Tomáš Ratinger

Seasonal time series with missing observations

Applications of Mathematics, Vol. 41 (1996), No. 1, 41–55

Persistent URL: http://dml.cz/dmlcz/134312

SEASONAL TIME SERIES WITH MISSING OBSERVATIONS

TOMÁŠ RATINGER, Praha

(Received July 27, 1994)

Summary. Popular exponential smoothing methods dealt originally only with equally spaced observations. When time series contains gaps, smoothing constants have to be adjusted. Cipra et al., following Wright's approach of irregularly spaced observations, have suggested ad hoc modification of smoothing constants for the Holt-Winters smoothing method. In this article the fact that the underlying model of the Holt-Winters method is a certain seasonal ARIMA is used. Minimum mean square error smoothing constants are derived and compared with those of Cipra.

Keywords: Holt-Winters method, missing observations, seasonal ARIMA model

AMS classification: 62M10, (62M20, 90A20, 93E14)

1. INTRODUCTION

Exponential smoothing methods, especially Holt's and Holt-Winters methods, have achieved great popularity in practical time series analysis. The reason is that they are easy to understand and to compute. Although they are ad hoc methods, they produce satisfactory forecasts comparable with, for example, ARIMA models. Originally, these methods have been designed for time series with equally spaced observations. If time series contains missing observations, methods have to be adjusted. Wright [6] proposed to treat the problem of missing observations as irregularly spaced data, and modified in that sense the nonseasonal exponential smoothing methods. Cipra et al. [5] suggested an extension of Wright's approach to the (seasonal) Holt-Winters method. Utilizing the fact that exponential smoothing techniques are optimal provided the investigated process follows a certain ARIMA model, Aldrin and Damsleth [3] derived optimal smoothing constants for non seasonal methods for a single gap of missing observations. This paper extends their approach to the Holt-Winters method. In the next section we introduce the seasonal ARIMA model corresponding to the Holt-Winters additive method. Since we deal with non-stationary time series we cannot use the infinite linear process representation, which is essential for further explanation. Section 3 shows how to overcome this problem. Optimal smoothing constants are derived in Section 4. Sections 5 and 6 are devoted to the properties of smoothing constants and to their comparison with those derived by means of the irregularly spaced observation approach.

2. Holt-Winters additive method

Suppose that $z_1, \ldots, z_n$ are observations of a time series following the seasonal additive model

$$z_{t+j} = T_{t+j} + S_{t+j} + \varepsilon_{t+j}, \tag{2.1}$$

where the trend component is linear, $T_{t+j} = \mu_{t+j} = \mu_t + \beta_t j$, and the s seasonal factors $S_i = S_{i+s} = \ldots$ (for $i = 1, \ldots, s$) are restricted add to zero. As usual, error components ε_t represent a zero mean process of independent identically distributed random variables (white noise). Three possibly different smoothing constants are considered in the *Holt-Winters Additive Seasonal Forecast Procedure*: one to update the level, one for the slope, and one for the seasonal components (see [2], pp. 167–170). An estimate of $\mu_{\varepsilon+1}$ can be constructed as a linear combination of the most recent observation, z_{t+1}, adjusted by its seasonal factor and the trend estimate

$$\hat{\mu}_{t+1} = \alpha_1(z_{t+1} - \hat{S}_{t+1-s}) + (1-\alpha_1)(\hat{\mu}_t + \hat{\beta}_t). \tag{2.2}$$

Similarly, the estimate of the slope β_{t+1} is taken as a weighted average of the most recent and the previous estimates of the slope

$$\hat{\beta}_{t+1} = \alpha_2(\hat{\mu}_{t+1} - \hat{\mu}_t) + (1-\alpha_2)\hat{\beta}_t. \tag{2.3}$$

Finally, the estimate of the seasonal coefficient S_{t+1} is a weighted average of the most recent estimate of the seasonal factor and its previous estimate

$$\begin{aligned} \hat{S}_{t+1,s} &= \hat{S}_{t+1} = \alpha_3(z_{t+1} - \hat{\mu}_{t+1}) + (1-\alpha_3)\hat{S}_{t+1-s}, \\ \hat{S}_{t+1,j} &= \hat{S}_{t,j+1} = \hat{S}_{t+1-s+j}, \quad j = 1, \ldots, s-1. \end{aligned} \tag{2.4}$$

After obtaining the estimates for the trend, the slope and the seasonal components, we can calculate the forecast of the future value z_{t+1} from the time origin t as

$$\hat{z}_t(m) = \hat{\mu}_t + \hat{\beta}_t m + \hat{S}_{t+m-s} \quad \text{for } m = 1, 2, \ldots, s \tag{2.5}$$

$$\hat{z}_t(m) = \hat{\mu}_t + \hat{\beta}_t m + \hat{S}_{t+m-2s} \quad \text{for } m = s+1, \ldots, 2s \tag{2.6}$$

and so on. $\hat{S}$ denotes the most recent estimate of the corresponding seasonal component. Previous equations (2.2)–(2.4) can be transformal to the error correction form

$$\begin{aligned}
\hat{\mu}_t &= \alpha_1\big(z_t - \hat{z}_{t-1}(1)\big) + \hat{\mu}_{t-1} + \hat{\beta}_{t-1},\\
\hat{\beta}_t &= \alpha_1\alpha_2\big(z_t - \hat{z}_{t-1}(1)\big) + \hat{\beta}_{t-1},\\
\hat{S}_t &= \hat{S}_{t,j} = \alpha_3(1-\alpha_1)\big(z_t - \hat{z}_{t-1}(1)\big) + \hat{S}_{t-s,s},\\
\hat{S}_{t,j} &= \hat{S}_{t-1,j+1} = \hat{S}_{t-s+j}, \quad j = 1,\dots,s-1.
\end{aligned} \tag{2.7}$$

Although the updating equations do not necessarily restrict seasonal factors to add to zero, the appropriate normalization at each time period leads to the fulfillment of these constraints. The observed time series $(z_1,\dots,z_n)$ can be viewed as a particular realization of a stochastic process. The ARIMA model implied by the Holt-Winters additive method is of the form

$$\begin{aligned}
(1-B)(1-B^s)z_t &= \Big(1 - \sum_{j=1}^{s+1}\theta_j B_j\Big)\varepsilon_t\\
\text{with}\quad \theta_1 &= 1 - \alpha_1(1+\alpha_2),\\
\theta_j &= -\alpha_1\alpha_2, \quad j = 2,\dots,s-1,\\
\theta_s &= (1-\alpha_3) - \alpha_1(\alpha_2 - \alpha_3),\\
\theta_{s+1} &= -(1-\alpha_1)(1-\alpha_3),
\end{aligned} \tag{2.8}$$

where B represents the backshift operator (see [1]). In order to show this denote

$$\begin{gathered}
\varepsilon_t = z_t - \hat{z}_{t-1}(1),\\
A_1 = \alpha_1, \quad A_2 = \alpha_1\alpha_2, \quad C_j = 0, \quad j = 1,\dots,s-1, \quad C_s = \alpha_3(1-\alpha_1).
\end{gathered} \tag{2.9}$$

Then z_t can be expressed as

$$z_t = \hat{\mu}_{t-1} + \hat{\beta}_{t-1} + \hat{S}_{t-1,1} + \varepsilon_t. \tag{2.10}$$

Since

$$\begin{aligned}
(1-B)^2\hat{\mu}_t &= \big(A_1 + (A_2 - A_1)B\big)\varepsilon_t,\\
(1-B)\hat{\beta}_t &= A_2\varepsilon_t,\\
(1-B^s)\hat{S}_t &= \varepsilon_t\sum_{j=1}^{s} C_j B^{j-1},
\end{aligned} \tag{2.11}$$

we get

$$
\begin{aligned}
(2.12)\quad (1-B)(1-B^s)z_t &= \Big(\sum_{j=1}^{s} B^{j-1}\Big)(1-B)^2\hat{\mu}_{t-1} \\
&\quad + (1-B^s)(1-B)\hat{\beta}_{t-1} + (1-B)(1-B^s)\hat{S}_{t-1} \\
&\quad + (1-B^s)(1-B)\varepsilon_t \\
&= \varepsilon_t\Big[A_1(B-B^{s+1}) + A_2\sum_{j=1}^{s} B^j + \sum_{j=1}^{s} C_j B^j \\
&\quad - \sum_{j=2}^{s+1} C_{j-1}B^j + 1 - B - B^s + B^{s+1}\Big],
\end{aligned}
$$

which yields the coefficients of B^j given by (2.7), (2.8).

3. Linear filter representation

Due to reasons which will be apparent later, it would be useful to have the seasonal ARIMA model (2.7), (2.8) in its linear filter representation

$$
z_t = \sum_{j\geqslant 0} \Psi_j \varepsilon_{t-j}. \tag{3.1}
$$

Since the underlying process is not stationary, the infinite expansion (3.1) does not make much sense. Nevertheless, since we need several first summands on the right-hand side of (3.1) we might only modify the understanding of that equation.

After some algebraic treatment it is not difficult to show that

$$
z_t = \sum_{j\geqslant 0} \Psi_j \varepsilon_{t-j} = \sum_{j=0}^{h} \Psi_j \varepsilon_{t-j} + y_{([\frac{t}{s}]s-t)} \tag{3.2}
$$

with initial values $y_0, y_1, \ldots, y_{-s+1}$ orthogonal to ε_i, $i = t-h, \ldots, t$ (the values $y_0, y_1, \ldots, y_{-s+1}$ depend on t and h). The function $[\,]$ denotes the integer part of the argument. In what follows we will understand linear filter representation (3.1) exactly in this sense. Now, a comparison of coefficients at ε in (2.7), (2.8) and (3.1) yields

$$
\Psi_0 = 1, \quad \Psi_{j+ns} = \lambda_j + n\lambda_{s+1}, \tag{3.3}
$$

where

$$\lambda_0 = 1 + \theta_{s+1},$$
$$\lambda_k = 1 - \sum_{j=1}^{k} \theta_j, \quad k = 1, \ldots, s+1$$

for $j = 0, \ldots, s-1$ and $n = 1, 2, 3, \ldots$. From (2.7), (2.8) we obtain the coefficients Ψ in terms of the smoothing constants α_1, α_2, α_3

$$\Psi_i = \begin{cases} 1, & i = 0, \\ \alpha_1(1 + i\alpha_2), & i > 0,\ i \neq ns,\ n = 0, 1, \ldots \\ \alpha_1(1 + i\alpha_2) + \alpha_3(1 - \alpha_1), & i = ns,\ n = 1, 2, \ldots. \end{cases} \tag{3.4}$$

4. The single gap case

Assume that $z_1, \ldots, z_{t-k-1}, z_t$ are observed, while $z_{t-k}, \ldots, z_{t-1}$ are missing. When there are no missing observations, the linear m-step-ahead (in MSE sense) forecast from the origin t is given by

$$\hat{z}_t(m) = E[z_{t+m} | z_{t-k-1}, \ldots] = \sum_{j=m}^{h} \Psi_j \varepsilon_{t+m-j} + y_{([\frac{t}{s}]s-t)} = \sum_{j \geqslant m} \Psi_j \varepsilon_{t+m-j} \tag{4.1}$$

(see [2]). When data are missing we can calculate the $(k+1)$-step-ahead forecast of z_t from the time origin $t-k-1$ with forecast errors given by

$$z_t - \hat{z}_{t-k-1}(k+1) = \sum_{j=0}^{k} \Psi_j \varepsilon_{t-j} = \delta_t. \tag{4.2}$$

The forecast $\hat{z}_t(m)$ can be expressed in the form

$$\hat{z}_t(m) = c_0 \delta_t + \sum_{j \geqslant 1} c_j \varepsilon_{t-k-j}, \tag{4.3}$$

where the coefficients c_0, c_j, $j > 0$, have to be determined. By (3.1) for z_{t+m}, the forecast error can be calculated as

$$\begin{aligned} z_{t+m} - \hat{z}_t(m) &= \sum_{j \geqslant 0} \Psi_j \varepsilon_{t+m-j} - c_0 \delta_t - \sum_{j \geqslant 1} c_j \varepsilon_{t-k-j} \\ &= \sum_{j=0}^{m-1} \Psi_j \varepsilon_{t+m-j} + \sum_{j \geqslant 1} (\Psi_{m+k+j} - c_j) \varepsilon_{t-k-j} \\ &\quad + \sum_{j=0}^{k} (\Psi_{m+j} - c_0 \Psi_j) \varepsilon_{t-j}. \end{aligned} \tag{4.4}$$

To ensure that the forecast minimizes the squared deviation the coefficients c's are given by

$$(4.5)\qquad c_0 = \frac{\sum_{j=0}^{k} \Psi_j \Psi_{m+j}}{\sum_{j=0}^{k} \Psi_j^2}, \quad c_j = \Psi_{k+m+j}.$$

Hence,

$$(4.6)\qquad \begin{aligned} \sum_{j\geqslant 1} c_j \varepsilon_{t-k-j} &= \sum_{j\geqslant 1} \Psi_{k+m+j}\varepsilon_{t-k-j} \\ &= \sum_{j\geqslant m+k+1} \Psi_j \varepsilon_{t-k-1+m+k+1-j} = \hat{z}_{t-k-1}(m+k+1), \end{aligned}$$

which implies the formula for the m-step-ahead forecast as a linear combination of $(m+k+1)$-step-ahead forecast from the time origin before the gap and the error of the forecast of the first observation after the gap constructed from the available observations before the gap

$$(4.7)\qquad \hat{z}_t(m) = c_0\big(z_t - \hat{z}_{t-k-1}(k+1)\big) + \hat{z}_{t-k-1}(m+k+1).$$

Let us define

$$(4.8)\qquad \varphi_j = \alpha_1(1 + j\alpha_2) \quad \text{for } j > 0,\ \varphi_0 = 1,\ \eta = \alpha_3(1-\alpha_1).$$

Then we have

$$(4.9)\qquad \Psi_j = \begin{cases} \varphi_j & \text{for } j \neq ns, \\ = \varphi_j + \eta & \text{for } j = ns,\ n = 1, 2, \ldots. \end{cases}$$

It can be easily derived that

$$(4.10)\qquad \begin{aligned} \sum_{j=1}^{k} \varphi_j &= k\alpha_1 + \alpha_1\alpha_2\frac{k(k+1)}{2}, \\ \sum_{j=1}^{k} \varphi_j^2 &= \alpha_1^2\Big(k + 2\alpha_2\frac{k(k+1)}{2} + \alpha_2^2\frac{k(k+1)(2k+1)}{6}\Big). \end{aligned}$$

After substituting (4.9) into the sum of the squared Ψ_j we obtain

$$\begin{aligned}
(4.11) \qquad \sum_{j=0}^{k} \Psi_j^2 &= 1 + \sum_{j=1}^{k} \varphi_j^2 + 2\eta \sum_{j=1}^{[\frac{k}{s}]} \varphi_{js} + \left[\frac{k}{s}\right]\eta^2 \\
&= 1 + \alpha_1^2 k\Big(1 + \alpha_2(k+1) + \alpha_2^2 \frac{(k+1)(2k+1)}{6}\Big) \\
&\quad + 2\alpha_3(1-\alpha_1)\left(\alpha_1\left[\frac{k}{s}\right] + \alpha_1\alpha_2 \frac{s}{2}\left[\frac{k}{s}\right]\left(\left[\frac{k}{s}\right]+1\right)\right) \\
&\quad + \alpha_3^2(1-\alpha_1)^2\left[\frac{k}{s}\right].
\end{aligned}$$

According to the formulas (3.4) and (4.8) we have

$$(4.12) \qquad \Psi_{j+m} = \begin{cases} \Psi_j + m\alpha_1\alpha_2, & j \neq n_1 s - m,\ j \neq n_3 s, \\ \Psi_j - \eta + m\alpha_1\alpha_2, & j \neq n_1 s - m,\ j = n_3 s, \\ \Psi_j + \eta + m\alpha_1\alpha_2, & j = n_1 s - m, \end{cases}$$

for $m \neq n_2 + s$ and

$$\Psi_{j+m} = \Psi_j + m\alpha_1\alpha_2$$

for $m = n_2 s$ where n_1, n_2, n_3, are positive integers. Similarly, we can express the product $\Psi_j\Psi_{j+m}$ in the form

$$(4.13) \qquad \Psi_j\Psi_{j+m} = \begin{cases} \Psi_j^2 + m\alpha_1\alpha_2\Psi_j, & j \neq n_1 - m,\ j = n_3 s, \\ \Psi_j^2 + m\alpha_1\alpha_2\Psi_j - \eta\Psi_j, & j = n_1 s, \\ \Psi_j^2 + m\alpha_1\alpha_2 + \Psi_j + \eta\Psi_j, & j = ns - m, \end{cases}$$

for $m \neq n_2 + s$ and

$$\Psi_j\Psi_{j+m} = \Psi_j^2 + m\alpha_1\alpha_2\Psi_j$$

for $m = n_2 s$. If $m = n_2 s$, then we have

$$\begin{aligned}
(4.14) \qquad \sum_{j=0}^{k} \Psi_j\Psi_{j+m} &= \Psi_m + \sum_{j=1}^{k} \Psi_j^2 + m\alpha_1\alpha_2 \sum_{j=1}^{k} \Psi_j \\
&= \alpha_1 + \sum_{j=1}^{k} \Psi_j^2 + m\alpha_1\alpha_2\Big(1 + \sum_{j=1}^{k} \Psi_j\Big) + \eta.
\end{aligned}$$

We proceed similarly if m differs from the integer product of a seasonal length s $(m \neq n_2 s)$:

$$\sum_{j=0}^{k} \Psi_j \Psi_{j+m} = \alpha_1 + \sum_{j=1}^{k} \Psi_j^2 + m\alpha_1\alpha_3\Big(1 + \sum_{j=1}^{k} \Psi_j\Big) + \eta\Big(- \sum_{j=1}^{[\frac{k}{s}]} \Psi_{js} + \sum_{j=[\frac{m}{s}]+1}^{[\frac{k+m}{s}]} \Psi_{js-m}\Big). \tag{4.15}$$

The last term in this equation can be further simplified to

$$\begin{aligned}
&\sum_{j=[\frac{m}{s}]+1}^{[\frac{k+m}{s}]} \Psi_{js-m} - \sum_{j=1}^{[\frac{k}{s}]} \Psi_{js} \\
&= \sum_{j=[\frac{m}{s}]+1}^{[\frac{k+m}{s}]} \varphi_{js} - \sum_{j=1}^{[\frac{k}{s}]} \Psi_{js} - m\alpha_1\alpha_2\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big]\Big) \\
&= \alpha_1\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big] - \Big[\frac{k}{s}\Big]\Big) + \alpha_1\alpha_2\frac{s}{2}\Big\{\Big[\frac{k+m}{s}\Big]\Big(\Big[\frac{k+m}{s}\Big] + 1\Big) \\
&\quad - \Big[\frac{m}{s}\Big]\Big(\Big[\frac{m}{s}\Big] + 1\Big) - \Big[\frac{k}{s}\Big]\Big(\Big[\frac{k}{s}\Big] + 1\Big)\Big\} - \eta\Big[\frac{k}{s}\Big] \\
&\quad - m\alpha_1\alpha_2\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big]\Big) \\
&= \alpha_1\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big] - \Big[\frac{k}{s}\Big]\Big) \\
&\quad + \alpha_1\alpha_2\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big]\Big)\Big\{s\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{k}{s}\Big]\Big) - m\Big\} - \eta\Big[\frac{k}{s}\Big].
\end{aligned} \tag{4.16}$$

Note that this expression equals $-\eta[k/s]$ for $m = ns$, $n = 1, 2, \ldots$, and

$$s\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{k}{s}\Big]\Big) - m = s\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{k}{s}\Big] - \frac{m}{s}\Big) \tag{4.17}$$

does not depend on the forecasting horizon m; it only corresponds to the time within the period s. Let us denote the first two terms in the last row of (4.16) by τ. Then we can rewrite (4.15) as

$$\sum_{j=0}^{k} \Psi_j \Psi_{j+m} = \alpha_1 + \sum_{j=1}^{k} \Psi_j^2 + m\alpha_1\alpha_2\Big(1 + \sum_{j=1}^{k} \Psi_j\Big) + \eta\tau - \eta^2\Big[\frac{k}{s}\Big]. \tag{4.18}$$

After combining the last equation with (4.14) we obtain the general formula for the sum of $\Psi_j\Psi_{j+m}$ products

$$\begin{aligned}\sum_{j=0}^{k}\Psi_j\Psi_{j+m} = \alpha_1 + \sum_{j=1}^{k}\Psi_j^2 + \eta\tau - \eta^2\left[\frac{k}{s}\right] + m\alpha_1\alpha_2\sum_{j=0}^{k}\Psi_j \\ + \eta + \eta^2\left[\frac{k}{s}\right] \quad \text{for } m = ns \\ + 0 \qquad \text{elsewhere,}\end{aligned} \tag{4.19}$$

$(n = 1, 2, \ldots)$. Hence, c_0 can be written as

$$c_0 = u + mv + w_{s,m}, \tag{4.20}$$

where the parameters u, v and w are given by

$$u = 1 + \frac{\eta\tau - \eta^2\frac{k}{s} - (1-\alpha_1)}{\sum\limits_{j=0}^{k}\Psi_j^2}, \tag{4.21}$$

$$v = \alpha_1\alpha_2\frac{\sum\limits_{j=0}^{k}\Psi_j}{\sum\limits_{j=0}^{k}\Psi_j^2},$$

$$w_{s,m} = \begin{cases}\dfrac{\eta + \eta^2\left[\frac{k}{s}\right]}{\sum\limits_{j=0}^{k}\Psi_j^2} & \text{for } m = ns,\ n = 1, 2, \ldots \\ 0 & \text{elsewhere.}\end{cases}$$

As we are looking for optimal smoothing constants A_1 (for the level), A_2 (for the slope) and A_3 (for the seasonal component), in the case of a gap of the length k we replace the smoothing constants α_1, α_2, α_3 in (2.7) by them. In accordance with the forecasting formula (2.5), we have

$$\begin{aligned}\hat{z}_t(m) &= A_1\delta_t\hat{\mu}_{t-k-1} + (k+1)\hat{\beta}_{t-k-1} + m\Big(\hat{\beta}_{t-k-1} + \frac{A_1A_2}{k+1}\delta_t\Big) \\ &\quad + A_3(1-A_1)\delta_t + \hat{S}_{t-([\frac{k}{s}]+1)s} \quad \text{for } m = ns,\ n = 1, 2, \ldots, \\ &\quad + \hat{S}_{t+m-([\frac{k+m}{s}]+1)s} \qquad \text{elsewhere,} \\ &= \delta_t\Big(A_1 + m\frac{A_1A_2}{k+1} + \iota_{m,s}A_3(1-A_1)\Big) \\ &\quad + \hat{\mu}_{t-k-1} + (m+k+1)\hat{\beta}_{t-k-1} + \hat{S}_{t+m-([\frac{k+m}{s}]+1)s} \\ &= \Big(A_1 + m\frac{A_1A_2}{k+1} + \iota_{m,s}A_3(1-A_1)\Big)\delta_t + \hat{z}_{t-k-1}(m+k+1),\end{aligned} \tag{4.22}$$

where $\iota_{m,s} = 1$ whenever $m = ns$, $n = 1, 2, \ldots$, and 0 elsewhere.

The comparison of this result with (4.7) and (4.20) yields straightforward relationships between the optimal smoothing constants and c_0

$$A_1 = u, \quad v = \frac{A_1 A_2}{k+1}, \quad w_{m,s} = A_3(1 - A_1). \tag{4.23}$$

Therefore, the optimal smoothing constants are

$$\begin{aligned} A_1 = \alpha_1 \Big\{ & 1 + \alpha_1 k \Big(1 + \alpha_2(k+1) + \alpha_2^2 \frac{(k+1)(2k+1)}{6} \Big) \\ & + \alpha_3(1-\alpha_1)\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big] + \Big[\frac{k}{s}\Big] \Big) \\ & + \alpha_2 s \Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big] \Big)\Big(\Big[\frac{k}{s}\Big] + 1 + \Big[\frac{m}{s}\Big] - \frac{m}{s} \Big) \Big\} \Big/ \sum_{j=0}^{k} \Psi_j^2, \end{aligned} \tag{4.24}$$

$$A_2 = \alpha_2 \frac{\alpha_1}{A_1} \frac{(k+1) \sum_{j=0}^{k} \Psi_j}{\sum_{j=0}^{k} \Psi_j^2} \tag{4.25}$$

where

$$\sum_{j=0}^{k} \Psi_j = 1 + k\alpha_1 + \alpha_1 \alpha_2 \frac{k(k+1)}{2} + \Big[\frac{k}{s}\Big] \alpha_3 (1 - \alpha_1), \tag{4.26}$$

and

$$A_3 = \alpha_3 \frac{1 + \alpha_3(1-\alpha_1)[\frac{k}{s}]}{1 + \alpha_3^2(1-\alpha_1)[\frac{k}{s}]}. \tag{4.27}$$

5. Properties of optimal smoothing constants

The optimal smoothing constants are functions of the gap length k. Since $A_1(0) = \alpha_1$, $A_2(0) = \alpha_2$, $A_3(0) = \alpha_3$, when there are no missing observations the Holt-Winters method is unmodified.

In order to analyze the case of the infinite gap we can use the following asymptotic properties:

$$\sum_{j=1}^{k} \varphi_j = k\alpha_1 + \alpha_1\alpha_2 \frac{k(k+1)}{2} = O(k^2), \tag{5.1}$$

$$\sum_{j=1}^{k} \varphi_j^2 = k\alpha_1^2 + \alpha_1^2\alpha_2 k(k+1) + \alpha_1^2\alpha_2^2 \frac{k(k+1)(2k+1)}{6} = O(k^3), \tag{5.2}$$

$$\sum_{j=1}^{k} \Psi_j = 1 + \sum_{j=1}^{k} \varphi_j + \eta\Big[\frac{k}{s}\Big] = O(k^2), \tag{5.3}$$

$$\sum_{j=1}^{k} \Psi_j^2 = 1 + \sum_{j=1}^{k} \varphi_j^2 + 2\eta \sum_{j=1}^{[\frac{k}{s}]} \varphi_{js} + \eta^2 \Big[\frac{k}{s}\Big] = O(k^3), \tag{5.4}$$

$$\begin{aligned} \tau = {} & \alpha_1\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{k}{s}\Big] - \Big[\frac{m}{s}\Big]\Big) \\ & + \alpha_1\alpha_2 s\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big]\Big)\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{k}{s}\Big] - \frac{m}{s}\Big) = O(k). \end{aligned} \tag{5.5}$$

Hence, from equations (4.24) and (4.27) we get limits for optimal smoothing constants A_1 and A_2:

$$\lim_{k\to\infty} A_1(k) = 1, \quad \lim_{k\to\infty} A_3(k) = 1. \tag{5.6}$$

As the asymptotic behaviour of A_2 is connected it follows from (4.25) and (4.27) that

$$\begin{aligned} A_2 = {} & \frac{\alpha_1\alpha_2}{A_1} \frac{(k+1)\sum_{j=0}^{k} \Psi_j}{\sum_{j=0}^{k} \Psi_j^2} = \alpha_2(k+1)\Big(1 + k\alpha_1 + \alpha_1\alpha_2 \frac{k(k+1)}{2} + \eta\Big[\frac{k}{s}\Big]\Big) \\ & \times \Bigg\{1 + \alpha_1 k \left(1 + \alpha_2(k+1)\Big(1 + \alpha_2 \frac{2k+1}{6}\Big)\right) \\ & + \eta\Big(\Big[\frac{k+m}{s}\Big] + \Big[\frac{k}{s}\Big] - \Big[\frac{m}{s}\Big]\Big) + \eta\alpha_2 s \\ & \times\Big(\Big[\frac{k+m}{s}\Big] - \Big[\frac{m}{s}\Big]\Big)\Big(\Big[\frac{m}{s}\Big] + \Big[\frac{k}{s}\Big] + 1 - \frac{m}{s}\Big)\Bigg\}^{-1}. \end{aligned} \tag{5.7}$$

The numerator and the denominator are of the same order $O(k^3)$. The constants at k^3 are $\frac{1}{2}\alpha_1\alpha_2^2$ for the numerator and $\frac{1}{3}\alpha_1\alpha_2^2$ for the denominator. Therefore, the limit of A_2 for the infinite gap is given by

$$\lim_{k\to\infty} A_2(k) = \frac{\frac{1}{2}\alpha_1\alpha_2^2}{\frac{1}{3}\alpha_1\alpha_2^2} = \frac{3}{2}. \tag{5.8}$$

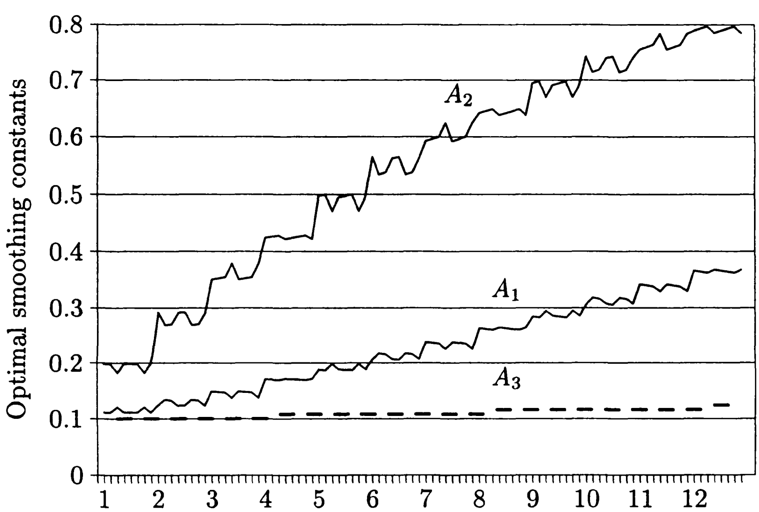
It also follows from these results that the limits of the discount coefficients ($\omega_p = 1 - \alpha_p$, $p = 1, 2, 3$, see [2] p. 85) are

$$\lim_{k\to\infty} \omega_1(k) = 0, \quad \lim_{k\to\infty} \omega_2(k) = -\frac{1}{2}, \quad \text{and} \quad \lim_{k\to\infty} \omega_3(k) = 0. \tag{5.9}$$

This result is in accordance with the result of Aldrin & Damsleth [3] for Holt's method, where the negative value of the discount coefficient for slope has been also advocated. The behaviour of the smoothing constants in the dependence on the length k and the forecasting horizon m is shown in graphs. The smoothing constant A_2 increases sharply for lower values of k. It is also obvious how the seasonal pattern of the underlying process is transmitted into the smoothing constants. In the figures (Fig. 1, Fig. 2) the numbers along the horizontal axes represents the size of the gap k, while blank ticks correspond to the forecasting horizon m ($m = 1, \ldots, 2s$) for each k.

Optimal smoothing constants (Winter's additive method)

$s = 4$, $\alpha_1 = .1$, $\alpha_2 = .1$, $\alpha_3 = .1$



$k = 1, \ldots, 3s$, $m = 1, \ldots, 2s$

Fig. 1

Optimal smoothing constants (Winter's additive method)

$s = 12$, $\alpha_1 = .1$, $\alpha_2 = .1$, $\alpha_3 = .1$

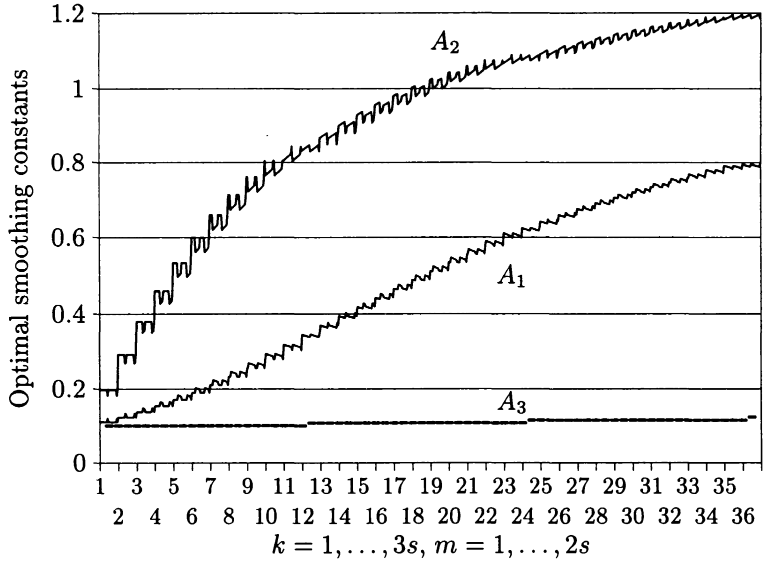


Fig. 2

6. Comparison to Wright's type smoothing constants

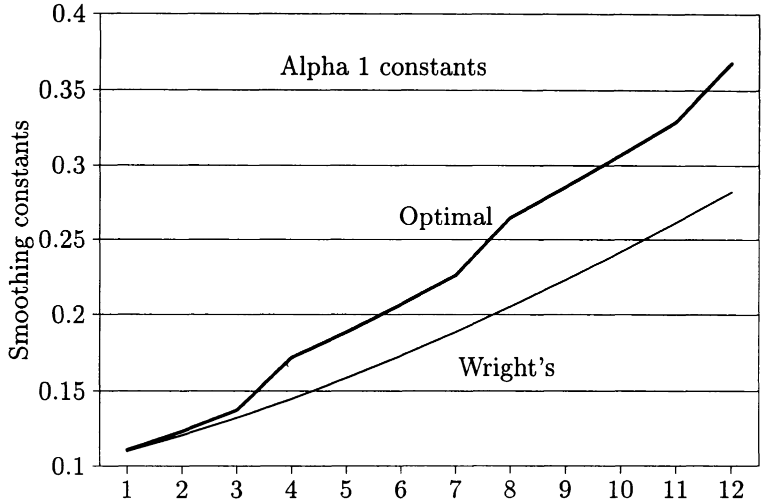
Cipra, Rubio and Trujillo [5] have extended Wright's approach of irregularly spaced data to the Holt-Winters seasonal method. According to their results the smoothing constants for a single gap of a length k are given by

$$\begin{aligned} A_1^{(w)} &= \frac{\alpha_1}{(1-\alpha_1)^{k+1} + \alpha_1}, \\ A_2^{(w)} &= \frac{\alpha_2}{(1-\alpha_2)^{k+1} + \alpha_2}, \\ A_3^{(w)} &= \frac{\alpha_3}{(1-\alpha_3)^{[\frac{k}{s}]+1} + \alpha_3}. \end{aligned} \tag{6.1}$$

All smoothing constants of Wright's type converge to 1, if k goes to infinity. In contrast to our procedure, $A_1^{(w)}$ does not depend on α_2, α_3 and $A_2^{(w)}$ does not depend on α_1, α_3, and both are independent of the position of the forecasting horizon within the seasonal period. The comparison of our MMSE smoothing constants and those derived by Cipra et al. is illustrated for various lengths of the gap in the pictures Fig. 3–Fig. 5. It is evident from the graphs that the methods differ substantially for smoothing constants relating to the slope.

Optimal and Wright's smoothing constants (Winter's additive method)

$s = 4$, $\alpha_1 = .1$, $\alpha_2 = .1$, $\alpha_3 = .1$

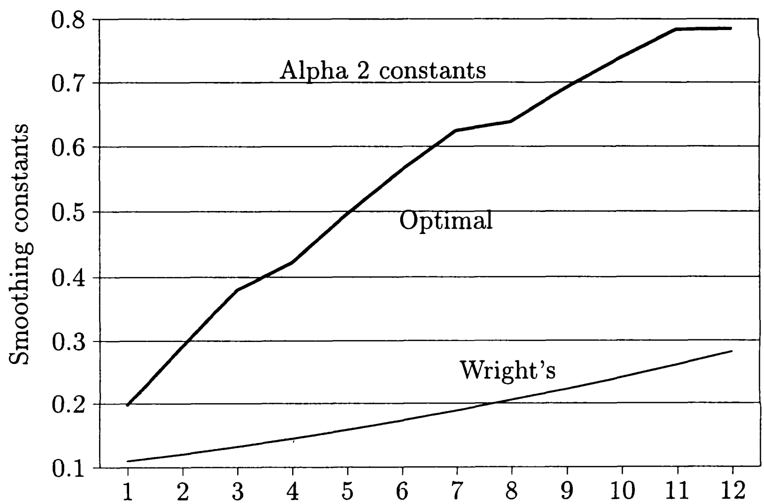


$k = 1, \ldots, 3s$, $m = s$

Fig. 3

Optimal and Wright's smoothing constants (Winter's additive method)

$s = 4$, $\alpha_1 = .1$, $\alpha_2 = .1$, $\alpha_3 = .1$



$k = 1, \ldots, 3s$, $m = s$

Fig. 4

Optimal and Wright's smoothing constants (Winter's additive method)

$s = 4, \alpha_1 = .1, \alpha_2 = .1, \alpha_3 = .1$

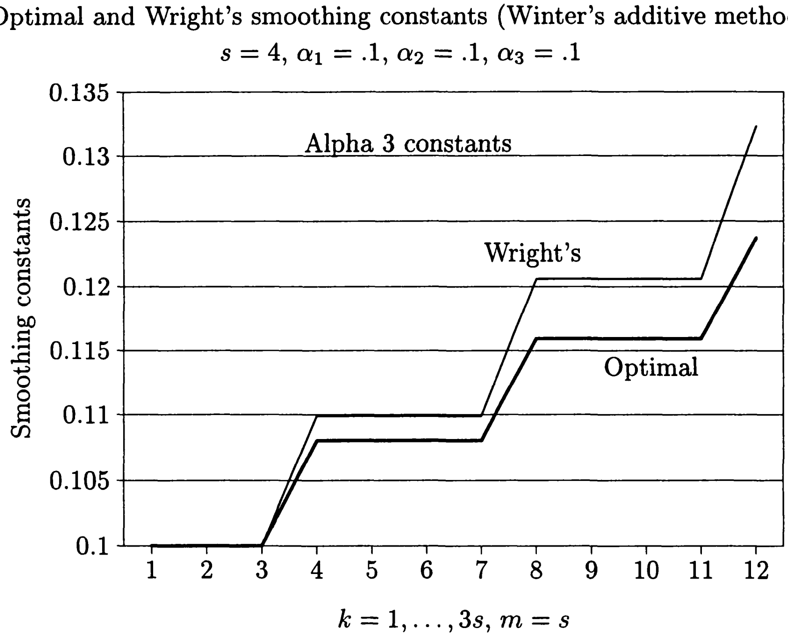


Fig. 5

References

[1] *B. Abraham, J. Ledolter*: Forecast functions implied by autoregressive integrated moving average models and other related forecast procedures. International Statistical Review *54* (1986), 51–66.

[2] *B. Abraham, J. Ledolter*: Statistical Methods for Forecasting. Wiley, New York, 1983.

[3] *M. Aldrin, E. Damsleth*: Forecasting non-seasonal time series with missing observations. Journal of Forecasting *8* (1989), 97–116.

[3] *P.J. Brockwell, R.A. Davis*: Time Series Theory and Methods. Springer, New York, 1991.

[5] *T. Cipra, A. Rubio, J. Trujillo*: Holt-Winters method with missing observations. Management Science. To be published.

[6] *D.J. Wright*: Forecasting data published at irregular time intervals using an extension of Holt's method. Management Science *32* (1986), 499–510.

Author's address: *Tomáš Ratinger*, VÚZE Praha, Mánesova 75, 120 58 Praha 2, Czech Republic.