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REISSNER-MINDLIN MODEL FOR PLATES OF VARIABLE THICKNESS. SOLUTION BY MIXED-INTERPOLATED ELEMENTS

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Summary. Hard clamped and hard simply supported elastic plate is considered. The mixed finite element analysis combined with some interpolation, proposed by Brezzi, Fortin and Stenberg, is extended to the case of variable thickness and anisotropic material.

Keywords: Reissner-Mindlin plate model, mixed-interpolated elements

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INTRODUCTION

One of the simplest mathematical models of plate bending, which refine the well-known Kirchhoff theory, is the Reissner-Mindlin model. It replaces the classical hypothesis on the invariance of the fiber normal to the middle surface of the plate by introducing a new variable, the vector of rotation of the fibers normal to the midplane.

Although the formulation of the potential energy is straightforward, the standard variational solution deteriorates as the thickness t tends to zero. The reason is, that the potential energy of (transversal) shear stresses has a factor t, whereas the energy of remaining stresses a factor t^3. Brezzi, Fortin, Bathe and Stenberg [1], [2], [3], [4] proposed a remedy, using a mixed formulation by introducing a scaled shear force and an interpolation operator.

In the present paper, we extend the theory and error analysis of [2 – §VII. 3] to plates of variable thickness and anisotropic material. We consider "moderately" varying thickness functions, which are Lipschitz continuous, bounded from below and from above and have bounded derivatives. Two cases of boundary conditions are considered simultaneously, namely so called hard clamped and hard simply supported plates.

1. Formulation of the Reissner-Mindlin model

Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ with a polygonal boundary $\partial\Omega$. Let the plate occupy a three-dimensional domain $\Omega \times \big(-t(x_1,x_2), t(x_1,x_2)\big)$, where the (half-)thickness t belongs to the set

$$\mathscr{U}_{ad} = \Big\{ t \in C^{(0),1}(\bar{\Omega}) \text{ (i.e., Lipschitz function) } | \quad t_{\min} \leqslant t(x_1,x_2) \leqslant t_{\max}, \ \Big|\frac{\partial t}{\partial x_1}\Big| \leqslant C_1, \ \Big|\frac{\partial t}{\partial x_2}\Big| \leqslant C_2 \Big\}$$

where

$$0 < t_{\min} < t_{\max} < +\infty$$

and C_1, C_2, $t_{\min}$, $t_{\max}$ are given positive constants.

The fundamental hypothesis assumes that the "horizontal" displacements u_1 and u_2 have the form

$$u_i = -x_3\beta_i(x_1,x_2), \quad i = 1,2 \tag{1}$$

and the "vertical" displacement u_3 has the form

$$u_3 = w(x_1,x_2).$$

Assume zero body forces, and the external surface load

$$\boldsymbol{f} = (0,0,f)^{\mathrm{T}},$$

acting on the upper surface $x_3 = t(x_1,x_2)$.

In the following, we shall use Greek subscripts within the range $\{1,2\}$ and the summation convention for repeated subscripts. From (1) we easily obtain the components of the small strain tensor

$$\begin{gathered} e_{11} = -x_3\partial\beta_1/\partial x_1, \quad e_{22} = -x_3\partial\beta_2/\partial x_2, \quad e_{33} = 0, \\ e_{12} = -\frac{1}{2}x_3(\partial\beta_1/\partial x_2 + \partial\beta_2/\partial x_1), \\ e_{13} = \frac{1}{2}(\partial w/\partial x_1 - \beta_1), \quad e_{23} = \frac{1}{2}(\partial w/\partial x_2 - \beta_2). \end{gathered} \tag{2}$$

We assume the following generalized Hooke's law

$$\sigma_{\alpha\beta} = c_{\alpha\beta\gamma\delta} e_{\gamma\delta}, \quad (\alpha, \beta = 1, 2) \tag{3}$$

$$\sigma_{\alpha 3} = \mathscr{E}_{\alpha\beta} e_{\beta 3}, \tag{4}$$

where the coefficients $c_{\alpha\beta\gamma\delta}$, $\mathscr{E}_{\alpha\beta}$ are constant,

$$c_{\alpha\beta\gamma\delta} = c_{\gamma\delta\alpha\beta} = c_{\beta\alpha\gamma\delta}, \tag{5}$$

$$c_{\alpha\beta\gamma\delta}\tau_{\alpha\beta}\tau_{\gamma\delta} \geqslant c_0 \tau_{\alpha\beta}\tau_{\alpha\beta} \tag{6}$$

holds for all symmetric matrices $(\tau_{\alpha\beta})$, with some positive c_0; $\mathscr{E}$ is a diagonal matrix with positive entries. (For isotropic materials $\mathscr{E} = \lambda I$, $\lambda = Ek/(1+\sigma)$, where E is the Young's modulus and σ the Poisson's ratio, k is a correction factor).

The total potential energy of the plate is then

$$\begin{aligned} \Pi &= \int\limits_{\Omega\times(-t,t)} \Big[\frac{1}{2}\sigma_{\alpha\beta}e_{\alpha\beta} + \sigma_{\alpha 3}e_{\alpha 3}\Big]\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 - \int\limits_{\Omega} fw\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \frac{1}{3}\int\limits_{\Omega} t^3 c_{\alpha\beta\gamma\delta}\frac{\partial\beta_\alpha}{\partial x_\beta}\frac{\partial\beta_\gamma}{\partial x_\delta}\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\quad + \frac{1}{2}\int\limits_{\Omega} t\mathscr{E}_{\alpha\beta}\Big(\frac{\partial w}{\partial x_\alpha} - \beta_\alpha\Big)\Big(\frac{\partial w}{\partial x_\beta} - \beta_\beta\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int\limits_{\Omega} fw\,\mathrm{d}x_1\,\mathrm{d}x_2. \end{aligned} \tag{7}$$

Remark 1.1. For isotropic materials we have

$$\begin{aligned} \Pi = \frac{E}{3(1-\sigma^2)}\int\limits_{\Omega} t^3 &[(\partial\beta_1/\partial x_1 + \sigma\partial\beta_2/\partial x_2)\partial\beta_1/\partial x_1 \\ &+ (\partial\beta_2/\partial x_2 + \sigma\partial\beta_1/\partial x_1)\partial\beta_2/\partial x_2 \\ &+ \frac{1}{2}(1-\sigma)(\partial\beta_1/\partial x_2 + \partial\beta_2/\partial x_1)^2]\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &+ \frac{1}{2}\lambda\int\limits_{\Omega} t|\nabla w - \beta|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int\limits_{\Omega} fw\,\mathrm{d}x_1\,\mathrm{d}x_2. \end{aligned} \tag{8}$$

Introducing the bilinear form

$$\tilde{a}(t;\beta,\eta) = \frac{2}{3}\int\limits_{\Omega} t^3 c_{\alpha\beta\gamma\delta}(\partial\beta_\alpha/\partial x_\beta)(\partial\eta_\gamma/\partial x_\delta)\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

we may write the relation (7) in the following form

$$\Pi = \frac{1}{2}\tilde{a}(t;\beta,\beta) + \frac{1}{2}\int\limits_{\Omega} t(\nabla w - \beta)^{\mathrm{T}}\mathscr{E}(\nabla w - \beta)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int\limits_{\Omega} fw\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{9}$$

(Recall that the Kirchhoff model sets $\beta = \nabla w$, so that

$$\tilde{a}(t; \beta, \beta) = \tilde{a}(t; \nabla w, \nabla w)$$

coincides with the bending energy of the plate.)

We shall consider only the two following basic cases of boundary conditions:

(i) "*hard clamped*" edge of the plate

$$\beta = 0 \quad \text{and} \quad w = 0 \quad \text{on } \partial\Omega;$$

(ii) "*hard simply supported*" edge of the plate

$$M_\nu(\beta) = 0, \; \beta \cdot \tau = 0 \quad \text{and} \quad w = 0 \quad \text{on } \partial\Omega,$$

where τ denotes the unit tangential vector with respect to $\partial\Omega$, $\beta \cdot \tau = \beta_\alpha \tau_\alpha$, $M_\nu(\beta) = c_{\alpha\beta\gamma\delta}\nu_\alpha\nu_\beta\partial\beta_\gamma/\partial x_\delta$ and ν is the unit outward normal vector.

Thus the principle of minimum potential energy implies the minimization of $\Pi(\beta, w)$ on the set

$$\begin{aligned} [H_0^1(\Omega)]^2 \times H_0^1(\Omega), &\quad \text{in case (i)}, \\ V \times H_0^1(\Omega), &\quad \text{in case (ii)}, \end{aligned}$$

where

$$V = \{\beta \in [H^1(\Omega)]^2 \mid \beta \cdot \tau = 0 \quad \text{on } \partial\Omega\}.$$

Henceforth we denote by

$$(u, v) = \int_\Omega uv \, \mathrm{d}x_1 \, \mathrm{d}x_2, \quad \|u\|_0 = (u, u)^{1/2}$$

the inner product and the norm in the space $L^2(\Omega)$. The same notation will be used for vector functions from $[L^2(\Omega)]^2$. The norm in $H^1(\Omega)$ will be denote by $\|u\|_1$. In $H_0^1(\Omega)$ we shall use the equivalent norm

$$|u|_1 = \|\nabla u\|_0.$$

It is readily seen, that by

$$[u, v] = (t\mathscr{E}u, v)$$

another inner product in $[L^2(\Omega)]^2$ is defined, provided $t \in \mathscr{U}_{ad}$.

We introduce the operators

$$\underline{\operatorname{rot}}\, q = (\partial q/\partial x_2, -\partial q/\partial x_1)^{\mathrm{T}},$$
$$\operatorname{rot} \eta = \partial \eta_2/\partial x_1 - \partial \eta_1/\partial x_2),$$

and the space

$$H_0(\operatorname{rot}; \Omega) = \{v \in [L^2(\Omega)]^2 \mid \operatorname{rot} v \in L^2(\Omega),\ v \cdot \tau = 0 \text{ on } \partial\Omega\}$$

with the norm

$$\|v\|_{\mathrm{r}} = (\|v\|_0^2 + \|\operatorname{rot} v\|_0^2)^{1/2}. \tag{10}$$

Lemma 1.1. *Let either $\eta \in H_0(\operatorname{rot};\Omega)$, $q \in H^1(\Omega)$ or $\eta \in [L^2(\Omega)]^2$, $\operatorname{rot}\eta \in L^2(\Omega)$ and $q \in H_0^1(\Omega)$.*

Then

$$(\operatorname{rot}\eta, q) = (\eta, \underline{\operatorname{rot}}\, q).$$

In particular,

$$(\nabla u, \underline{\operatorname{rot}}\, q) = 0 \tag{11}$$

holds if $u \in H_0^1(\Omega)$, $q \in H^1(\Omega)$.

P r o o f. We have

$$\begin{aligned}(\operatorname{rot}\eta, q) &= \int_\Omega (-\eta_2 \partial q/\partial x_1 + \eta_1 \partial q/\partial x_2)\,\mathrm{d}x + \int_{\partial\Omega} (\eta_2\nu_1 - \eta_1\nu_2) q\,\mathrm{d}s \\ &= (\eta, \underline{\operatorname{rot}}\, q) + \int_{\partial\Omega} (\eta\cdot\tau) q\,\mathrm{d}s\end{aligned}$$

and the last integral vanishes, as either $\eta\cdot\tau = 0$ or $q = 0$ on $\partial\Omega$. If $u \in H_0^1(\Omega)$, then $\nabla u \in H_0(\operatorname{rot},\Omega)$ and $\operatorname{rot}\nabla u = 0$. □

Let $\varepsilon_{\alpha\beta}(\eta)$ be the symmetric part of the matrix $(\partial\eta_\alpha/\partial x_\beta)$.

Lemma 1.2. *For all $\eta \in V$ the inequality*

$$\int_\Omega c_{\alpha\beta\gamma\delta} \frac{\partial \eta_\alpha}{\partial x_\beta}\frac{\partial\eta_\gamma}{\partial x_\delta}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_\Omega c_{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}(\eta)\varepsilon_{\gamma\delta}(\eta)\,\mathrm{d}x_1\,\mathrm{d}x_2 \geqslant C\|\eta\|_1^2$$

holds with some positive constant C.

Proof. From (5) and (6) we easily obtain that the left-hand side is bounded below by

$$c_0 \int_\Omega \sum_{\alpha,\beta=1}^{2} \varepsilon_{\alpha\beta}^2(\eta)\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{12}$$

Let us consider the subspace of rigid body (2D) displacements

$$\mathscr{P} = \{\boldsymbol{v} = (a_1 - bx_2, a_2 + bx_1)^{\mathrm{T}} = \boldsymbol{a} + b\boldsymbol{k}\times x,\ a \in \mathbb{R}^2,\ b \in \mathbb{R}\}, \tag{13}$$

where $\boldsymbol{k}$ is the unit vector of x_3-axis.

Let us verify that

$$\mathscr{P} \cap V = \{0\}. \tag{14}$$

In fact, denoting $\boldsymbol{a}^\perp = (-a_2, a_1)^{\mathrm{T}} = \boldsymbol{k} \times \boldsymbol{a}$, we may write

$$\boldsymbol{v} \cdot \tau = -\nu \cdot \boldsymbol{a}^\perp - bx \cdot \nu = \nu \cdot (-\boldsymbol{a}^\perp - bx).$$

Let the origin coincide with the vertex $A \in \partial\Omega$ (see Fig. 1). Then for $\boldsymbol{v} \in \mathscr{P} \cap V$ we obtain

$$-\nu^{(1)} \cdot \boldsymbol{a}^\perp = 0, \quad -\nu^{(2)} \cdot \boldsymbol{a}^\perp = 0$$

so that $\boldsymbol{a}^\perp = 0$ and $\boldsymbol{a} = 0$. Inserting $\boldsymbol{x} \in \overline{BC}$, we have

$$b\nu^{(3)} \cdot \boldsymbol{x} = 0, \quad \nu^{(3)} \cdot \boldsymbol{x} \neq 0$$

so that $b = 0$ follows. Consequently, (14) holds.

Having (14), we may apply the general result on inequalities of Korn's type [6 – Lemma 11.3.2], which says that (12) is the square of an equivalent norm in V. □

Lemma 1.3. *Positive constants c_1, c_2, c_3 exist such that the inequality*

$$\tilde{a}(t;\beta,\beta) + [\nabla w - \beta, \nabla w - \beta] \geqslant \frac{c_1 t_{\min}^3}{c_2 + c_3 t_{\min}^2}(\|\beta\|_1^2 + |w|_1^2) \tag{15}$$

holds for all $w \in H_0^1(\Omega)$ and $\beta \in V$, $t \in \mathscr{U}_{ad}$.

Proof. Lemma 1.2 and the definition of $\mathscr{U}_{ad}$ yield

$$\tilde{a}(t;\beta,\beta) \geqslant \frac{2}{3} t_{\min}^3 \int_\Omega c_{\alpha\beta\gamma\delta} \frac{\partial \beta_\alpha}{\partial x_\beta} \frac{\partial \beta_\gamma}{\partial x_\delta}\,\mathrm{d}x_1\,\mathrm{d}x_2 \geqslant c_K t_{\min}^3 \|\beta\|_1^2 \quad \forall \beta \in V. \tag{16}$$

Second, we have

$$[\nabla w - \beta, \nabla w - \beta] \geqslant c_E t_{\min} \|\nabla w - \beta\|_0^2.$$

Obviously, we may write ($t_{\min} \equiv t_m$)

$$(17)\qquad \begin{aligned} \frac{1}{2}|w|_1^2 &\leqslant \|\nabla w - \beta\|_0^2 + \|\beta\|_0^2 \\ &\leqslant t_m^{-3} c_K^{-1} \tilde{a}(t; \beta, \beta) + t_m^{-1} c_E^{-1} [\nabla w - \beta, \nabla w - \beta] \\ &\leqslant A(t_m^{-3} c_K^{-1} + t_m^{-1} c_E^{-1}) \end{aligned}$$

where A denotes the left-hand side of (15). Combining (16) and (17), we arrive at

$$\|\beta\|_1^2 + |w|_1^2 \leqslant A(2c_K t_m^2 + 3c_E)/(c_E c_K t_m^3)$$

so that (15) follows. □

Remark 1.2. Since $[H_0^1(\Omega)]^2 \subset V$, the inequality (15) holds for all $w \in H_0^1(\Omega)$, $\beta \in [H_0^1(\Omega)]^2$ and $t \in \mathscr{U}_{ad}$, as well.

Proposition 1.1. *Let Ω be a convex polygon. Then the mapping*

$$B\colon (\eta, \zeta) \to (\nabla \zeta - \eta)$$

is surjective from $[H_0^1(\Omega)]^2 \times H_0^1(\Omega)$ onto $H_0(\text{rot}; \Omega)$ or from $V \times H_0^1(\Omega)$ onto $H_0(\text{rot}; \Omega)$.

The *proof* has been given in [2 – Propos. VII. 3.2, p. 298] for $\eta \in [H_0^1(\Omega)]^2$. Since $[H_0^1(\Omega)]^2 \subset V$ and for $\eta \in V$

$$(\nabla \zeta - \eta) \cdot \tau = \partial \zeta / \partial s - \eta \cdot \tau = 0 \quad \text{on } \partial\Omega,$$

the assertion holds for $\eta \in V$, as well.

Proposition 1.2. *Any element $\gamma \in [L^2(\Omega)]^2$ can be written in a unique way as*

$$\gamma = t\mathscr{E}\nabla\psi + \underline{\text{rot}}\, p,$$

with $\psi \in H_0^1(\Omega)$, $p \in H^1(\Omega)/\mathbb{R}$.

Proof. (cf. [2 – Propos. VII. 3.4]). Let us set $\xi = \operatorname{div} \gamma \in H^{-1}(\Omega)$. Consider the solution $\psi \in H_0^1(\Omega)$ of the Dirichlet problem

$$\operatorname{div}(t\mathscr{E}\nabla\psi) = \xi.$$

Defining $\alpha := \gamma - t\mathscr{E}\nabla\psi$, we realize that $\operatorname{div}\alpha = 0$, so that $\alpha = \underline{\text{rot}}\, p$ for some $p \in H^1(\Omega)/\mathbb{R}$ (see [5 – Theorem I.3.1, p. 37]). □

Approximations, based on the minimum of potential energy (9) and finite elements, cannot be recommended. They deteriorate as the thickness tends to zero. Theoretically, such phenomenon is caused by the fact, that the "constant of coercivity" in (15) is $O(t_{\min}^3)$.

To overcome the troubles, we apply three following steps [2]. First, we define a scaled potential energy

$$\Pi_o = t_{\min}^{-3}\Pi.$$

Second, we introduce a mixed variational formulation with the help of a "scaled shear force"

$$\gamma := t\mathscr{E}(\nabla w - \beta)/t_{\min}^3. \tag{18}$$

(Note that γ is proportional to the shearing force, i.e., to the integral

$$\int_{-t}^{t} \sigma_{\alpha 3}\,\mathrm{d}x_3).$$

We preserve the first part of Π_o, and denote it by

$$\frac{1}{2}a(t;\beta,\beta) := \frac{1}{2}t_{\min}^{-3}\tilde{a}(t;\beta,\beta).$$

The second part — energy of shear stresses — will be transformed as in the principle of Hellinger-Reissner (see [6 – §5.4]). Thus we arrive at a new functional

$$\mathscr{R}(\beta, w;\gamma) = \frac{1}{2}a(t;\beta,\beta) + (\gamma, \nabla w - \beta) - \frac{1}{2}t_m^3\Big(\frac{1}{t}\mathscr{E}^{-1}\gamma,\gamma\Big) - t_m^{-3}(f,w).$$

(For brevity, we denote $t_m := t_{\min}$). The solution of our problem $\{\beta, w\}$ and γ is a saddle point of $\mathscr{R}$ and we have the following conditions of optimality

$$a(t;\beta,\eta) + (\gamma,\nabla\zeta - \eta) - t_m^{-3}(f,\zeta) = 0 \quad \forall\{\eta,\zeta\} \in [H_0^1(\Omega)]^2 \times H_0^1(\Omega), \quad \text{or } \forall\{\eta,\zeta\} \in V \times H_0^1(\Omega), \tag{19}$$

$$(\nabla w - \beta, \delta) - t_m^3\Big(\frac{1}{t}\mathscr{E}^{-1}\gamma,\delta\Big) = 0 \quad \forall \delta \in [L^2(\Omega)]^2. \tag{20}$$

Let us insert the decomposition of γ by Proposition 1.2 and the analogous decomposition of δ. Thus we obtain the following system for ψ, β, p, w:

$$[\nabla\psi, \nabla\zeta] = t_m^{-3}(f, \zeta) \quad \forall\zeta \in H_0^1(\Omega), \tag{21}$$

$$a(t; \beta, \eta) - (\underline{\operatorname{rot}}\, p, \eta) = [\nabla\psi, \eta] \quad \forall\eta \in [H_0^1(\Omega)]^2 \text{ or } \forall\eta \in V; \tag{22}$$

$$-(\beta, \underline{\operatorname{rot}}\, q) = t_m^3\Big(\frac{1}{t}\mathscr{E}^{-1}\,\underline{\operatorname{rot}}\, p, \underline{\operatorname{rot}}\, q\Big) \quad \forall q \in H^1(\Omega)/\mathbb{R}, \tag{23}$$

$$[\nabla w, \nabla\chi] = [\beta, \nabla\chi] + (f, \chi) \quad \forall\chi \in H_0^1(\Omega). \tag{24}$$

Note that (21) and (24) is a Dirichlet problem for ψ and w, respectively. The system (22)–(23) represents a "Stokes-like" problem for β and p, "penalized" by the term on the right-hand side of (23). Indeed, to see this, we replace the vector-functions η by the rotated vectors $\eta^{\perp}$, so that for instance

$$-(\underline{\operatorname{rot}}\, p, \eta) = -(p, \operatorname{rot}\eta) = (p, \operatorname{div}\eta^{\perp}).$$

On the basis of (21)-(24) a discretization by finite element method can be proposed and an error estimate derived [4]. Such a numerical method, however, is not used in practice, even if a Stokes solver is available. The solution for small thickness displays troubles, since the condition $(\nabla w - \beta) = 0$ is enforced too much.

Therefore, we apply (third step) a kind of numerical integration in the second term of the potential energy. Such an approach is nearer to the engineering practice. The details will be shown in the next Section.

3. Solution by mixed-interpolated finite elements

In the present section we apply an example [2] of the so-called mixed-interpolated elements for Reissner–Mindlin plates, which were proposed by Bathe, Brezzi and Fortin in [1] and analyzed by Brezzi, Fortin and Stenberg in [3]. The error analysis will be extended to the plates of variable thickness, anisotropic materials and hard simply supported edges.

Let us consider a regular family of triangulations $\{\mathrm{T}_h\}$, $h \to 0$, of the domain Ω.

Let k be a non-negative integer, s a positive integer; we denote by $\mathscr{L}_s^k$ the space of piecewise polynomials on T_h of degree $\leqslant s$, which belong to $H^k(\Omega)$, $\big(H^0(\Omega) \equiv L^2(\Omega)\big)$. Let B_3 be the space of "bubble functions" on T_h of the third degree, i.e.,

$$B_3 = \{v | v|_K \in P_3(K) \cap H_0^1(K) \text{ for all triangles } K \in \mathrm{T}_h\}.$$

Let H_h be the intersection of

$$(\mathscr{L}_2^1 \oplus B_3)^2$$

with $[H_0^1(\Omega)]^2$ or V, respectively, (Crouzeix–Raviart element),

$$W_h = \mathscr{L}_2^1 \cap H_0^1(\Omega), \quad Q_h = \mathscr{L}_1^0/\mathbb{R},$$
$$\Gamma_h = (RT_1)^\perp \cap H_0(\mathrm{rot};\Omega),$$

where RT_1 denotes the space of Raviart–Thomas elements of the first degree and $(.)^\perp$ denotes the rotation by $\pi/2$, defined by $a^\perp = (-a_2, a_1)^T = \boldsymbol{k} \times \boldsymbol{a}$.

Recall that [2 – p. 116] the restrictions on the triangles are defined by

$$RT_1(K) = \big(P_1(K)\big)^2 + xP_1(K) \quad \forall K \in \mathrm{T}_h$$

and $RT_1 \subset H(\mathrm{div};\Omega)$, i.e., the degrees of freedom have been chosen in order to ensure continuity of the flux $\gamma \cdot \nu$ at interfaces of elements.

Note that $\nabla W_h \subset \Gamma_h$.

Following [2 – p. 312] we define the interpolation $\Pi_h \colon H_h \to (RT_1)^\perp$ by means of

$$\int_e (\eta_h - \Pi_h\eta_h)\cdot \tau\mu_1 \,\mathrm{d}s = 0 \quad \forall \mu_1 \in P_1(e) \tag{25}$$

for all sides $e \in \partial K \in \mathrm{T}_h$, and

$$\int_K (\eta_h - \Pi_h\eta_h)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0 \quad \forall K \in \mathrm{T}_h. \tag{26}$$

We can show, that $\Pi_h \colon H_h \to \Gamma_h$, i.e. the interpolant $\Pi_h\eta_h$ satisfies the boundary condition $\Pi_h\eta_h \cdot \tau = 0$. In fact, the traces of $\gamma\cdot\tau$ for $\gamma \in (RT_1)^\perp$ on any $e \in \partial K \cap \partial\Omega$ are linear polynomials [2 – Prop. III.3.2, p. 116], so that (25) is sufficient to guarantee the zero trace of $\gamma \cdot \tau$.

Moreover, we have the estimate

$$\|\Pi_h\eta_h\|_0 \leqslant \|\eta_h\|_1 \quad \forall \eta_h \in H_h \tag{27}$$

(see [2 – p. 313 and Prop. III.3.9, p. 132]) and

$$\big(q_h, \mathrm{rot}(\Pi_h\eta_h - \eta_h)\big) = 0 \quad \forall q_h \in Q_h,\ \eta_h \in H_h \tag{28}$$

(see [2 – Prop. III.3.7, p. 129]).

Lemma 3.1. [2 – Propos. VII.3.10, p. 315]. *For any $g_h \in \Gamma_h$ there exist $\psi_h \in W_h$ and a unique $p_h \in Q_h$ such that*

$$[g_h, \delta_h] = [\nabla\psi_h, \delta_h] + (p_h, \mathrm{rot}\,\delta_h) \quad \forall \delta_h \in \Gamma_h.$$

Proof. Let us consider the following mixed problem: find $\alpha_h \in \Gamma_h$ and $p_h \in Q_h$ such that

$$[\alpha_h, \delta_h] - (p_h, \operatorname{rot} \delta_h) = 0 \quad \forall \delta_h \in \Gamma_h, \tag{29}$$

$$(\operatorname{rot} \alpha_h, q_h) = (\operatorname{rot} g_h, q_h) \quad \forall q_h \in Q_h. \tag{30}$$

Since the pair of spaces Γ_h, Q_h fulfils the so-called inf-sup condition [2 – p. 138], the problem has a unique solution. By virtue of the fact, that $\operatorname{rot}(g_h - \alpha_h) \in Q_h$, (30) implies $\operatorname{rot}(g_h - \alpha_h) = 0$. Then a $\psi_h \in W_h$ exists such that $g_h - \alpha_h = \nabla\psi_h$ (see [2 – Corol. III.3.2, p. 117] and [5 – Theorem I.3.1, p. 37]). Then from (29) we obtain for all $\delta_h \in \Gamma_h$

$$[g_h, \delta_h] = [\nabla\psi_h, \delta_h] + [\alpha_h, \delta_h] = [\nabla\psi_h, \delta_h] + (p_h, \operatorname{rot} \delta_h).$$

□

In order to overcome the above mentioned "shear locking" effect of numerical solution, instead of the minimization of the functional Π_o it is suitable to consider

$$\min_{\{\beta_h, w_h\} \in H_h \times W_h} \left\{ \frac{1}{2} a(t; \beta_h, \beta_h) + \frac{1}{2} t_m^{-3} [\nabla w_h - \Pi_h \beta_h, \nabla w_h - \Pi_h \beta_h] - t_m^{-3} (f, w_h) \right\}. \tag{31}$$

Since by means of Lemma 1.2, (16) and (27) we can prove that the functional is strictly convex and continuous, there exists a unique minimizer. The optimality conditions of the problem (31) are

$$a(t; \beta_h, \eta_h) - t_m^{-3} [\nabla w_h - \Pi_h \beta_h, \Pi_h \eta_h] = 0 \quad \forall \eta_h \in H_h, \tag{32}$$

$$[\nabla w_h - \Pi_h \beta_h, \nabla \varphi_h] = (f, \varphi_h) \quad \forall \varphi_h \in W_h \tag{33}$$

Proposition 3.1. *Let $\{\beta_h, w_h\}$ be the solution of (32)–(33), i.e., the minimizer of (31). Then the function*

$$g_h = t_m^{-3} (\nabla w_h - \Pi_h \beta_h) \tag{34}$$

belongs to Γ_h and can be decomposed according to the formula

$$g_h = \nabla\psi_h + \alpha_h, \tag{35}$$

where $\psi_h \in W_h$ and α_h belongs to the orthocomplement of ∇W_h in Γ_h (with respect to the inner product $[.,.]$).

Moreover, there exists a unique $p_h \in Q_h$ such that $\{\beta_h, w_h, \psi_h, \alpha_h, p_h\} \in H_h \times W_h \times W_h \times \Gamma_h \times Q_h$ satisfy the following system

$$[\nabla\psi_h, \nabla\varphi_h] = t_m^{-3}(f, \varphi_h) \quad \forall \varphi_h \in W_h, \tag{36}$$

$$a(t; \beta_h, \eta_h) - (p_h, \operatorname{rot} \eta_h) = [\nabla\psi_h, \Pi_h \eta_h] \quad \forall \eta_h \in H_h, \tag{37}$$

$$(\operatorname{rot} \beta_h, q_h) + t_m^3(\operatorname{rot} \alpha_h, q_h) = 0 \quad \forall q_h \in Q_h, \tag{38}$$

$$[\alpha_h, \delta_h] - (p_h, \operatorname{rot} \delta_h) = 0 \quad \forall \delta_h \in \Gamma_h, \tag{39}$$

$$[\nabla w_h, \nabla\varphi_h] = [\Pi_h \beta_h, \nabla\varphi_h] + (f, \varphi_h) \quad \forall \varphi_h \in W_h. \tag{40}$$

Proof. The system (32)–(33) is equivalent to the following three equations (a mixed formulation, cf. (19)–(20))

$$a(t; \beta_h, \eta_h) - [g_h, \Pi_h \eta_h] = 0 \quad \forall \eta_h \in H_h, \tag{41}$$

$$[g_h, \nabla\varphi_h] = t_m^{-3}(f, \varphi_h) \quad \forall \varphi_h \in W_h, \tag{42}$$

$$[g_h, \delta_h] = t_m^{-3}[\nabla w_h - \Pi_h \beta_h, \delta_h] \quad \forall \delta_h \in \Gamma_h. \tag{43}$$

Using Lemma 3.1 and substituting $\delta_h := \nabla\varphi_h$, $\varphi_h \in W_h$ into (42), we obtain (36). The same Lemma in (41) and (28) yield (37). From (43) we derive with the help of Lemma 3.1 for $\delta_h := \nabla\varphi_h$, $\varphi_h \in W_h$

$$[\nabla\psi_h, \nabla\varphi_h] = t_m^{-3}[\nabla w_h - \Pi_h \beta_h, \nabla\varphi_h].$$

If we substitute here from (36), we arrive at (40).

Let us choose any $\delta_h \in (\nabla W_h)^\perp$, i.e., $\delta_h \in \Gamma_h$ such that $[\delta_h, \nabla\varphi_h] = 0 \ \forall \varphi_h \in W_h$. Then Lemma 3.1 and (43) imply that

$$(p_h, \operatorname{rot} \delta_h) = -t_m^{-3}[\Pi_h \beta_h, \delta_h] \quad \forall \delta_h \in (\nabla W_h)^\perp. \tag{44}$$

The function $\alpha_h \in \Gamma_h$ fullfils (39), as follows from (35) and Lemma 3.1 by comparison. Then (44) can be rewritten as

$$[\alpha_h, \delta_h] = -t_m^{-3}[\Pi_h \beta_h, \delta_h] \quad \forall \delta_h \in (\nabla W_h)^\perp. \tag{45}$$

From Lemma 3.1, however, we deduce that for any $\delta_h \in (\nabla W_h)^\perp$ there exists a unique $q_h \in Q_h$, such that

$$[\delta_h, \chi_h] = (q_h, \operatorname{rot} \chi_h) \quad \forall \chi_h \in (\nabla W_h)^\perp. \tag{46}$$

Since from (39) $\alpha_h \in (\nabla W_h)^\perp$ follows, we may write, using (46), (45) and (28),

$$\begin{aligned}(q_h, \operatorname{rot} \alpha_h) &= [\delta_h, \alpha_h] = -t_m^{-3}[\Pi_h \beta_h, \delta_h] \\ &= -t_m^{-3}(q_h, \operatorname{rot} \Pi_h \beta_h) = -t_m^{-3}(q_h, \operatorname{rot} \beta_h) \quad \forall \delta_h \in (\nabla W_h)^\perp.\end{aligned}$$

Consequently, (38) is satisfied for all $q_h \in Q_h$, since for any $q_h \in Q_h$ there exists $\delta_h \in (\nabla W_h)^\perp$ such that (46) holds. In fact, the equation (46) is uniquely solvable for δ_h. (For a fixed parameter h, all norms are equivalent, so that $\|\operatorname{rot} \chi_h\|_0 \leqslant \|\chi_h\|_1 \leqslant C\|\chi_h\|_0$, a.s.o.) The uniqueness of the decomposition (35) is obvious.

The existence and uniqueness of $p_h \in Q_h$ follows from the proof of Lemma 3.1.

□

Proposition 3.2. *The system* (36)–(40) *has a unique solution in* $H_h \times W_h \times W_h \times \Gamma_h \times Q_h$.

Proof. Since the existence was proved in Proposition 3.1, it remains to verify the uniqueness. Let $\{\Delta\beta_h, \Delta w_h, \Delta\psi_h, \Delta\alpha_h, \Delta p_h\}$ be the difference of two solutions. Let us drop out the subscripts h in what follows.

From (3.6) $\Delta\psi = 0$ follows immediately. We have therefore

$$\begin{aligned} a(t; \Delta\beta, \eta) - (\Delta p, \operatorname{rot} \eta) &= 0 \quad \forall \eta \in H_h, \\ (\operatorname{rot} \Delta\beta, q) + t_m^3(\operatorname{rot} \Delta\alpha, q) &= 0 \quad \forall q \in Q_h, \\ t_m^3[\Delta\alpha, \delta] - t_m^3(\Delta p, \operatorname{rot} \delta) &= 0 \quad \forall \delta \in \Gamma_h. \end{aligned}$$

Inserting $\eta := \Delta\beta$, $q = \Delta p$, $\delta = \Delta\alpha$ and summing these three equations, we arrive at

$$a(t; \Delta\beta, \Delta\beta) + t_m^3[\Delta\alpha, \Delta\alpha] = 0,$$

so that $\Delta\beta = 0$ and $\Delta\alpha = 0$ follows by (16).

Form (40) we derive

$$[\nabla\Delta w, \nabla\Delta w] = [\Pi\Delta\beta, \nabla\Delta w] = 0,$$

so that $\Delta w = 0$.

Finally, (37) yields that

$$(\Delta p, \operatorname{rot} \eta) = 0 \quad \forall \eta \in H_h.$$

Since the pair $\{H_h, Q_h\}$ satisfies the inf-sup condition for the "Stokes-like" problem, [2 – chapt. VI, p. 205, (2.13)], we have

$$k_0\|\Delta p\|_{0/R} \leqslant \sup_{\eta \in H_h} (\Delta p, \operatorname{rot} \eta)/\|\eta\|_1 = 0$$

and $\Delta p = 0$ follows in Q_h. □

Theorem 3.1. *Let β, w, γ be the solution of* (19)–(20). *Let γ be decomposed by means of Proposition* 1.2 *and let us set*

$$\alpha = \frac{1}{t}\mathscr{E}^{-1}\underline{\mathrm{rot}}p. \tag{47}$$

Let $\{\beta_h, w_h, \psi_h, \alpha_h, p_h\}$ be the solution of (36)–(40).
Then positive constants C_0, C exist, independent of $t \in \mathscr{U}_{ad}$ and such that

$$\begin{aligned}
&\|\beta_h - \beta\|_1 + \|w_h - w\|_1 + \|\psi_h - \psi\|_1 + \|\alpha_h - \alpha\|_0 + \|p_h - p\|_{0/R} \\
&\leqslant C_0\{\inf_{\eta_h \in H_h} \|\beta - \eta_h\|_1 + \inf_{\zeta_h \in W_h} \|w - \zeta_h\|_1 + \inf_{\varphi_h \in W_h} \|\psi - \varphi_h\|_1 \\
&\quad + \inf_{q_h \in Q_h} \|p - q_h\|_{0/R} + \inf_{\delta_h \in \Gamma_h} \|\alpha - \delta_h\|_\Gamma + \|\beta - \Pi_h\beta\|_0 \\
&\quad + \sup_{\eta_h \in H_h} [\nabla\psi, \eta_h - \Pi_h\eta_h]/\|\eta_h\|_1\} \\
&\leqslant Ch^s\{\|\beta\|_{s+1} + \|w\|_{s+1} + \|\psi\|_{s+1} + \|p\|_{s/R} + \|\alpha\|_s + \|\operatorname{rot}\alpha\|_s\}, \quad 1 \leqslant s \leqslant 2.
\end{aligned}$$

P r o o f. (i) It is easy to see that the functions β, w, ψ, p, α satisfy the following system

$$[\nabla\psi, \nabla\varphi] = t_m^{-3}(f, \varphi) \quad \forall \varphi \in H_0^1(\Omega) \tag{48}$$

$$a(t; \beta, \eta) - (p, \operatorname{rot}\eta) = [\nabla\psi, \eta] \quad \forall \eta \in [H_0^1(\Omega)]^2 \text{ or } \forall \eta \in V, \tag{49}$$

$$(\operatorname{rot}\beta, q) + t_m^3(\operatorname{rot}\alpha, q) = 0 \quad \forall q \in L^2(\Omega), \tag{50}$$

$$[\alpha, \delta] - (p, \operatorname{rot}\delta) = 0 \quad \forall \delta \in H_0(\mathrm{rot}; \Omega), \tag{51}$$

$$[\nabla w, \nabla\varphi] = [\beta, \nabla\varphi] + (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega). \tag{52}$$

In fact, (48) and (52) follows from (21) and (24), respectively. Using Lemma 1.1 in (22) (note that both $(H_0^1)^2$ and V are subspaces of $H_0(\mathrm{rot}, \Omega)$), we arrive at (49). The equation (50) follows from (23), (47), starting with $q \in C_0^\infty(\Omega)$ and employing Lemma 1.1 and the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$. We easily derive (51) from (47) and Lemma 1.1.

Let us define "intermediate" approximations

$$\{\beta_h^I, w_h^I, \psi_h^I, p_h^I, \alpha_h^I\} \in H_h \times W_h \times W_h \times Q_h \times \Gamma_h,$$

(close to β, w, ψ, p, α) in the following way.

We set (cf. (36))

$$\psi_h^I = \psi_h \tag{53}$$

and define β_h^I, $\tilde{p}_h^I$ as the solution of the "Stokes–like" discrete problem in $H_h \times Q_h$

$$a(t;\beta_h^I,\eta_h) + (\tilde{p}_h^I, \operatorname{rot}\eta_h) = a(t;\beta,\eta_h) \quad \forall \eta_h \in H_h, \tag{54}$$

$$(\operatorname{rot}\beta_h^I, q_h) = (\operatorname{rot}\beta, q_h) \quad \forall q_h \in Q_h. \tag{55}$$

For $\{\alpha_h^I, p_h^I\} \in \Gamma_h \times Q_h$ we define another mixed problem

$$[\alpha_h^I, \delta_h] - (p_h^I, \operatorname{rot}\delta_h) = 0 \quad \forall \delta_h \in \Gamma_h, \tag{56}$$

$$-t_m^3(\operatorname{rot}\alpha_h^I, q_h) = (\operatorname{rot}\beta_h^I, q_h) \quad \forall q_h \in Q_h. \tag{57}$$

Finally, we define w_h^I by the equation

$$[\nabla w_h^I, \nabla\varphi_h] = [\Pi_h\beta, \nabla\varphi_h] + (f,\varphi) \quad \forall \varphi_h \in W_h. \tag{58}$$

The differences $\{\beta_h - \beta_h^I, \dots, \alpha_h - \alpha_h^I\}$ satisfy the following system

$$\begin{aligned} a(t;\beta_h - \beta_h^I, \eta_h) - (p_h - p_h^I, \operatorname{rot}\eta_h) &= a(t;\beta - \beta_h^I, \eta_h) \\ - (p - p_h^I, \operatorname{rot}\eta_h) - [\nabla\psi, \eta_h - \Pi_h\eta_h] &+ [\nabla\psi_h - \nabla\psi, \Pi_h\eta_h] \quad \forall \eta_h \in H_h, \end{aligned} \tag{59}$$

$$\big(\operatorname{rot}(\beta_h - \beta_h^I), q_h\big) + t_m^3\big(\operatorname{rot}(\alpha_h - \alpha_h^I), q_h\big) = 0 \quad \forall q_h \in Q_h, \tag{60}$$

$$[\alpha_h - \alpha_h^I, \delta_h] - (p_h - p_h^I, \operatorname{rot}\delta_h) = 0 \quad \forall \delta_h \in \Gamma_h, \tag{61}$$

$$[\nabla(w_h - w_h^I), \nabla\varphi_h] = [\Pi_h\beta_h - \Pi_h\beta, \nabla\varphi_h] \quad \forall \varphi_h \in W_h. \tag{62}$$

From (21), (53) and (36) we get

$$\|\psi - \psi_h\|_1 \leqslant C \inf_{\varphi_h \in W_h} \|\psi - \varphi_h\|_1. \tag{63}$$

Let us insert $\eta_h := \beta_h - \beta_h^I \equiv \Delta\beta_h$ into (59), $q_h = p_h - p_h^I \equiv \Delta p_h$ into (60) and $\delta_h := (\alpha_h - \alpha_h^I) = \Delta\alpha_h$ into (61). Summing these three equations, we obtain

$$\begin{aligned} a(t;\Delta\beta_h,\Delta\beta_h) + [\Delta\alpha_h,\Delta\alpha_h] &= a(t;\beta - \beta_h^I, \Delta\beta_h) \\ - (p - p_h^I, \operatorname{rot}\Delta\beta_h) - [\nabla\psi, \Delta\beta_h - \Pi_h\Delta\beta_h] &+ [\nabla(\psi_h - \psi), \Pi_h\Delta\beta_h]. \end{aligned}$$

Using (16), (27) and the obvious estimates, we may write

$$\begin{aligned} C_K\|\Delta\beta_h\|_1^2 + C_E t_{\min}\|\Delta\alpha_h\|_0^2 \leqslant C_3\{&\|\beta - \beta_h^I\|_1 + \|p - p_h^I\|_{0/R} \\ + \|\psi - \psi_h\|_1 + &\sup_{\eta_h \in H_h}[\nabla\psi, \eta_h - \Pi_h\eta_h]/\|\eta_h\|_1\}\|\Delta\beta_h\|_1. \end{aligned} \tag{64}$$

Next, let us show that β_h^I is an optimal approximation of β, i.e.,

$$\|\beta - \beta_h^I\|_1 \leqslant C \inf_{\eta_h \in H_h} \|\beta - \eta_1\|_1. \tag{65}$$

Indeed, substituting from (49) into (54) and from (50) into (55), we have

$$\begin{aligned} a(t; \beta_h^I, \eta_h) + (\tilde{p}_h^I - p, \operatorname{rot} \eta_h) &= [\nabla \psi, \eta_h] \quad \forall \eta_h \in H_h, \\ (\operatorname{rot} \beta_h^I, q_h) &= -t_m^3 (\operatorname{rot} \alpha, q_h) \quad \forall q_h \in Q_h. \end{aligned}$$

Obviously, this is an approximation of (49)–(50), where p is replaced by $(p - \Theta)$, and the "continuous" solution is $\{\beta, \Theta\}$, $\Theta = 0$. Since $\{H_h, Q_h\}$ is a stable pair for Stokes problem [2 – chapt. VI and Thm. II.2.1, p. 60, (2.36)], we have

$$\|\beta - \beta_h^I\|_1 + \|\Theta - \tilde{p}_h^I\|_{0/R} \leqslant C\big(\inf_{\eta_h \in H_h} \|\beta - \eta_h\|_1 + \inf_{q_h \in Q_h} \|q_h\|_{0/R}\big),$$

so that (65) holds.

Let us show that $\{\alpha_h^I, p_h^I\}$ is an optimal approximation of $\{\alpha, p\}$. By virtue of (55), the right-hand side of (57) is $(\operatorname{rot} \beta, q_h)$. The solution $\{\alpha, p\}$ of the limit problem $(h \to 0)$ exists and is unique, as follows from [2 – Theorem II.1.1, p. 42].

Since the pair $\{\Gamma_h, Q_h\}$ satisfies the inf–sup condition [2 – p. 135–139 and (1.25)–(1.26), p. 139], we have

$$\|\alpha_h^I - \alpha\|_\Gamma \leqslant C \inf_{\delta_h \in \Gamma_h} \|\alpha - \delta_h\|_\Gamma \tag{66}$$

$$\|p_h^I - p\|_{0/R} \leqslant C\big(\inf_{\Gamma_h} \|\alpha - \delta_h\|_\Gamma + \inf_{Q_h} \|p - q_h\|_{0/R}\big). \tag{67}$$

The triangle inequality, (63), (64), (65) and (67) imply

$$\|\beta - \beta_h\|_1 \leqslant \|\beta - \beta_h^I\|_1 + \|\beta_h^I - \beta_h\|_1 \leqslant C_4 \{\mathscr{V}\}, \tag{68}$$

where

$$\begin{aligned} \{\mathscr{V}\} \equiv \inf_{H_h} \|\beta - \eta_h\|_1 + \inf_{\Gamma_h} \|\alpha - \delta_h\|_\Gamma + \inf_{Q_h} \|p - q_h\|_{0/R} \\ + \inf_{W_h} \|\psi - \varphi_h\|_1 + \sup_{H_h} [\nabla \psi, \eta_h - \Pi_h \eta_h] / \|\eta_h\|_1. \end{aligned}$$

From (64) we also obtain

$$C_E t_{\min} \|\Delta \alpha_h\|_0^2 \leqslant C_4 \{\mathscr{V}\} \|\Delta \beta_h\|_1 \leqslant (C_4 \{\mathscr{V}\})^2,$$

so that

$$\|\Delta\alpha_h\|_0 \leqslant C_5\{\mathscr{V}\}.$$

Using (66), we derive

$$(69) \qquad \|\alpha - \alpha_h\|_0 \leqslant \|\alpha - \alpha_h^I\|_\Gamma + \|\Delta\alpha_h\|_0 \leqslant C \inf_{\Gamma_h} \|\alpha - \delta_h\|_\Gamma + C_5\{\mathscr{V}\} \leqslant C_6\{\mathscr{V}\}.$$

Let us employ the inf–sup conditon for the "Stokes-like" problem [2 – pp. 213 and 205 (2.14)] in the equation (59). Using also (27), (68), (67) and (63), we have

$$\begin{aligned} k_0\|\Delta p_h\|_{0/R} &\leqslant \sup_{\eta_h \in H_h} (\Delta p_h, \operatorname{rot}\eta_h)/\|\eta_h\|_1 \\ &\leqslant C\big\{\|\Delta\beta_h\|_1 + \|\beta - \beta_h^I\|_1 + \|p - p_h^I\|_{0/R} + \|\psi - \psi_h\|_1 \\ &\quad + \sup_{H_h}[\nabla\psi, \eta_h - \Pi_h\eta_h]/\|\eta_h\|_1\big\} \leqslant C_7\{\mathscr{V}\}. \end{aligned}$$

The triangle inequality and (67) yield

$$(70) \qquad \|p - p_h\|_{0/R} \leqslant \|p - p_h^I\|_{0/R} + \|\Delta p_h\|_{0/R} \leqslant C_8\{\mathscr{V}\}.$$

Let us define the orthogonal projection $P_h\colon H_0^1(\Omega) \to W_h$ by means of the inner product $[\nabla u, \nabla v]$. Then

$$(71) \qquad \|w - w_h\|_1 \leqslant \|w - P_h w\|_1 + \|P_h w - w_h^I\|_1 + \|w_h^I - w_h\|_1.$$

Inserting $\varphi_h := \Delta w_h = w_h - w_h^I$ into (62), we may write

$$(72) \qquad \|w_h - w_h^I\|_1 \leqslant C\|\Pi_h(\beta_h - \beta)\|_0 \leqslant \widehat{C}\|\beta_h - \beta\|_1$$

[2 – p. 220, (4.2)].

By definition of P_h and (52)

$$[\nabla P_h w, \nabla\varphi_h] = [\nabla w, \nabla\varphi_h] = [\beta, \nabla\varphi_h] + (f, \varphi_h).$$

Subtracting (58), we obtain

$$[\nabla(P_h w - w_h^I), \nabla\varphi_h] = [\beta - \Pi_h\beta, \nabla\varphi_h].$$

Inserting $\varphi_h = P_h w - w_h^I$, we arrive at

$$(73) \qquad \|P_h w - w_h^I\|_1 \leqslant C\|\beta - \Pi_h\beta\|_0.$$

The optimality of the projection implies

$$(74)\qquad \|w - P_h w\|_1 \leqslant C \inf_{W_h} \|w - \varphi_h\|_1.$$

Combining (71)–(74) and (68), we are led to

$$(75)\qquad \begin{aligned}\|w - w_h\|_1 &\leqslant C\{\|\beta - \beta_h\|_1 + \|\beta - \Pi_h\beta\|_0 + \inf_{W_h}\|w - \varphi_h\|_1\}\\ &\leqslant C_9(\{\mathscr{V}\} + \inf_{W_h}\|w - \varphi_h\|_1 + \|\beta - \Pi_h\beta\|_0).\end{aligned}$$

Finaly, the first inequality of Theorem 3.1 follows from (68), (75), (63), (70) and (69).

(ii) It remains to estimate the individual terms on the right-hand side. We have [2 – Proposition III.3.9, p. 132 and (3.43) – p. 125]

$$(76)\qquad \|\beta - \Pi_h\beta\|_0 \leqslant Ch^m\|\beta\|_m, \quad 1 \leqslant m \leqslant 2.$$

The Crouzeix–Raviart elements satisfy [2 – Propos. III.2.2, p. 106]

$$(77)\qquad \inf_{\eta_h \in H_h} \|\beta - \eta_h\|_1 \leqslant Ch^{m-1}\|\beta\|_m, \quad 1 \leqslant m \leqslant 3.$$

We also have

$$(78)\qquad \inf_{W_h}\|w - \zeta_h\|_1 + \inf_{W_h}\|\psi - \varphi_h\|_1 \leqslant Ch^{m-1}(\|w\|_m + \|\psi\|_m),$$

provided $1 \leqslant m \leqslant 3$ and the family of triangulation $\{\mathscr{T}_h\}$ is regular,

$$(79)\qquad \inf_{Q_h}\|p - q_h\|_{0/R} \leqslant Ch^m\|p\|_{m/R}, \quad 1 \leqslant m \leqslant 2.$$

From [2 – Propositions III.3.6 and III.3.8 – p. 128, 130] we have

$$(80)\qquad \inf_{\Gamma_h}\|\alpha - \delta_h\|_\Gamma \leqslant Ch^m(|\alpha|_m + |\operatorname{rot}\alpha|_m), \quad 1 \leqslant m \leqslant 2.$$

Let $\overline{\psi}$ be the projection of $t\mathscr{E}\nabla\psi$ onto the subspace $(\mathscr{L}_0^0)^2$ of piecewise constant functions in $[L^2(\Omega)]^2$. Then (26) implies

$$(t\mathscr{E}\nabla\psi, \eta_h - \Pi_h\eta_h) = (t\mathscr{E}\nabla\psi - \overline{\psi}, \eta_h - \Pi_h\eta_h).$$

Moreover, we have the classical estimate

$$\|t\mathscr{E}\nabla\psi - \overline{\psi}\|_0 \leqslant Ch|t\mathscr{E}\nabla\psi|_1 \leqslant \widetilde{C}h\|\psi\|_2.$$

Using also (76), we may write

$$(81)\qquad \sup_{\eta_h \in H_h} [\nabla\psi, \eta_h - \Pi_h\eta_h]/\|\eta_h\|_1 \leqslant \widetilde{C}h\|\psi\|_2 C^* h = Ch^2\|\psi\|_2.$$

Combining (76)–(81), we obtain the second inequality of the Theorem. □

Corollary. *From* (18) *and* (34) *it follows, that* $t\mathscr{E}g_h$ *is an approximation of the "scaled shear force"* γ. *We have the following error estimate*

$$\|\gamma - t\mathscr{E}g_h\|_0 \leqslant C(\|\psi - \psi_h\|_1 + \|\alpha - \alpha_h\|_0) = O(h^s), \quad 1 \leqslant s \leqslant 2.$$

In fact, using Proposition 1.2, *Proposition* 3.1 – (35) *and* (47), *we have*

$$\gamma - t\mathscr{E}g_h = (t\mathscr{E}\nabla\psi + \underline{\mathrm{rot}}\, p) - t\mathscr{E}(\nabla\psi_h + \alpha_h) = t\mathscr{E}\big((\nabla\psi - \nabla\psi_h) + (\alpha - \alpha_h)\big).$$

Consequently, the error estimate follows from Theorem 3.1.

Theorem 3.2. *Let* β, w, γ *be the solution of* (19)–(20), *let* γ *be decomposed by means of Proposition 1.2 and let us set* $\alpha = t^{-1}\mathscr{E}^{-1}\underline{\mathrm{rot}}p$. *Assume no additional regularity of the solution, i.e., let* $\beta \in [H^1(\Omega)]^2$, $w \in H_0^1(\Omega)$, $\psi \in H_0^1(\Omega)$, $p \in H^1(\Omega)/\mathbb{R}$, $\alpha \in [L^2(\Omega)]^2$, $\mathrm{rot}\,\alpha \in L^2(\Omega)$.

Let $\{\beta_h, w_h, \psi_h, \alpha_h, p_h\}$ *be the solution of (36)–(40). Then*

$$\lim_{h\to 0}(\|\beta_h - \beta\|_1 + \|w_h - w\|_1 + \|\psi_h - \psi\|_1 + \|\alpha_h - \alpha\|_0 + \|p_h - p\|_{0/R}) = 0.$$

P r o o f. Let us use the first inequality from Theorem 3.1.

Assume $\beta \in [H_0^1(\Omega)]^2$. Since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ for any $\kappa > 0$ there exists $\beta_\kappa \in [C_0^\infty(\Omega)]^2$ such that $\|\beta - \beta_\kappa\|_1 < \kappa/2$. Denoting by r_h the interpolation from $[C(\bar\Omega)]^2$ into H_h, we have (see (77))

$$\|\beta_\kappa - r_h\beta_\kappa\|_1 \leqslant Ch^2\|\beta_\kappa\|_3.$$

Consequently, we obtain

$$\inf_{\eta_h\in H_h} \|\beta - \eta_h\|_1 \leqslant \|\beta - r_h\beta_\kappa\|_1 \to 0, \quad \text{as } h \to 0. \tag{82}$$

The same argument can be applied to the case $\beta \in V$, since $V \cap [C^\infty(\bar\Omega)]^2$ is dense in V.

In a parallel way, we can show that

$$\inf_{\zeta_h\in W_h} \|w - \zeta_h\|_1 \to 0, \quad \inf_{\varphi_h\in W_h} \|\psi - \varphi_h\|_1 \to 0. \tag{83}$$

Let us introduce the projection $\pi_h\colon H^1(\Omega) \to \mathscr{L}_1^0$ in $L^2(\Omega)$. The we have

$$\|p - \pi_h p\|_{0/R} = \inf_{c\in R} \|p - \pi_h p + c\|_0 \leqslant \|p - \pi_h p\|_0 \leqslant \tilde{C}h|p|_1 \leqslant Ch\|p\|_{1/R},$$

since $|p|_1$ is equivalent to the standard norm in $H^1(\Omega)/R$.

Consequently, we obtain

$$\inf_{q_h \in Q_h} \|p - q_h\|_{0/R} \leqslant \|p - \pi_h p\|_{0/R} \leqslant Ch\|p\|_{1/R}. \tag{84}$$

Form (47) $\alpha \in [L^2(\Omega)]^2$ follows and (50) implies that

$$\operatorname{rot} \alpha = -t_m^{-3} \operatorname{rot} \beta \in L^2(\Omega).$$

As $\operatorname{rot} \alpha = -\operatorname{div} \alpha^{\perp}$, we have $\alpha^{\perp} \in H(\operatorname{div}, \Omega)$. The space $[C^\infty(\bar{\Omega})]^2$ is dense in $H(\operatorname{div}, \Omega)$, as follows form [5 – Theorem 2.4, p. 27]. Consequently, a function $\alpha_\kappa^\perp \in [C^\infty(\bar{\Omega})]^2$ exists, such that

$$\begin{aligned}(\|\alpha^\perp - \alpha_\kappa^\perp\|_0^2 &+ \|\operatorname{div}(\alpha^\perp - \alpha_\kappa^\perp)\|_0^2)^{1/2} \\ &= (\|\alpha - \alpha_\kappa\|_0^2 + +\|\operatorname{rot}(\alpha - \alpha_\kappa)\|_0^2)^{1/2} = \|\alpha - \alpha_\kappa\|_\Gamma \leqslant \kappa/2.\end{aligned} \tag{85}$$

Then $\Pi_h \alpha_\kappa \in \Gamma_h$ and

$$\|\alpha_\kappa - \Pi_h \alpha_\kappa\|_\Gamma \leqslant Ch(|\alpha_\kappa|_1 + |\operatorname{rot} \alpha_\kappa|_1), \tag{86}$$

follows from [2 – Propos. III.3.6 and Propos. III.3.8, pp. 128, 130]. Combining (85) and (86), we arrive at

$$\begin{aligned}\inf_{\delta_h \in \Gamma_h} \|\alpha - \delta_h\|_\Gamma &\leqslant \|\alpha - \Pi_h \alpha_\kappa\|_\Gamma \\ &\leqslant \|\alpha - \alpha_\kappa\|_\Gamma + \|\alpha_\kappa - \Pi_h \alpha_\kappa\|_\Gamma \to 0, \quad \text{as } h \to 0.\end{aligned} \tag{87}$$

Form (76) we see that

$$\|\beta - \Pi_h \beta\|_0 \leqslant Ch\|\beta\|_1 \tag{88}$$

Since

$$(t\mathscr{E}\nabla\psi, \eta_h - \Pi_h \eta_h) \leqslant \|t\mathscr{E}\nabla\psi\|_0 \|\eta_h - \Pi_h \eta_h\|_0 \leqslant C\|\psi\|_1 h \|\eta_h\|_1,$$

we arrive at

$$\sup_{\eta_h \in H_h} [\nabla\psi, \eta_h - \Pi_h \eta_h]/\|\eta_h\|_1 \leqslant Ch\|\psi\|_1. \tag{89}$$

Collecting (82), (83), (84), (87), (88) and (89), we obtain the assertion of the Theorem. □

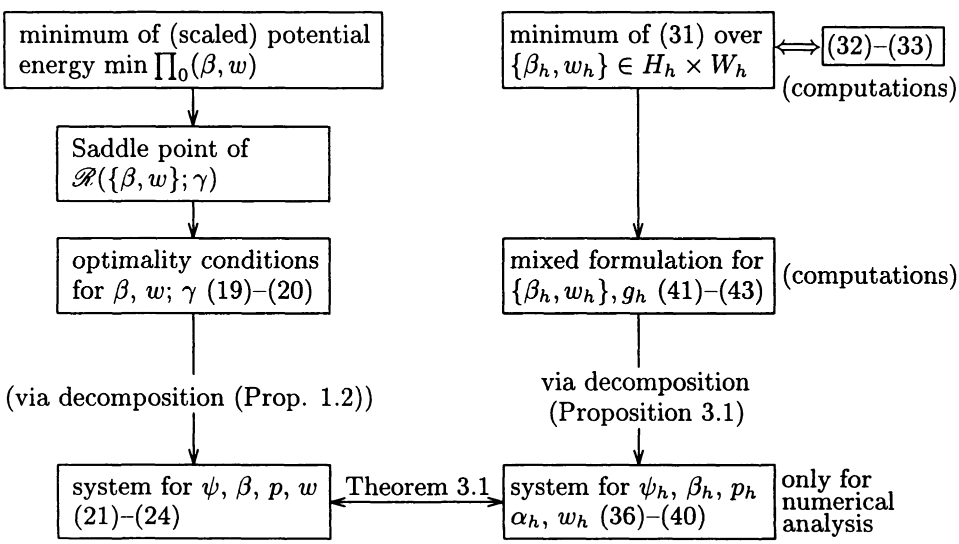
R e m a r k 3.1. One can derive an error estimate for $\|\beta - \beta_h\|_1 + \|w - w_h\|_1$ in a direct way, starting with the minimization problem (31) and treating the "perturbation" by the interpolation operator Π_h in the shear stress energy as numerical integration. By means of the First Strang Lemma [7 – Theorem 26.1, p. 192], we can get

$$\|\beta - \beta_h\|_1 + \|w - w_h\|_1 \leqslant Ch(\|\beta\|_1 + \|w\|_1).$$

We conclude that the more sophisticated error analysis of Theorem 3.1 is better, as it yields error estimates $0(h^2)$ optimal (with respect to the choice of quadratic polynomials in H_h and W_h).

R e m a r k 3.2. To elucidate the relations between various continuous and discrete problems, we display the following table.

Continuous formulations *Discrete formulations*



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