) + \tau'_B v(E)) = f_1(v) + f_2(v),$$

where

$$f_1(v) = \alpha\tau_A v(G) + \beta\tau'_B v(E),$$
$$f_2(v) = \alpha\tau'_A v(F) + \beta\tau_B v(D),$$

Let us introduce

$$t_0 = (\tau_A + \tau'_B)/2, \quad t_1 = (\tau_A - \tau'_B)/2, \tag{5.15}$$

so that

$$f_1(v) = t_0(\alpha v(G) + \beta v(E)) - t_1(\beta v(E) - \alpha v(G)) \equiv f_{11}(v) + f_{12}(v).$$

Introducing further

$$\delta = (\alpha + \beta)/2, \quad \gamma = (\beta - \alpha)/2,$$

we may write

$$|f_{11}(v)| \leqslant |\alpha v(G) + \beta v(E)| \leqslant |\delta|\,|v(G) + v(E)| + |\gamma|\,|v(E) - v(G)|. \tag{5.16}$$

Let us estimate $|\delta|$. We have

$$|\alpha + \beta| = \left|\frac{b + a}{ab}\right|. \tag{5.17}$$

Using (5.11), (5.12) and (5.15), we obtain

$$(5.18)\qquad \begin{aligned}|a+b| &= |\xi_A+\xi_B| \leqslant |\tau_A\xi_G+\tau'_B\xi_E| + |\tau'_A\xi_F+\tau_B\xi_D| \\ &= |t_0(\xi_G+\xi_E)-t_1(\xi_E-\xi_G)| + |\tilde{t}_0(\xi_F+\xi_D)+t_1(\xi_D-\xi_F)|\end{aligned}$$

where

$$\tilde{t}_0 = (\tau'_A+\tau_B)/2, \quad \tilde{t}_0 \in [0,1].$$

It is easy to find the following bounds

$$(5.19)\qquad \begin{aligned}&|\xi_G+\xi_E| + |\xi_F+\xi_D| \leqslant Ch^2\|x\|_{2,\infty},\\ &|\xi_E-\xi_G| + |\xi_D-\xi_F| \leqslant Ch\|x\|_{1,\infty}\end{aligned}$$

Let us find an estimate for $|t_1|$. We have

$$\begin{aligned}2t_1 &= \frac{-y_D}{y_D-y_E} + \frac{-y_F}{y_G-y_F} = \frac{y_Ey_F-y_Dy_G}{(y_D-y_E)(y_G-y_F)}\\ &= \frac{\frac{\partial y}{\partial X}(\theta_1)\frac{\partial y}{\partial Y}(\theta_2) - \frac{\partial y}{\partial X}(\theta_3)\frac{\partial y}{\partial Y}(\theta_4)}{\left[\frac{\partial y}{\partial Y}(\theta_3)+\frac{\partial y}{\partial X}(\theta_1)\right]\left[\frac{\partial y}{\partial Y}(\theta_4)+\frac{\partial y}{\partial X}(\theta_2)\right]} \equiv \frac{n(h)}{d(h)},\end{aligned}$$

where θ_i are some points on the square grid in accordance with the Taylor formula. It is readily seen (again from the Taylor formula) that

$$|n(h)| \leqslant Ch\|y\|^2_{2,\infty},$$

whereas

$$\lim_{h\to 0} d(h) = \left[\frac{\partial y}{\partial X}(0) + \frac{\partial y}{\partial Y}(0)\right]^2 > 0.$$

Consequently,

$$(5.20)\qquad |t_1| \leqslant Ch\|y\|^2_{2,\infty}$$

holds for sufficiently small h.

Combining (5.18)-(5.20), we get

$$(5.21)\qquad \begin{aligned}|a+b| &\leqslant |\xi_G+\xi_E| + |\xi_F+\xi_D| + |t_1|(|\xi_E-\xi_G| + |\xi_D-\xi_F|)\\ &\leqslant Ch^2(\|x\|_{2,\infty} + \|y\|^2_{2,\infty}\|x\|_{1,\infty}) = C_1h^2.\end{aligned}$$

Substituting (5.21) into (5.17) yields that

$$|\delta| \leqslant C\frac{h^2}{|ab|}.$$

Lemma 3.2 implies

$$(5.22)\qquad |\gamma| \leqslant \tfrac{1}{2}(|\beta| + |\alpha|) \leqslant C(|b|^{-1} + |a|^{-1}).$$

Since

$$\max(\xi^2, \eta^2, |\xi\eta|) \leqslant h_K^2$$

holds for all $(x, y) \in K$, we have

$$|v(G) + v(E)| \leqslant |v(G)| + |v(E)| \leqslant h_K^2 + h_{K'}^2 \quad .$$

Consequently,

$$(5.23)\qquad |\delta|\,|v(G) + v(E)| \leqslant Ch^2(h_K^2 + h_{K'}^2)|ab|^{-1}.$$

We show that

$$(5.24)\qquad (h_K^2 + h_{K'}^2)|ab|^{-1} \leqslant C$$

for sufficiently small h. Making use of (5.21) and [20, Lemma 2.2], we obtain

$$b \geqslant |a| - C_1 h^2 \geqslant |a| - C_2 \text{ meas } K.$$

Then

$$\frac{b}{h_K} \geqslant \frac{|a|}{h_K} - C_2 \frac{\text{meas } K}{h_K} \geqslant C^{-1} - C_2 \frac{\text{meas } K}{h_K} \geqslant \tfrac{1}{2} C^{-1} > 0$$

follows from (5.13) for sufficiently small h. An analogous estimate holds for $|a|/h_{K'}$ and therefore

$$\frac{h_K^2 + h_{K'}^2}{|a|b} \leqslant \tilde{C}\Big(\frac{h_K}{b} + \frac{h_{K'}}{|a|}\Big) \leqslant C.$$

Next, for $v \in \{\xi^2, \eta^2, \xi\eta\}$ we obtain

$$(5.25)\qquad |v(E) - v(G)| \leqslant Ch^3,$$

employing the estimates (5.19).

Combining (5.22), (5.25) and using [20, Lemma 2.2] and (5.13), we obtain

$$(5.26)\quad |\gamma|\,|v(E) - v(G)| \leqslant Ch^3(b^{-1} + |a|^{-1}) \leqslant C_1 h(\text{meas } K/|a| + \text{meas } K'/b)$$
$$C_2 h(h_K^2/|a| + h_{K'}^2/b) \leqslant C_3 h(h_K + h_{K'}) \leqslant Ch^2.$$

Finally,

$$|f_{11}(v)| \leqslant Ch^2$$

follows from (5.16), (5.23), (5.24) and (5.26).

Next we employ (5.20), Lemma 3.2 and (5.13) to derive

$$\begin{aligned}|f_{12}(v)| &\leqslant |t_1|(|\beta|\,|v(E)| + |\alpha|\,|v(G)|)\\ &\leqslant Ch(h_{K'}^2/b + h_K^2/|a|) \leqslant C_1 h(h_{K'} + h_K) \leqslant C_2 h^2.\end{aligned}$$

Consequently, we obtain

$$|f_1(v)| \leqslant |f_{11}(v)| + |f_{12}(v)| \leqslant Ch^2.$$

The estimate of $|f_2(v)|$ is completely analogous, as follows from (5.19) and the relation

$$\tau_A' - \tau_B = 1 - \tau_A - \tau_B = \tau_B' - \tau_A = -2t_1.$$

To find an estimate for a general function $v \in W_q^3(\Omega)$, we employ the following projection operator $\Pi\colon W_1^2(U) \to$ span $\{\xi^2, \eta^2, \xi\eta\}$,

$$\Pi v = (2 \text{ meas } U)^{-1}\Big[\xi^2 \int_U \frac{\partial^2 v}{\partial x^2}\, \mathrm{d}x\, \mathrm{d}y + 2\xi\eta \int_U \frac{\partial^2 v}{\partial x \partial y}\, \mathrm{d}x\, \mathrm{d}y + \eta^2 \int_U \frac{\partial^2 v}{\partial y^2}\, \mathrm{d}x\, \mathrm{d}y\Big].$$

Obviously, we have

$$f(v) = f(\Pi v) + f(v - \Pi v), \tag{5.27}$$

$$\begin{aligned}|f(\Pi v)| \leqslant \tfrac{1}{2}(\text{ meas } U)^{-1}\Big[|f(\xi^2)| \int_U \Big|\frac{\partial^2 v}{\partial x^2}\Big|\, \mathrm{d}x\, \mathrm{d}y + 2|f(\xi\eta)| \int_U \Big|\frac{\partial^2 v}{\partial x \partial y}\Big|\, \mathrm{d}x\, \mathrm{d}y \\ + |f(\eta^2)| \int_U \Big|\frac{\partial^2 v}{\partial y^2}\Big|\, \mathrm{d}x\, \mathrm{d}y\Big] \leqslant Ch^2 |v|_{2,\infty,U},\end{aligned} \tag{5.28}$$

using the above result.

For $f(v - \Pi v)$ we employ a special version of the Bramble-Hilbert theorem [5, Theorem 3]. Let us introduce

$$m = \min(|a|, b), \quad g(v) = mf(v - \Pi v).$$

Obviously,

$$\max\Big(\frac{m}{|a|}, \frac{m}{b}\Big) = 1, \quad m \leqslant 1 \quad \text{for sufficiently small } h,$$

so that by virtue of Lemma 3.2 we may write

$$|g(v)| \leqslant m|f(v)| + m|f(\Pi v)| \leqslant C|v|_{0,\infty,U} + Ch^2 |v|_{2,\infty,U}$$

and

$$h^2 \leqslant Ch_K^2 \leqslant C(\operatorname{diam} U)^2,$$

follows from (5.13).

Secondly,

$$g(p) = mf(p - \Pi p) = 0 \quad \forall p \in P_2(U)$$

holds due to the definition of Π.

Then [5, Theorem 3] yields that

$$|g(v)| \leqslant C(\operatorname{diam} U)^{3-2/q}|v|_{3,q,U} \tag{5.29}$$

holds when $q > 2$. Since

$$\operatorname{diam} U \leqslant 2 \max h_K \leqslant Ch \quad \text{and} \quad \frac{h}{m} \leqslant C$$

follows form (5.13), the estimate (5.29) implies

$$|f(v - \Pi v)| \leqslant m^{-1}|g(v)| \leqslant Ch^{2-2/q}|v|_{3,q,U}. \tag{5.30}$$

Combining (5.27), (5.28) and (5.30), we arrive at

$$|f(v)| \leqslant Ch^{2-2/q}(h^{2/q}|v|_{2,\infty,U} + |v|_{3,q,U}) \leqslant ch^{2-2/q}\|v\|_{3,q,U}.$$

Here we used the inequality

$$|v|_{2,\infty,U} \leqslant C(\operatorname{diam} U)^{-2/q}\|v\|_{3,q,U}.$$

□

Consider again the weak solution u of (5.1). Since $u(Z) = \pi_h u(Z)$ for all nodes, we have by Lemma 5.2 and the definition (3.2) that

$$\|G_h u(Z) - G_h(\pi_h u)(Z)\| \leqslant Ch^{2-2/q}\|u\|_{3,q,U}$$

for all $Z \in N_h \cap \Omega$. Hence, for the second term in (5.4) we get

$$\|G_h u - G_h(\pi_h u)\|_{0,2,\Omega_0} \leqslant Ch^2\|u\|_{3,q}, \tag{5.31}$$

using the same technique as in the proof of Theorem 3.8.

For the last term of (5.4) we obtain under the assumptions of Lemma 5.1 that

$$\|G_h(\pi_h u - u_h)\|_{0,2,\Omega_0} \leqslant \|G_h(\pi_h u - u_h)\|_{0,2} \leqslant C|\pi_h u - u_h|_{1,2}. \tag{5.32}$$

A sufficient condition for the $\mathscr{O}(h^2)$-convergence of the right-hand side is the inequality

$$a(u - \pi_h u, v_h) \leqslant C_1(u) h^2 |v_h|_{1,2} \quad \forall v_h \in V_h^0. \tag{5.33}$$

In fact, using the positive definiteness (5.2), (5.3) and (5.33), we may write

$$\begin{aligned} C_0 |u_h - \pi_h u|_{1,2}^2 &\leqslant a(u_h - \pi_h u, u_h - \pi_h u) \\ &= a(u - \pi_h u, u_h - \pi_h u) \leqslant C_1(u) h^2 |u_h - \pi_h u|_{1,2}. \end{aligned}$$

Combining this with (5.32), we get

$$\|G_h(\pi_h u - u_h)\|_{0,2,\Omega_0} \leqslant \tilde{C}(u) h^2. \tag{5.34}$$

Then (5.4), (5.5), (5.31) and (5.34) yield that

$$\|\operatorname{grad} u - G_h u_h\|_{0,2,\Omega_0} \leqslant C(u) h^2,$$

in other words, the weighted averaged gradient of the finite element solution is locally superconvergent.

Thus the whole problem reduces to the crucial inequality (5.33).

Following Levine [20] for $d = 2$, we obtain a class of superconvergent triangulations (the so-called quasiuniform triangulations) satisfying the inequality (5.33) with $C_1(u) = C\|u\|_{3,2}$. (Then $C(u) = C\|u\|_{3,q}$.) They are generated by smooth distortions (W^2_∞-diffeomorphism) from uniform triangulations of square grids of mesh-size h. The distorted triangulations have the following properties:

(i) precisely six elements meet at every node internal to the triangulation,

(ii) any quadrilateral formed by two adjacent elements is "almost parallelogram", as the distance of the midpoints of the diagonals is $\mathscr{O}(h^2)$.

Example 5.3. In optimal shape design, one meets domains (see e.g. [4])

$$\Omega(v) = \{(y_1, y_2) \in R^2 \mid 0 < y_1 < v(y_2),\ 0 < y_2 < 1\},$$

where $v \in W^2_\infty([0,1])$. This domain is usually approximated by a polygonal domain $\Omega(v_h)$, and $\Omega(v)$ can be transformed to the square $\hat{\Omega} = (0,1) \times (0,1)$ by the mapping

$$\begin{aligned} y_1 &= Y_1 v(Y_2), \quad (Y_1, Y_2) \in \hat{\Omega}, \\ y_2 &= Y_2. \end{aligned}$$

Let $n > 1$ be an integer, $h = 1/n$. Then the square uniform mesh of $\hat{\Omega}$ with the mesh-size h, which is triangulated by diagonals of slope $+1$, is mapped onto a superconvergent mesh of the domain $\Omega(v_h)$, if a positive C exists such that $v(y_2) \geqslant C$ for every $y_2 \in [0,1]$.

R e m a r k 5.4. In the forthcoming paper [13], we shall present other interior error estimates, where the global regularity assumption $u \in W_q^3(\Omega)$ is replaced by an interior regularity and some local error estimates up to the boundary.

6. Numerical tests

Let $u(x_1, x_2) = x_1 x_2 (1 - x_1)(1 - x_2)$ be the solution of the Poisson equation on the unit square $\Omega = (0,1) \times (0,1)$, i.e., $N_h^0 = \emptyset$. Let $u_h \in V_h$ be the finite element approximation of u over the triangulations of Figure 7. Since Δu is a quadratic polynomial, we were able to calculate the right-hand side of the associated Gram system exactly using an integration formula which is exact for all cubic polynomials. The Gram system of simultaneous equations was solved by a direct method. Thus the function u_h was computed exactly (except the rounding errors).

For $w = (w_1, w_2)^T \in (L^\infty(\Omega))^2$ we set

$$\|w\|_{\infty,\Omega_0} = \sum_{i=1}^{2} \operatorname*{ess\,sup}_{x \in \Omega_0} |w_i(x)|.$$

where $\Omega_0 = (0.15, 0.85) \times (0.15, 0.85)$. The averaged gradient was calculated by the formula (3.2). Note that continuous piecewise linear functions can be easily evaluated at any point $B_i \in \overline{\Omega}$. Table 1 illustrates the $\|.\|_{\infty,\Omega_0}$-norm of the following errors

$$e_h = \operatorname{grad}(u - u_h), \quad E_h = \operatorname{grad} u - G_h u_h,$$

even though the theory was done for the case $q < \infty$. Recall that $\operatorname{grad} u_h$ is piecewise constant whereas $G_h u_h$ is piecewise linear. We observe that the practical accuracy of the weighted averaged gradient seems to be almost $\mathcal{O}(h^2)$. Moreover, for all considered triangulations and all $Z \in \Omega_0 \cap N_h$ we have got

$$\|\operatorname{grad} u(Z) - G_h u(Z)\| \approx 10^{-12} \tag{6.1}$$

which is almost zero in computer arithmetics. This result follows from the formulae (3.13) and (3.14) which hold also for any biquadratic polynomial from $Q_2(\overline{\Omega})$. Hence, as in the proof of Lemma 3.6 we find that

$$G_h p(Z) = \operatorname{grad} p(Z) \quad \forall Z \in N_h \; \forall p \in Q_2(\overline{\Omega}),$$

which explains the result (6.1).

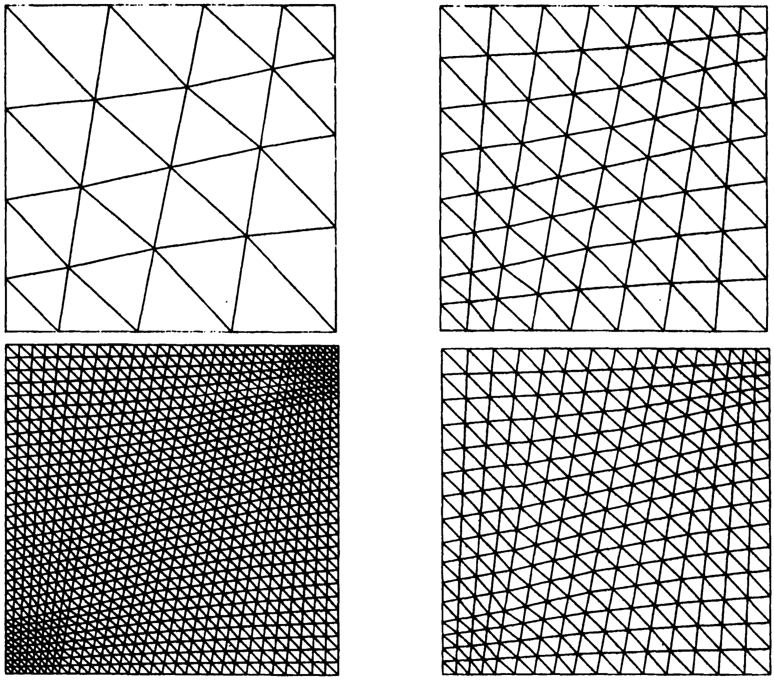

Figure 7

For "more nonuniform" meshes than in Figure 7 we usually observe only the $\mathcal{O}(h^{1+\varepsilon})$-superconvergence for some $\varepsilon \in (0,1)$.

h	$\lVert e_h \rVert_{\infty,\Omega_0}$	$\lVert E_h \rVert_{\infty,\Omega_0}$
0.3969	0.03903	0.01321
0.2111	0.02610	0.00450
0.1088	0.01492	0.00134
0.0552	0.00811	0.00043

Table 1

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