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CONVERGENCE OF NUMERICAL METHODS FOR SYSTEMS OF NEUTRAL FUNCTIONAL-DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Summary. A general class of numerical methods for solving initial value problems for neutral functional-differential-algebraic systems is considered. Necessary and sufficient conditions under which these methods are consistent with the problem are established. The order of consistency is discussed. A convergence theorem for a general class of methods is proved.

Keywords: neutral functional-differential-algebraic systems, consistency, convergence

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1. Introduction

Let I be a fixed interval of real line. By $C^i(I,\mathbb{R}^p)$ we denote the space of function with continuous derivatives up to the order i on I into $\mathbb{R}^p$; $C(I,\mathbb{R}^p) = C^0(I,\mathbb{R}^p)$. Let $J=[a,b]$, $\mathscr{A} = C(J,\mathbb{R}^p)$, $\mathscr{B} = C(J,\mathbb{R}^m)$, $\mathscr{Y} = \mathscr{A}\times\mathscr{A}\times\mathscr{B}$. Later, we will use also the following spaces: $\tilde{\mathscr{A}} = \tilde{C}(J,\mathbb{R}^p)$, $\tilde{\mathscr{B}} = \tilde{C}(J,\mathbb{R}^m)$, $\mathscr{U} = \mathscr{A}\times\tilde{\mathscr{A}}\times\tilde{\mathscr{B}}$, where $\tilde{C}(J,\mathbb{R}^m)$ or $\tilde{C}(J,\mathbb{R}^p)$ denotes the class of piecewise continuous functions from J into $\mathbb{R}^m$ or $\mathbb{R}^p$, respectively. These spaces are considered with the standard uniform topology.

For given $f\in C(\mathscr{Y},\mathscr{A})$, $g\in C(\mathscr{Y},\mathscr{B})$ and $y_0\in\mathbb{R}^p$, we consider the system of functional-differential-algebraic equations (FDAEs) of the form

$$(1)\qquad \begin{cases} y'(t) = f(y,y',z)(t), & y(a)=y_0,\\ z(t) = g(y,y',z)(t), & \end{cases}\qquad t\in J,$$

with unknown functions $y \in C^1(J, \mathbb{R}^p)$ and $z \in \mathscr{B}$. We assume that f and g are Volterra mappings, i.e. for any $t \in J$ the conditions

$$y(s) = \bar{y}(s), \quad Y(s) = \bar{Y}(s), \quad z(s) = \bar{z}(s) \quad \text{for } a \leqslant s \leqslant t$$

imply

$$f(y, Y, z)(t) = f(\bar{y}, \bar{Y}, \bar{z})(t) \quad \text{and} \quad g(y, Y, z)(t) = g(\bar{y}, \bar{Y}, \bar{z})(t).$$

Special cases of system (1) arise in many applications, among other in the modelling of engineering problems, for example in the simulation of electrical networks, mechanical systems, physical chemistry and in control applications (see [1–3]).

The aim of this paper is a numerical solution of problem (1). First, let us make some general comments about problem (1). Let $\varrho(D)$ denote the spectral radius of a square matrix D and let

$$\|u\|_t = \max_{a \leqslant s \leqslant t} \|u(s)\|$$

with $\|\cdot\|$ being a norm in $\mathbb{R}^n$. It can be proved (see [11]) that there is a unique solution of (1) if the Lipschitz conditions

$$\|f(x, y, z)(t) - f(\tilde{x}, \tilde{y}, \tilde{z})(t)\| \leqslant c_1 \|x - \tilde{x}\|_t + c_2 \|y - \tilde{y}\|_t + c_3 \|z - \tilde{z}\|_t,$$

$$\|g(x, y, z)(t) - g(\tilde{x}, \tilde{y}, \tilde{z})(t)\| \leqslant \bar{c}_1 \|x - \tilde{x}\|_t + \bar{c}_2 \|y - \tilde{y}\|_t + \bar{c}_3 \|z - \tilde{z}\|_t$$

hold for some $c_i, \bar{c}_i \geqslant 0$, $i = 1, 2, 3$, and $\varrho(D) < 1$ for

$$D = \begin{bmatrix} c_2 & c_3 \\ \bar{c}_2 & \bar{c}_3 \end{bmatrix}.$$

It is known that the difficulty in solving the problem of differential-algebraic equations (DAEs) depends on their index. We observe that the conditions assumed in the present paper imply the index of (1) to be at most one. For instance, let g appearing in (1) not depend on y', i.e.

$$g(y, y', z)(t) = g(y, z)(t), \quad t \in J.$$

Now, if we assume that the derivatives g_y and g_z exist, then differentiating the second equation in (1) we have

$$\begin{aligned} z'(t) &= \left(g_y(y, z)y'\right)(t) + \left(g_z(y, z)z'\right)(t) \\ &= \left(g_y(y, z)f(y, y', z)\right)(t) + \left(g_z(y, z)z'\right)(t), \quad t \in J. \end{aligned}$$

If $(I - g_z)^{-1}$ exists and is bounded, then

$$z'(t) = \left[\left(I - g_z(y, z)\right)^{-1} \left(g_y(y, z)f(y, y', z)\right)\right](t), \quad t \in J.$$

This means that (1) takes the form

$$(2)\qquad \begin{cases} y'(t) = f(y, y', z)(t), & y(a) = y_0, \\ z'(t) = \bar{g}(y, y', z)(t), & z(a) = z_0. \end{cases} \qquad t \in J$$

In this way we get a system of FDAEs without the algebraic part, i.e. a system of the index 0. This means, according to [1], [8], that the index of the original problem equals one (in [1], [8], the index of DAE is defined as the number of times the algebraic part of the system has to be differentiated to obtain an ordinary differential equation).

On the other hand, under the assumption $\varrho(D) < 1$, system (1) can be resolved with respect to y' and z. It means that there exist operators $\tilde{f} \in C(\mathscr{A}, \mathscr{A})$ and $\tilde{g} \in C(\mathscr{A}, \mathscr{A})$ such that (1) is equivalent to the explicit system

$$(3)\qquad \begin{cases} y'(t) = \tilde{f}(y)(t), & y(a) = y_0, \\ z(t) = \tilde{g}(y)(t), \end{cases} \qquad t \in J.$$

This proves that system (1) can be considered as a system of the index 1 because it is reduced to the simple system (3) which again can be solved by first solving the differential equation and then calculating z from the second equation of this system.

However, in practice, it is impossible to follow the way described above because the operators $\tilde{f}$ and $\tilde{g}$ are not known. Due to this fact, to solve (1) numerically we have to use the original form of system (1) employing the known operators f and g.

Before describing a class of numerical methods for solving problem (1), we observe that, at the present time, it is well recognized that the numerical methods for differential-delay equations have to be constructed in such a way that they make it possible to compute the values of approximate solutions at any point of the interval where they are defined. This is also the case for differential-delay-algebraic systems of equations. This is in contrast with the case of ordinary differential equations and DAEs for which it is quite enough to compute approximations only at the mesh points.

Assume that $(y^\star, z^\star)$ is a unique solution of problem (1). Let $Y^\star$ denote the derivative of $y^\star$. Take $h = (b-a)/N$ and $t_n = a + nh$, $n = 0, 1, \ldots, N$. By y^h, Y^h and z^h we will denote approximations of $y^\star$, $Y^\star$ and $z^\star$, respectively.

Put $\tilde{y} = y_0$, $\tilde{z} = z^\star(a)$ and $\tilde{Y} = y^{\star\prime}(a)$, so $\tilde{Y}$ and $\tilde{z}$ from a solution of the system of equations

$$\begin{aligned} \tilde{Y} &= f(\tilde{y}, \tilde{Y}, \tilde{z})(a), \\ \tilde{z} &= g(\tilde{y}, \tilde{Y}, \tilde{z})(a). \end{aligned}$$

The values $\tilde{Y}$ and $\tilde{z}$ have to be found first, then we construct y^h, Y^h and z^h by the procedure

$$(4) \qquad \begin{cases} y^h(a) = \tilde{y}, \\ y^h(t_n + rh) = y^h(t_n) + hF(y^h, Y^h, z^h; h, r)(t_n), \\ Y^h(a) = \tilde{Y}, \\ Y^h(t_n + rh) = G(y^h, Y^h, z^h; h, r)(t_n), \\ z^h(a) = \tilde{z}, \\ z^h(t_n + rh) = Q(y^h, Y^h, z^h; h, r)(t_n) \end{cases}$$

for $r \in (0,1]$ and $n = 0, 1, \ldots, N-1$; $F(y^h, Y^h, z^h; h, 0)(t) = \Theta_p$, where Θ_p denotes the zero vector in $\mathbb{R}^p$. The operators $F, G\colon \mathscr{U} \times \mathscr{H} \to \tilde{\mathscr{A}}$, $Q\colon \mathscr{U} \times \mathscr{H} \to \tilde{\mathscr{B}}$ are given, where $\mathscr{H} = H \times [0,1]$ with $H = [0, h_0]$ for some $h_0 > 0$. We assume that F, G and Q are Volterra mappings with respect to y^h, Y^h, z^h for any h and r.

The aim of this paper is to give conditions under which method (4) will converge to the solution $(y^\star, z^\star)$ of (1). The order of convergence will be discussed, too.

This paper extends the results of [5] and [9] formulated for ordinary differential-algebraic problems to the general form of the neutral functional-differential-algebraic systems.

2. Consistency

Now we are going to discuss the problem of consistency of method (4) with problem (1). Let us introduce

Definition 1. Method (4) is consistent with (1) on the solution $(y^\star, z^\star) \in C^1(J, \mathbb{R}^p) \times \mathscr{B}$ if there exist functions $\varepsilon_i\colon H \to \mathbb{R}_+ = [0, \infty)$, $i = 1, 2, 3$, $\varepsilon_i(h) \to 0$, $i = 2, 3$, $h^{-1}\varepsilon_1(h) \to 0$ as $h \to 0$, such that for $(t, h, r) \in J_h \times H \times [0,1]$, $J_h = [a, b-h]$, we have

1° $\|hF(y^\star, Y^\star, z^\star; h, r)(t) + y^\star(t) - y^\star(t + rh)\| \leqslant \varepsilon_1(h)$,

2° $\|G(y^\star, Y^\star, z^\star; h, r)(t) - Y^\star(t + rh)\| \leqslant \varepsilon_2(h)$,

3° $\|Q(y^\star, Y^\star, z^\star; h, r)(t) - z^\star(t + rh)\| \leqslant \varepsilon_3(h)$.

The order of consistency is q if, in addition,

$$\varepsilon_1(h) = O(h^{q+1}) \quad \text{and} \quad \varepsilon_i(h) = O(h^q), \ \ i = 2, 3 \ \text{ as } \ h \to 0.$$

Theorem 1. *Assume that $f \in C(\mathscr{Y}, \mathscr{A})$, $g \in C(\mathscr{Y}, \mathscr{B})$ and the mappings $F, G\colon \mathscr{U} \times \mathscr{H} \to \tilde{\mathscr{A}}$, $Q\colon \mathscr{U} \times \mathscr{H} \to \tilde{\mathscr{B}}$ are continuous with respect to the fourth argument* *uniformly with respect to the ones. Then, for $t \in J$ and $r \in [0,1]$, the relations*

$$(5)\qquad \begin{cases} F(y^\star, Y^\star, z^\star; 0, r)(t) = rf(y^\star, Y^\star, z^\star)(t), \\ G(y^\star, Y^\star, z^\star; 0, r)(t) = f(y^\star, Y^\star, z^\star)(t), \\ Q(y^\star, Y^\star, z^\star; 0, r)(t) = g(y^\star, Y^\star, z^\star)(t) \end{cases}$$

are necessary and sufficient conditions for the consistency of method (4) *with system* (1).

Proof. To prove the theorem use the Taylor formula and Definition 1 to get the assertion. □

Remark 1. If in Theorem 1 one assumes a bit more, namely that $Y^\star$ satisfies the Hölder condition with an exponent $\gamma \in (0,1]$, then the order of consistency of (4) equals at least γ.

To get the consistency of higher order of method (4) one needs to assume higher regularity of the solution $(y^\star, z^\star)$. To discuss this question we extend the approach of [14] to our case. We introduce

Assumption A. Assume the following conditions are satisfied:

1° $f_h \in C(\mathscr{U}, \tilde{\mathscr{A}})$ and $g_h \in C(\mathscr{U}, \tilde{\mathscr{B}})$ are approximations of $f \in C(\mathscr{Y}, \mathscr{A})$ and $g \in C(\mathscr{Y}, \mathscr{B})$ such that, for any $(y, Y, z) \in C^1(J, \mathbb{R}^p) \times C(J, \mathbb{R}^p) \times C(J, \mathbb{R}^m)$, the relations

$$\|f_h(y, Y, z)(t) - f(y, Y, z)(t)\| \leqslant \delta_1(h),$$
$$\|g_h(y, Y, z)(t) - g(y, Y, z)(t)\| \leqslant \delta_2(h)$$

hold for some $\delta_i \colon H \to \mathbb{R}_+$, $i = 1, 2$; and f_h, g_h are Volterra mappings;

2° there exist nonnegative constants M_i, P_i, $i = 1, 2, 3$, such that

$$\|f_h(x_1, x_2, x_3)(t) - f_h(y_1, y_2, y_3)(t)\| \leqslant \sum_{i=1}^{3} M_i \|x_i - y_i\|_t;$$

$$\|g_h(x_1, x_2, x_3)(t) - g_h(y_1, y_2, y_3)(t)\| \leqslant \sum_{i=1}^{3} P_i \|x_i - y_i\|_t,$$

3° $\varrho(A) < 1$ for $A = \begin{bmatrix} M_2 & M_3 \\ P_2 & P_3 \end{bmatrix}$.

Assumption B. Assume that

1° $\Phi \colon \mathscr{U} \times \mathscr{H} \to \tilde{\mathscr{A}}$, $\chi \colon \mathscr{U} \times \mathscr{H} \to \tilde{\mathscr{B}}$ are Volterra mappings, $\Phi(y, Y, z; h, \cdot)(t)$ is of class C^1 and we denote its derivative by Φ_5;

2° there exist $\bar{\delta}_i\colon H \to \mathbb{R}_+$, $i = 1, 2, 3$, such that the inequalities

$$\|y^\star(t) + hr\Phi(y^\star, Y^\star, z^\star; h, r)(t) - y^\star(t + rh)\| \leqslant \bar{\delta}_1(h), \quad r \in [0, 1],$$
$$\|\Phi(y^\star, Y^\star, z^\star; h, r)(t) + r\Phi_5(y^\star, Y^\star, z^\star; h, r)(t) - Y^\star(t + rh)\| \leqslant \bar{\delta}_2(h), \quad r \in [0, 1],$$
$$\|\chi(y^\star, Y^\star, z^\star; h, r)(t) - z^\star(t + rh)\| \leqslant \bar{\delta}_3(h), \quad r \in [0, 1]$$

hold for $t \in J_h$;

3° $a_j \in C^1([0, 1], \mathbb{R})$, $a_j(0) = 0$, $\bar{a}_j \in C([0, 1], \mathbb{R})$, $b_j \in [0, 1]$, $\bar{b}_j \in [0, 1]$, $b_i \neq b_j$, $\bar{b}i \neq \bar{b}_j$ for $i \neq j$ and $i, j = 1, 2, \ldots, k$.

Under Assumptions A and B, we define

$$F(y, Y, z; h, r)(t) = \sum_{j=1}^{k} a_j(r) f_h(\bar{y}, \bar{Y}, \bar{z})(t + b_j h), \tag{6a}$$

$$G(y, Y, z; h, r)(t) = \begin{cases} \sum_{j=1}^{k} a'_j(r) f_h(\bar{y}, \bar{Y}, \bar{z})(t + b_j h) & \text{for } \ 0 < r < 1, \\ f_h(\bar{y}, \bar{Y}, \bar{z})(t + rh) & \text{for } \ r = 0, \ r = 1, \end{cases} \tag{6b}$$

$$Q(y, Y, z; h, r)(t) = \begin{cases} \sum_{j=1}^{k} \bar{a}_j(r) g_h(\bar{y}, \bar{Y}, \bar{z})(t + \bar{b}_j h) & \text{for } \ 0 < r < 1, \\ g_h(\bar{y}, \bar{Y}, \bar{z})(t + rh) & \text{for } \ r = 0, \ r = 1, \end{cases} \tag{6c}$$

where

$$\bar{y}(s) = \begin{cases} y(s) & \text{for } \ a \leqslant s \leqslant t, \\ y(t) + (s - t)\Phi\Big(y, Y, z; h, \dfrac{s - t}{h}\Big)(t) & \text{for } \ t < s \leqslant t + h, \end{cases} \tag{6d}$$

(6e)
$$\bar{Y}(s) = \begin{cases} Y(s) & \text{for } a \leqslant s \leqslant t, \\ \Phi\Big(y, Y, z; h, \dfrac{s - t}{h}\Big)(t) + \dfrac{s - t}{h}\Phi_5\Big(y, Y, z; h, \dfrac{s - t}{h}\Big)(t) & \text{for } t < s < t + h, \\ f_h(\bar{y}, \bar{Y}, \bar{z})(t + h) & \text{for } s = t + h, \end{cases}$$

$$\bar{z}(s) = \begin{cases} z(s) & \text{for } \ a \leqslant s \leqslant t, \\ \chi\Big(y, Y, z; h, \dfrac{s - t}{h}\Big)(t) & \text{for } \ t < s < t + h, \\ g_h(\bar{y}, \bar{Y}, \bar{z})(t + h) & \text{for } \ s = t + h. \end{cases} \tag{6f}$$

The following theorem deals with the order of consistency of method (4) with system (1) for the increment functions F, G and Q described above.

Theorem 2. *Let $(y^\star, z^\star) \in C^{k+1}(J, \mathbb{R}^p) \times C^k(J, \mathbb{R}^m)$ be a solution of (1). Let Assumptions A and B be satisfied, and*

$$\sum_{j=1}^{k} a_j(r) b_j^{i-1} = \frac{r^i}{i}, \quad i = 1, 2, \ldots, k, \tag{7a}$$

$$\sum_{j=1}^{k} \bar{a}_j(r) \left(\bar{b}_j\right)^{i-1} = r^{i-1}, \quad i = 1, 2, \ldots, k. \tag{7b}$$

Then there exist constants $C_1, C_2, C_3 > 0$ such that

$$\begin{cases} \|hF(y^\star, Y^\star, z^\star; h, r)(t) + y^\star(t) - y^\star(t + rh)\| \leqslant C_1 h\Delta(h) + O(h^{k+1}), \\ \|G(y^\star, Y^\star, z^\star; h, r)(t) - Y^\star(t + rh)\| \leqslant C_2\Delta(h) + O(h^k), \\ \|Q(y^\star, Y^\star, z^\star; h, r)(t) - z^\star(t + rh)\| \leqslant C_3\Delta(h) + O(h^k) \end{cases} \tag{8}$$

as $h \to 0$, where

$$\Delta(h) = \sum_{i=1}^{3} \bar{\delta}_i(h) + \sum_{i=1}^{2} \delta_i(h).$$

If $\Delta(h) = O(h^k)$ as $h \to 0$, then the order of consistency of (4) is equal to k.

P r o o f. First of all we note that method (4) with F, G and Q of the form (6) is well defined. Moreover, the conditions (5) are satisfied.

Let $t \in J_h$, $c \in [0, 1]$ and $\bar{y}^\star$, $\bar{Y}^\star$ and $\bar{z}^\star$ be defined by formulas (6d)–(6f) with y, Y and z replaced by $y^\star$, $Y^\star$ and $z^\star$, respectively. According to condition 2° of Assumption B, we have

$$\begin{aligned} &\sup_{[a,t+ch]} \|\bar{y}^\star(s) - y^\star(s)\| \\ &\qquad = \sup_{[t,t+ch]} \|\bar{y}^\star(s) - y^\star(s)\| \\ &\qquad = \sup_{[0,c]} \|y^\star(t) + hs\Phi(y^\star, Y^\star, z^\star; h, s)(t) - y^\star(t + sh)\| \\ &\qquad \leqslant \bar{\delta}_1(h). \end{aligned} \tag{9}$$

Similarly, for $e \in [0, 1)$ we have

$$\begin{aligned} &\sup_{[a,t+eh]} \|\bar{Y}^\star(s) - Y^\star(s)\| \\ &\qquad = \sup_{[0,e]} \|\Phi(y^\star, Y^\star, z^\star; h, s)(t) + s\Phi_5(y^\star, Y^\star, z^\star; h, s)(t) - Y^\star(t + sh)\| \\ &\qquad \leqslant \bar{\delta}_2(h), \end{aligned} \tag{10}$$

$$(11) \qquad \sup_{[a,t+eh]} \|\bar{z}^\star(s) - z^\star(s)\| \leqslant \bar{\delta}_3(h).$$

Put

$$u = \|\bar{Y}^\star(t+h) - Y^\star(t+h)\|, \qquad v = \|\bar{z}^\star(t+h) - z^\star(t+h)\|.$$

By conditions 1° and 2° of Assumption A, we get

$$\begin{aligned} u &= \|f_h(\bar{y}^\star, \bar{Y}^\star, \bar{z}^\star)(t+h) - f_h(y^\star, Y^\star, z^\star)(t+h) \\ &\quad + f_h(y^\star, Y^\star, z^\star)(t+h) - f(y^\star, Y^\star, z^\star)(t+h)\| \\ &\leqslant M_1\|\bar{y}^\star - y^\star\|_{t+h} + M_2\|\bar{Y}^\star - Y^\star\|_{t+h} + M_3\|\bar{z}^\star - z^\star\|_{t+h} + \delta_1(h) \\ &\leqslant M_1\bar{\delta}_1(h) + M_2\bar{\delta}_2(h) + M_3\bar{\delta}_3(h) + \delta_1(h) + M_2 u + M_3 v \end{aligned}$$

and

$$\begin{aligned} v &= \|g_h(\bar{y}^\star, \bar{Y}^\star, \bar{z}^\star)(t+h) - g_h(y^\star, Y^\star, z^\star)(t+h) \\ &\quad + g_h(y^\star, Y^\star, z^\star)(t+h) - g(y^\star, Y^\star, z^\star)(t+h)\| \\ &\leqslant P_1\bar{\delta}_1(h) + P_2\bar{\delta}_2(h) + P_3\bar{\delta}_3(h) + \delta_2(h) + P_2 u + P_3 v. \end{aligned}$$

The last two inequalities for u and v can be combined into

$$(12) \qquad U \leqslant AU + B(h),$$

where

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad B(h) = \begin{bmatrix} M_1\bar{\delta}_1(h) + M_2\bar{\delta}_2(h) + M_3\bar{\delta}_3(h) + \delta_1(h) \\ P_1\bar{\delta}_1(h) + P_2\bar{\delta}_2(h) + P_3\bar{\delta}_3(h) + \delta_2(h) \end{bmatrix}.$$

Here the inequality $[x_1, x_2]^T \leqslant [\bar{x}_1, \bar{x}_2]^T$ means that $x_i \leqslant \bar{x}_i$, $i = 1, 2$. It is known that $\varrho(A) < 1$ implies that the matrix $I - A$ is nonsingular and its inverse has nonnegative elements, so (12) yields $U \leqslant (I - A)^{-1}B(h)$. Combining this with (10) and (11), we find

$$(13) \qquad \sup_{[a,t+ch]} \|\bar{Y}^\star(s) - Y^\star(s)\| \leqslant \overline{M}_1\Delta(h),$$

$$(14) \qquad \sup_{[a,t+ch]} \|\bar{z}^\star(s) - z^\star(s)\| \leqslant \overline{P}_1\Delta(h),$$

where $\overline{M}_1$ and $\overline{P}_1$ are constants independent on h.

The Taylor formula for $y^\star \in C^{k+1}(J, \mathbb{R}^p)$ yields

$$\begin{aligned} \eta_3(h) &= \left\| h \sum_{j=1}^{k} a_j(r)\,[y^\star(t + b_j h)]' + y^\star(t) - y^\star(t + rh) \right\| \\ &= \left\| \sum_{i=1}^{k} h^i\,[y^\star(t)]^{(i)} \left[\sum_{j=1}^{k} a_j(r) \frac{b_j^{i-1}}{(i-1)!} - \frac{r^i}{i!} \right] \right\| + O(h^{k+1}) \quad \text{as} \quad h \to 0. \end{aligned}$$

According to condition (7a), we find $\eta_3(h) = O(h^{k+1})$ as $h \to 0$.

Basing on the conditions 1°–2° of Assumption A and (9), (13)–(14), we obtain

$$\begin{aligned}\eta_1(h) &= \left\| h\sum_{j=1}^{k} a_j(r)\left[f_h(\bar{y}^\star,\bar{Y}^\star,\bar{z}^\star)(t+b_jh) - f_h(y^\star,Y^\star,z^\star)(t+b_jh)\right]\right\| \\ &\leqslant h\sum_{j=1}^{k}|a_j(r)|\left[M_1\|\bar{y}^\star - y^\star\|_{t+b_jh} + M_2\|\bar{Y}^\star - Y^\star\|_{t+b_jh} + M_3\|\bar{z}^\star - z^\star\|_{t+b_jh}\right] \\ &\leqslant h\overline{M}_2\Delta(h), \quad \overline{M}_2 \text{ is a nonnegative constant,}\end{aligned}$$

$$\begin{aligned}\eta_2(h) &= \left\| h\sum_{j=1}^{k} a_j(r)\left[f_h(y^\star,Y^\star,z^\star)(t+b_jh) - f(y^\star,Y^\star,z^\star)(t+b_jh)\right]\right\| \\ &\leqslant h\sum_{j=1}^{k}|a_j(r)|\delta_1(h).\end{aligned}$$

Hence

$$\|hF(y^\star,Y^\star,z^\star;h,r)(t) + y^\star(t) - y^\star(t+rh)\| \leqslant \eta_1(h) + \eta_2(h) + \eta_3(h),$$

so we have found the first estimate of (8).

By differentiating (7a) with respect to r we have

$$\sum_{j=1}^{k} a_j'(r)b_j^{i-1} = r^{i-1}, \quad i = 1,2,\ldots,k.$$

Using this, the Taylor formula, (9), (13)–(14) and (7b), we obtain

$$\|G(y^\star,Y^\star,z^\star;h,r)(t) - Y^\star(t+rh)\| \leqslant \begin{cases}\delta_1(h) & \text{for } r = 0,\\ \overline{P}_3\Delta(h) + O(h^k) & \text{for } 0 < r < 1,\\ \overline{M}_3\Delta(h) & \text{for } r = 1,\end{cases}$$

$$\|Q(y^\star,Y^\star,z^\star;h,r)(t) - z^\star(t+rh)\| \leqslant \begin{cases}\delta_2(h) & \text{for } r = 0,\\ \overline{P}_4\Delta(h) + O(h^k) & \text{for } 0 < r < 1,\\ \overline{M}_4\Delta(h) & \text{for } r = 1\end{cases}$$

as $h \to 0$, where $\overline{M}_3$, $\overline{M}_4$, $\overline{P}_3$, $\overline{P}_4$ are nonnegative constants. From this we get the rest of the assertion of (8). Finally, we find that if $\Delta(h) = O(h^k)$ as $h \to 0$, then (8) implies that the order of consistency of (4) is equal to k.

The proof is complete. □

R e m a r k 2. By the property of the Vandermonde determinant, system (7a) has a unique solution a_j for any choice of distinct values of b_j (see [14]). It means that method (4) has sense for F, G and Q defined by (6).

R e m a r k 3. It follows from the proof of Theorem 2 that the assertion of this theorem remains valid if the condition 1° of Assumption A is true only for $y = y^\star$, $Y = Y^\star$, $z = z^\star$.

R e m a r k 4. Let $(y^\star, z^\star) \in C^{k+1}(J, \mathbb{R}^p) \times C^k(J, \mathbb{R}^m)$. If the derivatives $(y^\star)^{(k+1)}$ and $(z^\star)^{(k)}$ satisfy the Hölder condition with the exponent $\gamma \in (0,1]$, and $\Delta(h) = O(h^\nu)$ as $h \to 0$, then the order of consistency equals $\min(\nu, \gamma + k)$.

E x a m p l e s. Now we are going to give some examples of method (4) of the corresponding order with F, G and Q defined by (6).

(a). Put $k = 1$ and $b_1 = \bar{b}_1 = 0$. Then, according to (7), we have $a_1(r) = r$, $\bar{a}_1(r) = 1$. Let

$$\Phi(y, Y, z; h, r)(t) = f_h(y, Y, z)(t),$$
$$\chi(y, Y, z; h, r)(t) = g_h(y, Y, z)(t).$$

If $\delta_1(h) = O(h)$ and $\delta_2(h) = O(h)$, then $\bar{\delta}_1(h) = O(h^2)$, $\bar{\delta}_2(h) = O(h)$, $\bar{\delta}_3(h) = O(h)$ as $h \to 0$, and we have the method of order one.

(b). Put $k = 2$, $b_1 = 0$, $b_2 = 1$. Then $a_1(r) = r - \frac{1}{2}r^2$, $a_2(r) = \frac{1}{2}r^2$. If $\bar{b}_1 = 0$, $\bar{b}_2 = 1$, then $\bar{a}_i(r) = a_i'(r)$, $i = 1, 2$, while if $\bar{b}_1 = \frac{1}{2}$, $\bar{b}_2 = 1$, then $\bar{a}_1(r) = 2(1 - r)$, $\bar{a}_2(r) = -1 + 2r$. Let

$$\Phi(y, Y, z; h, r)(t) = f_h(y, Y, z)(t) + \frac{1}{2} r \left[f_h(\bar{\bar{y}}, \bar{\bar{Y}}, \bar{\bar{z}})(t + h) - f_h(y, Y, z)(t) \right],$$

$$\chi(y, Y, z; h, r)(t) = g_h(y, Y, z)(t) + r \left[g_h(\bar{\bar{y}}, \bar{\bar{Y}}, \bar{\bar{z}})(t + h) - g_h(y, Y, z)(t) \right],$$

with

$$\bar{\bar{y}}(s) = \begin{cases} y(s) & \text{for } a \leqslant s \leqslant t, \\ y(t) + (s - t) f_h(y, Y, z)(t) & \text{for } t < s \leqslant t + h, \end{cases}$$

$$\bar{\bar{Y}}(s) = \begin{cases} Y(s) & \text{for } a \leqslant s \leqslant t, \\ f_h(y, Y, z)(t) & \text{for } t < s < t + h, \\ f_h(\bar{\bar{y}}, \bar{\bar{Y}}, \bar{\bar{z}})(t + h) & \text{for } s = t + h, \end{cases}$$

$$\bar{\bar{z}}(s) = \begin{cases} z(s) & \text{for } a \leqslant s \leqslant t, \\ g_h(y, Y, z)(t) & \text{for } t < s < t + h, \\ g_h(\bar{\bar{y}}, \bar{\bar{Y}}, \bar{\bar{z}})(t + h) & \text{for } s = t + h. \end{cases}$$

If $\delta_1(h) = O(h^2)$ and $\delta_2(h) = O(h^2)$ as $h \to 0$, then we have the method of order two.

3. Convergence

In this section, the convergence of the method (4) will be proved. The order of convergence will be discussed, too.

Theorem 3. *Assume that:*

1° $f \in C(\mathscr{Y}, \mathscr{A})$, $g \in C(\mathscr{Y}, \mathscr{B})$, $f, G: \mathscr{U} \times \mathscr{H} \to \tilde{\mathscr{A}}$, $Q: \mathscr{U} \times \mathscr{H} \to \tilde{\mathscr{B}}$ all are *Volterra mappings, F, G, Q are continuous with respect to the fifth argument,*

2° *there exists a unique solution $(y^\star, z^\star)$ of (1),*

3° *method (4) is consistent with problem (1) on the solution $(y^\star, z^\star)$,*

4° *there exist constants L_i, $K_i \geqslant 0$, $i = 1, 2, 3$, such that*

$$\|F(u_1, u_2, u_3; h, r)(t) - F(\bar{u}_1, \bar{u}_2, \bar{u}_3; h, r)(t)\| \leqslant \sum_{i=1}^{3} L_i \|u_i - \bar{u}_i\|_{t+h},$$

$$\|G(u_1, u_2, u_3; h, r)(t) - G(\bar{u}_1, \bar{u}_2, \bar{u}_3; h, r)(t)\| \leqslant \sum_{i=1}^{3} K_i \|u_i - \bar{u}_i\|_{t+h}$$

hold for all u_1, $\bar{u}_1 \in \mathscr{A}$, u_2, $\bar{u}_2 \in \tilde{\mathscr{A}}$, u_3, $\bar{u}_3 \in \tilde{\mathscr{B}}$,

5° $\|Y^h(a) - Y^\star(a)\| \leqslant \varepsilon_2(h)$,

6° *there exist constants $d_i \geqslant 0$, $i = 1, 2, 3$, such that*

$$\begin{aligned} &\|Q(u_1, u_2, u_3; h, r)(t) - Q(\bar{u}_1, \bar{u}_2, \bar{u}_3; h, r)(t)\| \\ &\qquad \leqslant d_1 \|u_1 - \bar{u}_1\|_{t+h} + d_2 \|u_2 - \bar{u}_2\|_{t+h} + d_3 \|u_3 - \bar{u}_3\|_{t+h} \end{aligned}$$

holds for $u_1 \in \mathscr{A}$, $u_2 \in \tilde{\mathscr{A}}$, $u_3 \in \tilde{\mathscr{B}}$,

7° $\varrho(\bar{A}) < 1$ *for* $\bar{A} = \begin{bmatrix} K_2 & K_3 \\ d_2 & d_3 \end{bmatrix}$. *Then the method (4) is convergent to $(y^\star, z^\star)$ and there exist constants $\gamma_1, \gamma_2, \gamma_3 \geqslant 0$ such that, for $\varepsilon(h) = h^{-1}\varepsilon_1(h) + \varepsilon_2(h) + \varepsilon_3(h)$, the estimates*

$$(15) \qquad \begin{cases} \sup\limits_{t \in J} \|y^h(t) - y^\star(t)\| \leqslant \gamma_1 \varepsilon(h), \\ \sup\limits_{t \in J} \|Y^h(t) - Y^\star(t)\| \leqslant \gamma_2 \varepsilon(h), \\ \sup\limits_{t \in J} \|z^h(t) - z^\star(t)\| \leqslant \gamma_3 \varepsilon(h) \end{cases}$$

hold for h sufficiently small.

Proof. Put

$$\begin{aligned} v^h(t) &= \|y^h(t) - y^\star(t)\|, & V_n^h &= \sup_{[a, t_n]} v^h(\tau), \\ w^h(t) &= \|Y^h(t) - Y^\star(t)\|, & W_n^h &= \sup_{[a, t_n]} w^h(\tau), \\ x^h(t) &= \|z^h(t) - z^\star(t)\|, & X_n^h &= \sup_{[a, t_n]} x^h(\tau). \end{aligned}$$

Using the conditions 2°–4°, we have

$$\begin{aligned} w^h(t_n + rh) &= \|G(y^h, Y^h, z^h; h, r)(t_n) - G(y^\star, Y^\star, z^\star; h, r)(t_n) \\ &\quad + G(y^\star, Y^\star, z^\star; h, r)(t_n) - Y^\star(t_n + rh)\| \\ &\leqslant K_1 V_{n+1}^h + K_2 W_{n+1}^h + K_3 X_{n+1}^h + \varepsilon_2(h) \equiv \zeta_{n+1}^h, \end{aligned}$$

so

$$\sup_{[t_n, t_{n+1}]} w^h(\tau) \leqslant \zeta_{n+1}^h.$$

This and condition 5° yield

$$W_n^h \leqslant K_1 V_n^h + K_2 W_n^h + K_3 X_n^h + \varepsilon_2(h), \quad n = 0, 1, \dots, N. \tag{16}$$

Similarly, we have

$$\begin{aligned} v^h(t_n + rh) &= \|y^h(t_n) + hF(y^h, Y^h, z^h; h, r)(t_n) - y^\star(t_n) - hF(y^\star, Y^\star, z^\star; h, r)(t_n) \\ &\quad + y^\star(t_n) + hF(y^\star, Y^\star, z^\star; h, r)(t_n) - y^\star(t_n + rh)\| \\ &\leqslant V_n^h + h\left[L_1 V_{n+1}^h + L_2 W_{n+1}^h + L_3 X_{n+1}^h\right] + \varepsilon_1(h) \\ &\equiv \xi_{n+1}^h, \quad n = 0, 1, \dots, N-1, \end{aligned}$$

so

$$\sup_{[t_n, t_{n+1}]} v^h(\tau) \leqslant \xi_{n+1}^h,$$

and

$$V_{n+1}^h \leqslant V_n^h + hL_1 V_{n+1}^h + hL_2 W_{n+1}^h + hL_3 X_{n+1}^h + \varepsilon_1(h) \quad \text{for} \quad n = 0, 1, \dots, N-1. \tag{17}$$

Now we need an estimate for x^h. Condition 6° yields

$$\begin{aligned} x^h(t_n + rh) &= \|Q(y^h, Y^h, z^h; h, r)(t_n) - Q(y^\star, Y^\star, z^\star; h, r)(t_n) \\ &\quad + Q(y^\star, Y^\star, z^\star; h, r)(t_n) - z^\star(t_n + rh)\| \\ &\leqslant d_1 V_{n+1}^h + d_2 W_{n+1}^h + d_3 X_{n+1}^h + \varepsilon_3(h), \quad n = 0, 1, \dots, N-1; \end{aligned}$$

hence

$$X_{n+1}^h \leqslant d_1 V_{n+1}^h + d_2 W_{n+1}^h + d_3 X_{n+1}^h + \varepsilon_3(h), \quad n = 0, 1, \dots, N-1. \tag{18}$$

Writing (16) and (18) in the vector notation, we obtain

$$\varphi_{n+1}^h \leqslant \bar{A} \varphi_{n+1}^h + \vartheta_{n+1}^h, \quad n = 0, 1, \dots, N-1, \tag{19}$$

where

$$\varphi_n^h = \begin{bmatrix} W_n^h \\ X_n^h \end{bmatrix}, \qquad \vartheta_n^h = \begin{bmatrix} K_1 V_n^h + \varepsilon_2(h) \\ d_1 V_n^h + \varepsilon_3(h) \end{bmatrix}.$$

By this and condition 7°, we have

$$\varphi_{n+1}^h \leqslant (I - \bar{A})^{-1} \vartheta_{n+1}^h, \quad n = 0, 1, \ldots, N-1,$$

which proves that there exist nonnegative constants a_1, a_2, b_1, b_2 such that the relations

$$\text{(20)} \qquad \begin{cases} W_{n+1}^h \leqslant a_1 V_{n+1}^h + a_2 \left[\varepsilon_2(h) + \varepsilon_3(h)\right], \\ X_{n+1}^h \leqslant b_1 V_{n+1}^h + b_2 \left[\varepsilon_2(h) + \varepsilon_3(h)\right] \end{cases}$$

hold for $n = 0, 1, \ldots, N-1$.

Combining the last two inequalities with (17) we see that

$$V_{n+1}^h \leqslant V_n^h + h\bar{L}_1 V_{n+1}^h + h\bar{L}_2 \varepsilon(h), \quad n = 0, 1, \ldots, N-1,$$

where $\bar{L}_1$ and $\bar{L}_2$ are nonnegative constants. Hence, for sufficiently small h such that $h\bar{L}_1 < 1$, we have

$$V_n^h \leqslant \gamma_1 \varepsilon(h), \quad n = 0, 1, \ldots, N.$$

Combining this with (20) we have estimate (15). The proof is complete. □

Remark 5. If method (4) is consistent of order q, then the convergence is also of order q, so

$$\sup_{t \in J} \|y^h(t) - y^\star(t)\| = O(h^q), \quad \sup_{t \in J} \|Y^h(t) - Y^\star(t)\| = O(h^q),$$
$$\sup_{t \in J} \|z^h(t) - z^\star(t)\| = O(h^q)$$

as $h \to 0$.

Remark 6. It is easy to prove that

$$K_2 + d_3 < 2 \quad \text{and} \quad K_2 + d_3(1 - K_2) + K_3 d_2 < 1$$

imply condition 7°, i.e. $\varrho(\bar{A}) < 1$ holds.

Remark 7. Notice that the values $\tilde{y}$, $\tilde{Y}$ and $\tilde{z}$ appearing in (4) usually are found approximately. In this case, the convergence of order q can be reached only if these approximations are also of order q.

4. Comments

Up to now we have considered system (1) in which the operators f and g require only the values of the unknown functions y, y' and z for arguments $t \in J$. A more general type of FDAEs is that in which the values of the unknown functions appear in f and g for arguments t on the left hand side of the left endpoint a of the interval J. Below, we consider this case, and we show that it can be transformed to a problem of type (1). To do this some additional notation will be required.

Let $J_0 = [a - a_0, a]$, $J_1 = [a - a_1, a]$, $\bar{J} = J \cup J_0$, $\tilde{J} = J \cup J_1$, a_0, $a_1 \geqslant 0$, and let $\varphi \in C^1(J_0, \mathbb{R}^p)$ and $\psi \in C(J_1, \mathbb{R}^m)$ be given. By $\mathscr{C}^i(\bar{J}, \mathbb{R}^p)$, $i = 0, 1$, we mean the class of all functions $x \in C(\bar{J}, \mathbb{R}^p)$ which are identical with $\varphi^{(i)}$ on J_0; $\mathscr{C}(\bar{J}, \mathbb{R}^p) = \mathscr{C}^0(\bar{J}, \mathbb{R}^p)$. Similarly, $\mathscr{C}_2(\tilde{J}, \mathbb{R}^m)$ denotes the class of all functions $x \in C(\tilde{J}, \mathbb{R}^m)$ which are identical with ψ on J_1. Put $\mathscr{X} = \mathscr{C}(\bar{J}, \mathbb{R}^p) \times \mathscr{C}^1(\bar{J}, \mathbb{R}^p) \times \mathscr{C}_2(\tilde{J}, \mathbb{R}^m)$.

Now we consider the system

$$(21) \qquad \begin{cases} y'(t) = \mathscr{F}(y, y', z)(t), \\ z(t) = \mathscr{G}(y, y', z)(t), \end{cases} \quad t \in J$$

for some $\mathscr{F} \in C(\mathscr{X}, \mathscr{A})$ and $\mathscr{G} \in C(\mathscr{X}, \mathscr{B})$. System (21) will be supplied by the initial conditions

$$(22) \qquad \begin{cases} y(t) = \varphi(t), & t \in J_0, \\ z(t) = \psi(t), & t \in J_1 \end{cases}$$

satisfying the consistency conditions of the form

$$(23) \qquad \begin{cases} \varphi'(a) = \mathscr{F}(\varphi, \varphi', \psi)(a), \\ \psi(a) = \mathscr{G}(\varphi, \varphi', \psi)(a). \end{cases}$$

Observe that the mappings $\mathscr{F}$ and $\mathscr{G}$ may be of different form; for example systems of functional-differential-delay-algebraic equations and also integro-differential-algebraic equations of Volterra type are special cases of (21). System (1) is also a special case of (21), in this case we have $a_0 = a_1 = 0$.

As was mentioned earlier, problem (21) can be reduced to the case for which $a_0 = a_1 = 0$, which means the case when the initial conditions have the form

$$(24) \qquad \begin{cases} y(a) = y_0, \quad y'(a) = y_1, \\ z(a) = z_0 \end{cases}$$

for given $y_0, y_1 \in \mathbb{R}^p$ and $z_0 \in \mathbb{R}^m$ such that

$$y_1 = \mathscr{F}(y_0, y_1, z_0)(a), \qquad z_0 = \mathscr{G}(y_0, y_1, z_0)(a).$$

Indeed, for $y \in C^1(J, \mathbb{R}^p)$ and $z \in \mathscr{B}$ satisfying the conditions

$$y(a) = \varphi(a), \qquad y'(a) = \varphi'(a), \qquad z(a) = \psi(a)$$

we define operators T_1 and T_2 by the relations

$$\tag{25} \begin{cases} (T_1 y)(t) = \begin{cases} \varphi(t), & t \in J_0, \\ y(t), & t \in J, \end{cases} \\ (T_2 z)(t) = \begin{cases} \psi(t), & t \in J_1, \\ z(t), & t \in J. \end{cases} \end{cases}$$

Now, problem (21)–(23) can be replaced by an equivalent one of the form

$$\tag{26} \begin{cases} y'(t) = \mathscr{F}(T_1 y, (T_1 y)', T_2 z)(t), \\ z(t) = \mathscr{G}(T_1 y, (T_1 y)', T_2 z)(t), \end{cases} \quad t \in J,$$

$$\tag{27} \begin{cases} y(a) = \varphi(a), \quad y'(a) = \varphi'(a), \\ z(a) = \psi(a). \end{cases}$$

Notice that if (y, z) is a solution of (26)–(27), then $(T_1 y, T_2 z)$ is the corresponding solution of (21)–(23). In this way we see that problem (21)–(23) is reduced to problem (1).

In literature we can also find problems of fully implicit type. To conclude the paper we briefly consider the case when the algebraic equation of problem (1) is replaced by

$$g(t, y(\alpha(t)), z(t)) = \Theta_m$$

with $\alpha \in C(J, J)$, $\alpha(t) \leqslant t$, $t \in J$ and $g \colon J \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^m$. To use method (4) for finding a numerical solution of such a problem, we define the mapping Q by the relation

$$Q(u, v, w; h, r)(t) = w(t) - P^{-1} g(t, u(\alpha(t)), w(t)),$$

where P is a nonsingular $m \times m$ matrix. Assume that the derivative of g with respect to the last variable exists and denote it by g_3. If

$$\tag{i} \|P^{-1}[g(t, u, v) - g(t, \bar{u}, v)]\| \leqslant d_1 \|u - \bar{u}\|,$$

$$\tag{ii} \|v - \bar{v} - P^{-1}[g(t, u, v) - g(t, u, \bar{v})]\| \leqslant d_3 \|v - \bar{v}\|,$$

then it is easy to see that condition 6° of Theorem 3 holds. Indeed, we have

$$\begin{aligned} &\|Q(u_1, u_2, u_3; h, r)(t) - Q(\bar{u}_1, \bar{u}_2, \bar{u}_3; h, r)(t)\| \\ &\quad = \|u_3(t) - \bar{u}_3(t) - P^{-1}[g(t, u_1(\alpha(t)), u_3(t)) - g(t, u_1(\alpha(t)), \bar{u}_3(t)) \\ &\qquad + g(t, u_1(\alpha(t)), \bar{u}_3(t)) - g(t, \bar{u}_1(\alpha(t)), \bar{u}_3(t))]\| \\ &\quad \leqslant d_1 \|u_1 - \bar{u}_1\|_t + d_3 \|u_3 - \bar{u}_3\|_t, \quad t \in J. \end{aligned}$$

Using the mean value theorem one finds that condition (ii) holds if

$$\sup_{t,u,v} \|I_m - P^{-1} g_3(t,u,v)\| \leqslant d_3;$$

here I_m denotes the unit $m \times m$ matrix.

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