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MEAN SQUARE APPROXIMATION BY OPTIMAL PERIODIC INTERPOLATION

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Summary. Following the research of Babuška and Práger, the author studies the approximation power of periodic interpolation in the mean square norm thus extending his own former results.

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0. Introduction

Babuška [1] introduced the concept of the periodic Hilbert space for studying optimal quadrature formulas, Práger [8] continued these investigations and related these problems to the minimum norm interpolation (optimal periodic interpolation) in periodic Hilbert spaces. These ideas have been further developed in a number of papers [2, 3, 4, 5, 6, 7]. In this paper we will study the approximation power of optimal periodic interpolation in the mean square norm an thereby extended results of [4].

1. Periodic Hilbert spaces

We denote by $\mathcal{C}$ the algebra of continuous complex-valued functions on $\mathbb{R}$ with period 2π. $\mathcal{C}$ is a Banach algebra with respect to the uniform norm $\|u\|_\infty = \sup_{x\in\mathbb{R}} |u(x)|$.

Next we denote by $\mathcal{L}^2$ the Hilbert space of square integrable periodic functions with an inner product

$$(u,v) = \frac{1}{2\pi}\int_0^{2\pi} u(x)v(x)^* \,\mathrm{d}x.$$

The exponentials are given by $e_k(x) = \mathrm{e}^{ikx}$, $k \in \mathbb{Z}$. The finite Fourier transform of $u \in \mathcal{L}^2$ is given by

$$(u,e_k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e_k(x)^* \,\mathrm{d}x, \qquad k \in \mathbb{Z}.$$

Its inversion is the Fourier series of u, $\sum\limits_{k=-\infty}^{\infty} (u,e_k)e_k$, which converges in the $\mathcal{L}^2$-norm to u.

Next we introduce the Wiener algebra $\mathcal{A}$ of functions $u \in \mathcal{L}^2$ having absolutely convergent Fourier series. $\mathcal{A}$ is a Banach algebra with respect to the norm

$$\|u\|_a = \sum_{k=-\infty}^{\infty} |(u,e_k)|.$$

Clearly, $\mathcal{A}$ is a subalgebra of $\mathcal{C}$. We have the estimates

$$\|u\| \leqslant \|u\|_\infty \leqslant \|u\|_a \qquad (x \in \mathbb{R}, u \in \mathcal{A})$$

and the inclusions

$$A \subset \mathcal{C} \subset \mathcal{L}^2.$$

To introduce the periodic Hilbert space $\mathcal{H}_d$ we need a biinfinite symmetric positive l_1-sequence $d = (d_k)$, i.e., we have

$$d_{-k} = d_k > 0 \quad (k \in \mathbb{Z}), \qquad \sum_{k=-\infty}^{\infty} d_k < \infty.$$

It is also convenient to assume the monotonicity condition

$$d_k > d_{k+1} \qquad (k \geqslant 0).$$

We define by $\mathcal{H}_d$ the linear space of all functions $u \in \mathcal{L}^2$ satisfying

$$\sum_{k=-\infty}^{\infty} \frac{1}{d_k} |(u, e_k)|^2 < \infty.$$

The inner product of $\mathcal{H}_d$ is defined by

$$(u, v)_d = \sum_{k=-\infty}^{\infty} \frac{1}{d_k} (u, e_k)(v, e_k)^*.$$

We list some properties of $\mathcal{H}_d$. Each $\mathcal{H}_d$ is a subspace of the Wiener algebra and therefore also of $\mathcal{C}$, the imbeddings being continuous:

$$\mathcal{H}_d \subset \mathcal{A} \subset \mathcal{C}.$$

We denote by τ the algebra of trigonometric polynomials. For $m \geqslant 0$ and $u \in \mathcal{L}^2$ we denote by

$$S_m(u) = \sum_{k=-m}^{m} (u, e_k) e_k$$

the Fourier partial sum polynomial of u of order m. It is immediate that

$$\tau \subset \mathcal{H}_d$$

and

$$\lim_{m \to \infty} \|u - S_m(u)\|_d = 0, \qquad u \in \mathcal{H}_d.$$

Moreover, $\mathcal{H}_d$ as well as its norm $\|u(\cdot - a)\|_d = \|u\|_d$ are translation invariant ($u \in \mathcal{H}_d$, $a \in \mathbb{R}$). The characterizing sequence $d = (d_k)$ defines the kernel function

$$g(x) = \sum_{k=-\infty}^{\infty} d_k e_k(x)$$

which is an element of $\mathcal{H}_d \cdot \mathcal{H}_d$ is a reproducing kernel Hilbert space with the kernel function $g(y, x) = K(y, x)$. We have

$$u(x) = (u, g(\cdot - x))_d (u \in \mathcal{H}_d, \qquad x \in \mathbb{R}),$$

which implies the estimate

$$|u(x)| \leqslant \|u\|_d \cdot \|g\|_d, \qquad \|g\|_d = \sqrt{g(0)}.$$

We conclude this section by presenting two examples.

E x a m p l e 1. The sequence d is given by

$$d_0 = 1, \qquad d_k = k^{-2r} \qquad (k \neq 0).$$

Here r is a positive integer. The periodic Hilbert space is the periodic Sobolev space $\mathcal{W}^r$. The kernel function is given by

$$g(x) = 1 + (-1)^r B_{2r}(x)$$

where $B_q(x)$ denotes the Bernoulli function (polynomial) of degree q.

E x a m p l e 2. In this case the sequence d is given by

$$d_k = \mathrm{e}^{-|k|b} \qquad (k \in \mathbb{Z})$$

where b is a positive real number. The periodic Hilbert space consists of restrictions of functions to the real axis which are holomorphic in the strip $|\operatorname{Im}(z)| < b$. The kernel function is in this case the well known Poisson kernel

$$g(x) = \frac{\sin h(b)}{\cos h(b) - \cos(x)}.$$

2. Optimal periodic interpolation

We first treat the problem of interpolation with shifts of the kernel function g which are related to the knots x_j, $0 \leqslant j < n$, satisfying

$$0 \leqslant x_0 < x_1 < \ldots < x_{n-1} < 2\pi.$$

It follows from the theory of trigonometric interpolation that for any data y_j, $0 \leqslant j < n$, there is a trigonometric polynomial $w \in \tau$ satisfying the interpolation conditions

$$w(x_j) = y_j, \qquad 0 \leqslant j < n.$$

If $n = 2m + 1$, then w can be made unique by assuming that w has order m, i.e., $w \in \tau_m$. If $n = 2m$ then $w \in \tau_m$ exists and can be made unique by deleting $\cos(mx)$ or $\sin(mx)$ depending on the position of knots.

The space of interpolating functions related to g is given by

$$\langle g(\cdot - x_0), g(\cdot - x_1), \ldots, g(\cdot - x_{n-1}) \rangle .$$

Proposition 2.1. *The space* $\langle g(\cdot - x_0), g(\cdot - x_1), \ldots, g(\cdot - x_{n-1})\rangle$ *has dimension* n.

Proof. Recall that

$$u(x) = (u, g(\cdot - x))_d \qquad (u \in \mathcal{H}_d,\ x \in \mathbb{R}).$$

Let $w_k \in \tau_m$ be a trigonometric polynomial satisfying

$$w_k(x_j) = \delta_{j,k}, \qquad 0 \leqslant j, k < n.$$

Thus

$$(w_k, g(\cdot - x_j))_d = \delta_{j,k}, \qquad 0 \leqslant j, k < n,$$

which yields the linear independence of the shifted functions $g(\cdot - x_0), g(\cdot - x_1), \ldots,$ $g(\cdot - x_{n-1})$. □

Let Q_n denote the unique orthogonal projector having the n-dimensional space $\langle g(\cdot - x_0), g(\cdot - x_1), \ldots, g(\cdot - x_{n-1})\rangle$ as its range:

$$\mathcal{R}(Q_n) = \langle g(\cdot - x_0), g(\cdot - x_1), \ldots, g(\cdot - x_{n-1})\rangle .$$

For $u \in \mathcal{H}_d$ the function $Q_n(u)$ is the unique best approximation of u in $\langle g(\cdot - x_0),$ $g(\cdot - x_1), \ldots, g(\cdot - x_{n-1})\rangle$.

Proposition 2.2. *Given* $u \in \mathcal{H}_d$, *the best approximant* $Q_n(u)$ *of* u *in* $\langle g(\cdot - x_0),$ $g(\cdot - x_1), \ldots, g(\cdot - x_{n-1})\rangle$ *is also the unique interpolant of* u *at* x_j, $0 \leqslant j < n$, *with the minimum norm in* $\mathcal{H}_d$:

(i) $Q_n(u)(x_j) = u(x_j)$, $0 \leqslant j < n$;

(ii) $\|Q_n(u)\|_d = \min\{\|v\|_d \colon v(x_j) = u(x_j),\ 0 \leqslant j < n\}$.

Proof. The characterization of the best approximation in Hilbert spaces yields the equation

$$(u - Q_n(u), g(\cdot - x_j))_d = 0 \qquad (0 \leqslant j < n).$$

Taking into account

$$u(x) = (u, g(\cdot - x))_d$$

we obtain

$$Q_n(u)(x_j) = u(x_j), \qquad 0 \leqslant j < n.$$

This proves (i).

As a consequence there exists a Lagrange basis

$$\begin{gathered} h_0, \dots, h_{n-1} \quad \text{of} \quad \langle g(\cdot - x_0), g(\cdot - x_1), \dots, g(\cdot - x_{n-1})\rangle : \\ \langle g(\cdot - x_0), \dots, g(\cdot x_{n-1})\rangle = \langle h_0, \dots, h_{n-1}\rangle\,, \\ h_k(x_j) = \delta_{j,k}, \qquad 0 \leqslant j, k < n. \end{gathered}$$

Using the Lagrange basis of $\langle g(\cdot - x_0), \dots, g(\cdot - x_{n-1})\rangle$ it follows from $u(x_j) = v(x_j)$, $0 \leqslant j < n$, that $Q_n(u) = Q_n(v)$. Taking into account

$$(v - Q_n(v), Q_n(v))_d = 0$$

we can conclude

$$(v, v)_d = (Q_n(v), Q_n(v))_d + (v - Q_n(v), v - Q_n(v))_d \geqslant (Q_n(u), Q_n(u))_d$$

with equality if and only if $v = Q_n(u)$. This proves (ii). □

3. Optimal periodic interpolation on uniform meshes

In this section we treat the much more explicit case of uniformly distributed knots:

$$x_j = \frac{2\pi}{n}j, \qquad 0 \leqslant j < n.$$

It is easily established that the space of interpolants

$$\langle g(\cdot - x_0), \dots, g(\cdot - x_{n-1})\rangle$$

is translation invariant with respect to the mesh size $\frac{2\pi}{n}$:

$$w \in \langle g(\cdot - x_0), \dots, g(\cdot - x_{n-1})\rangle \Rightarrow w\Big(\cdot - \frac{2\pi}{n}\Big) \in \langle g(\cdot - x_0), \dots, g(\cdot - x_{n-1})\rangle\,.$$

Thus the Lagrange basis $h_0, \dots, h_{n-1}$ of $\langle g(\cdot - x_0(, \dots, g(\cdot - x_{n-1})\rangle$ is obtained by translation:

$$h_k = h_0(\cdot - x_k), \qquad 0 \leqslant k < n.$$

Thus the interpolation projector Q_n possesses the Lagrange representation

$$Q_n(u)(x) = \sum_{j=0}^{n-1} u(x_j) h_0(x - x_k).$$

We will now use the discrete Fourier transform to obtain an alternative representation of the optimal periodic interpolant $Q_n(u)$. Recall that the discrete Fourier transform of u is defined by

$$c_{k,n}(u) = \frac{1}{n}\sum_{j=0}^{n-1} u(x_j)e_k(x_j)^*, \qquad 0 \leqslant k < n.$$

The inverse Fourier transform is given by

$$u(x_j) = \sum_{k=0}^{n-1} c_{k,n}(u)e_k(x_j), \qquad 0 \leqslant j < n.$$

The fundamental property of the discrete Fourier transform is the convolution theorem. Let w be the discrete convolution of u and v:

$$w(x_k) = \sum_{j=0}^{n-1} u(x_j)v(x_k - x_j) = \sum_{j=0}^{n-1} u(x_j)v(x_{k-j}).$$

Then the convolution theorem yields

$$n \cdot c_{k,n}(u)c_{k,n}(v) = c_{k,n}(w), \qquad 0 \leqslant k < n.$$

Next we are looking for a trigonometric polynomial

$$a \in \langle e_0, \ldots, e_{n-1}\rangle$$

such that the optimal periodic interpolant $Q_n(u)$ is given by

$$Q_n(u)(x) = \sum_{j=0}^{n-1} a(x_j)g(x - x_j).$$

The interpolation conditions yield the convolution equation

$$u(x_k) = \sum_{j=0}^{n-1} a(x_j)g(x_k - x_j).$$

Now the convolution theorem implies

$$c_{k,n}(u) = n \cdot c_{k,n}(a)c_{k,n}(g).$$

Since $g \in \mathcal{A}$ aliasing is applicable and we get

$$c_{k,n}(g) = \sum_{r=-\infty}^{\infty} d_{k+rn} > 0.$$

Thus we have

$$c_{k,n}(a) = \frac{c_{k,n}(u)}{n \cdot c_{k,n}(g)}, \qquad 0 \leqslant k < n.$$

Using the inverse discrete Fourier transform we obtain

$$a(x_j) = \sum_{k=0}^{n-1} \frac{c_{k,n}(u)}{n \cdot c_{k,n}(g)} e_k(x_j), \qquad 0 \leqslant j < n.$$

It is obvious that the trigonometric polynomial is given by

$$a(x) = \sum_{k=0}^{n-1} \frac{c_{k,n}(u)}{n \cdot c_{k,n}(g)} e_k(x).$$

Proposition 3.1. *Let $u \in \mathcal{H}_d$. Then the optimal periodic interpolant $Q_n(u)$ with respect to the uniform knots $x_j = \frac{2\pi}{n} j$, $0 \leqslant j < n$, is given by*

$$Q_n(u)(x) = \sum_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \frac{c_{k,n}(u)}{n \cdot c_{k,n}(g)} e_k(x_j) \right) g(x - x_j).$$

Note that for $u = h_0$ we get

$$c_{k,n}(h_0) = \frac{1}{n}, \qquad 0 \leqslant k < n.$$

This implies the representation formula for the fundamental Lagrange function h_0 of optimal periodic interpolation

$$h_0(x) = \sum_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} \frac{1}{n^2 \cdot c_{k,n}(g)} e_k(x_j) \right) g(x - x_j).$$

Remark 3.1. It follows from the proof of Proposition 3.1 that this result remains valid for any $g \in C$ if and only if

$$c_{k,n}(g) \neq 0, \qquad 0 \leqslant k < n.$$

See Locher [7], Cheney [2]. If these conditions are violated for some k, a modified approach is possible. See Delvos [3].

4. Exponential interpolants

We apply the representation formula of Proposition 3.1 to the exponentials e_k. In this connection it is appropriate to list some properties of the projector Q_n of optimal periodic interpolation. Since the kernel g is real valued we have

$$Q_n(f^*) = Q_n(f)^*.$$

As a special case we have

$$Q_n(e_{-k} = Q_n(e_k)^*, \qquad k \in \mathbb{Z}.$$

Moreover, for the sequel it is convenient to assume

$$n = 2m + 1.$$

Proposition 4.1. *The exponential interpolants are given by*

$$Q_n(e_k)(x) = \frac{1}{c_{k,n}(g)} \sum_{r=-\infty}^{\infty} d_{k+rn} e_{k+rn}(x), \qquad 0 \leqslant k < n.$$

Proof. Recall that for arbitrary u we have

$$Q_n(u)(x) = \sum_{j=0}^{n-1} a(x_j) g(x - x_j)$$

with

$$a(x_j) = \sum_{l=0}^{n-1} \frac{c_{l,n}(u)}{n \cdot c_{l,n}(g)} e_l(x_j), \qquad 0 \leqslant j < n.$$

Replacing u by e_k the orthogonality properties of the discrete Fourier transform yield

$$c_{l,n}(e_k) = 0, \quad l - k \notin n\mathbb{Z},$$
$$c_{l,n}(e_k) = 1, \quad l - k \in n\mathbb{Z}.$$

Thus we get

$$a(x_j) = \frac{1}{n \cdot c_{k,n}(g)} e_k(x_j), \quad 0 \leqslant j < n.$$

Now we can conclude

$$Q_n(e_k)(x) = \frac{1}{c_{k,n}(g)} \cdot \frac{1}{n} \sum_{j=0}^{n-1} g(x - x_j) e_k(x_j) = \frac{c_{-k,n}(g(x - \cdot))}{c_{k,n}(g)}.$$

Recall the formula of aliasing for $w \in \mathcal{A}$:

$$c_{k,n}(w) = \sum_{r=-\infty}^{\infty} (w, e_{k+rn}).$$

Since

$$g(x-t) = \sum_{s=-\infty}^{\infty} d_s e_s(x) e_s(-t) = \sum_{s=-\infty}^{\infty} d_{-s} e_{-s}(x) e_s(t)$$

we get

$$c_{-k,n}(g(x-\cdot)) = \sum_{r=-\infty}^{\infty} d_{k-rn} e_{k-rn}(x) = \sum_{r=-\infty}^{\infty} d_{k+rn} e_{k+rn}(x),$$

which completes the proof in view of

$$Q_n(e_k)(x) = \frac{c_{-k,n}(g(x-\cdot))}{c_{k,n}(g)}.$$

□

Proposition 4.2. *Let $n = 2m+1$. Then $Q_n(u)$ possesses the discrete Fourier representation*

$$Q_n(u) = \sum_{k=-m}^{m} c_{k,n}(u) Q_n(e_k).$$

Proof. The trigonometric polynomial

$$T_m(u) = \sum_{k=-m}^{m} c_{k,n}(u)(e_k)$$

satisfies the interpolation conditions

$$v(x_j) = u(x_j), \qquad 0 \leqslant j < n.$$

Since $Q_n(T_m(u)) = Q_n(u)$ the linearity of Q_n completes the proof. □

Proposition 4.3. *The exponential interpolants are orthogonal:*
(i) $(Q_n(e_k), Q_n(e_l))_d = 0,\ 0 < |k-l| < n.$
(ii) $(Q_n(e_k), Q_n(e_l)) = 0,\ 0 < |k-l| < n.$
Moreover, the relations

(iii) $(Q_n(e_k), Q_n(e_k))_d = \frac{1}{c_{k,n}(g)^2} \sum\limits_{r=-\infty}^{\infty} d_{k+rn}$, $0 \leqslant k < n$,

(iv) $(Q_n(e_k), Q_n(e_k)) = \frac{1}{c_{k,n}(g)^2} \sum\limits_{r=-\infty}^{\infty} (d_{k+rn})^2$, $0 \leqslant k < n$

hold.

P r o o f. The relations of Proposition 4.3 follow from Fourier expansions of the exponential interpolants and the definition of the inner product of the periodic Hilbert space. □

5. Convergence of the exponential interpolants

We start with investigating the approximation order of the exponential interpolant in the norm of $\mathcal{H}_d$. For this purpose we introduce the quantities

$$D_{r,n} = c_{r,n}(g) - d_r = \sum_{l\neq 0}^{\infty} d_{r+ln}, \quad 0 \leqslant r \leqslant m,\ n = 2m+1.$$

Recall that

$$d_{-k} = d_k > 0 \quad (k \in \mathbb{Z}), \qquad d_k > d_{k+1} \quad (k \geqslant 0).$$

Then we have

$$D_{r,n} \leqslant \sum_{l=1}^{\infty} d_{lm} =: D_n, \qquad 0 \leqslant r \leqslant m,\ n = 2m+1.$$

It is obvious that

$$D_n > D_{n+1}, \qquad \lim_{n\to\infty} D_n = 0.$$

In many cases we have

$$D_n \leqslant \alpha \cdot d_m$$

where α is a constant independent of n. In particular this is true for our examples.

E x a m p l e 1.

$$D_n = \frac{1}{m^{2r}} \sum_{s=1}^{\infty} s^{-2r} = d_m \sum_{s=1}^{\infty} s^{-2r}.$$

E x a m p l e 2.

$$D_n = \sum_{s=1}^{\infty} \mathrm{e}^{-smb} \leqslant \frac{\mathrm{e}^{-mb}}{1-\mathrm{e}^{-b}} = d_m \frac{1}{1-\mathrm{e}^{-b}}.$$

Proposition 5.1. *The asymptotic relation*

$$\|e_k - Q_n(e_k)\|_d = \mathcal{O}\big(\sqrt{D_n}\big) \qquad (n \to \infty)$$

holds.

Proof. We assume without loss of generality that $0 \leqslant k \leqslant m$, $n = 2m + 1$. Recall that

$$Q_n(e_k) = \frac{1}{c_{k,n}(g)} \sum_{r=-\infty}^{\infty} d_{k+rn} e_{k+rn}(x).$$

Then we obtain

$$\begin{aligned}
(\|e_k - Q_n(e_k)\|_d)^2 &= \frac{1}{d_k}\Big|1 - \frac{d_k}{c_{k,n}(g)}\Big|^2 + \frac{1}{c_{k,n}(g)^2}\sum_{r\neq 0} d_{k+rn} \\
&= \frac{1}{d_k}\Big|\frac{D_{k,n}}{c_{k,n}(g)}\Big|^2 + \frac{1}{c_{k,n}(g)^2} D_{k,n} \\
&\leqslant \frac{D_n^2}{d_k^3} + \frac{D_n}{d_k^2} = \mathcal{O}\Big(\frac{D_n}{d_k^2}\Big) = \mathcal{O}(D_n)
\end{aligned}$$

as $n \to \infty$. □

Proposition 5.2. *The estimate*

$$\|e_k - Q_n(e_k)\| \leqslant \sqrt{2}\frac{D_n}{d_k}$$

holds for $|k| \leqslant m$, $n = 2m + 1$. In particular we have

$$\|e_k - Q_n(e_k)\| = \mathcal{O}(D_n)(n \to \infty).$$

Proof. As in the proof of Proposition 5.1 we have for $0 \leqslant k \leqslant m$ and $n = 2m+1$

$$\begin{aligned}
\|e_k - Q_n(e_k)\|^2 &= \Big|1 - \frac{d_k}{c_{k,n}(g)}\Big|^2 + \frac{1}{c_{k,n}(g)^2}\sum_{r\neq 0}(d_{k+rn})^2 \\
&= \Big|\frac{D_{k,n}}{c_{k,n}(g)}\Big|^2 + \frac{1}{c_{k,n}(g)^2}(D_{k,n})^2 \\
&\leqslant \frac{(D_n)^2}{d_k^2} + \frac{1}{d_k^2}(D_n)^2 = \frac{2 \cdot D_n^2}{d_k^2}.
\end{aligned}$$

□

Using the discrete Fourier representation of the optimal periodic interpolant $Q_n(u)$ we extend Proposition 5.2 to trigonometric polynomials.

Proposition 5.3. *Let $u \in \tau_m$ be a trigonometric polynomial of order m. Then the estimate*

$$\|u - Q_n(u)\| \leqslant \sqrt{2} D_n \|u\|_{d^2}$$

holds with $n = 2m+1$ and $(d^2)_k = (d_k)^2$.

Proof. It follows from Proposition 4.1 that

$$(e_k - Q_n(e_k), e_1 - Q_n(e_1)) = 0, \qquad 0 < |k-1| < n.$$

Taking into account Proposition 4.2 and Proposition 5.2 we can conclude

$$\begin{aligned} \|u - Q_n(u)\|^2 &= \sum_{k=-m}^{m} (u, e_k)[e_k - Q_n(e_k)]\|^2 \\ &= \sum_{k=-m}^{m} |(u, e_k)|^2 \|e_k - Q_n(e_k)\|^2 \\ &\leqslant \left[\sum_{k=-m}^{m} \frac{1}{d_k^2} |(u, e_k)|^2 \right] \cdot 2 \cdot D_n^2, \end{aligned}$$

i.e., we have shown

$$\|u - Q_n(u)\|^2 \leqslant 2 \cdot D_n^2 \|u\|_{d^2}.$$

□

6. Convergence in periodic Hilbert spaces

We start with a qualitative result. Recall first that

$$S_m(u) = \sum_{k=-m}^{m} (u, e_k) e_k$$

satisfies

$$\lim_{m \to \infty} \|u - S_m(u)\|_d = 0, \qquad u \in \mathcal{H}_d.$$

Proposition 6.1. *Let $u \in \mathcal{H}_d$. Then*

$$\lim_{n \to \infty} \|u - Q_n(u)\|_d = 0.$$

Proof. Given $\varepsilon > 0$ there exists $r \in \mathbb{N}$ such that

$$\|u - S_r(u)\|_d < \varepsilon.$$

It follows from Proposition 5.3, that there exists a $q \in \mathbb{N}$ depending only on r such that

$$\|S_r(u) - Q_n(S_r(u))\|_d < \varepsilon, \qquad n \geqslant q.$$

Since Q_n is an orthogonal projector on $\mathcal{H}_d$ we can conclude

$$\begin{aligned}\|u - Q_n(u)\|_d &\leqslant \|u - S_r(u)\|_d + \|S_r(u) - Q_n(S_r(u))\|_d + \|Q_n(u - S_r(u))\|_d \\ &\leqslant 2\|u - S_r(u)\|_d + \|S_r(u) - Q_n(S_r(u))\|_d.\end{aligned}$$

i.e., we have

$$\|u - Q_n(u)\|_d \leqslant 3\varepsilon, \qquad n \geqslant q.$$

This completes the proof of Proposition 6.1. □

Proposition 6.2. *Let $u \in \mathcal{H}_{d^2}$ and $n = 2m + 1$. Then*

$$\|Q_n(u - S_m(u))\| \leqslant D_n \|u - S_m(u)\|_{d^2}.$$

Proof. Put

$$v = u - S_m(u).$$

Then we have

$$(v, e_k) = 0, \qquad |k| \leqslant m.$$

Since

$$Q_n(v) = \sum_{k=-m}^{m} c_{k,n}(v) Q_n(e_k)$$

it follows from Proposition 4.3 that

$$\|Q_n(v)\|^2 = \sum_{k=-m}^{m} |c_{k,n}(v)|^2 \|Q_n(e_k)\|^2.$$

Recall that

$$(Q_n(e_k), Q_n(e_k)) = \frac{1}{c_{k,n}(g)^2} \sum_{r=-\infty}^{\infty} (d_{k+rn})^2, \qquad 0 \leqslant k < n,$$

which yields

$$\|Q_n(e_k)\|^2 \leqslant 1.$$

This shows

$$\|Q_n(v)\|^2 \leqslant \sum_{k=-m}^{m} |c_{k,n}(v)|^2.$$

Due to

$$(v, e_k) = 0, \qquad |k| \leqslant m$$

we obtain

$$\begin{aligned}
\sum_{k=-m}^{m} |c_{k,n}(v)|^2 &= \sum_{k=-m}^{m} \left| \sum_{r\neq 0} (v, e_{k+rn}) \right|^2 \\
&= \sum_{k=-m}^{m} \left| \sum_{r\neq 0} \frac{(v, e_{k+rn})}{d_{k+rn}} d_{k+rn} \right|^2 \\
&\leqslant \sum_{k=-m}^{m} \left(\sum_{r\neq 0} \frac{|(v, e_{k+rn})|^2}{(d_{k+rn})^2} \cdot \sum_{s\neq 0} (d_{k+sn})^2 \right) \\
&\leqslant \sum_{k=-m}^{m} \sum_{r\neq 0} \frac{|(v, e_{k+rn})|^2}{(d_{k+rn})^2} (D_n)^2 \\
&= (\|v\|_{d^2})^2 (D_n)^2,
\end{aligned}$$

i.e., we have shown that

$$\|Q_n(v)\|^2 \leqslant \sum_{k=-m}^{m} |c_{k,n}(v)|^2 \leqslant (\|v\|_{d^2})^2 (D_n)^2.$$

□

Proposition 6.3. *Let $u \in \mathcal{H}_{d^2}$. Then*

$$\|u - S_m(u)\| \leqslant D_n \|u - S_m(u)\|_{d^2}.$$

Proof. If $u \in \mathcal{H}_{d^2}$ then

$$\begin{aligned}
\|u - S_m(u)\|^2 &= \sum_{|k|>m} \frac{|(u, e_k)|^2}{(d_k)^2} (d_k)^2 \\
&\leqslant \sum_{|k|>m} \frac{|(u, e_k)|^2}{(d_k)^2} (d_m)^2 \\
&\leqslant \sum_{|k|>m} \frac{|(u, e_k)|^2}{(d_k)^2} (D_n)^2,
\end{aligned}$$

i.e., we have shown

$$\|u - S_m(u)\| \leqslant D_n \|u - S_m(u)\|_{d^2}.$$

□

We conclude with the main quantitative result which extends Proposition 5.3 to the periodic Hilbert space $\mathcal{H}_{d^2}$.

Proposition 6.4. *Let $u \in \mathcal{H}_{d^2}$. Then*

$$\|u - Q_n(u)\| \leqslant 4D_n \|u\|_{d^2}.$$

Proof. Taking into account Proposition 5.3 and Proposition 6.2 we can conclude

$$\begin{aligned}\|u - Q_n(u)\| &\leqslant \|u - S_m(u)\| + \|S_m(u) - Q_n(S_m(u)\| + \|Q_n(u - S_m(u))\| \\ &\leqslant D_n \|u - S_m(u)\|_{d^2} + \sqrt{2} D_n \|S_m(u)\|_{d^2} + D_n \|u - S_m(u)\|_{d^2} \\ &\leqslant 4D_n \|u\|_{d^2}.\end{aligned}$$

□

We apply Proposition 6.4 to obtain quantitative bounds for the mean square error of optimal periodic interpolation in our specific examples.

Example 1.

$$D_n = \frac{1}{m^{2r}} \sum_{s=1}^{\infty} s^{-2r} = d_m \sum_{s=1}^{\infty} s^{-2r}.$$

If u is a function of the periodic Sobolev space $\mathcal{W}^{2r}$ then Proposition 6.4 yields

$$\|u - Q_n(u)\| = \mathcal{O}(m^{-2r}), \qquad n = 2m + 1 \to \infty.$$

Example 2.

$$D_n = \sum_{s=1}^{\infty} \mathrm{e}^{-sbm} \leqslant \frac{\mathrm{e}^{-mb}}{1 - \mathrm{e}^{-b}} = d_m \frac{1}{1 - \mathrm{e}^{-b}}.$$

In this case u has to satisfy the condition

$$\sum_{k=-\infty}^{\infty} |(u, e_k)|^2 \cdot \mathrm{e}^{2b|k|} < \infty.$$

Then Proposition 6.4 implies

$$\|u - Q_n(u)\| = \mathcal{O}(\mathrm{e}^{-mb}), \qquad n = 2m + 1 \to \infty.$$

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