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ON THE SOLVABILITY OF SOME MULTI-POINT BOUNDARY VALUE PROBLEMS

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Summary. Let $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions and let $e(t) \in L^1[0,1]$. Let $\xi_i, \tau_j \in (0,1)$, $c_i, a_j \in \mathbb{R}$, all of the c_i's, (respectively, a_j's) having the same sign, $i = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1$ be given. This paper is concerned with the problem of existence of a solution for the multi-point boundary value problems

$$\text{(E)} \qquad x''(t) = f(t, x(t), x'(t)) + e(t), \qquad t \in (0,1)$$

$$\text{(BC)}_{mn} \qquad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$$

and

$$\text{(E)} \qquad x''(t) = f(t, x(t), x'(t)) + e(t), \qquad t \in (0,1)$$

$$\text{(BC)}'_{mn} \qquad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j),$$

Conditions for the existence of a solution for the above boundary value problems are given using Leray-Schauder Continuation theorem.

Keywords: multi-point boundary value problems, four point boundary value problems, Leray-Schauder Continuation theorem, a priori bounds

AMS classification: 34B10, 34B15

1. Introduction

Let $f: [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions and $e: [0,1] \to \mathbb{R}$ be a function in $L^1[0,1]$, $c_i, a_j \in \mathbb{R}$, with all of the c_i's, (respectively, a_j's), having the same sign, $\xi_i, \tau_j \in (0,1), i = 1,2,\ldots,m-2, j = 1,2,\ldots,n-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1$. The main purpose of this paper is to get results on the solvability of the following boundary value problems (BVPs for short)

$$\text{(E)} \qquad x''(t) = f(t, x(t), x'(t)) + e(t), \qquad t \in (0,1)$$

$$(\text{BC})_{mn} \qquad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$$

and

$$\text{(E)} \qquad x''(t) = f(t, x(t), x'(t)) + e(t), \qquad t \in (0,1)$$

$$(\text{BC})'_{mn} \qquad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j)$$

The results are motivated by the so called "nonlocal" BVPs studied by Il'in and Moiseev [5], [6]. Using the Mawhin's version of "Leray-Schauder Continuation theorem" ([8]), we prove the existence of a solution of the BVPs (E)–$(\text{BC})_{mn}$ and (E)–$(\text{BC})'_{mn}$. This method reduces the problem of existence of solutions of a BVP to the problem of establishing a priori bounds for the set of solutions of a family of these problems. Hence our main purpose is to give conditions on f which imply the needed a priori bounds.

It is well known (see [5], [6]) that if a function $x \in C^1$ satisfies the boundary condition $(\text{BC})_{mn}$ or $(\text{BC})'_{mn}$ and c_i, a_j, $i = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$ are as above, then there exist $\zeta \in [\xi_1, \xi_{m-2}]$, $\eta \in [\tau_1, \tau_{n-2}]$ such that

$$x(0) = \gamma x'(\zeta), \qquad x(1) = \alpha x(\eta)$$

or

$$x(0) = \gamma x'(\zeta), \qquad x'(1) = \alpha x'(\eta)$$

respectively with $\gamma = \sum_{i=1}^{m-2} c_i$, $\alpha = \sum_{j=1}^{n-2} a_j$. Hence for every solution x of the BVPs (E)–$(\text{BC})_{mn}$ or (E)–$(\text{BC})'_{mn}$ there exist $\zeta \in [\xi_1, \xi_{m-2}]$, $\eta \in [\tau_1, \tau_{n-2}]$ such that x is a solution of the following four point BVPs

$$\text{(E)} \qquad x''(t) = f(t, x(t), x'(t)), \qquad t \in [0,1]$$

$$(\text{BC})_4 \qquad x(0) = \gamma x'(\zeta), \qquad x(1) = \alpha x(\eta).$$

or

$$\text{(E)} \qquad x''(t) = f(t, x(t), x'(t)), \qquad t \in [0,1]$$
$$\text{(BC)}'_4 \qquad x(0) = \gamma x'(\zeta), \qquad x'(1) = \alpha x'(\eta)$$

respectively. We shall prove that all solutions of the BVPs (E_λ)–$(\mathrm{BC})_4$ and (E_λ)–$(\mathrm{BC})'_4$ are a priori bounded, with bounds independent of ζ and η, where (E_λ) stands for the equation $x'' = \lambda f + \lambda e$. Then, it is obvious, that these a priori bounds are also a priori bounds for the solutions of the BVP (E)–$(\mathrm{BC})_{mn}$ and (E)–$(\mathrm{BC})'_{mn}$. Recently Gupta, Ntouyas and Tsamatos studied in [3] and [4] the above BVP when $\gamma = 0$. Here we extend the results for general γ. For some recent results on the three point BVPs see [1], [2], [7].

We use the classical spaces $C[0,1]$, $C^k[0,1]$, $L^k[0,1]$, and $L^\infty[0,1]$ of continuous, k-times continuously differentiable, measurable real valued functions whose k-th power of the absolute value is Lebesgue integrable on $[0,1]$, or measurable functions that are essentially bounded on $[0,1]$. We also use the Sobolev spase $W^{2,k}(0,1)$, $k = 1, 2$ defined by

$$W^{2,k}(0,1) = \{x\colon [0,1] \to \mathbb{R} \;\; x, x' \text{ abs. cont. on } [0,1] \;\text{ with } x'' \in L^k[0,1]\}$$

with the usual norm. We denote the norm in $L^k[0,1]$ by $\|.\|_k$, and the norm in $L^\infty[0,1]$ by $\|.\|_\infty$.

2. Main results

2a. The boundary value problem (E)–$(\mathrm{BC})_{mn}$

We study first the BVP (E)–$(\mathrm{BC})_{mn}$. We begin with the following definition:

Definition 2.1. A function $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Caratheodory's conditions if (i) for each $(x, y) \in \mathbb{R}^2$, the function $t \in [0,1] \to f(t,x,y) \in \mathbb{R}$ is measurable on $[0,1]$, (ii) for a.e. $t \in [0,1]$, the function $(x,y) \in \mathbb{R}^2 \to f(t,x,y) \in \mathbb{R}$ is continuous on $\mathbb{R}^2$, and for each $r > 0$, there exists $g_r \in L^1[0,1]$ such that $|f(t,x,y)| \leqslant g_r(t)$ for a.e. $t \in [0,1]$ and $(x,y) \in \mathbb{R}^2$ with $\sqrt{x^2+y^2} \leqslant r$.

Lemma 2.2. *Let ζ, $\eta \in (0,1)$ be given and $x(t) \in W^{2,1}(0,1)$ be such that $x(0) = \gamma x'(\zeta)$, $x(1) = \alpha x(\eta)$. Then*

$$\|x\|_\infty \leqslant A\|x'\|_\infty, \quad \|x'\|_\infty \leqslant B\|x''\|_1$$

where

$$A=\begin{cases}1, & \text{if} \quad \alpha\leqslant 0\\ L, & \text{if} \quad \alpha>0,\, \alpha\neq 1,\\ 1+|\gamma|, & \text{if} \quad \alpha=1\end{cases}$$

and

$$B=\begin{cases}1, & \text{if} \quad \alpha\leqslant 0,\, \gamma=0\\ \frac{1}{1-Q}, & \text{if} \quad \alpha\leqslant 0,\, \gamma\neq 0\\ \frac{1}{1-S}, & \text{if} \quad \alpha>0,\, \alpha\neq 1\\ 1, & \text{if} \quad \alpha=1\end{cases}$$

where for $\alpha>0$, $\alpha\neq 1$, $M=\min\{\alpha,\frac{1}{\alpha}\}<1$, $L=\min\{\frac{1}{1-M}, 1+\frac{1-\eta}{|1-\alpha|}, 1+\frac{|\alpha|(1-\eta)}{|1-\alpha|}, 1+|\gamma|\}$, $S=\min\{\frac{|1-\alpha|}{1-\eta}L, \frac{|1-\alpha|}{\alpha(1-\eta)}L, \frac{1}{|\gamma|}L\}$, $Q=\min\{\frac{1-\alpha}{1-\eta}, \frac{1-\alpha}{|\alpha|(1-\eta)}, \frac{1}{|\gamma|}\}$ *if* $\alpha<0$ *and for* $\alpha=0$, $Q=\frac{1}{|\gamma|}$ *provided* $Q<1$, *and* $S<1$.

Proof. We consider the following cases:

Case 1. $\alpha\leqslant 0$**.** In this case $x(1)\cdot x(\eta)\leqslant 0$ and accordingly there exists a $\theta\in[\eta,1]$ such that $x(\theta)=0$. Hence it follows that $\|x\|_\infty\leqslant\|x'\|_\infty$. Also if $\gamma=0$, we have from $x(0)=0$ and $x(\theta)=0$ that there exists a $z\in(0,\theta)$ such that $x'(z)=0$. Accordingly, we get that $\|x'\|_\infty\leqslant\|x''\|_1$. Suppose, now, $\alpha<0$ and $\gamma\neq 0$. Next we see from Mean Value Theorem there exists an $\omega\in(\eta,1)$ such that

$$(\alpha-1)x(\eta)=x(1)-x(\eta)=(1-\eta)x'(\omega)$$

and hence

$$x(\eta)=\frac{1-\eta}{\alpha-1}x'(\omega). \tag{2.2}$$

Also, since $x(1)=\alpha x(\eta)$ we get

$$x(1)=\frac{\alpha(1-\eta)}{\alpha-1}x'(\omega). \tag{2.3}$$

It then follows from the relations

$$x'(t)=x'(\omega)+\int_\omega^t x''(s)\,\mathrm{d}s=\frac{\alpha-1}{1-\eta}x(\eta)+\int_\omega^t x''(s)\,\mathrm{d}s, \tag{2.4}$$

$$x'(t)=x'(\omega)+\int_\omega^t x''(s)\,\mathrm{d}s=\frac{\alpha-1}{\alpha(1-\eta)}x(1)+\int_\omega^t x''(s)\,\mathrm{d}s \tag{2.5}$$

and

$$x'(t)=x'(\zeta)+\int_0^t x''(s)\,\mathrm{d}s=\frac{1}{\gamma}x(0)+\int_0^t x''(s)\,\mathrm{d}s \tag{2.6}$$

that

$$\|x'\|_\infty \leqslant \frac{1}{1-Q}\|x''\|_1, \tag{2.7}$$

where $Q = \min\{\frac{1-\alpha}{1-\eta}, \frac{1-\alpha}{|\alpha|(1-\eta)}, \frac{1}{|\gamma|}\}$ if $Q < 1$. Finally, for $\alpha = 0$, $\gamma \neq 0$ it is easy to see from (2.4), (2.6) that $Q = \frac{1}{|\gamma|}$ since we require that $Q < 1$ and $\frac{1}{1-\eta} > 1$.

Case 2. $\alpha > 0$, $\alpha \neq 1$. We first consider the relations

$$x(t) = x(1) + \int_1^t x'(s)\,\mathrm{d}s = \alpha x(\eta) + \int_1^t x'(s)\,\mathrm{d}s$$

and

$$x(t) = x(\eta) + \int_\eta^t x'(s)\,\mathrm{d}s = \frac{1}{\alpha}x(1) + \int_\eta^t x'(s)\,\mathrm{d}s$$

Since, now, $M = \min\{\alpha, \frac{1}{\alpha}\} < 1$, we get from the above relations that

$$\|x\|_\infty \leqslant \frac{1}{1-M}\|x'\|_\infty.$$

Next, we use the equations (2.2) and (2.3) to get the relations

$$x(t) = x(1) + \int_1^t x'(s)\,\mathrm{d}s = \frac{\alpha(1-\eta)}{\alpha-1}x'(\omega) + \int_1^t x'(s)\,\mathrm{d}s$$

and

$$x(t) = x(\eta) + \int_\eta^t x'(s)\,\mathrm{d}s = \frac{1-\eta}{\alpha-1}x'(\omega) + \int_\eta^t x'(s)\,\mathrm{d}s.$$

Also

$$x(t) = x(0) + \int_0^t x'(s)\,\mathrm{d}s = \gamma x'(\zeta) + \int_0^t x'(s)\,\mathrm{d}s.$$

It is then immediate that

$$\|x\|_\infty \leqslant L\|x'\|_\infty,$$

where $L = \min\{\frac{1}{1-M}, 1 + \frac{1-\eta}{|\alpha-1|}, 1 + \frac{|\alpha|(1-\eta)}{|\alpha-1|}, 1 + |\gamma|\}$.

Further, we see using the relations (2.4), (2.5) and (2.6) that

$$\|x'\|_\infty \leqslant \frac{1}{1-S}\|x''\|_1, \tag{2.8}$$

where $S = \min\{\frac{|1-\alpha|}{1-\eta}L, \frac{|1-\alpha|}{\alpha(1-\eta)}L, \frac{1}{|\gamma|}L\}$ if $S < 1$.

Case 3. $\alpha = 1$. Since $x(1) = x(\eta)$ there exists an $\omega \in (\eta, 1)$ with $x'(\omega) = 0$. It is then immediate that $\|x'\|_\infty \leqslant \|x''\|_1$. Also since $x(t) = x(0) + \int_0^t x'(s)\,\mathrm{d}s = \gamma x'(\zeta) + \int_0^t x'(s)\,\mathrm{d}s$, it is immediate that $\|x\|_\infty \leqslant (1 + |\gamma|)\|x'\|_\infty$.

This completes the proof of the lemma. □

Theorem 2.3. *Let $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^1[0,1]$ such that*

$$|f(t,u,v)| \leqslant p(t)|u| + q(t)|v| + r(t) \tag{2.9}$$

for a.e. $t \in [0,1]$ and all $(u,v) \in \mathbb{R}^2$. Also let $\eta \in (0,1)$ be given and $\alpha, \gamma \in \mathbb{R}$ with $1 + \gamma \neq \alpha(\gamma + \eta)$. Moreover we assume that $Q < 1$ and $S < 1$.

(I) *If $\alpha \leqslant 0$, $\gamma = 0$ then the* BVP (E)–(BC)$_4$ *has at least one solution in $C^1[0,1]$ provided*

$$\|p\|_1 + \|q\|_1 < 1. \tag{2.10}$$

(II) *If $\alpha \leqslant 0$ and $\gamma \neq 0$ then the* BVP (E)–(BC)$_4$ *has at least one solution in $C^1[0,1]$ provided*

$$\|p\|_1 + \|q\|_1 < 1 - Q.$$

(III) *If $\alpha > 0$, $\alpha \neq 1$ then the* BVP (E)–(BC)$_4$ *has at least one solution in $C^1[0,1]$ provided*

$$L\|p\|_1 + \|q\|_1 < 1 - S \tag{2.11}$$

(IV) *If $\alpha = 1$ then the* BVP (E)–(BC)$_4$ *has at least one solution in $C^1[0,1]$ provided*

$$(1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1. \tag{2.12}$$

Proof. Let X be the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We denote a linear mapping $L\colon D(L) \subset X \to Y$ by setting

$$D(L) = \{x \in W^{2,1}(0,1)\colon x(0) = \gamma x'(\zeta), x(1) = \alpha x(\eta)\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear mapping $N\colon X \to Y$ by setting

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,1].$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping $L\colon D(L) \subset X \to Y$, is one-to-one mapping. Next, the linear mapping $K: Y \to X$, defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s)y(s)\,\mathrm{d}s + \gamma \int_0^\zeta y(s)\,\mathrm{d}s + \frac{\gamma + t}{1+\gamma-\alpha(\gamma+\eta)}\left[\alpha \int_0^\eta (\eta - s)y(s)\,\mathrm{d}s - \int_0^1 (1-s)y(s)\,\mathrm{d}s + \gamma(\alpha-1)\int_0^\zeta y(s)\,\mathrm{d}s\right], \qquad t \in [0,1].$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli Theorem that KN maps bounded subsets of X into relatively compact subsets of X. Hence $KN: X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the BVP (E)–$(\mathrm{BC})_4$ if and only if x is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$

We apply the Leray-Schauder Continuation theorem (see, e.g. [8], Corollary IV.7) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the BVP (E)–$(\mathrm{BC})_4$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$(\mathrm{E})_\lambda \qquad x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad t \in (0,1)$$
$$x(0) = \gamma x'(\zeta), \qquad x(1) = \alpha x(\eta)$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

(I) Assume that $\alpha \leqslant 0$, $\gamma = 0$. From Lemma 2.2 we have

$$\|x\|_\infty \leqslant \|x'\|_\infty \leqslant \|x''\|_1$$

Let, now, $x(t)$ be a solution of (E_λ) for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = \gamma x'(\zeta)$, $x(1) = \alpha x(\eta)$. We then get from (E_λ) that

$$\begin{aligned}\|x''\|_1 &= \lambda \|f(t, x(t), x'(t)) + e(t)\|_1 \\ &\leqslant \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + \|r\|_1 + \|e\|_1 \\ &\leqslant (\|p\|_1 + \|q\|_1)\|x''\|_1 + \|r\|_1 + \|e\|_1\end{aligned}$$

It follows from the assumption (2.8) that there is a constant c, independent of $\lambda \in [0,1]$, such that

$$\|x''\|_1 \leqslant c.$$

It is now immediate that the set of solutions of the family of equations (E_λ) is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

(II) Assume that $\alpha \leqslant 0, \gamma \neq 0$. Then we have, by Lemma 2.2 that

$$\|x\|_\infty \leqslant \|x'\|_\infty, \|x'\|_\infty \leqslant \frac{1}{1-Q}\|x''\|_1.$$

We then get from (E_λ) that

$$\begin{aligned}\|x''\|_1 &= \lambda\|f(t,x(t),x'(t)) + e(t)\|_1 \\ &\leqslant \|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1 + \|e\|_1 \\ &\leqslant [\|p\|_1 + \|q\|_1]\frac{1}{1-Q}\|x''\|_1 + \|r\|_1 + \|e\|_1.\end{aligned}$$

We proceed as in case (I).

The process for the other cases is similar to the previous cases and we omit the details. This completes the proof of the theorem. □

Theorem 2.4. *Let $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^1[0,1]$ such that*

$$|f(t,u,v)| \leqslant p(t)|u| + q(t)|v| + r(t) \tag{2.13}$$

for a.e. $t \in [0,1]$ and all $(u,v) \in \mathbb{R}^2$. Let $c_i, a_j \in \mathbb{R}$, with all of the c_i's, (respectively, a_j's), having the same sign, $\xi_i, \tau_j \in (0,1)$, $i = 1,2,\dots,m-2$, $j = 1,2,\dots,n-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \dots < \tau_{n-2} < 1$ be given. Suppose that $1 + \left(\sum_{i=1}^{m-2} c_i\right)\left(1 - \sum_{j=1}^{n-2} a_j\right) - \sum_{j=1}^{n-2} a_j\tau_j \neq 0$. Let $\gamma = \sum_{i=1}^{m-2} c_i$ and $\alpha = \sum_{j=1}^{n-2} a_j$. Moreover we assume that $Q^{mn} < 1$, and $S^{mn} < 1$, where $M = \min\{\alpha, \frac{1}{\alpha}\} < 1$,

$$\begin{aligned} L^{mn} &= \min\left\{\frac{1}{1-M}, 1 + \frac{1-\tau_1}{|1-\alpha|}, 1 + \frac{|\alpha|(1-\tau_1)}{|1-\alpha|}, 1 + |\gamma|\right\}, \\ S^{mn} &= \min\left\{\frac{|1-\alpha|}{1-\tau_{n-2}}L, \frac{|1-\alpha|}{\alpha(1-\tau_{n-2})}L, \frac{1}{|\gamma|}L\right\} \\ Q^{mn} &= \min\left\{\frac{1-\alpha}{1-\tau_{n-2}}, \frac{1-\alpha}{|\alpha|(1-\tau_{n-2})}, \frac{1}{|\gamma|}\right\}.\end{aligned}$$

(I) *If* $\alpha \leqslant 0$, $\gamma = 0$ *then the* BVP (E)–$(\mathrm{BC})_{mn}$ *has at least one solution in* $C^1[0,1]$ *provided*

$$\|p\|_1 + \|q\|_1 < 1. \tag{2.14}$$

(II) *If* $\alpha \leqslant 0$ *and* $\gamma \neq 0$ *then the* BVP (E)–$(\mathrm{BC})_4$ *has at least one solution in* $C^1[0,1]$ *provided*

$$\|p\|_1 + \|q\|_1 < 1 - Q^{mn}. \tag{2.15}$$

(III) *If* $\alpha > 0$, $\alpha \neq 1$ *then the* BVP (E)–$(\mathrm{BC})_{mn}$ *has at least one solution in* $C^1[0,1]$ *provided*

$$L^{mn}\|p\|_1 + \|q\|_1 < 1 - S^{mn}. \tag{2.16}$$

(IV) *If* $\alpha = 1$ *then the* BVP (E)–$(\mathrm{BC})_{mn}$ *has at least one solution in* $C^1[0,1]$ *provided*

$$(1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1. \tag{2.17}$$

Proof. As we have remarked in the introduction, we study the multi-point BVP using the a priori estimates that can be obtained for a four-point BVP. This is because for every solution $x(t)$ of the BVP (E)–$(\mathrm{BC})_{mn}$, there exist $\eta \in [\xi_1, \xi_{m-2}]$, $\zeta \in [\tau_1, \tau_{n-2}]$, depending on, $x(t)$, such that $x(t)$ is also a solution of the BVP (E)–$(\mathrm{BC})_4$ with $\gamma = \sum\limits_{i=1}^{m-2} c_i$ and $\alpha = \sum\limits_{j=1}^{n-2} a_j$. The proof is quite similar to the proof of Theorem 2.3 and uses the a priori estimates obtained in the proof of Theorem 2.3 for the set of solutions of the family of equations (E_λ)–$(\mathrm{BC})_4$. We note that it was shown that the set of solutions of the family of equations (E_λ)–$(BC)_4$ was, a priori, bounded by a constant independent of $\lambda \in [0,1]$ and both $\eta, \zeta \in (0,1)$, and this fact is the key point needed in the proof of Theorem 2.4.

Let X be the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We denote a linear mapping $L\colon D(L) \subset X \to Y$ by setting

$$D(L) = \left\{ x \in W^{2,1}(0,1) \colon x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j) \right\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear mapping $N\colon X \to Y$ by setting

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,1].$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping $L\colon D(L) \subset X \to Y$, is one-to-one mapping. Next, the linear mapping $K\colon Y \to X$, defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s)y(s)\,\mathrm{d}s + ct + k, \qquad t \in [0,1]$$

where c and k are given by,

$$\left[1 + \left(\sum_{i=1}^{m-2} c_i\right)\left(1 - \sum_{j=1}^{n-2} a_j\right) - \sum_{j=1}^{n-2} a_j\tau_j\right] c = \left(\sum_{j=1}^{n-2} a_j - 1\right)\left(\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} y(s)\,\mathrm{d}s\right) + \sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s)\,\mathrm{d}s - \int_0^1 (1-s)y(s)\,\mathrm{d}s$$

and

$$\left[1 + \left(\sum_{i=1}^{m-2} c_i\right)\left(1 - \sum_{j=1}^{n-2} a_j\right) - \sum_{j=1}^{n-2} a_j\tau_j\right] k = \sum_{i=1}^{m-2} c_i \sum_{j=1}^{n-2} a_j \int_0^{\tau_j} (\tau_j - s)y(s)\,\mathrm{d}s - \sum_{i=1}^{m-2} c_i \int_0^1 (1-s)y(s)\,\mathrm{d}s + \left(1 - \sum_{j=1}^{n-2} a_j\tau_j\right)\sum_{i=1}^{m-2} c_i \int_0^{\xi_i} y(s)\,\mathrm{d}s$$

is such that for $y \in Y$, $Ky \in D(L)$and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli Theorem that KN maps bounded subsets of X into relatively compact subsets of X. Hence $KN\colon X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the BVP (E)–$(\mathrm{BC})_{mn}$ if and only if x is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$

We apply the Leray-Schauder Continuation theorem (see, e.g. [8], Corollary IV.7) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the BVP (E)–$(\mathrm{BC})_{mn}$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$(\mathrm{E})_\lambda \qquad x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad t \in (0,1)$$

$$(\mathrm{BC})_{mn} \qquad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), \qquad x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

Let, now, $x(t)$ be a solution of (E_λ)–$(\mathrm{BC})_{mn}$ for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i)$, $x(1) = \sum_{j=1}^{n-2} a_j x(\tau_j)$. Accordingly, there exist $\zeta \in [\xi_1, \xi_{m-2}]$ and $\eta \in [\tau_1, \tau_{n-2}]$ depending on $x(t)$, such that $x(t)$ is a solution of the four point BVP

$$x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad t \in (0,1)$$
$$x(0) = \gamma x'(\zeta), \qquad x(1) = \alpha x(\eta)$$

It then follows, as in the proof of Theorem 2.4 that there is a constant c, independent of $\lambda \in [0,1]$, and $\eta \in [\xi_1, \xi_{m-2}]$, $\zeta \in [\tau_1, \tau_{n-2}]$ such that

$$\|x\|_\infty \leqslant c_1 \|x'\|_\infty \leqslant c_2 \|x''\|_1 \leqslant c,$$

where c_1, c_2 are constants independent of λ, η, ζ according to the cases (I), (II) or (III). Thus the set of solutions of the family of equations (E_λ)–$(\mathrm{BC})_{mn}$ is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in [0,1]$.

It is important to remark that the assumptions of Theorem 2.4, ensure that the needed a priori bounds are independent of $\zeta \in [\xi_1, \xi_{m-2}]$ and $\eta \in [\tau_1, \tau_{n-2}]$. This completes the proof of the theorem. □

2B. The boundary value problem (E)–$(\mathrm{BC})'_{mn}$.

In this section we study, by a similar way, the BVP (E)–$(\mathrm{BC})'_{mn}$.

Lemma 2.5. *Let $\eta \in (0,1)$ and $\gamma, \alpha \in \mathbb{R}$ be given. Let $x(t) \in W^{2,1}(0,1)$ be such that $x(0) = \gamma x'(\zeta), x'(1) = \alpha x'(\eta)$. Then*

$$\|x\|_\infty \leqslant (1 + |\gamma|)\|x'\|_\infty, \ \ \|x'\|_\infty \leqslant A\|x''\|_1,$$

where

$$A = \begin{cases} 1, & \text{if} \quad \alpha \leqslant 0 \\ \frac{1}{1-M}, & \text{if} \quad \alpha > 0,\, \alpha \neq 1 \end{cases}$$

and $M = \min\{\alpha, \frac{1}{\alpha}\} < 1$.

P r o o f. First we have from the relation

$$x(t) = x(0) + \int_0^1 x'(s)\,\mathrm{d}s = \gamma x'(\zeta) + \int_0^1 x'(s)\,\mathrm{d}s$$

that

$$\|x\|_\infty \leqslant (1 + |\gamma|)\|x'\|_\infty$$

Next, when $\alpha \leqslant 0$ there exists a $\theta \in [\eta, 1]$ such that $x'(\theta) = 0$ from which we get that $\|x'\|_\infty \leqslant \|x''\|_1$. Now, if $\alpha > 0$ and $\alpha \neq 1$ we see from the relations

$$x'(t) = x'(1) + \int_1^t x''(s)\,\mathrm{d}s = \alpha x'(\eta) + \int_1^t x''(s)\,\mathrm{d}s$$
$$x'(t) = x'(\eta) + \int_\eta^t x''(s)\,\mathrm{d}s = \frac{1}{\alpha} x'(1) + \int_\eta^t x''(s)\,\mathrm{d}s$$

that

$$\|x'\|_\infty \leqslant M\|x'\|_\infty + \|x''\|_1$$

and hence

$$\|x'\|_\infty \leqslant \frac{1}{1-M}\|x''\|_1.$$

This completes the proof of the lemma. □

Theorem 2.6. *Let $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^1[0,1]$ such that*

$$|f(t,u,v)| \leqslant p(t)|u| + q(t)|v| + r(t)$$

for a.e. $t \in [0,1]$ and all $(u,v) \in \mathbb{R}^2$. Also let $\eta, \zeta \in (0,1)$ be given and $\alpha, \gamma \in \mathbb{R}$ with $\alpha \neq 1$.

(I) *If $\alpha \leqslant 0$ then the* BVP (E)–(BC)$'_4$ *has at least one solution in $C^1[0,1]$ provided*

$$(1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1. \tag{2.18}$$

(II) *If $\alpha > 0$, $\alpha \neq 1$ then the* BVP (E)–(BC)$'_4$ *has at least one solution in $C^1[0,1]$ provided*

$$(1 + |\gamma|)\|p\|_1 + \|q\|_1 < 1 - M. \tag{2.19}$$

Proof. Let X be the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We denote a linear mapping $L\colon D(L) \subset X \to Y$ by setting

$$D(L) = \{x \in W^{2,1}(0,1)\colon x(0) = \gamma x'(\zeta), x'(1) = \alpha x'(\eta)\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear mapping $N\colon X \to Y$ by setting

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,1].$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping $L\colon D(L) \subset X \to Y$, is one-to-one mapping. Next, the linear mapping $K\colon Y \to X$, defined for $y \in Y$ by

$$(Ky)(t) = \int_0^t (t-s)y(s)\,\mathrm{d}s + \frac{\gamma + t}{1-\alpha}\left[\alpha \int_0^\eta y(s)\,\mathrm{d}s - \int_0^1 y(s)\,\mathrm{d}s\right] + \gamma \int_0^\zeta y(s)\,\mathrm{d}s, \qquad t \in [0,1].$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$, and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli Theorem that KN maps bounded subsets of X into relatively compact subsets of X. Hence $KN\colon X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the BVP (E)–(BC)$'_4$ if and only if x is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$

We apply the Leray-Schauder Continuation theorem (see, e.g. [8], Corollary IV.7) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the BVP (E)–(BC)$'_4$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$(\mathrm{E})_\lambda \qquad x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad t \in (0,1)$$
$$x(0) = \gamma x'(\zeta), \qquad x'(1) = \alpha x'(\eta)$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

Assume that $\alpha \leqslant 0$. From Lemma 2.5 we have

$$\|x\|_\infty \leqslant (1+|\gamma|)\|x'\|_\infty, \|x'\|_\infty \leqslant \|x''\|_1$$

Let, now, $x(t)$ be a solution of (E_λ) for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = \gamma x'(\zeta), x'(1) = \alpha x'(\eta)$. We then get from (E_λ) that

$$\begin{aligned}\|x''\|_1 &= \lambda \|f(t, x(t), x'(t)) + e(t)\|_1 \\ &\leqslant \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + \|r\|_1 + \|e\|_1 \\ &\leqslant ((1+|\gamma|)\|p\|_1 + \|q\|_1)\|x''\|_1 + \|r\|_1 + \|e\|_1\end{aligned}$$

It follows from the assumption (2.13) that there is a constant c, independent of $\lambda \in [0,1]$, such that

$$\|x''\|_1 \leqslant c.$$

It is now immediate that the set of solutions of the family of equations (E_λ) is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

The case $\alpha > 0$, $\alpha \neq 1$ is similar and simple.

This completes the proof of the theorem. □

Theorem 2.7. *Let $f\colon [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^1[0,1]$ such that*

$$|f(t,u,v)| \leqslant p(t)|u| + q(t)|v| + r(t)$$

for a.e. $t \in [0,1]$ and all $(u,v) \in \mathbb{R}^2$. Let $c_i, a_j \in \mathbb{R}$, with all of the c_i's (respectively, a_j's), having the same sign, $\xi_i, \tau_j \in (0,1)$, $i = 1,2,\ldots,m-2$, $j = 1,2,\ldots,n-2$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \ldots < \tau_{n-2} < 1$ be given. Suppose that $1 - \sum_{j=1}^{n-2} a_j \neq 0$.

Then for any given $e(t)$ in $L^1(0,1)$, the mn-point BVP (E)–$(\mathrm{BC})'_{mn}$ has at least one solution in $C^1[0,1]$.

Proof. As we have remarked in the introduction, we study the mn-point BVP using the a priori estimates that can be obtained for a four point BVP. This is because for every solution $x(t)$ of the BVP (E)–$(\mathrm{BC})_{mn}$ there exist $\zeta \in [\xi_1, \xi_{m-2}]$, $\eta \in [\tau_1, \tau_{n-2}]$, depending on $x(t)$, such that $x(t)$ is also a solution of the BVP (E)–$(\mathrm{BC})'_4$ with $\gamma = \sum_{i=1}^{m-2} c_i$ and $\alpha = \sum_{j=1}^{n-2} a_j \neq 1$. The proof is quite similar to the proof of Theorem 2.6 and uses the a priori estimates obtained in the proof of

Theorem 2.3 for the set of solutions of the family of equations (E_λ)–$(BC)'_4$. We note that it was shown that the set of solutions of the family of equations (E_λ)–$(BC)'_4$ was, a priori, bounded by a constant independent of $\lambda \in [0,1]$ and $\eta \in (0,1)$, and this fact is the key point needed in the proof of Theorem 2.7.

Let X be the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We denote a linear mapping $L\colon D(L) \subset X \to Y$ by setting

$$D(L) = \left\{ x \in W^{2,1}(0,1) \colon x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), x'(1) = \sum_{j=1}^{n-2} a_j x'(\tau_j) \right\},$$

and for $x \in D(L)$,

$$Lx = x''.$$

We also define a nonlinear mapping $N\colon X \to Y$ by setting

$$(Nx)(t) = f(t, x(t), x'(t)), \qquad t \in [0,1].$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping $L\colon D(L) \subset X \to Y$, is one-to-one mapping. Next, the linear mapping $K\colon Y \to X$, defined for $y \in Y$ by

$$(Ky)(t) = \frac{t + \sum_{i=1}^{m-2} c_i}{1 - \sum_{j=1}^{n-2} a_j} \left[\sum_{j=1}^{n-2} a_j \int_0^{\tau_j} y(s)\,\mathrm{d}s - \int_0^1 y(s)\,\mathrm{d}s \right] + \sum_{i=1}^{m-2} c_i \int_0^{\xi_i} y(s)\,\mathrm{d}s + \int_0^t (t-s) y(s)\,\mathrm{d}s, \qquad t \in [0,1]$$

is such that for $y \in Y$, $Ky \in D(L)$ and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli Theorem that KN maps bounded subsets of X into relatively compact subsets of X. Hence $KN\colon X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the BVP (E)–$(BC)'_{mn}$ if and only if x is a solution to the operator equation

$$Lx = Nx + e.$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke.$$

We apply the Leray-Schauder Continuation theorem (see, e.g. [8], Corollary IV.7) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the BVP (E)–$(BC)'_{mn}$.

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$(E)_\lambda \qquad x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad t \in (0,1)$$

$$(BC)'_{mn} \qquad x(0) = \sum_{i=1}^{m-2} c_i x'(\xi_i), x'(1) = \sum_{j=1}^{n-2} a_i x'(\tau_j)$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

Let, now, $x(t)$ be a solution of (E_λ)–$(BC)'_m$ for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = \sum\limits_{i=1}^{m-2} c_i x'(\xi_i)$, $x'(1) = \sum\limits_{j=1}^{n-2} a_j x'(\tau_j)$. Accordingly, there exist $\zeta \in [\xi_1, \xi_{m-2}]$, $\eta \in [\tau_1, \tau_{n-2}]$ depending on $x(t)$, such that $x(t)$ is a solution of the four point BVP

$$x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t), \qquad t \in (0,1)$$
$$x(0) = \gamma x'(\zeta), \qquad x'(1) = \alpha x'(\eta)$$

It then follows, as in the proof of Theorem 2.6 that there is a constant c, independent of $\lambda \in [0,1]$, and $\zeta \in [\xi_1, \xi_{m-2}]$, $\eta \in [\tau_1, \tau_{n-2}]$ such that

$$\|x\|_\infty \leqslant c_1 \|x'\|_\infty \leqslant c_2 \|x''\|_1 \leqslant c.$$

Thus the set of solutions of the family of equations (E_λ)–$(BC)'_{mn}$ is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in [0,1]$.

This completes the proof of the theorem. □

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