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Zdzisław Jackiewicz; Rosemary Anne Renaut; Marino Zennaro
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EXPLICIT TWO-STEP RUNGE-KUTTA METHODS

Zdzislaw Jackiewicz, Rosemary Anne Renaut, Tempe,
Marino Zennaro, Trieste

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Summary. The explicit two-step Runge-Kutta (TSRK) formulas for the numerical solution of ordinary differential equations are analyzed. The order conditions are derived and the construction of such methods based on some simplifying assumptions is described. Order barriers are also presented. It turns out that for order $p \leqslant 5$ the minimal number of stages for explicit TSRK method of order p is equal to the minimal number of stages for explicit Runge-Kutta method of order $p-1$. Numerical results are presented which demonstrate that constant step size TSRK can be both effectively and efficiently used in an Euler equation solver. Furthermore, a comparison with a variable step size formulation shows that in these solvers the variable step size formulation offers no advantages compared to the constant step size implementation.

1. Introduction

Consider the initial value problem for systems of ordinary differential equations (ODEs)

$$(1.1)\qquad \begin{cases} y'(x) = f(y(x)), & x \in [a,b], \\ y(a) = y_0, \end{cases}$$

where $f\colon \mathbb{R}^q \to \mathbb{R}^q$ is assumed to be sufficiently smooth. We will investigate the explicit two-step Runge-Kutta (TSRK) methods for the numerical solution of (1.1). These methods form a subclass of general linear methods considered by Butcher [4]

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and are defined by

$$(1.2)\qquad \begin{cases} y_{i+1} = (1-\theta)y_i + \theta y_{i-1} + h\sum_{j=1}^{m}(v_j f(Y_{i-1}^j) + w_j f(Y_i^j)), \\ Y_{i-1}^j = y_{i-1} + h\sum_{s=1}^{j-1} a_{js} f(Y_{i-1}^s), \\ Y_i^j = y_i + h\sum_{s=1}^{j-1} a_{js} f(Y_i^s), \end{cases}$$

$i = 1, 2, \dots, N-1$, $Nh = b - a$, where the starting values y_0 and y_1 are assumed to be given. Here, $h > 0$ is a constant stepsize and y_i is an approximation to the solution y of (1.1) at the gridpoint $x_i = a + ih$, $i = 0, 1, \dots, N$. It is convenient to represent (1.2) by the following table of coefficients

$$\begin{array}{c|c} c & A \\ \hline & v^T \\ \theta & w^T \end{array} = \begin{array}{c|ccccc} c_1 = 0 & & & & & \\ c_2 & a_{21} & & & & \\ c_3 & a_{31} & a_{32} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ c_m & a_{m1} & a_{m2} & \cdots & a_{m\,m-1} & \\ \hline & v_1 & v_2 & \cdots & v_{m-1} & v_m \\ \theta & & & & & \\ & w_1 & w_2 & \cdots & w_{m-1} & w_m \end{array}$$

where $c_i = \sum_{j=1}^{i-1} a_{ij}$, $i = 1, 2, \dots, m$. Observe that when advancing from t_i to t_{i+1} it is not necessary to compute Y_{i-1}^j since these values have already been computed in the previous step. The method requires only m evaluations of the function f associated with the computations of the internal stages Y_i^j related to the current step. The method (1.2) is convergent if it is consistent and zero stable, see [19]. The method is consistent if

$$\sum_{j=1}^{m}(v_j + w_j) = 1 + \theta,$$

and zero-stable if $\theta \in (-1, 1]$, compare [9].

It will be shown that TSRK methods of order p require the same number of evaluations of the function f as the Runge-Kutta (RK) methods of order $p-1$, at least for $p \leqslant 5$. This increased efficiency makes them attractive candidates for the numerical integration of large systems of ODEs arising from semidiscretizations of parabolic and hyperbolic partial differential equations, and they have been studied in this context by Renaut [12, 13] and Verwer [16–18]. The two-stage TSRK methods of order three and the three-stage TSRK methods of order four corresponding to $\theta = 0$ have been also studied by Byrne and Lambert [5]. Implicit and the semi implicit TSRK methods of order up to four were investigated by Jackiewicz, Renaut and Feldstein [9]. Another application of TSRK methods is the estimation of local discretization error of RK methods. This aspect was stressed by Jackiewicz and Zennaro [10] for explicit continuous RK methods and by Bellen, Jackiewicz and Zennaro [1] for singly-implicit methods introduced by Burrage [2].

In the next section we list the order conditions up to order five for TSRK methods and formulate a theorem which form a basis for construction of these methods. In Section 3 the TSRK methods up to the order four are presented. These methods contain as special cases the methods by Byrne and Lambert [5] and Renaut [12]. In Section 4 we investigate the necessity of simplifying assumptions given in Theorem 2 in Section 2 for TSRK methods of order five with four stages, and in Section 5 families of such methods are constructed. This construction is based on the simplifying assumptions mentioned above and on the Butcher Lemma presented at the beginning of Section 5. The results of Section 3, 4, and 5 are summarized in Section 6 in the form of the theorem about order barriers for explicit TSRK methods of order $p \leqslant 5$, and a conjecture about the order barrier for general order p is formulated. The order barrier for $p = 5$ gives a positive answer to the question posed by Renaut [12] about the minimum number of stages of TSRK methods of order five. (There is an incorrect entry corresponding to the number of stages $m = 4$ in Table 2 in Renaut recent paper [13]). The paper is concluded with Section 7 in which the results of some numerical experiments for subsonic flow past a cylinder are presented. These results are obtained using a finite-volume method for the solution of the Euler equations. A comparison of results for a variable step-size and a constant step-size implementation is made. We conclude that in such situations there is no need to use the variable-step formulation and hence the results of this paper can be used to derive high order constant step size TSRK suitable for inclusion in Euler equation solvers.

2. Order conditions and construction of TSRK methods

It is possible to generate order conditions for TSRK methods (1.2) in many different ways. Renaut [12] obtained conditions up to the order four in elementary way using Taylor's series expansion. However, this approach is very tedious for higher orders and other approaches were developed. Jackiewicz, Renaut and Feldstein generated order conditions for (1.2) using the theory of Hairer and Wanner [6] for multistep-multistage-multiderivative methods for ODEs, and Renaut [13] used the composition theorem of Hairer and Wanner [7]. These order conditions can be also generated using the approach by Burrage [3] developed for general multivalue methods for ODEs. A simple and convenient mechanical rule to generate the order conditions for TSRK methods defined on nonuniform grid was derived recently by Jackiewicz and Zennaro [10]. Denote by τ the unique tree of order one and by $t = [t_1, t_2, \ldots, t_s]$ the tree formed from $t_1, t_2, \ldots, t_s$ by adding new root and joining it with the roots of $t_1, t_2, \ldots, t_s$. Then the mechanical rule presented in [10] and adapted to uniform grid reads:

Theorem 1. (Jackiewicz and Zennaro [10]). *The order condition corresponding to τ is*

$$(v^T + w^T)u = 1 + \theta,$$

and the order condition corresponding to $t = [t_1, t_2, \ldots, t_s]$ is

$$v^T \bar{\psi}^v(t_j) + w^T \bar{\psi}^w(t_j) = \frac{1 - (-1)^{\varrho(t)}\theta}{\gamma(t)},$$

where

$$\bar{\psi}^v(t_j) = \prod_{j=1}^{s} \Big(A\bar{\psi}^v(t_j) + \frac{(-1)^{\varrho(t_j)}u}{\gamma(t_j)} \Big),$$

and

$$\bar{\psi}^w(t_j) = \prod_{j=1}^{s} (A\bar{\psi}^w(t_j)),$$

are elementary weights corresponding to v and w and the trees t_j, $j = 1, 2, \ldots, s$.

In this theorem u stands for the vector $[1, 1, \ldots, 1]^T$ of appropriate dimension and the products denote componentwise multiplication of vectors. The symbols $\varrho(t)$ and $\gamma(t)$ denote the order and the density of the tree t (compare Butcher [4]).

The order conditions generated using Theorem 1 are given in Table 1 for $p \leqslant 4$ and in Table 2 for $p = 5$. We denoted by $t_{p,i}$, $i = 1, 2, \ldots, n_p$, the rooted trees of order p, where n_p stands for the number of such trees.

Denote by T the set of all rooted trees and by T_P the set of primary trees defined by $[\tau^r]$, $r = 0, 1, \ldots$, $[\tau^0] := \tau$. The set $T_S := T - T_P$ is called the set of secondary trees. We will call the order conditions corresponding to $t \in T_P$ the primary conditions and the order conditions corresponding to $t \in T_S$ the secondary conditions. Denote by $e^{(1)}$ the vector $[1, 0, \ldots, 0]^T$ of appropriate dimension. We have the following result which is the adaptation of Theorem 4 in [10] to uniform grid.

Theorem 2. (Jackiewicz and Zennaro [10]). *Assume that*

$$v + w = (1 + \theta)e^{(1)}, \tag{2.1}$$

Table 1. Conditions up to order four for TSRK methods

	t	Order condition
		$\underline{\varrho = 1}$
	$t_{1,1} = \tau$	$(v^T + w^T)u = 1 + \theta$
		$\underline{\varrho = 2}$
	$t_{2,1} = [\tau]$	$(v^T + w^T)c - v^T u = \frac{1-\theta}{2}$
		$\underline{\varrho = 3}$
	$t_{3,1} = [\tau^2]$	$(v^T + w^T)c^2 - v^T(2c - u) = \frac{1+\theta}{3}$
	$t_{3,2} = [_2\tau]_2$	$(v^T + w^T)Ac - v^T(c - \frac{1}{2}u) = \frac{1+\theta}{6}$
		$\underline{\varrho = 4}$
	$t_{4,1} = [\tau^3]$	$(v^T + w^T)c^3 - v^T(3c^2 - 3c + u) = \frac{1-\theta}{4}$
	$t_{4,2} = [\tau[\tau]]$	$(v^T + w^T)(c \cdot Ac) - v^T(c^2 + Ac - \frac{3}{2}c + \frac{1}{2}u) = \frac{1-\theta}{8}$
	$t_{4,3} = [_2\tau^2]_2$	$(v^T + w^T)Ac^2 - v^T(2Ac - c + \frac{1}{3}u) = \frac{1-\theta}{12}$
	$t_{4,4} = [_3\tau]_3$	$(v^T + w^T)A^2c - v^T(Ac - \frac{1}{2}c + \frac{1}{6}u) = \frac{1-\theta}{24}$

Table 2. Conditions of order five for TSRK methods

	t	Order condition
	$t_{5,1} = [\tau^4]$	$(v^T + w^T)c^4 - v^T(4c^3 - 6c^2 + 4c - u) = \frac{1+\theta}{5}$
	$t_{5,2} = [\tau^2[\tau]]$	$(v^T + w^T)(c^2 \cdot Ac) - v^T(c^3 + 2c \cdot Ac - \frac{5}{2}c^2 - Ac + 2c - \frac{1}{2}u) = \frac{1+\theta}{10}$
	$t_{5,3} = [\tau[\tau^2]]$	$(v^T + w^T)(c \cdot Ac^2) - v^T(2c \cdot Ac + Ac^2 - c^2 - 2Ac + \frac{4}{3}c - \frac{1}{3}u) = \frac{1+\theta}{15}$
	$t_{5,4} = [\tau[_2\tau]_2]$	$(v^T + w^T)(c \cdot A^2c) - v^T(c \cdot Ac + A^2c - \frac{1}{2}c^2 - Ac + \frac{2}{3}c - \frac{1}{6}u) = \frac{1+\theta}{30}$
	$t_{5,5} = [[\tau]^2]$	$(v^T + w^T)(Ac)^2 - v^T(2c \cdot Ac - c^2 - Ac + c - \frac{1}{4}u) = \frac{1+\theta}{20}$
	$t_{5,6} = [_2\tau^3]_2$	$(v^T + w^T)Ac^3 - v^T(3Ac^2 - 3Ac + c - \frac{1}{4}u) = \frac{1+\theta}{20}$
	$t_{5,7} = [_2\tau[\tau]]_2$	$(v^T + w^T)A(c \cdot Ac) - v^T(A^2c + Ac^2 - \frac{3}{2}Ac + \frac{1}{2}c - \frac{1}{8}u) = \frac{1+\theta}{40}$
	$t_{5,8} = [_3\tau^2]_3$	$(v^T + w^T)A^2c^2 - v^T(2A^2c - Ac + \frac{1}{3}c - \frac{1}{12}u) = \frac{1+\theta}{60}$
	$t_{5,9} = [_4\tau]_4$	$(v^T + w^T)A^3c - v^T(A^2c - \frac{1}{2}Ac + \frac{1}{6}c - \frac{1}{24}u) = \frac{1+\theta}{120}$

and that v satisfies all the secondary conditions up to the order $p-1$ of the one-step RK method with modified right-hand side as follows

$$v^T \bar{\psi}^w(t) = \frac{\varrho(t)}{\gamma(t)} v^T c^{\varrho(t)-1}, \tag{2.2}$$

$t \in T_S$, $\varrho(t) \leqslant p-1$. Then all the secondary conditions up to the order p reduce to the primary conditions of the same order.

It follows from this theorem that TSRK method (1.2) has order p if (2.1) and (2.2) hold and the primary conditions up to the order p are satisfied. This simplifies considerably the construction of TSRK methods. For example, to obtain the method of order five without using Theorem 2 we have to solve the system of 17 nonlinear equations corresponding to order conditions up to order five. On the other hand, using Theorem 2 the number of equations is reduced to 9. Moreover, they are of simpler structure than the order conditions given in Tables 1 and 2.

TSRK methods constructed using Theorem 2 are presented in Section 3 and 5.

3. TSRK methods of order $p \leqslant 4$

In this section we present TSRK methods up to the order four. These methods are listed below.

1. $p=1$, $m=1$:

There is only one order condition

$$v_1 + w_1 = 1 + \theta,$$

and the two-parameter family of methods is

$$\begin{array}{c|c} 0 & 0 \\ \hline \theta & \begin{array}{c} v_1 \\ 1+\theta-v_1 \end{array} \end{array}$$

Putting $\theta = 0$ and $v_1 = 0$ we obtain the Euler method. For $v_1 = \frac{\theta-1}{2}$ the above method has order two.

2. $p=2$, $m=1$: The order conditions take the form

$$\begin{cases} v_1 + w_1 = 1 + \theta, \\ -v_1 = \dfrac{1-\theta}{2}, \end{cases}$$

and the one-parameter family of methods is

$$\begin{array}{c|c} 0 & 0 \\ \hline & \frac{\theta-1}{2} \\ \theta & \\ & \frac{3+\theta}{2} \end{array}$$

This method attains order three for $\theta = 5$ but the resulting formula is not zero-stable.

3. $p = 3$, $m = 2$: We will first show the necessity of simplifying condition (2.1). Denote by $(t_{p,i})$ the order condition corresponding to the tree $t_{p,i}$. Subtracting $2(t_{3,2})$ from $(t_{3,1})$ and taking into account that $Ac = 0$ we obtain

$$(v^T + w^T)c^2 = 0.$$

Since $c_2 \neq 0$ (otherwise we could reduce the method to TSRK method of order 3 with $m = 1$) it follows that

$$v_2 + w_2 = 0$$

which proves (2.1).

Now Theorem 2 leads to the system of equations

$$\begin{cases} v + w = (1+\theta)[1,0]^T \\ -v^T u = \dfrac{1-\theta}{2}, \\ v^T u - 2v^T c = \dfrac{1+\theta}{3}. \end{cases}$$

The solution to this system is

$$\begin{aligned} v_1 &= \frac{\theta-1}{2} - \frac{\theta-5}{12\,c_2}, \quad & v_2 &= \frac{\theta-5}{12\,c_2}, \\ w_1 &= \frac{3+\theta}{2} + \frac{\theta-5}{12\,c_2}, \quad & w_2 &= -v_2, \end{aligned}$$

and the TSRK method takes the form

$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2 & \\ \hline & \frac{\theta-1}{2} - \frac{\theta-5}{12\,c_2} & \frac{\theta-5}{12\,c_2} \\ \theta & & \\ & \frac{3+\theta}{2} + \frac{\theta-5}{12\,c_2} & \frac{5-\theta}{12\,c_2} \end{array}$$

Putting $\theta = 0$ these methods reduce to TSRK methods considered by Byrne and Lambert [5] (compare also [12]).

4. $p = 4$, $m = 3$: We will first show that if TSRK method has order four then $m \geqslant 3$. Assume to the contrary that $m = 2$. Consider $(t_{4,2})$ and $(t_{4,3})$. Since $v_2 + w_2 = 0$ (compare the case $p = 3$, $m = 2$) these conditions take the form

$$-v^T\Big(c^2 - \frac{3}{2}c + \frac{1}{2}u\Big) = \frac{1-\theta}{8},$$
$$-v^T\Big(-c + \frac{1}{3}u\Big) = \frac{1-\theta}{12},$$

and it follows that $v^T c^2 = 0$. Since $c_2 \neq 0$ we have $v_2 = 0$. Consider now $(t_{2,1})$ and $(t_{4,1})$. These conditions take the form

$$-v_1 = \frac{1-\theta}{2},$$
$$-v_1 = \frac{1-\theta}{4},$$

and are satisfied only if $\theta = 1$ and $v_1 = 0$. But then the conditions of order three cannot be satisfied. This proves that $m \geqslant 3$.

We will show next that the conditions (2.1) and (2.2) are in fact necessary for TSRK method of order $p = 4$ with $m = 3$. Taking $(t_{4,3}) - 2(t_{4,4})$ and $2(t_{4,1}) - 6(t_{4,2}) + 3(t_{4,3})$ we obtain

$$(v^T + w^T)(Ac^2 - 2A^2c) = 0,$$
$$(v^T + w^T)(2c^3 - 6c \cdot Ac + 3Ac^2) = 0,$$

or

$$a_{32}c_2^2(v_3 + w_3) = 0,$$
$$2c_2^3(v_2 + w_2) + (2c_3^3 - 6a_{32}c_2c_3 + 3a_{32}c_2^2)(v_3 + w_3) = 0.$$

The main determinant of this system is $2c_2^5 a_{32}$. If $c_2 = 0$ then we could combine firs, and second stage and obtain TSRK method of order four with two stages which is impossible. If $a_{32} = 0$ then $Ac = 0$, $Ac^2 = 0$, $c \cdot Ac = 0$, and the system $(t_{3,1}) - 2(t_{3,2})$ and $2(t_{4,1}) - 6(t_{4,2}) + 3(t_{4,3})$ takes the form

$$(v^T + w^T)c^2 = 0,$$
$$(v^T + w^T)c^3 = 0.$$

The main determinant of this system is $\Delta = c_2^2c_3^2(c_3 - c_2)$. It is easy to check that if $c_2 = 0$ or $c_3 = 0$ or $c_3 = c_2$ then the method would reduce to TSRK method (order four with $m = 2$ which is impossible. Hence, $\Delta \neq 0$ and

$$v_2 + w_2 = 0,$$
$$v_3 + w_3 = 0.$$

These relations are also satisfied if $a_{32} \neq 0$ which proves (2.1). Taking $\frac{1}{2}(t_{4,1}) - (t_{4,2})$ we get

$$v^T Ac = \frac{1}{2} v^T c^2$$

which proves the necessity of (2.2).

Now Theorem 2 leads to the system of equations

$$\begin{cases} v + w = (1+\theta)[1,0,0]^T, \\ v^T Ac = \dfrac{1}{2} v^T c^2, \\ -v^T u = \dfrac{1-\theta}{2}, \\ \quad v^T u - 2v^T c = \dfrac{1+\theta}{3}, \\ -v^T u + 3v^T c - 3v^T c^2 = \dfrac{1-\theta}{4}. \end{cases}$$

To solve this system we select c and compute $v^T u$, $v^T c$, $v^T c^2$ and then v from the last three equations, we compute A from the second equation and the relation $Au = c$ (this gives $a_{21} = c_2$, $a_{31} = c_3 - a_{32}$), and we compute w from the first equation (this gives $w_1 = 1 + \theta - v_1$, $w_2 = -v_2$, $w_3 = -v_3$). We distinguish three cases:

Case 1: 0, c_2, c_3 all different and $c_2 \neq \frac{4}{5-\theta}$. The solution is

$$\begin{aligned} v_2 &= \frac{4 - (5-\theta)c_3}{12\, c_2(c_3 - c_2)}, \\ v_3 &= \frac{(5-\theta)c_2 - 4}{12\, c_3(c_3 - c_2)}, \\ v_1 &= \frac{\theta - 1}{2} - v_2 - v_3, \\ a_{32} &= -\frac{1}{6 v_3 c_2}. \end{aligned}$$

This case was analyzed before by Renaut [13]. Putting $\theta = 0$ these methods reduce to TSRK methods considered by Byrne and Lambert [5].

Case 2: $c_2 = \frac{4}{5-\theta}$, $c_3 = 0$, v_3—arbitrary nonzero number. The solution is

$$\begin{aligned} v_2 &= -\frac{(\theta - 5)^2}{48}, \\ v_1 &= \frac{\theta - 1}{2} - v_2 - v_3, \\ a_{32} &= \frac{\theta - 5}{24\, v_3}. \end{aligned}$$

Case 3: $c_2 = c_3 = \frac{4}{5-\theta}$, v_3—arbitrary nonzero number. The solution is

$$\begin{aligned} v_1 &= \frac{\theta - 1}{2} + \frac{(\theta - 5)^2}{48}, \\ v_2 &= -\frac{(\theta - 5)^2}{48} - v_3, \\ a_{32} &= \frac{\theta - 5}{24\, v_3}. \end{aligned}$$

4. The necessity of simplifying assumptions (2.1) and (2.2) for TSRK methods of order five

We will first show the necessity of (2.2) assuming that (2.1) is already satisfied. The conditions (2.2) for TSRK method of order five take the form

$$\left\{ \begin{aligned} v^T A c - \frac{1}{2} v^T c^2 &= 0, \\ v^T (c \cdot Ac) - \frac{1}{2} v^T c^3 &= 0, \\ v^T A^2 c - \frac{1}{3} v^T c^3 &= 0, \\ v^T A^2 c - \frac{1}{6} v^T c^3 &= 0. \end{aligned} \right. \tag{4.1}$$

Define the matrix X by

$$X = \begin{bmatrix} 0 & 0 & 0 & -4 & 0 & 6 & -4 & 1 & \frac{1}{5} \\ 0 & -2 & 0 & -1 & 1 & \frac{5}{2} & -2 & \frac{1}{2} & \frac{1}{10} \\ 0 & -2 & -1 & 0 & 2 & 1 & -\frac{4}{3} & \frac{1}{3} & \frac{1}{15} \\ -1 & -1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{2}{3} & \frac{1}{6} & \frac{1}{30} \\ 0 & -2 & 0 & 0 & 1 & 1 & -1 & \frac{1}{4} & \frac{1}{20} \\ 0 & 0 & -3 & 0 & 3 & 0 & -1 & \frac{1}{4} & \frac{1}{20} \\ -1 & 0 & -1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & \frac{1}{8} & \frac{1}{40} \\ -2 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & \frac{1}{12} & \frac{1}{60} \\ -1 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{6} & \frac{1}{24} & \frac{1}{120} \end{bmatrix},$$

where the elements of ith row are coefficients of

$$(v^T A^2 c, v^T (Ac \cdot c), v^T Ac^2, v^T c^3, v^T Ac, v^T c^2, v^T c, v^T u, 1 + \theta)$$

appearing in the order conditions corresponding to $t_{5,i}$, compare Table 2. Similarly, define the matrix Y by

$$Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the elements of each row correspond to simplifying conditions (4.1). Consider also conditions of order four. Since $v + w = (1+\theta)e^{(1)}$, these conditions take the form

$$\begin{aligned} -v^T(3c^2 - 3c + u) &= \frac{1-\theta}{4}, \\ -v^T\Big(c^2 + Ac - \frac{3}{2}c + \frac{1}{2}u\Big) &= \frac{1-\theta}{8}, \\ -v^T\Big(2Ac - c + \frac{1}{3}u\Big) &= \frac{1-\theta}{12}, \\ -v^T\Big(Ac - \frac{1}{2}c + \frac{1}{6}u\Big) &= \frac{1-\theta}{24}. \end{aligned}$$

In analogy to X and Y define the matrix Z corresponding to the above conditions by

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -3 & 3 & -1 & \frac{1}{4}\cdot\frac{1-\theta}{1+\theta} \\ 0 & 0 & 0 & 0 & -1 & -1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{8}\cdot\frac{1-\theta}{1+\theta} \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & -\frac{1}{3} & \frac{1}{12}\cdot\frac{1-\theta}{1+\theta} \\ 0 & 0 & 0 & 0 & -1 & 0 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{24}\cdot\frac{1-\theta}{1+\theta} \end{bmatrix}.$$

It can be checked that

$$\operatorname{rank}\begin{bmatrix} X \\ Z \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = 6.$$

This means that the simplifying conditions (4.1) are linearly dependent on the order conditions of order four and five, hence they are automatically satisfied.

We will show next the necessity of (2.1). If $m = 3$ this follows using the same arguments as in the case $p = 4$, $m = 3$ considered in Section 3. To show this for $m = 4$ we will first show that TSRK method of order five with $m = 3$ does not exist. In this case $\frac{1}{6}v^Tc^3 = v^TA^2c = 0$ and the system of primary order conditions takes the form

$$\begin{cases} -2v^T u = 1 - \theta, \\ 3v^T u - 6v^T c = 1 + \theta, \\ -4v^T u + 12v^T c - 12v^T c^2 = 1 - \theta, \\ 5v^T u - 20v^T c + 30v^T c^2 = 1 + \theta. \end{cases}$$

It is easy to check that this system does not have a solution for any $\theta \in (-1, 1]$, which proves the above claim. Assume now that $m = 4$. Taking $(t_{5,8}) - 2(t_{5,9})$ we obtain

$$(v^T + w^T)(A^2c^2 - 2A^3c) = 0,$$

or

$$a_{43}a_{32}c_2^2(v_4 + w_4) = 0.$$

Hence, $c_2 = 0$ or $a_{32} = 0$ or $a_{43} = 0$ or $v_4 + w_4 = 0$. If $c_2 = 0$ then we could combine the first and second stage and obtain TSRK method of order five with $m = 3$ which is impossible. If $v_4 + w_4 = 0$ then using the same arguments as in the case $p = 4$ and $m = 3$ considered in Section 3 it follows that $v_3 + w_3 = 0$ and $v_2 + w_2 = 0$. Assume now that $a_{32} = 0$. Then $A^2c = 0$, $A^2c^2 = 0$, $A(c \cdot Ac) = 0$, and $(t_{4,3}) - 2(t_{4,4})$ and $2(t_{5,6}) - 6(t_{5,7}) + 3(t_{5,8})$ take the form

$$\begin{cases} (v^T + w^T)Ac^2 = 0, \\ (v^T + w^T)Ac^3 = 0, \end{cases}$$

or

$$\begin{cases} (a_{42}c_2^2 + a_{43}c_3^2)(v_4 + w_4) = 0, \\ (a_{42}c_2^3 + a_{43}c_3^3)(v_4 + w_4) = 0. \end{cases}$$

If $v_4 + w_4 \neq 0$ then

$$\begin{cases} a_{42}c_2^2 + a_{43}c_3^2 = 0, \\ a_{42}c_2^3 + a_{43}c_3^3 = 0. \end{cases}$$

The main determinant of this system is

$$\Delta_1 = c_2^2c_3^2(c_3 - c_2).$$

Assume that $\Delta_1 = 0$. Then $c_2 = 0$ or $c_3 = 0$ or $c_2 = c_3$. If $c_2 = 0$ then the method would reduce to TSRK of order five with $m = 3$ which is impossible. If $c_3 = 0$ then $a_{31} = 0$ (recall that $a_{32} = 0$) and we could combine first and third stage which is impossible. If $c_2 = c_3$ then $a_{31} = a_{21} = c_2$ (recall that $a_{32} = 0$) and again the method reduces to TSRK method or order five with $m = 3$ which is impossible. Consequently, $\Delta_1 \neq 0$. This means that $a_{42} = 0$ and $a_{43} = 0$. Hence, $Ac = 0$,

$Ac^2 = 0$, $c \cdot Ac = 0$, $c^2 \cdot Ac = 0$, and $(t_{3,1}) - 2(t_{3,2})$, $2(t_{4,1}) - 6(t_{4,2}) + 3(t_{4,3})$ and $(t_{5,1}) - 4(t_{5,2}) + 4(t_{5,5})$ gives

$$\begin{cases} (v^T + w^T)c^2 = 0, \\ (v^T + w^T)c^3 = 0, \\ (v^T + w^T)c^4 = 0. \end{cases}$$

It is easy to check that $c_2 \neq 0$, $c_3 \neq 0$, $c_4 \neq 0$, $c_2 \neq c_3$, $c_2 \neq c_4$, $c_3 \neq c_4$, otherwise the method would reduce to TSRK method of order five with $m = 3$ which is impossible. Hence, $v_4 + w_4 = 0$, and using the same arguments as in the case $p = 4$, $m = 3$ in Section 3, it follows that $v_2 + w_2 = 0$ and $v_3 + w_3 = 0$. This proves that if $a_{32} = 0$ then $v_i + w_i = 0$, $i = 2, 3, 4$.

Assume now that $a_{32} \neq 0$ and $a_{43} = 0$ and consider $(t_{4,3}) - 2(t_{4,4})$ and $(t_{5,5}) - 2(t_{5,4}) + (t_{5,8})$. Since $c_2 \neq 0$ this system can be written as

$$\begin{cases} a_{32}(v_3 + w_3) + a_{42}(v_4 + w_4) = 0, \\ a_{32}(a_{32} - 2c_3)(v_3 + w_3) + a_{42}(a_{42} - 2c_4)(v_4 + w_4) = 0, \end{cases}$$

and its main determinant is

$$\Delta_2 = a_{32}a_{42}(a_{42} - 2c_4 - a_{32} + 2c_3).$$

Assume that $\Delta_2 = 0$. If $a_{42} = 0$ then $v_3 + w_3 = 0$ and taking $2(t_{4,1}) - 6(t_{4,2}) + 3(t_{4,3})$ and $(t_{5,1}) - 4(t_{5,2}) + 4(t_{5,5})$ we obtain

$$\begin{cases} (v_2 + w_2)c_2^3 + (v_4 + w_4)c_4^3 = 0, \\ (v_2 + w_2)c_2^4 + (v_4 + w_4)c_4^4 = 0. \end{cases}$$

Hence, since it must be $c_2 \neq c_4$ we have $v_2 + w_2 = 0$ and $v_4 + w_4 = 0$. Assume now that

$$a_{42} - 2c_4 - a_{32} + 2c_3 = 0,$$

and consider $(t_{4,3}) - 2(t_{4,4})$ and $(t_{5,6}) + 3(t_{5,5}) - 3(t_{5,3})$. This system takes the form

$$\begin{cases} a_{32}(v_3 + w_3) + a_{42}(v_4 + w_4) = 0, \\ a_{32}c_2^2(c_2 + 3a_{32} - 3c_3)(v_3 + w_3) + a_{42}c_2^2(c_2 + 3a_{42} - 3c_4)(v_4 + w_4) = 0, \end{cases}$$

and its main determinant is

$$\Delta_3 = 3a_{32}a_{42}c_2^2(a_{42} - c_4 - a_{32} + c_3).$$

If $\Delta_3 \neq 0$ then $v_4 + w_4 = 0$ and $v_3 + w_3 = 0$ and using the same arguments as in the case $p = 4$ and $m = 3$ we have also $v_2 + w_2 = 0$. Assume next that $\Delta_3 = 0$. We have already shown that $c_2 \neq 0$ and $a_{32} \neq 0$ and that $a_{42} = 0$ implies that $v_i + w_i = 0$, $i = 2, 3, 4$. Hence, $a_{42} - c_4 - a_{32} + c_3 = 0$, and since also $a_{42} - 2c_4 - a_{32} + 2c_3 = 0$, it follows that $c_3 = c_4$. Hence, $a_{42} = a_{32}$ and $a_{41} = a_{31}$ (recall that $a_{43} = 0$), and the method reduces to TSRK method of order five with $m = 3$ which is impossible.

Summing up the above discussion it follows that the conditions $v_i + w_i = 0$, $i = 2, 3, 4$ are always satisfied for TSRK method of order five with $m = 4$.

5. Construction of TSRK methods of order five with four stages

We will need the following lemma due to Butcher [4].

Lemma 1. (Butcher [4]). *Let P and Q be 3×3 matrices such that*

$$PQ = \begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $s_{11}s_{22} - s_{12}s_{21} \neq 0$. Then either the third row of P is the zero vector or the third column of Q is the zero vector.

We have shown in Section 4 that simplifying assumptions (2.1) and (2.2) are in fact necessary for TSRK method of order five with $m = 4$. Hence, the Theorem 2 leads to the system of equations

$$v + w = (1+\theta)[1, 0, 0, 0]^T, \tag{5.1}$$

$$\begin{cases} -2v^T u = 1 - \theta, \\ 3v^T u - 6v^T c = 1 + \theta, \\ -4v^T u + 12v^T c - 12v^T c^2 = 1 - \theta, \\ 5v^T u - 20v^T c + 30v^T c^2 - 20v^T c^3 = 1 + \theta, \end{cases} \tag{5.2}$$

and

$$\begin{cases} v^T Ac = \frac{1}{2}v^T c^2, \\ v^T (c \cdot Ac) = \frac{1}{2}v^T c^3, \\ v^T Ac^2 = \frac{1}{3}v^T c^3, \\ v^T A^2 c = \frac{1}{6}v^T c^3. \end{cases} \tag{5.3}$$

Solving (5.2) for $v^T u$, $v^T c$, $v^T c^2$, and $v^T c^3$ we obtain

$$(5.4)\qquad \begin{cases} v^T u = -\dfrac{1}{2}(1-\theta), \\ v^T c = -\dfrac{1}{12}(5-\theta), \\ v^T c^2 = -\dfrac{1}{3}, \\ v^T c^3 = -\dfrac{1}{120}(31+\theta), \end{cases}$$

and (5.3) can be rewritten as

$$(5.5)\qquad \begin{cases} v^T Ac = -\dfrac{1}{6}, \\ v^T(c\cdot Ac) = -\dfrac{1}{240}(31+\theta), \\ v^T Ac^2 = -\dfrac{1}{360}(31+\theta), \\ v^T A^2 c = -\dfrac{1}{720}(31+\theta). \end{cases}$$

Let us introduce the matrices

$$P = \begin{bmatrix} v_2 & v_3 & v_4 \\ v_2 c_2 & v_3 c_3 & v_4 c_4 \\ \sum_i v_i a_{i2} - v_2(\alpha - \beta c_2) & \sum_i v_i a_{i3} - v_3(\alpha - \beta c_3) & \sum_i v_i a_{i4} - v_4(\alpha - \beta c_4) \end{bmatrix}$$

and

$$Q = \begin{bmatrix} c_2 & c_2^2 & \sum_j a_{2j}c_j - \frac{1}{2}c_2^2 \\ c_3 & c_3^2 & \sum_j a_{3j}c_j - \frac{1}{2}c_3^2 \\ c_4 & c_4^2 & \sum_j a_{4j}c_j - \frac{1}{2}c_4^2 \end{bmatrix}$$

where the summations run from 1 to 4 and where α and β are constants determined by solving the system of equations

$$\begin{cases} (5-\theta)\alpha - 4\beta = 2, \\ 120\alpha - 3(31+\theta)\beta = 31+\theta. \end{cases}$$

Assuming that $\theta^2 + 26\theta + 5 \neq 0$, that is $\theta \neq -13 \pm \sqrt{164}$, this system has a unique solution given by

$$(5.6)\qquad \begin{cases} \alpha = \dfrac{-2(31+\theta)}{3(\theta^2+26\theta+5)}, \\ \beta = -\dfrac{\theta^2+26\theta+85}{3(\theta^2+26\theta+5)}. \end{cases}$$

It can be checked using (5.4) and (5.5) that

$$PQ = \begin{bmatrix} -\frac{1}{2}(5-\theta) & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{120}(31+\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $\theta \neq -13 \pm \sqrt{164}$ implies that

$$\begin{vmatrix} -\frac{1}{2}(5-\theta) & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{120}(31+\theta) \end{vmatrix} \neq 0,$$

it follows from Lemma 1 that the last row of P or the last column of Q must be equal to zero. Since $c_1 = 0$ we have

$$\sum_j a_{2j}c_j - \frac{1}{2}c_2^2 = -\frac{1}{2}c_2^2 \neq 0,$$

and it follows that the last row of P must be equal to zero. This gives

$$\sum_i v_i a_{ij} = v_j(\alpha - \beta c_j), \; j = 2, 3, 4. \tag{5.7}$$

In particular

$$\sum_i v_i a_{i4} - v_4(\alpha - \beta c_4) = -v_4(\alpha - \beta c_4) = 0.$$

If $v_4 = 0$ then the last condition of (5.5) reads

$$0 = v^T A^2 c = -\frac{1}{720}(31 + \theta),$$

and $\theta = -31$ which violates the condition of zero stability. Consequently, $v_4 \neq 0$ and $\alpha - \beta c_4 = 0$ which gives

$$c_4 = \frac{2(31+\theta)}{\theta^2 + 26\theta + 85}. \tag{5.8}$$

Observe that c_4 is well defined for any $\theta \in (-1, 1]$.

It can be checked using (5.4) that the condition (5.7) implies the first and third condition of (5.5) and that the second and fourth conditions of (5.5) are equivalen Moreover, the fourth condition of (5.5) can be written in the form

$$v_3(\alpha - \beta c_3)a_{32}c_2 = -\frac{1}{720}(31 + \theta). \tag{5.9}$$

In particular $v_3 \neq 0$, $c_3 \neq c_4$, $c_2 \neq 0$, and $a_{32} \neq 0$.

Summing up the above discussion we have the following algorithm for the construction of TSRK method of order five with $m = 4$.

Step 1. Choose $\theta \in (-1, 1]$, $\theta \neq -13 + \sqrt{164}$, and compute α and β from (5.6) and c_4 from (5.8).

Step 2. Choose c_2 and c_3 and compute v_i, $i = 1, 2, 3, 4$, from the system of equations (5.4).

Step 3. Compute a_{32} from (5.9). This gives

$$a_{32} = -\frac{31 + \theta}{720(\alpha - \beta c_3) v_3 c_2}.$$

Step 4. Compute a_{42} and a_{43} from (5.7), $j = 2, 3$. This leads to

$$a_{42} = \frac{v_2(\alpha - \beta c_2) - v_3 a_{32}}{v_4},$$
$$a_{43} = \frac{v_3(\alpha - \beta c_3)}{v_4}.$$

Step 5. Compute

$$a_{21} = c_2, \quad a_{31} = c_3 - a_{32}, \quad a_{41} = c_4 - a_{42} - a_{43}.$$

Step 6. Compute w from $v + w = (1 + \theta)[1, 0, 0, 0]^T$. This gives

$$w_1 = 1 + \theta - v_1, \quad w_2 = -v_2, \quad w_3 = -v_3, \quad w_4 = -v_4.$$

Below we list the five cases for which the program described above can be carried out.

Case 1: $0, c_2, c_3, c_4 = \alpha/\beta$ all distinct, and

$$10(5 - \theta)c_2 c_4 - 40(c_2 + c_4) + 31 + \theta \neq 0,$$
$$10(5 - \theta)c_2 c_3 - 40(c_2 + c_3) + 31 + \theta \neq 0.$$

The solution is

$$v_2 = \frac{10(5 - \theta)c_2 c_4 - 40(c_3 + c_4) + 31 + \theta}{120\, c_2(c_2 - c_3)(c_4 - c_2)},$$
$$v_3 = \frac{10(5 - \theta)c_2 c_4 - 40(c_2 + c_4) + 31 + \theta}{120\, c_3(c_2 - c_3)(c_3 - c_4)},$$
$$v_4 = \frac{10(5 - \theta)c_2 c_3 - 40(c_2 + c_3) + 31 + \theta}{120\, c_4(c_3 - c_4)(c_4 - c_2)},$$
$$v_1 = \frac{1}{2}(\theta - 1) - v_2 - v_3 - v_4.$$

Case 2: $c_2 = c_3 = \frac{40\,c_4-31-\theta}{10(5-\theta)c_4-40}$, $c_4 = \alpha/\beta$, $v_3 \neq 0$. The solution is

$$v_2 = \frac{4-(5-\theta)c_4}{12\,c_2(c_4-c_2)} - v_3,$$
$$v_4 = \frac{(5-\theta)c_2-4}{12\,c_4(c_4-c_2)},$$
$$v_1 = \frac{1}{2}(\theta-1) - v_2 - v_3 - v_4.$$

Case 3: $c_2 = \frac{40\,c_4-31-\theta}{10(5-\theta)c_4-40}$, $c_3 = 0$, $c_4 = \alpha/\beta$, $v_3 \neq 0$. The solution is

$$v_2 = \frac{4-(5-\theta)c_2}{12\,c_2(c_4-c_2)},$$
$$v_4 = \frac{(5-\theta)c_2-4}{12\,c_4(c_4-c_2)},$$
$$v_1 = \frac{1}{2}(\theta-1) - v_2 - v_3 - v_4.$$

Case 4: $c_2 = c_4 = \alpha/\beta$, $c_3 = \frac{40c_4-31-\theta}{10(5-\theta)c_4-40}$, $v_4 \neq 0$. The solution is

$$v_2 = \frac{4-(5-\theta)c_3}{12\,c_4(c_3-c_4)} - v_4,$$
$$v_3 = \frac{(5-\theta)c_4-4}{12\,c_3(c_3-c_4)},$$
$$v_1 = \frac{1}{2}(\theta-1) - v_2 - v_3 - v_4.$$

Case 5: $v_2 = 0$, $c_2 \neq 0$, $c_3 = \frac{40\,c_4-31-\theta}{10(5-\theta)c_4-40}$, $c_4 = \alpha/\beta$. The solution is

$$v_3 = \frac{4-(5-\theta)c_4}{12\,c_3(c_4-c_3)},$$
$$v_4 = \frac{(5-\theta)c_3-4}{12\,c_4(c_4-c_3)},$$
$$v_1 = \frac{1}{2}(\theta-1) - v_3 - v_4.$$

The case 5 overlaps with the cases 1, 2, and 4. Observe that the remaining cases $c_3 = c_4$ or $v_3 = 0$ or $v_4 = 0$ do not have solutions.

Choosing $\theta = 0$, $c_2 = \frac{1}{4}$, $c_3 = \frac{1}{2}$ the case 1 leads to the following example of TSRK method of order five:

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{4} & \frac{1}{4} & & & \\
\frac{1}{2} & \frac{1}{64} & \frac{31}{64} & & \\
\frac{62}{85} & \frac{2500522}{17809625} & \frac{2081836}{17809625} & \frac{8408192}{17809625} & \\
\hline
 & -\frac{1}{248} & -\frac{8}{489} & \frac{32}{117} & -\frac{3561925}{4729608} \\
0 & & & & \\
 & \frac{249}{248} & \frac{8}{489} & -\frac{32}{117} & \frac{3561925}{4729608}
\end{array}
$$

6. Order barriers for explicit TSRK methods

Denote by TFSEN(p) the minimal number of stages required for explicit fixed stepsize TSRK method of order p. The following theorem presents such order barriers for $p \leqslant 5$.

Theorem 3. *The minimal number of stages required by explicit fixed stepsize TSRK method of order $p \leqslant 5$ is given by*

$$
\begin{aligned}
\text{TFSEN}\,(1) &= 1,\\
\text{TFSEN}\,(2) &= 1,\\
\text{TFSEN}\,(3) &= 2,\\
\text{TFSEN}\,(4) &= 3,\\
\text{TFSEN}\,(5) &= 4.
\end{aligned}
$$

P r o o f. This theorem follows from the results presented in Section s 3, 4, and 5. □

The result TFSEN(5) $= 4$ corrects an entry in Table 2 in Renaut recent paper [13]. It follows from Theorem 3 that for $2 \leqslant p \leqslant 5$ we have TFSEN(p) $=$ EN($p-1$), where EN($p-1$) is the minimal number of stages required for explicit RK method of order $p-1$. Since Theorem 2 leads to the system of equations for explicit TSRK method of order p, which is similar to the system of order conditions for explicit RK methods of order $p-1$, we suspect that this is true for any $p \geqslant 2$. We formulate this as the conjecture.

Conjecture. *The following relation holds*

$$\text{TFSEN}(p) = \text{EN}(p-1)$$

for any $p \geqslant 2$.

Order barriers for variable-step explicit TSRK methods were investigated recently by Jackiewicz and Zennaro [10]. They are related to order barriers for continuous explicit RK methods studied by Owren and Zennaro [11].

7. Numerical examples

We have applied some of the TSRK methods of order five presented in Section 5 to a problem in the paper by Hull et al. [8]. These tests clearly demonstrate the order $p = 5$ convergence. Furthermore, the break down in order in the neighborhood of the singular point $\theta = -13 + \sqrt{164}$ was seen. These results alone, however, do not validate the use of the TSRK.

Here we consider instead a practical implementation of the TSRK in an Euler equation solver and compare with the variable step (VS) formulation of Jackiewicz and Zennaro [10]. The VSTSRK is defined by

$$y_{i+1} = (1 - \xi\theta)y_i + \xi\theta y_{i-1} + h_{i-1}\sum_{j=1}^{m}(v_j f(Y_{i-1}^j) + \xi w_j f(Y_i^j)) \tag{7.1}$$

where Y_i^j and Y_{i-1}^j are defined as in (1.2) but with h replaced by h_{i-1} and ξh_{i-1}, respectively, and $\xi = h_i/h_{i-1}$, where h_i is the timestep used to advance to y_{i+1}. In order to maintain the advantage of the TSRK we must choose $\theta = 0$ in (7.1).

Suppose we wish to solve an ODE system of equations which arises from the discretization of a hyperbolic system of equations with added artificial viscosity. In this context A-stability is not important. Rather, it is appropriate to look for methods with large interval of stability along the imaginary axis for which the stability region is not too slender near the imaginary axis. Some low-order methods with appropriate stability properties were presented and tested in [14]. Note, of course, that the stability properties of the VSTSRK are dependent on the range of ξ values that are determined to be acceptable. In particular $\xi = 1$ corresponds to a constant step-size implementation and hence the stability region of the VSTSRK for $\xi = 1$ is the usual stability region of the constant stepsize method.

In order to implement the VSTSRK in a local time stepping Euler solver a mechanism for adjusting the local time step has to be employed. For one-step methods this mechanism, based on the maximum CFL numbers, the wave speed in the given cell and its volume, is well tested [15]. In the two-step situation the maximum CFL number is determined by the size of the stability region, which in turn depends on ξ. The other parameters used in this step-size adjustment are independent of the RK scheme and hence the same mechanism can be used to adjust the time step for the VSTSRK provided some information about how the maximum CFL number changes with the size of ξ is supplied. In the numerical tests reported here an assumption was made that the maximum CFL number of the VSTSRK is given by a constant value, the maximum CFL for the constant step-size implementation. This assumes that $\xi \approx 1$ throughout and hence stability is determined by the constant step method. If $\xi \not\approx 1$ throughout this assumption would cause instability. As the latter did not occur we know that the local stepsizes are not changing very rapidly.

The experiments performed were for non-lifting subsonic and transonic flow past a cylinder. This flow is modelled by the solution of the finite volume representation of the Euler equations:

$$\frac{\partial}{\partial t} \iint_S w \, \mathrm{d}S + \int_{\mathrm{d}S} (f \, \mathrm{d}y - g \, \mathrm{d}x) = 0,$$

where

$$w = \begin{pmatrix} \varrho \\ \varrho u \\ \varrho v \\ \varrho E \end{pmatrix}, \quad f = \begin{pmatrix} \varrho u \\ \varrho u^2 + P \\ \varrho uv \\ \varrho uH \end{pmatrix}, \quad g = \begin{pmatrix} \varrho v \\ \varrho uv \\ \varrho v^2 + P \\ \varrho vH \end{pmatrix},$$

$E = \frac{P}{(\gamma-1)\varrho} + \frac{1}{2}(u^2 + v^2)$ and $H = E + P/\varrho$. Here P, ϱ, E, H, u, and v represent the pressure, density, energy, enthalpy and cartesian velocity components, respectively. Subsonic flow was calculated for Mach number 0.35 on an O mesh with 80 intervals in the chordwise direction and 40 intervals in the normal direction. For Mach number 0.45 a coarser 40×20 grid was used. All details of the method, enthalpy damping and dissipation were held constant throughout the comparison.

In Figure 1 we show the convergence history of the residual for Mach = .35 for both variable step size and constant stepsize implementations of two 3-stage schemes. We see that these results are so close that the convergence history is not affected by the implementation, despite the fact that the variable step size formulation amounts to an order reduction of the TSRK (see [10]). This order reduction is small enough not to be noticeable because the accuracy of the spatial discretization alone is at most two and hence dominates the error. Note also, that in the situation where the stepsizes are not adjusted greatly, VSTSRK are more expensive than TSRK since the coefficients $\{v_i, w_i\}$ are recalculated each step. Hence VSTSRK provide no advantage over TSRK for steady state solutions of the Euler equation. We do not repeat the additional results in [14] which show that TSRK offer gains in efficiency compared to the usual RK.

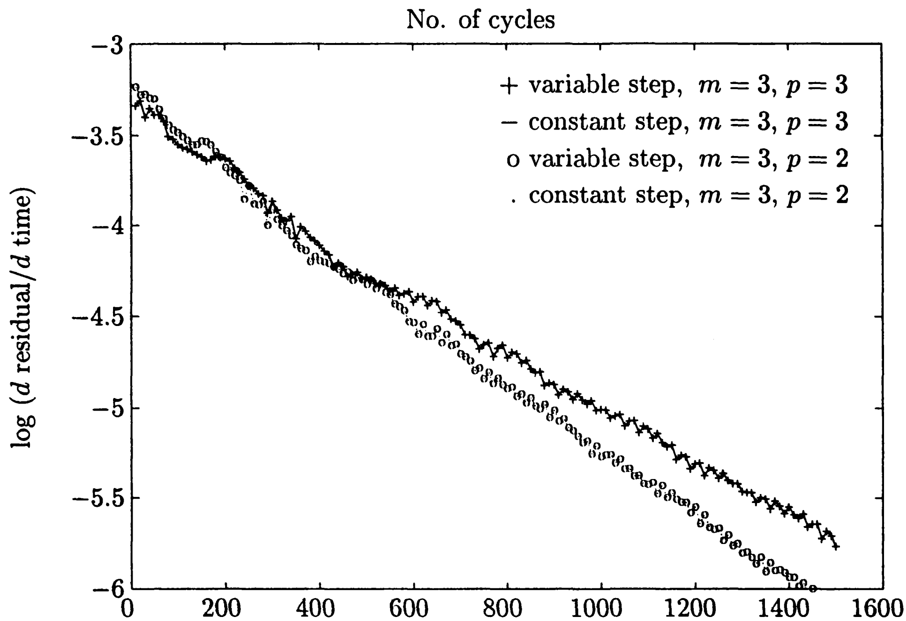

Figure 1. Converging solutions, Mach = .35

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Authors' addresses: *Z. Jackiewicz*, *R. A. Renaut*, Department of Mathematics, Arizona State University, Tempe, Arizona, 85287, USA; and *M. Zennaro*, Dipartimento di Scienze Matematiche, Universitá di Trieste, I-34100 Trieste, Italy.