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EFFICIENT NUMERICAL SOLUTION OF MIXED FINITE ELEMENT DISCRETIZATIONS BY ADAPTIVE MULTILEVEL METHODS

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Summary. We consider mixed finite element discretizations of second order elliptic boundary value problems. Emphasis is on the efficient iterative solution by multilevel techniques with respect to an adaptively generated hierarchy of nonuniform triangulations. In particular, we present two multilevel solvers, the first one relying on ideas from domain decomposition and the second one resulting from mixed hybridization. Local refinement of the underlying triangulations is done by efficient and reliable a posteriori error estimators which can be derived by a defect correction in higher order ansatz spaces or by taking advantage of superconvergence results. The performance of the algorithms is illustrated by several numerical examples.

Keywords: elliptic boundary value problems, mixed finite element methods, adaptive multilevel techniques

AMS classification: 65N15, 65N30, 65N55

1. INTRODUCTION

In this paper, we are concerned with adaptive multilevel techniques for mixed finite element discretizations of second order elliptic boundary value problems. Mixed finite element methods which are based on the mixed or dual formulation are frequently used in such cases where the dual formulation does provide a more appropriate solution concept than that obtained by the standard primal formulation. We refer to the monograph of Brezzi and Fortin [11] and the survey article of Roberts and Thomas [23] for a detailed discussion and an extensive bibliography. For the efficient numerical solution we focus on multilevel iterative methods and adaptive techniques involving local refinement of the triangulations based on appropriate a posteriori error estimators. We remark that such techniques are well developed for the standard

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conforming finite element methods. In particular, we refer to the recent survey articles of Xu [27], Yserentant [29] and Zhang [30] concerning multilevel methods and to the pioneering work of Babuska and Rheinboldt [2], [3] and the recent papers of Bornemann et al. [7] and Verfürth [25] for the issue of error estimation. However, less work has been done in the framework of mixed methods. We mention a multigrid approach proposed by Brenner [10] and additive as well as multiplicative Schwarz iterations developed by Cai, Goldstein and Pasciak [12], Ewing and Wang [15], [16], [17] and Vassilevski and Wang [24]. As far as efficient and reliable a posteriori error estimators are concerned, we are only aware of a recent paper by Braess and Verfürth [9] on residual based techniques.

The paper is organized as follows: In Section 2, we briefly recall the mixed formulation and mixed discretization of second order elliptic boundary value problems. Then, in Section 3 we present a multilevel preconditioned cg-iteration acting on an appropriate subspace with a hierarchical type preconditioner that can be derived by subspace decompositions similar to those that have been used by Cai et al. in [12]. Local refinement of the triangulations relies on an error estimator obtained by the principle of defect correction in higher order mixed ansatz spaces and a localization by suitable hierarchical two-level splittings of these ansatz spaces. Section 4 is devoted to an alternative adaptive multilevel method based on the technique of mixed hybridization. In view of its equivalence with an extended nonconforming approximation, we use a preconditioned cg-iteration with a BPX-type multilevel preconditioner designed for nonconforming $P1$ approximations. In this case, an L^2 error estimator is used for local grid refinement that can be motivated by a superconvergence result for mixed hybridization. Finally, in Section 5 we present some illustrative numerical results.

2. Mixed formulation and mixed discretization

We consider the following boundary value problem for a linear second order elliptic differential operator

$$
\begin{aligned}
-\operatorname{div}(a\nabla u) + cu &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma := \partial\Omega
\end{aligned}
\tag{2.1}
$$

where Ω is a bounded, polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$ and $f \in L^2(\Omega)$. We further assume the coefficients to be a piecewise continuous, symmetric matrix-valued function $a = (a_{ij})_{i,j=1}^2$ and a piecewise continuous function c satisfying for

$x \in \Omega$

$$\alpha_0|\xi|^2 \leqslant \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \leqslant \alpha_1|\xi|^2,\ \xi \in \mathbb{R}^2,\ 0 < \alpha_0 \leqslant \alpha_1, \tag{2.2}$$
$$c_0 \leqslant c(x) \leqslant c_1,\ 0 \leqslant c_0 \leqslant c_1.$$

We remark that only for simplicity we have chosen homogeneous Dirichlet boundary conditions in (2.1). Other types of boundary conditions can be treated as well.

For $D \subseteq \Omega$ we denote by $(\cdot,\cdot)_{k,D}$, $k \geqslant 0$, the standard inner products on the Sobolev spaces $H^k(\Omega)$ and $(H^k(\Omega))^2$ and by $|\cdot|_{k,D}$ and $\|\cdot\|_{k,D}$ the associated Sobolev seminorms and norms, respectively. The subindex D will be omitted in case $D = \Omega$. Moreover, we refer to $H(\mathrm{div};\Omega)$ as the Hilbert space

$$H(\mathrm{div};\Omega) := \{\mathbf{q} \in (L^2(\Omega))^2;\ \mathrm{div}\,\mathbf{q} \in L^2(\Omega)\}$$

equipped with the inner product

$$(\mathbf{p},\mathbf{q})_{H(\mathrm{div};\Omega)} := (\mathbf{p},\mathbf{q})_0 + (\mathrm{div}\,\mathbf{p}, \mathrm{div}\,\mathbf{q})_0$$

and the associated graph norm $\|\cdot\|_{H(\mathrm{div};\Omega)} = (\cdot,\cdot)^{\frac{1}{2}}_{H(\mathrm{div};\Omega)}$.

Introducing the flux $\mathbf{j} = -a\nabla u$ as an additional unknown, the second order equation (2.1) can be formally written as a first order system whose variational formulation is commonly referred to as the mixed formulation of (2.1):

Find $(\mathbf{j}, u) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} a(\mathbf{j},\mathbf{q}) + b(\mathbf{q},u) &= 0,\ \ \mathbf{q} \in H(\mathrm{div};\Omega), \\ b(\mathbf{j},v) - c(u,v) &= l(v),\ \ v \in L^2(\Omega) \end{aligned} \tag{2.3}$$

where the bilinear forms a, b, c and the functional l are given by

$$\begin{aligned} a(\mathbf{q}_1,\mathbf{q}_2) &:= \int_\Omega a^{-1}\mathbf{q}_1\mathbf{q}_2\,\mathrm{d}\mathbf{x}, \quad \mathbf{q}_\nu \in H(\mathrm{div};\Omega),\ 1 \leqslant \nu \leqslant 2, \\ b(\mathbf{q},v) &:= -\int_\Omega \mathrm{div}\,\mathbf{q}v\,\mathrm{d}\mathbf{x}, \quad \mathbf{q} \in H(\mathrm{div};\Omega),\ v \in L^2(\Omega), \\ c(v_1,v_2) &:= \int_\Omega c v_1 v_2\,\mathrm{d}\mathbf{x}, \quad v_\nu \in L^2(\Omega),\ 1 \leqslant \nu \leqslant 2, \\ l(v) &:= -\int_\Omega f v\,\mathrm{d}\mathbf{x}, \quad v \in L^2(\Omega). \end{aligned}$$

Denoting by $A\colon H(\mathrm{div};\Omega) \to H(\mathrm{div};\Omega)^*$, $B\colon H(\mathrm{div};\Omega) \to L^2(\Omega)$ and $C\colon L^2(\Omega) \to L^2(\Omega)$ the operators associated with the bilinear forms a, b and c, respectively, in operator form the system (2.3) can be stated as follows

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \mathbf{j} \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ -f \end{pmatrix}.$$

It is easy to see that a, b and c are continuous bilinear forms with a being symmetric and coercive on $\mathrm{Ker}\, B$, c being symmetric, positive semidefinite and b satisfying the standard inf-sup condition (Brezzi condition). Since $\mathrm{Im}\ B = L^2(\Omega)$, we have $\mathrm{Ker}\, B^T = \{0\}$ and consequently, it follows from [11; § II, Thm. 1.2] that (2.3) admits a unique solution.

The mixed fixed finite element discretization of (2.1) is based on its mixed formulation (2.3). We consider a simplicial triangulation $\mathcal{T}_h$ of Ω and denote by $\mathcal{N}_h$ and $\mathcal{E}_h$ the sets of vertices and edges. In particular, we refer to e_ν^K, $1 \leqslant \nu \leqslant 3$, as the edges of $K \in \mathcal{T}_h$. Further, for $D \subseteq K$ we denote by $P_k(D)$, $k \geqslant 0$, the set of polynomials of degree $\leqslant k$ on D.

The flux space $H(\mathrm{div};\Omega)$ will be approximated by

$$RT_0(\Omega;\mathcal{T}_h) := \{\mathbf{q}_h \in H(\mathrm{div};\Omega);\ \mathbf{q}_h|_K \in RT_0(K),\ K \in \mathcal{T}_h\}$$

where $RT_0(K)$ stands for the lowest order Raviart-Thomas element

$$RT_0(K) := (P_0(K))^2 + \mathbf{x}P_0(K), \quad \mathbf{x} := (x_1, x_2)^T.$$

Note that any $\mathbf{q}_h|_K \in RT_0(K)$ is uniquely determined by the following three degrees of freedom

$$\int\limits_{\partial K} n\mathbf{q}_h|_K p\, d\sigma, \quad p \in R_0(\partial K)$$

where n is the outer normal on ∂K and $R_0(\partial K) := \{p \in L^2(\partial K);\, p|_{e_\nu^K} \in P_0(e_\nu^K),\ 1 \leqslant \nu \leqslant 3\}$.

The conformity of the approximation is guaranteed by specifying the basis fields in such a way that continuity of the normal components

$$(n \cdot \mathbf{q}_h)\,|_{e\cap K} = -\,(n' \cdot \mathbf{q}_h)\,|_{e\cap K'} \quad K \cap K' = e \in \mathcal{E}_h \cap \Omega \tag{2.4}$$

is satisfied across interelement boundaries where n and n' stand for the outer normal on $e \cap \partial K$ and $e \cap \partial K'$, respectively.

Observing $\mathrm{div}\,\mathbf{q}_K \in P_0(K)$, $K \in \mathcal{T}_h$, a natural choice for the approximation of the primal variable u is to use piecewise constants leading to the ansatz space

$$W_0(\Omega;\mathcal{T}_h) := \{v_h \in L^2(\Omega);\, v_h|_K \in P_0(K),\ K \in \mathcal{T}_h\}.$$

Then, the mixed discretization of (2.1) requires the computation of a pair $(\mathbf{j}_h, u_h) \in RT_0(\Omega; \mathcal{T}_h) \times W_0(\Omega; \mathcal{T}_h)$ such that

$$\begin{aligned} a(\mathbf{j}_h, \mathbf{q}_h) + b(\mathbf{q}_h, u_h) &= 0, \;\; \mathbf{q}_h \in RT_0(\Omega; \mathcal{T}_h), \\ b(\mathbf{j}_h, v_h) - c(u_h, v_h) &= l(v_h), \;\; v_h \in W_0(\Omega; \mathcal{T}_h). \end{aligned} \tag{2.5}$$

It can be easily verified that the bilinear form $a|_{RT_0(\Omega;\mathcal{T}_h)\times RT_0(\Omega;\mathcal{T}_h)}$ is uniformly coercive on $\operatorname{Ker} B|_{RT_0(\Omega;\mathcal{T}_h)}$ and that the bilinear form $b|_{RT_0(\Omega;\mathcal{T}_h)\times W_0(\Omega;\mathcal{T}_h)}$ satisfies the inf-sup condition (Babuska-Brezzi condition). Moreover, we have $\operatorname{Ker} B^T|_{W_0(\Omega;\mathcal{T}_h)} = \{0\}$ and hence, the mixed discretization (2.5) is uniquely solvable (cf. e.g. [11; § II, Prop. 2.11]).

3. Adaptive multilevel iterative techniques: Algorithm I

In this section, we will construct an adaptive multilevel iterative solver for the efficient numerical solution of the mixed discretization. The iterative solution will be based on a preconditioned cg-iteration acting on an appropriate subspace. We will use a hierarchical preconditioner constructed by means of suitable multilevel splittings of the mixed ansatz spaces $RT_0(\Omega; \mathcal{T}_h)$ and $W_0(\Omega; \mathcal{T}_h)$. We note that such splittings have been used by Cai, Goldstein and Pasciak [12], Ewing and Wang [15], [16], [17] and Vassilevski and Wang [24]. Moreover, an efficient and reliable a posteriori error estimator will be developed by the concept of defect correction in higher order ansatz spaces and an appropriate localization by hierarchical two-level splittings of these ansatz spaces. We remark that this concept is widely used in case of conforming or nonconforming finite element approximations (cf. e.g. [6], [7], [14], [19], [25]).

We assume that $(\mathcal{T}_k)_{k=0}^{l}$ is a hierarchy of possibly highly nonuniform triangulations generated by the meanwhile standard refinement process due to Bank et al. [5] (cf. also [4], [8], [14], [19], [28]). In particular, a triangle $K \in \mathcal{T}_k$, $0 \leqslant k \leqslant l$, either remains unrefined or is subdivided into four congruent subtriangles (regular or red refinement) or is bisected into two subtriangles (irregular or green refinement). The subtriangles are referred to as regular or irregular triangles, respectively. The following refinement rules have to be observed:

(R 1) Each vertex of $\mathcal{T}_{k+1}$ that does not belong to $\mathcal{T}_k$ is a vertex of a regular triangle.
(R 2) Irregular triangles must not be further refined.
(R 3) Only triangles $K \in \mathcal{T}_k$ of level k, i.e., triangles that do not belong to $\mathcal{T}_{k-1}$, may be refined for the construction of $\mathcal{T}_{k+1}$.

As a consequence of the refinement rules, each triangle $K \in \mathcal{T}_k$, $1 \leqslant k \leqslant l$, is geometrically similar either to an element of $\mathcal{T}_0$ or to an irregularly refined triangle of $\mathcal{T}_0$. Moreover, the sequence $(\mathcal{T}_k)_{k=0}^{l}$ is locally quasiuniform and can be uniquely reconstructed from the initial triangulation $\mathcal{T}_0$ and the final triangulation $\mathcal{T}_l$. (For details we refer to the literature cited above).

In this section, for simplicity we assume $c|_K = c_K > 0$, $K \in \mathcal{T}_0$. The case $c_K = 0$ for some $K \in \mathcal{T}_0$ can be treated under some slight modifications. The mixed discretization with respect to the finest triangulation $\mathcal{T}_l$ leads to the linear system

$$\mathcal{A}_l z_l := \begin{pmatrix} A_l & B_l^T \\ B_l & -C_l \end{pmatrix} \begin{pmatrix} \mathbf{j}_l \\ u_l \end{pmatrix} = \begin{pmatrix} 0 \\ -f_l \end{pmatrix}. \tag{3.1}$$

Note that $\mathcal{A}_l$ considered as an operator on $\left(L^2(\Omega)\right)^2 \times L^2(\Omega)$ with $D(\mathcal{A}_l) = RT_0(\Omega; \mathcal{T}_l) \times W_0(\Omega; \mathcal{T}_l)$ is symmetric, but indefinite.

For the solution of (3.1) we set $z_l := z_l^h + z_l^p$ where $z_l^p := (\mathbf{j}_l^p, u_l^p)$ is a particular solution of the inhomogeneous equation

$$B_l \mathbf{j}_l^p - C_l u_l^p = -f_l. \tag{3.2}$$

Then, $z_l^h := \left(\mathbf{j}_l^h, u_l^h\right)$ satisfies

$$\mathcal{A}_l z_l^h = \begin{pmatrix} A_l & B_l^T \\ B_l & -C_l \end{pmatrix} \begin{pmatrix} \mathbf{j}_l^h \\ u_l^h \end{pmatrix} = \begin{pmatrix} g_l \\ 0 \end{pmatrix} \tag{3.3}$$

where $g_l = -\left(A_l \mathbf{j}_l^p + B_l^T u_l^p\right)$. We refer to Z_l as the subspace

$$Z_l := \{(\mathbf{q}_l, v_l) \in RT_0(\Omega; \mathcal{T}_l) \times W_0(\Omega; \mathcal{T}_l)\,;\; B_l \mathbf{q}_l - C_l v_l = 0\}$$

and introduce another operator-valued 2×2 matrix

$$\tilde{\mathcal{A}}_l := \begin{pmatrix} \tilde{A}_l & B_l^T \\ B_l & -C_l \end{pmatrix}$$

which will serve as a preconditioner for the iterative solution of (3.3) by a preconditioned cg-iteration.

We further note that elimination of u_l^h from the first equation in (3.3) yields

$$D_l \mathbf{j}_l^h = g_l, \quad D_l := A_l + B_l^T C_l^{-1} B_l. \tag{3.4}$$

Likewise, if we consider (3.3) with $\tilde{\mathcal{A}}_l$ instead of $\mathcal{A}_l$, the same holds true with D_l replaced by

$$\tilde{D}_l := \tilde{A}_l + B_l^T C_l^{-1} B_l. \tag{3.5}$$

The iterative solution of (3.3) will be based on the following results:

Lemma 3.1. *Let $\mathcal{A}_l$, $\tilde{\mathcal{A}}_l$, D_l, $\tilde{D}_l$ and Z_l be given as above. Then there holds:*

(i) *The operator $\tilde{\mathcal{A}}_l$ is symmetric, positive definite on the subspace Z_l.*

(ii) *If $z_l^\nu := (\mathbf{j}_l^\nu, u_l^\nu)$, $\nu \geqslant 1$, are the iterates obtained by a preconditioned cg-iteration applied to (3.3) with preconditioner $\tilde{\mathcal{A}}_l$ and startiterate $z_l^0 = (\mathbf{j}_l^0, u_l^0)$, then $z_l^0 \in Z_l$ implies $z_l^\nu \in Z_l$, $\nu \geqslant 1$.*

(iii) *If there exist constants $0 < \gamma \leqslant \Gamma$ such that*

$$\gamma \left(\tilde{D}_l \mathbf{q}_l, \mathbf{q}_l\right)_0 \leqslant (D_l \mathbf{q}_l, \mathbf{q}_l)_0 \leqslant \Gamma \left(\tilde{D}_l \mathbf{q}_l, \mathbf{q}_l\right)_0, \quad \mathbf{q}_l \in RT_0(\Omega; \mathcal{T}_l),$$

then we also have

$$\gamma \left(\tilde{\mathcal{A}}_l z_l, z_l\right)_0 \leqslant (\mathcal{A}_l z_l, z_l)_0 \leqslant \Gamma \left(\tilde{\mathcal{A}}_l z_l, z_l\right)_0, \quad z_l \in Z_l.$$

P r o o f. For $z_l = (\mathbf{q}_l, v_l) \in Z_l$ we have

$$(\mathcal{A}_l z_l, z_l)_0 = \left(A_l \mathbf{q}_l + B_l^T v_l, \mathbf{q}_l\right)_0 = (D_l \mathbf{q}_l, \mathbf{q}_l)_0 \geqslant 0. \tag{3.6}$$

If $(D_l \mathbf{q}_l, \mathbf{q}_l)_0 = 0$, then $\mathbf{q}_l = 0$ and hence $v_l = 0$ because of $C_l v_l = 0$ which proves (i). Since (3.6) also holds true with $\mathcal{A}_l$, A_l and D_l replaced by $\tilde{\mathcal{A}}_l$, $\tilde{A}_l$ and $\tilde{D}_l$, respectively, (iii) is readily established. Finally, (ii) can be easily verified by induction. □

The preconditioner $\tilde{\mathcal{A}}_l$ will be constructed by means of appropriate multilevel decompositions of $RT_0(\Omega; \mathcal{T}_l)$ and $W_0(\Omega; \mathcal{T}_l)$ similar to those used in [12]. These decompositions will also provide an efficient tool for the computation of z_l^p satisfying (3.2). In particular, we denote by $\varrho_k \colon RT_0(\Omega; \mathcal{T}_l) \longmapsto RT_0(\Omega; \mathcal{T}_k)$, $0 \leqslant k \leqslant l$, the interpolation operators defined locally by

$$\int_{\partial K} n \cdot (\varrho_k \mathbf{q}_l - \mathbf{q}_l)\, p_0 \, \mathrm{d}\sigma = 0, \quad p_0 \in R_0(\partial K), \ K \in \mathcal{T}_k,$$

and we further denote by Π_k the $L^2(\Omega)$ projections onto $W_0(\Omega; \mathcal{T}_k)$, $0 \leqslant k \leqslant l$. Note that the operators ϱ_k and Π_k are related by

$$\operatorname{div}(\varrho_k \mathbf{q}_l) = \Pi_k (\operatorname{div} \mathbf{q}_l), \quad \mathbf{q}_l \in RT_0(\Omega; \mathcal{T}_l) \tag{3.7}$$

(cf. e.g. [11; § III, Prop. 3.7]). Observing that ϱ_l and Π_l correspond to the identities on $RT_0(\Omega; \mathcal{T}_l)$ and $W_0(\Omega; \mathcal{T}_l)$ and setting $\varrho_{-1} := 0$, $\Pi_{-1} := 0$ we consider the direct subspace decompositions

$$RT_0(\Omega;\mathcal{T}_l) = \bigoplus_{k=0}^{l} \widetilde{RT}_0(\Omega;\mathcal{T}_k), \tag{3.8}$$

$$W_0(\Omega;\mathcal{T}_l) = \bigoplus_{k=0}^{l} \widetilde{W}_0(\Omega;\mathcal{T}_k), \tag{3.9}$$

where $\widetilde{RT}_0(\Omega;\mathcal{T}_k) := (\varrho_k - \varrho_{k-1})\, RT_0(\Omega;\mathcal{T}_l)$ and $\widetilde{W}_0(\Omega;\mathcal{T}_k) := (\Pi_k - \Pi_{k-1}) \times W_0(\Omega;\mathcal{T}_l)$, $0 \leqslant k \leqslant l$.

In view of (3.7) we have $\operatorname{div} \widetilde{RT}_0(\Omega;\mathcal{T}_k) = \widetilde{W}_0(\Omega;\mathcal{T}_k)$, $0 \leqslant k \leqslant l$. Moreover, for $1 \leqslant k \leqslant l$ and $K \in \mathcal{T}_{k-1}$ it is easy to see that $\int_K \tilde{w}_k \,\mathrm{d}x = 0$, $\tilde{w}_k \in \widetilde{W}_0(\Omega;\mathcal{T}_k)$ and, as a consequence of Gauss' theorem, $\int_{\partial K} n \cdot \tilde{\mathbf{q}}_k \,\mathrm{d}\sigma = 0$, $\tilde{\mathbf{q}}_k \in \widetilde{RT}_0(\Omega;\mathcal{T}_k)$. In the light of this last observation, for $1 \leqslant k \leqslant l$ we will further decompose the level k subspace $\widetilde{RT}_0(\Omega;\mathcal{T}_k)$. For that purpose we denote by $\mathcal{T}_k^{\mathrm{ref}}$ and $\mathcal{E}_k^{\mathrm{ref}}$ the set of all refined triangles $K \in \mathcal{T}_k$ and refined edges $e \in \mathcal{E}_k$. We further refer to $S_1(\Omega;\mathcal{T}_k)$ as the standard conforming finite element space of continuous, piecewise linear finite elements with respect to $\mathcal{T}_k$, $0 \leqslant k \leqslant l$. We recall Yserentant's hierarchical decomposition of $S_1(\Omega;\mathcal{T}_l)$ according to

$$S_1(\Omega;\mathcal{T}_l) = \bigoplus_{k=0}^{l} \tilde{S}_1(\Omega;\mathcal{T}_k)$$

where $\tilde{S}_1(\Omega;\mathcal{T}_k) := (I_k - I_{k-1})\, S_1(\Omega;\mathcal{T}_l)$, $0 \leqslant k \leqslant l$, and I_k stands for the interpolation operator $(I_k u_l)(p) := u_l(p)$, $p \in \mathcal{N}_k$, $0 \leqslant k \leqslant l$, and $I_{-1} := 0$. We define

$$\begin{aligned} M_0^1(\Omega;\mathcal{T}_k) &:= \left\{ \mathbf{q}_k \in \widetilde{RT}_0(\Omega;\mathcal{T}_k)\,;\, n \cdot \mathbf{q}_k|_{\partial K} = 0, \quad K \in \mathcal{T}_{k-1}^{\mathrm{ref}} \right\}, \quad 1 \leqslant k \leqslant l, \\ M_0^2(\Omega;\mathcal{T}_k) &:= \operatorname{curl} \tilde{S}_1(\Omega;\mathcal{T}_k), \quad 1 \leqslant k \leqslant l. \end{aligned}$$

The subspaces $M_0^\nu(\Omega;\mathcal{T}_k)$, $1 \leqslant \nu \leqslant 2$, do provide a direct subspace decomposition of $\widetilde{RT}_0(\Omega;\mathcal{T}_k)$

$$\widetilde{RT}_0(\Omega;\mathcal{T}_k) = M_0^1(\Omega;\mathcal{T}_k) \oplus M_0^2(\Omega;\mathcal{T}_k), \quad 1 \leqslant k \leqslant l. \tag{3.10}$$

We note that $M_0^1(\Omega;\mathcal{T}_k)$ admits the splitting

$$M_0^1(\Omega;\mathcal{T}_k) = \bigoplus_{K \in \mathcal{T}_{k-1}^{\mathrm{ref}}} M_0^1(K;\mathcal{T}_k) \tag{3.11}$$

where $M_0^1(K;\mathcal{T}_k)$, $K = \bigcup_{\nu=1}^{\nu_K} K_\nu$, $K_\nu \in \mathcal{T}_k$, $\nu_K = 2$ or $\nu_K = 4$, is the $(\nu_K - 1)$-dimensional subspace spanned by the level k basis fields associated with the interior level k edges $e \subset \partial K_\nu \cap \operatorname{int} K$, $1 \leqslant \nu \leqslant \nu_K$.

Moreover, for $M_0^2(\Omega;\mathcal{T}_k)$ we have the decomposition

$$M_0^2(\Omega;\mathcal{T}_k) = \bigoplus_{e\in\mathcal{E}_{k-1}^{\mathrm{ref}}} M_0^2(e;\mathcal{T}_k) \tag{3.12}$$

where $M_0^2(e;\mathcal{T}_k)$ is the one-dimensional subspace spanned by the curl of the level k basis function $\varphi_k^{m_e} \in \tilde{S}_1(\Omega;\mathcal{T}_k)$ associated with the midpoint m_e of a refined edge $e \in \mathcal{E}_{k-1}^{\mathrm{ref}}$.

It is well known that subspace decompositions give rise to multilevel Schwarz iterations of both additive and multiplicative type and to associated preconditioners (cf. e.g. [27], [29], [30]). Here, we consider a preconditioner of hybrid type similar to that in [12] which is additive with respect to the "vertical" decomposition (3.8) and multiplicative with respect to the "horizontal" decomposition (3.10). The preconditioner for D_l from (3.4) is given by

$$\begin{aligned} \tilde{D}_l^{-1}\mathbf{q}_l &:= \Big(P_0 + \sum_{k=1}^{l} P_k\Big) D_l^{-1}\mathbf{q}_l, \quad \mathbf{q}_l \in RT_0(\Omega;\mathcal{T}_l), \\ P_k &:= P_{M_0^1} + \Big(I - P_{M_0^1}\Big)\Big(\sum_{e\in\mathcal{E}_{k-1}^{\mathrm{ref}}} P_e\Big)\Big(I - P_{M_0^1}\Big), \quad 1 \leqslant k \leqslant l, \end{aligned} \tag{3.13}$$

where P_0, $P_{M_0^1}$ and P_e are the projections onto $RT_0(\Omega;\mathcal{T}_0)$, $M_0^1(\Omega;\mathcal{T}_k)$ and $M_0^2(e;\mathcal{T}_k)$, $1 \leqslant k \leqslant l$, with respect to the weighted inner product

$$\langle \mathbf{q}, \mathbf{p}\rangle_{H(\mathrm{div};\Omega)} := \int_\Omega a^{-1}\mathbf{q}\cdot\mathbf{p}\,d\mathbf{x} + \int_\Omega c^{-1}\,\mathrm{div}\,\mathbf{q}\cdot\mathrm{div}\,\mathbf{p}\,d\mathbf{x}.$$

The preconditioner can be cheaply implemented. In particular, the evaluation of P_0 requires the solution of a saddle point problem on the initial triangulation $\mathcal{T}_0$. Moreover, in view of (3.11) and (3.12), on levels $1 \leqslant k \leqslant l$ for the evaluation of $P_{M_0^1}$ we have to solve local, low-dimensional saddle point problems associated with $K \in \mathcal{T}_{k-1}^{\mathrm{ref}}$ whereas the evaluation of P_e only requires the solution of a scalar equation.

For the determination of a particular solution z_l^p of (3.2) we compute $(\mathbf{j}_0, u_0)$ as the solution of the Raviart-Thomas approximation of (2.1) on level $k = 0$. Further, denoting by B_K the restriction of B to the subspaces $M_0^1(K;\mathcal{T}_k)$, $K \in \mathcal{T}_{k-1}^{\mathrm{ref}}$, $1 \leqslant k \leqslant l$, we compute $\tilde{\mathbf{j}}_k^K \in M_0^1(K;\mathcal{T}_k)$ as the solution of the local problems $B_K\tilde{\mathbf{j}}_k^K = -[(\Pi_k - \Pi_{k-1}) f]|_K$ and set $\tilde{\mathbf{j}}_k := \sum_K \tilde{\mathbf{j}}_k^K$. Obviously, the pair $(\mathbf{j}_l^p, u_0)$ with $\mathbf{j}_l^p = \mathbf{j}_0 + \sum_{k=1}^{l} \tilde{\mathbf{j}}_k$ satisfies the inhomogeneous equation (3.2).

R e m a r k 3.1. For a vanishing Helmholtz term $c \equiv 0$ in (2.1) the subspace Z_l corresponds to the subspace $RT_0^0(\Omega;\mathcal{T}_l) := \{\mathbf{q}_l \in RT_0(\Omega;\mathcal{T}_l);\ \operatorname{div}\mathbf{q}_l = 0\}$ of divergence-free vector fields. Observing that $RT_0^0(\Omega;\mathcal{T}_l)$ can be identified with $\operatorname{curl} S_1(\Omega;\mathcal{T}_l)$, we may compute $\mathbf{j}_l^h$ as the projection of $-\mathbf{j}_l^p$ onto $\operatorname{curl} S_1(\Omega;\mathcal{T}_l)$ with respect to the weighted $L^2(\Omega)$ inner product $(\cdot,\cdot)_{0;a^{-1}} := \left(a^{-1}\cdot,\cdot\right)_0$. Note that this particular case has been treated by Ewing and Wang in [15].

As far as the convergence of the preconditioned cg-iteration is concerned, it can be shown that in the asymptotic range the condition number of $\tilde{D}_l^{-1}D_l$ and hence, also that of $\tilde{\mathcal{A}}_l^{-1}\mathcal{A}_l|_{Z_l}$ grows quadratically with the refinement level and thus behaves in the same way as Yserentant's hierarchical preconditioner in case of the standard conforming $P1$ approximation of (2.1) (cf. e.g. [28]).

Theorem 3.2. *There exist constants $0 < \gamma \leqslant \Gamma$ depending only on the shape regularity of $\mathcal{T}_0$ and on the bounds for the coefficients of the elliptic operator such that for all $\mathbf{q}_l \in RT_0(\Omega;\mathcal{T}_l)$*

$$\gamma\,(l+1)^{-2}\,|\!|\!|\mathbf{q}_l|\!|\!|^2_{H(\operatorname{div};\Omega)} \leqslant \left\langle \tilde{D}_l^{-1}D_l\mathbf{q}_l,\mathbf{q}_l\right\rangle_{H(\operatorname{div};\Omega)} \leqslant \Gamma|\!|\!|\mathbf{q}_l|\!|\!|^2_{H(\operatorname{div};\Omega)} \tag{3.14}$$

where $|\!|\!|\cdot|\!|\!|_{H(\operatorname{div};\Omega)} := \langle\cdot,\cdot\rangle^{1/2}_{H(\operatorname{div};\Omega)}$.

P r o o f. In view of (3.13) the lower bound in (3.14) can be established by the P.L. Lions' type estimate

$$\gamma_0\,(l+1)^{-2}\sum_{k=0}^{l}|\!|\!|\tilde{\mathbf{q}}_k|\!|\!|^2_{H(\operatorname{div};\Omega)} \leqslant |\!|\!|\mathbf{q}_l|\!|\!|^2_{H(\operatorname{div};\Omega)} \tag{3.15}$$

where $\mathbf{q}_l = \sum\limits_{k=0}^{l}\tilde{\mathbf{q}}_k$, $\tilde{\mathbf{q}}_k := (\varrho_k - \varrho_{k-1})\,\mathbf{q}_l$, $0 \leqslant k \leqslant l$, and by means of the estimate

$$\gamma_1\left(|\!|\!|P_{M_0^1}\tilde{\mathbf{q}}_k|\!|\!|^2_{H(\operatorname{div};\Omega)} + \sum_{e\in\mathcal{E}_{k-1}^{\mathrm{ref}}}\left|\!\left|\!\left|P_e\left(I - P_{M_0^1}\right)\tilde{\mathbf{q}}_k\right|\!\right|\!\right|^2_{H(\operatorname{div};\Omega)}\right) \leqslant |\!|\!|\tilde{\mathbf{q}}_k|\!|\!|^2_{H(\operatorname{div};\Omega)} \tag{3.16}$$

which holds true for all $\tilde{\mathbf{q}}_k \in \widetilde{RT}_0(\Omega;\mathcal{T}_k)$, $1 \leqslant k \leqslant l$.

On the other hand, taking advantage of (3.13) the upper bound in (3.14) follows from the strengthened Cauchy-Schwarz inequality

$$\langle\tilde{\mathbf{p}}_j,\tilde{\mathbf{q}}_k\rangle_{H(\operatorname{div};\Omega)} \leqslant \Gamma_0 2^{-(k-j)/2}|\!|\!|\tilde{\mathbf{q}}_k|\!|\!|_{H(\operatorname{div};\Omega)}|\!|\!|\tilde{\mathbf{p}}_j|\!|\!|_{H(\operatorname{div};\Omega)} \tag{3.17}$$

where $\tilde{\mathbf{p}}_j \in \widetilde{RT}_0(\Omega;\mathcal{T}_j)$, $\tilde{\mathbf{q}}_k \in \widetilde{RT}_0(\Omega;\mathcal{T}_k)$, $0 \leqslant j \leqslant k \leqslant l$, and from the estimate

$$(3.18)\quad \|\tilde{\mathbf{q}}_k\|^2_{H(\mathrm{div};\Omega)} \leqslant \Gamma_1 \left(\|P_{M_0^1}\tilde{\mathbf{q}}_k\|^2_{H(\mathrm{div};\Omega)} + \sum_{e\in\mathcal{E}^{\mathrm{ref}}_{k-1}} \left\| P_e \left(I - P_{M_0^1} \right) \tilde{\mathbf{q}}_k \right\|^2_{H(\mathrm{div};\Omega)} \right)$$

where $\tilde{\mathbf{q}}_k \in \widetilde{RT}_0(\Omega;\mathcal{T}_k)$.

We note that the P.L. Lions' type result (3.15) can be deduced from the stability estimate

$$\|\varrho_k \mathbf{q}_l\|^2_{H(\mathrm{div};\Omega)} \leqslant C\,(l-k+1)\,\|\mathbf{q}_l\|^2_{H(\mathrm{div};\Omega)}, \mathbf{q}_l \in RT_0(\Omega;\mathcal{T}_l).$$

Further, the strengthened Cauchy-Schwarz inequality (3.17) can be established by means of the decomposition $\tilde{\mathbf{q}}_k = \tilde{\mathbf{q}}_k^1 + \operatorname{curl}\tilde{\Phi}_k$, $\tilde{\mathbf{q}}_k^1 \in M_0^1(\Omega;\mathcal{T}_k)$, $\tilde{\Phi}_k \in \tilde{S}_1(\Omega;\mathcal{T}_k)$, $1 \leqslant k \leqslant l$. Finally the estimates (3.16) and (3.18) can be verified by similar arguments as used in [12]. For details we refer to [21]. □

Remark 3.2. The lower and upper bounds in (3.14) improve on those in [12] where a lower bound of order $O\left(2^{-(l+1)}\right)$ and an upper bound of order $O\,(l+1)$ could be established.

Local refinement of the triangulations will be performed based on a reliable and efficient a posteriori error estimator for the total error in the flux which can be derived by the principle of defect correction in higher order ansatz spaces and an appropriate localization by hierarchical two-level decompositions of these ansatz spaces (cf. e.g. [6], [7], [14] in case of conforming $P1$ approximations and [19], [26] for nonconforming $P1$ approximations).

We denote by $(\mathbf{j}, u) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ the unique solution of the mixed formulation (2.3) and by $(\tilde{\mathbf{j}}_0, \tilde{u}_0)$ an iterative approximation of the lowest order Raviart-Thomas approximation $(\mathbf{j}_0, u_0) \in RT_0(\Omega;\mathcal{T}_l) \times W_0(\Omega;\mathcal{T}_l)$. It can be easily seen that the total error $(\mathbf{j} - \tilde{\mathbf{j}}_0, u - \tilde{u}_0)$ satisfies the saddle point problem

$$(3.19)\qquad \begin{aligned} a(\mathbf{j}-\tilde{\mathbf{j}}_0,\mathbf{q}) + b(\mathbf{q}, u-\tilde{u}_0) &= r(\mathbf{q}),\ \ \mathbf{q} \in H(\mathrm{div};\Omega), \\ b(\mathbf{j}-\tilde{\mathbf{j}}_0, v) - c(u-\tilde{u}_0, v) &= (f - f^0, v)_0,\ \ v \in L^2(\Omega) \end{aligned}$$

where r stands for the residuum $r(\mathbf{q}) = -\left(a(\tilde{\mathbf{j}}_0,\mathbf{q}) + b(\mathbf{q},\tilde{u}_0)\right)$, $\mathbf{q} \in H(\mathrm{div};\Omega)$, and f^0 is the L^2 projection of f onto $W_0(\Omega;\mathcal{T}_l)$.

The defect problem (3.19) will be approximated with respect to the higher order mixed ansatz spaces

$$\begin{aligned} RT_1(\Omega;\mathcal{T}_l) &:= \left\{\mathbf{q}_l \in H(\mathrm{div};\Omega);\ \mathbf{q}_l|_K \in RT_1(K),\ K \in \mathcal{T}_l\right\}, \\ W_1(\Omega;\mathcal{T}_l) &:= \left\{v_l \in L^2(\Omega);\ v_l\big|_K \in P_1(K),\ K \in \mathcal{T}_l\right\} \end{aligned}$$

where $RT_1(K)$ stands for the Raviart-Thomas element

$$RT_1(K) := (P_1(K))^2 + \mathbf{x}P_1(K), \quad \mathbf{x} := (x_1, x_2)^T.$$

We note that any vector field $\mathbf{q} \in RT_1(K)$ is uniquely determined by the following eight degrees of freedom

$$\begin{aligned} &\int_{\partial K} n \cdot \mathbf{q} p \, d\sigma, \quad p \in R_1(\partial K), \\ &\int_K \mathbf{q} \cdot \mathbf{p} \, d\mathbf{x}, \quad \mathbf{p} \in (P_0(K))^2 \end{aligned} \tag{3.20}$$

where $R_1(\partial K) := \left\{p \in L^2(\partial K);\, p\big|_{e_\nu^K} \in P_1(e_\nu^K), \quad 1 \leqslant \nu \leqslant 3\right\}$.

In particular, the approximation of (3.19) requires the computation of a pair $(e_{\mathbf{j}}, e_u) \in RT_1(\Omega; \mathcal{T}_l) \times W_1(\Omega; \mathcal{T}_l)$ such that

$$\begin{aligned} a(e_{\mathbf{j}}, \mathbf{q}_l) + b(\mathbf{q}_l, e_u) &= r(\mathbf{q}_l), \ \mathbf{q}_l \in RT_1(\Omega; \mathcal{T}_l), \\ b(e_{\mathbf{j}}, v_l) - c(e_u, v_l) &= (f - f^0, v_l)_0, \ v_l \in W_1(\Omega; \mathcal{T}_l). \end{aligned} \tag{3.21}$$

Under the saturation assumption

$$|\!|\!| \mathbf{j} - \mathbf{j}_1 |\!|\!|_{H(\mathrm{div};\Omega)} \leqslant \beta |\!|\!| \mathbf{j} - \mathbf{j}_0 |\!|\!|_{H(\mathrm{div};\Omega)}, \quad \beta > 0 \tag{3.22}$$

and the additional assumption

$$|\!|\!| \mathbf{j} - \mathbf{j}_0 |\!|\!|_{H(\mathrm{div};\Omega)} \leqslant \delta |\!|\!| \mathbf{j} - \tilde{\mathbf{j}}_0 |\!|\!|_{H(\mathrm{div};\Omega)}, \quad \delta > 0,\ \delta\beta < 1, \tag{3.23}$$

which are supported by [11; § II, Prop. 2.6 and § III, Prop. 3.9], it is easily seen that

$$(1 + \delta\beta)^{-1} |\!|\!| e_{\mathbf{j}} |\!|\!|_{H(\mathrm{div};\Omega)} \leqslant |\!|\!| \mathbf{j} - \tilde{\mathbf{j}}_0 |\!|\!|_{H(\mathrm{div};\Omega)} \leqslant (1 - \delta\beta)^{-1} |\!|\!| e_{\mathbf{j}} |\!|\!|_{H(\mathrm{div};\Omega)}. \tag{3.24}$$

The practical computation of an error estimator for the total error in the flux relies on a hierarchical two-level splitting of the mixed ansatz spaces $RT_1(\Omega; \mathcal{T}_l)$ and $W_1(\Omega; \mathcal{T}_l)$. For that purpose, for $0 \leqslant \nu \leqslant 1$ we denote by Π^ν the L^2 projections onto $W_\nu(\Omega; \mathcal{T}_l)$ and by ϱ^ν the interpolation operators $\varrho^\nu \colon RT_1(\Omega; \mathcal{T}_l) \longmapsto RT_\nu(\Omega; \mathcal{T}_l)$ given locally by

$$\begin{aligned} \int_{\partial K} n \cdot \varrho^\nu \mathbf{q}_l p_\nu \, d\sigma &= \int_{\partial K} n \cdot \mathbf{q}_l p_\nu \, d\sigma, \quad p_\nu \in R_\nu(\partial K),\ 0 \leqslant \nu \leqslant 1, \\ \int_K \varrho^1 \mathbf{q}_l \cdot \mathbf{p} \, d\mathbf{x} &= \int_K \mathbf{q}_l \cdot \mathbf{p} \, d\mathbf{x}, \quad \mathbf{p} \in (P_0(K))^2. \end{aligned}$$

Then, we have the following two-level decompositions

$$(3.25)\quad RT_1(\Omega;\mathcal{T}_l) = RT_0(\Omega;\mathcal{T}_l) \oplus \widetilde{RT}_1(\Omega;\mathcal{T}_l), \quad \widetilde{RT}_1(\Omega;\mathcal{T}_l) := \left(\varrho^1 - \varrho^0\right) RT_1(\Omega;\mathcal{T}_l),$$

$$(3.26)\quad W_1(\Omega;\mathcal{T}_l) = W_0(\Omega;\mathcal{T}_l) \oplus \widetilde{W}_1(\Omega;\mathcal{T}_l), \quad \widetilde{W}_1(\Omega;\mathcal{T}_l) := \left(\Pi^1 - \Pi^0\right) W_1(\Omega;\mathcal{T}_l).$$

Recalling the hierarchical two-level splitting $S_2(\Omega;\mathcal{T}_l) = S_1(\Omega;\mathcal{T}_l) \oplus \tilde{S}_2(\Omega;\mathcal{T}_l)$ of the conforming finite element space of continuous, piecewise quadratic elements where $\tilde{S}_2(\Omega;\mathcal{T}_l)$ stands for the hierarchical surplus spanned by the basis functions associated with the midpoints of the edges (cf. e.g. [14]), the hierarchical surplus $\widetilde{RT}_1(\Omega;\mathcal{T}_l)$ can be further decomposed according to

$$(3.27)\qquad \widetilde{RT}_1(\Omega;\mathcal{T}_l) = \widetilde{RT}_1^0(\Omega;\mathcal{T}_l) \oplus \widetilde{RT}_1^1(\Omega;\mathcal{T}_l)$$

where

$$(3.28)\qquad \widetilde{RT}_1^0(\Omega;\mathcal{T}_l) := \operatorname{curl} \tilde{S}_2(\Omega;\mathcal{T}_l),$$

$$(3.29)\quad \widetilde{RT}_1^1(\Omega;\mathcal{T}_l) := \left\{ \mathbf{q}_l \in RT_1(\Omega;\mathcal{T}_l);\ \int_{\partial K} n \cdot \mathbf{q}_l p \, d\sigma = 0,\ p \in R_1(\partial K),\ K \in \mathcal{T}_l \right\}.$$

Note that $\widetilde{RT}_1^0(\Omega;\mathcal{T}_l)$ represents the divergence-free subspace of the hierarchical surplus.

On the other hand, denoting by $\widetilde{RT}_1(K;K_{\text{int}})$ the two-dimensional subspace spanned by the basis fields associated with the "interior" degrees of freedom (3.20), it follows that

$$(3.30)\qquad \widetilde{RT}_1(\Omega;\mathcal{T}_l) = \bigoplus_{K \in \mathcal{T}_l} \widetilde{RT}_1(K;K_{\text{int}}).$$

Based on the splittings (3.25), (3.26) and (3.27) an error estimator $\tilde{e}_{\mathrm{j}} = \tilde{e}_{\mathrm{j}}^0 + \tilde{e}_{\mathrm{j}}^*$ with $\tilde{e}_{\mathrm{j}}^* \in \widetilde{RT}_1^1(\Omega;\mathcal{T}_l)$ and $\tilde{e}_{\mathrm{j}}^0 \in \widetilde{RT}_1^0(\Omega;\mathcal{T}_l)$ will be determined in two steps:

Firstly, in view of (3.30) we compute a pair $(\tilde{e}_{\mathrm{j}}^*, \tilde{e}_u) \in \widetilde{RT}_1^1(\Omega;\mathcal{T}_l) \times \widetilde{W}_1(\Omega;\mathcal{T}_l)$ by the solutions of the local saddle point problems

$$\begin{aligned} a(\tilde{e}_{\mathrm{j}}^*|_K, \mathbf{q}_k) + b(\mathbf{q}_k, \tilde{e}_u|_K) &= r(\mathbf{q}_k), \quad \mathbf{q}_k \in \widetilde{RT}_1(K;K_{\text{int}}), \\ b(\tilde{e}_{\mathrm{j}}^*|_K, v_l|_K) - c(\tilde{e}_u|_K, v_l|_K) &= \left((\Pi^1 - \Pi^0)f, v_l|_K\right)_0, \quad v_l \in \widetilde{W}_1(\Omega;\mathcal{T}_l). \end{aligned}$$

Secondly, observing (3.28) we compute the divergence-free part $\tilde{e}_{\mathrm{j}}^0 \in \widetilde{RT}_1^0(\Omega;\mathcal{T}_l)$ of the error estimator by the solution of the local 3×3 problems

$$\int_K a^{-1} \tilde{e}_{\mathrm{j}}^0|_K \cdot \operatorname{curl} \varphi_K^{m_{e_\nu}} \, d\mathbf{x} = -\int_K a^{-1} \tilde{e}_{\mathrm{j}}^* \cdot \operatorname{curl} \varphi_K^{m_{e_\nu}} \, dx + r|_K(\operatorname{curl} \varphi_K^{m_{e_\nu}}), \quad 1 \leqslant \nu \leqslant 3$$

where $\tilde{e}_{\mathbf{j}}^0|_K \in \operatorname{span}\{\operatorname{curl}\varphi_K^{m_{e_\nu}}\,;\, 1 \leqslant \nu \leqslant 3\}$ and $\varphi_K^{m_{e_\nu}} \in \tilde{S}_2(\Omega;\mathcal{T}_l)$, $1 \leqslant \nu \leqslant 3$, stand for the basis functions associated with the midpoints m_{e_ν} of the three edges e_ν of $K \in \mathcal{T}_l$.

Then there holds:

Theorem 3.3. *Under the assumptions* (3.22), (3.23) *there exist positive constants* $0 < \kappa_0 \leqslant \kappa_1$ *depending on* β *and* δ *such that*

$$\kappa_0 |\!|\!| \tilde{e}_{\mathbf{j}} |\!|\!|_{H(\operatorname{div};\Omega)} \leqslant |\!|\!| \mathbf{j} - \tilde{\mathbf{j}}_0 |\!|\!|_{H(\operatorname{div};\Omega)} \leqslant \kappa_1 |\!|\!| \tilde{e}_{\mathbf{j}} |\!|\!|_{H(\operatorname{div};\Omega)}. \tag{3.31}$$

Proof. We will only sketch the proof and refer to [21] for details. Denoting by P the projection of $RT_1(\Omega;\mathcal{T}_l)$ onto $RT_0(\Omega;\mathcal{T}_l)$ and by $\tilde{P}_\nu$, $0 \leqslant \nu \leqslant 1$, the projections of $\widetilde{RT}_1(\Omega;\mathcal{T}_l)$ onto $\widetilde{RT}_1^{\nu}(\Omega;\mathcal{T}_l)$ with respect to $\langle\cdot,\cdot\rangle_{H(\operatorname{div};\Omega)}$, we have $\tilde{e}_{\mathbf{j}}^* = \tilde{P}_1 \hat{e}_{\mathbf{j}}$, $\tilde{e}_{\mathbf{j}}^0 = \tilde{P}_0\left(I - \tilde{P}_1\right)\hat{e}_{\mathbf{j}}$ where $\hat{e}_{\mathbf{j}} := P e_{\mathbf{j}}$. The assertion follows from the norm equivalences

$$\begin{gathered}\left(1-\eta_0^2\right)^{1/2} |\!|\!| e_{\mathbf{j}} |\!|\!|_{H(\operatorname{div};\Omega)} \leqslant |\!|\!| \hat{e}_{\mathbf{j}} |\!|\!|_{H(\operatorname{div};\Omega)} \leqslant |\!|\!| e_{\mathbf{j}} |\!|\!|_{H(\operatorname{div};\Omega)},\\ (1-\eta_1)\, |\!|\!| \hat{e}_{\mathbf{j}} |\!|\!|_{H(\operatorname{div};\Omega)} \leqslant |\!|\!| \tilde{e}_{\mathbf{j}} |\!|\!|_{H(\operatorname{div};\Omega)} \leqslant |\!|\!| \hat{e}_{\mathbf{j}} |\!|\!|_{H(\operatorname{div};\Omega)}\end{gathered}$$

which can be deduced by means of the Cauchy-Schwarz inequalities

$$\begin{aligned}&|\langle \mathbf{q}_1, \mathbf{q}_2\rangle_{H(\operatorname{div};\Omega)}| \leqslant \eta_0^2 |\!|\!| \mathbf{q}_1 |\!|\!|_{H(\operatorname{div};\Omega)} |\!|\!| \mathbf{q}_2 |\!|\!|_{H(\operatorname{div};\Omega)},\\ &\qquad\qquad\qquad\qquad \mathbf{q}_1 \in RT_1(\Omega;\mathcal{T}_l),\ \mathbf{q}_2 \in RT_0(\Omega;\mathcal{T}_l),\\ &|\langle \mathbf{q}_1, \mathbf{q}_2\rangle_{H(\operatorname{div};\Omega)}| \leqslant \eta_1^2 |\!|\!| \mathbf{q}_1 |\!|\!|_{H(\operatorname{div};\Omega)} |\!|\!| \mathbf{q}_2 |\!|\!|_{H(\operatorname{div};\Omega)},\\ &\qquad\qquad\qquad\qquad \mathbf{q}_\nu \in \widetilde{RT}_1^{\nu}(\Omega;\mathcal{T}_l),\ 0 \leqslant \nu \leqslant 1\end{aligned}$$

where $\eta_\nu^2 < 1$, $0 \leqslant \nu \leqslant 1$. □

Remark 3.3. Based on the estimates (3.31), as a refinement criterion we use a meanvalue strategy as described e.g. in [14].

4. Adaptive multilevel iterative techniques: Algorithm II

In this section, we will present another multilevel method which is based on the technique of mixed hybridization. This technique has been originally developed by Fraeijs de Veubeke [18] and further investigated by Arnold and Brezzi [1] (cf. also [11] and [23]). The idea is to eliminate the continuity constraints (2.4) for the normal components on the interelement boundaries from the ansatz space $RT_0(\Omega;\mathcal{T}_h)$. This leads to the nonconforming Raviart-Thomas approximation

$$RT_0^{-1}(\Omega;\mathcal{T}_h) := \left\{\mathbf{q}_h \in \left(L^2(\Omega)\right)^2 \, ; \, \mathbf{q}_h|_K \in RT_0(K), \;\, K \in \mathcal{T}_h\right\}.$$

Note that the dimension of $RT_0^{-1}(\Omega;\mathcal{T}_h)$ exceeds that of its conforming counterpart $RT_0(\Omega;\mathcal{T}_h)$ by the number of interelement boundaries, since now two basis fields are associated with each interior edge. Instead, the continuity constraints are taken care of by Lagrangian multipliers from the multiplier space

$$M_0(\mathcal{E}_h) := \left\{\mu_h \in \left(L^2(\mathcal{E}_h)\right)^2 \, ; \, \mu_h|_e \in P_0(e), \;\, e \in \mathcal{E}_h \cap \Omega, \;\, \mu_h|_e = 0, \;\, e \in \mathcal{E}_h \cap \Gamma\right\}.$$

Then, the nonconforming mixed discretization of (2.3) requires the computation of $(\mathbf{j}_h, u_h, \lambda_h) \in RT_0^{-1}(\Omega;\mathcal{T}_h) \times W_0(\Omega;\mathcal{T}_h) \times M_0(\mathcal{E}_h)$ such that

$$\begin{aligned} \hat{a}(\mathbf{j}_h, \mathbf{q}_h) + \hat{b}(\mathbf{q}_h, u_h) + \hat{d}(\lambda_h, \mathbf{q}_h) &= 0, \;\, \mathbf{q}_h \in RT_0^{-1}(\Omega;\mathcal{T}_h), \\ \hat{b}(\mathbf{j}_h, v_h) - \hat{c}(u_h, v_h) &= -(f, v_h)_0, \;\, v_h \in W_0(\Omega;\mathcal{T}_h), \\ \hat{d}(\mu_h, \mathbf{j}_h) &= 0, \;\, \mu_h \in M_0(\mathcal{E}_h) \end{aligned} \tag{4.1}$$

where the bilinear forms $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are given by

$$\hat{a} := \sum_{K\in\mathcal{T}_h} a|_K, \;\, \hat{b} := \sum_{K\in\mathcal{T}_h} b|_K, \;\, \hat{c} := c \text{ and}$$

$$\hat{d}(\mu_h, \mathbf{q}_h) := \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu_h n \cdot \mathbf{q}_h \, d\sigma, \quad \mu_h \in M_0(\mathcal{E}_h), \;\, \mathbf{q}_h \in RT_0^{-1}(\Omega;\mathcal{T}_h).$$

The distinctive feature of mixed hybridization is to eliminate the discrete flux $\mathbf{j}_h$ from (4.1) and to take advantage of the equivalence of the resulting variational equations in $(u_h, \lambda_h) \in W_0(\Omega;\mathcal{T}_h) \times M_0(\mathcal{E}_h)$ with an extended nonconforming approximation. In particular, we refer to $CR_1(\Omega;\mathcal{T}_h)$ as the lowest order nonconforming Crouzeix-Raviart ansatz space

$$\begin{aligned} CR_1(\Omega;\mathcal{T}_h) := \; \{v_h \in L^2(\Omega) \, ; \, v_h|_K \in P_1(K), \;\, K \in \mathcal{T}_h, \quad v_h|_K(m_e) = v_h|_{K'}(m_e), \\ e = K \cap K' \in \mathcal{E}_h \cap \Omega, v_h(m_e) = 0, e \in \mathcal{E}_h \cap \Gamma\} \end{aligned}$$

where m_e stands for the midpoint of an edge $e \in \mathcal{E}_h$.

We further denote by $B_3(\Omega;\mathcal{T}_l)$ the space of cubic bubble functions

$$B_3(\Omega;\mathcal{T}_h) := \; \left\{v_h \in L^2(\Omega) \, ; \, v_h|_K \in P_3(K), \;\, v_h|_{\partial K} = 0, \;\, K \in \mathcal{T}_h\right\},$$

and we set

$$N(\Omega;\mathcal{T}_h) := CR_1(\Omega;\mathcal{T}_h) \oplus B_3(\Omega;\mathcal{T}_h).$$

Then, if P_h and Π_h are the L^2 projections onto $W_0(\Omega;\mathcal{T}_h)$ and $M_0(\mathcal{E}_h)$, respectively, it can be shown that there exists a unique $\psi_h \in N(\Omega;\mathcal{T}_h)$ such that $u_h = P_h\psi_h$ and $\lambda_h = \Pi_h\psi_h$ (cf. e.g. [1; Lemmas 2.3, 2.4]). Moreover, ψ_h is the unique solution of the variational equation

$$a_{N_h}(\psi_h,\eta_h) = (P_h f,\eta_h)_0, \quad \eta_h \in N(\Omega;\mathcal{T}_h) \tag{4.2}$$

where the bilinear form $a_{N_h}\colon N(\Omega;\mathcal{T}_h) \times N(\Omega;\mathcal{T}_h) \longmapsto \mathbb{R}$ is given by

$$a_{N_h}(\xi_h,\eta_h) := \sum_{K\in\mathcal{T}_h} \int_K \left(\hat{P}_{a^{-1}}(a\nabla\xi_h)\cdot\nabla\eta_h + P_h(c\xi_h)\eta_h\right) \mathbf{dx}$$

and $\hat{P}_{a^{-1}}$ denotes the L^2 projection onto $RT_0^{-1}(\Omega;\mathcal{T}_h)$ with respect to the weighted L^2 inner product $(\cdot,\cdot)_{0,a^{-1}}$.

The extended nonconforming approximation (4.2) will be solved by a multilevel preconditioned cg-iteration. For that purpose we assume $(\mathcal{T}_k)_{k=0}^{l}$ to be a hierarchy of triangulations generated according to the refinement rules (R1), (R2) and (R3) as in Section 2. In particular, we suppose the coarsest triangulation $\mathcal{T}_0$ to be such that there exists a constant $\gamma > 0$ with

$$\alpha_0 - \gamma h_0^2 c_1 \geqslant 0 \tag{4.3}$$

where α_0 and c_1 are from (2.2) and $h_0 := \max\limits_{K\in\mathcal{T}_0} \operatorname{diam}(K)$. The construction of the preconditioner for (4.2) with respect to the finest triangulation $\mathcal{T}_l$ is based on the natural splitting of $N(\Omega;\mathcal{T}_l)$ into the standard nonconforming part $CR_1(\Omega;\mathcal{T}_l)$ involving the bilinear form

$$a_{CR_l}(u_l^{CR},v_l^{CR}) := \sum_{K\in\mathcal{T}_l} \int_K \left(a\nabla u_l^{CR}\cdot\nabla v_l^{CR} + c u_l^{CR} v_l^{CR}\right) dx \tag{4.4}$$

where u_l^{CR}, $v_l^{CR} \in CR_1(\Omega;\mathcal{T}_l)$, and the "bubble" part $B_3(\Omega;\mathcal{T}_l)$ giving rise to the bilinear form

$$a_{B_l}(w_l^B,z_l^B) := \sum_{K\in\mathcal{T}_l} \int_K \left(a\hat{P}(\nabla w_l^B)\cdot\hat{P}(\nabla z_l^B) + cP_{h_l}(w_l^B)P_{h_l}(z_l^B)\right) dx \tag{4.5}$$

where w_l^B, $z_l^B \in B_3(\Omega;\mathcal{T}_l)$ and $\hat{P}$ is the L^2 projection onto $RT_0^{-1}(\Omega;\mathcal{T}_h)$. We define $\tilde{a}_{N_l}\colon N(\Omega;\mathcal{T}_l) \times N(\Omega;\mathcal{T}_l) \longmapsto \mathbb{R}$ by

$$\tilde{a}_{N_l}(\xi_l,\eta_l) := a_{CR_l}(u_l^{CR},v_l^{CR}) + a_{B_l}(w_l^B,z_l^B)$$

where $\xi_l := u_l^{CR} + w_l^B$, $\eta_l := v_l^{CR} + z_l^B$, u_l^{CR}, $v_l^{CR} \in CR_1(\Omega;\mathcal{T}_l)$ and w_l^B, $z_l^B \in B_3(\Omega;\mathcal{T}_l)$.

Denoting by A_{CR_l} and A_{B_l} the operators associated with the bilinear forms a_{CR_l} and a_{B_l}, respectively, we note that A_{B_l} can be represented by a diagonal matrix and thus is easily invertible. On the other hand, the operator A_{CR_l} will be preconditioned by a multilevel preconditioner of BPX-type which has been developed by the authors in [20], [26] (note that a related BPX preconditioner for nonconforming $P1$ approximations has been proposed by Oswald [22]). If $\mathcal{T}_{l+1}$ is the triangulation obtained from $\mathcal{T}_l$ by regular refinement of all $K \in \mathcal{T}_l$, the idea is to identify $CR_1(\Omega;\mathcal{T}_l)$ with a closed subspace of $S_1(\Omega;\mathcal{T}_{l+1})$ by means of the pseudo-interpolant

$$\left(P_l^{CR} u_l\right)(p) = \begin{cases} u_l^{CR}(p), & \text{if } p = m_e \\ \nu_p^{-1} \sum\limits_{\nu=1}^{\nu_p} u_l^{CR}(m_{e_\nu}^p), & \text{if } p \neq m_e \end{cases}$$

where $m_{e_\nu}^p$, $1 \leqslant \nu \leqslant \nu_p$, are the midpoints of those edges $e \in \mathcal{E}_l \cap \Omega$ having $p \in \mathcal{N}_{l+1} \cap \Omega$ as a common vertex (cf. [13] for a related domain decomposition approach). We denote by C_{BPX} the standard BPX preconditioner with respect to the hierarchy $S_1(\Omega,\mathcal{T}_0) \subset \cdots S_1(\Omega;\mathcal{T}_l) \subset S_1(\Omega;\mathcal{T}_{l+1}) \supset P_l^{CR}(CR_1(\Omega;\mathcal{T}_l))$ of conforming ansatz spaces. Further, we refer to $I_{l+1}^S : S_1(\Omega;\mathcal{T}_{l+1}) \longmapsto CR_1(\Omega;\mathcal{T}_l)$ as the mapping defined by $\left(I_{l+1}^S u_{l+1}\right)(m_e) := u_{l+1}(m_e)$, $u_{l+1} \in S_1(\Omega;\mathcal{T}_{l+1})$. Then, a suitable BPX preconditioner for A_{CR_l} is given by $C_{NC} := I_{l+1}^S C_{BPX} \left(I_{l+1}^S\right)^*$.

The condition number of the preconditioned operator asymptotically behaves like $O(1)$ as follows from the following results (cf. [20; Thms. 3.3, 3.7]):

Theorem 4.1. *There exist positive constants $0 < \kappa_0 \leqslant \kappa_1$ and $0 < \eta_0 \leqslant \eta_1$ depending only on the shape regularity of $\mathcal{T}_0$ and on the bounds for the coefficients of the elliptic operator such that for all $\psi_l \in N(\Omega;\mathcal{T}_l)$ and $u_l \in CR_1(\Omega;\mathcal{T}_l)$*

$$\kappa_0 \tilde{a}_{N_l}(\psi_l,\psi_l) \leqslant a_{N_l}(\psi_l,\psi_l) \leqslant \kappa_1 \tilde{a}_{N_l}(\psi_l,\psi_l),$$
$$\eta_0 a_{CR_l}(u_l,u_l) \leqslant a_{CR_l}(C_{NC}^{-1} A_{CR_l} u_l, u_l) \leqslant \eta_1 a_{CR_l}(u_l,u_l).$$

Adaptivity by local refinement of the triangulations will be realized using an efficient and reliable a posteriori error estimator for the L^2-norm of the total error in the primal variable u which can be derived from a saturation assumption motivated by a superconvergence result for mixed hybridization. In particular, we denote by $\hat{u}_l \in CR_1(\Omega;\mathcal{T}_l)$ the nonconforming interpolation of the Lagrangian multiplier $\lambda_l \in M_0(\mathcal{E}_h)$ in the sense that $\Pi_l \hat{u}_l = \lambda_l$. Then, if $u \in H^2(\Omega)$ and $f \in H^1(\Omega)$, in the L^2-norm the nonconforming interpolation $\hat{u}_l$ does provide an approximation of u of order $O(h_l^2)$ compared to the approximation of order $O(h_l)$ of u by u_l (cf. [1;

Thm. 1.1], [11; § V, Thm. 3.1]). This superconvergence result supports the following saturation assumption

$$\|u - \hat{u}_l\|_0 \leqslant \beta \|u - u_l\|_0, \quad 0 \leqslant \beta < 1. \tag{4.6}$$

Theorem 4.2. *Let $\tilde{\psi}_l \in N(\Omega; \mathcal{T}_l)$ be an iterative approximation of the unique solution $\psi_l \in N(\Omega; \mathcal{T}_l)$ of* (4.2) *and let $\tilde{u}_l := P_l\tilde{\psi}_l$, $\tilde{\lambda}_l := \Pi_l\tilde{\psi}_l$. Further, suppose that $\hat{\tilde{u}}_l \in CR_1(\Omega; \mathcal{T}_l)$ is the nonconforming interpolation of $\tilde{\lambda}_l$. Then, under the saturation assumption* (4.6) *there holds*

$$\begin{aligned} &\|u - \tilde{u}_l\|_0 \leqslant (1-\beta)^{-1}\|\tilde{u}_l - \hat{\tilde{u}}_l\|_0 + 2\frac{1+\beta}{1-\beta}\|\psi_l - \tilde{\psi}_l\|_0, \\ &\|u - \tilde{u}_l\|_0 \geqslant (1+\beta)^{-1}\|\tilde{u}_l - \hat{\tilde{u}}_l\|_0 - 2\|\psi_l - \tilde{\psi}_l\|_0. \end{aligned} \tag{4.7}$$

Proof. cf. (4.7) in [20] . □

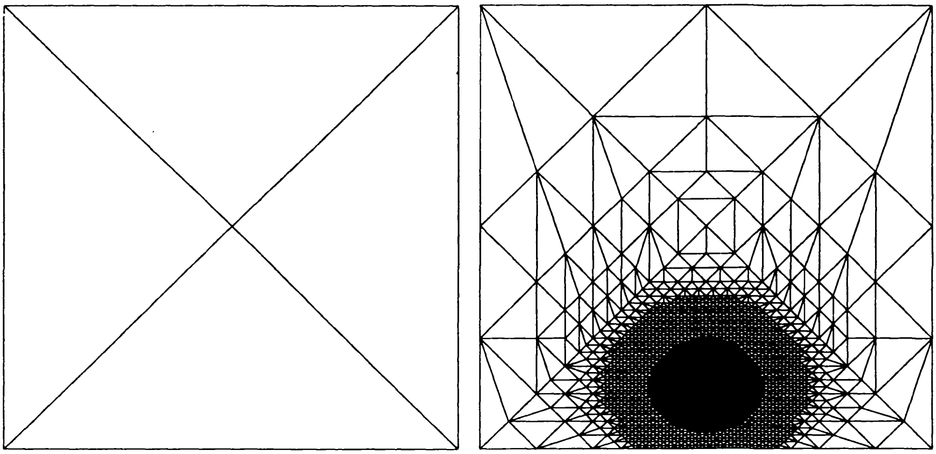
We remark that $\|\psi_l - \tilde{\psi}_l\|_0$ in (4.7) represents the iteration error which can be controlled by the iterative solution process. Therefore, we may use the cheaply computable local contributions $\|\tilde{u}_l - \hat{\tilde{u}}_l\|_0$, $K \in \mathcal{T}_l$, as indicators for local refinement of $\mathcal{T}_l$. We further refer to [20] for a characterization of the error estimator in terms of a weighted sum of the jumps of the approximation $\tilde{u}_l$ on the interelement boundaries.

5. Numerical results

We present numerical results for two specific examples illustrating the refinement processes and the performance of the a posteriori error estimators. As test examples we have chosen:

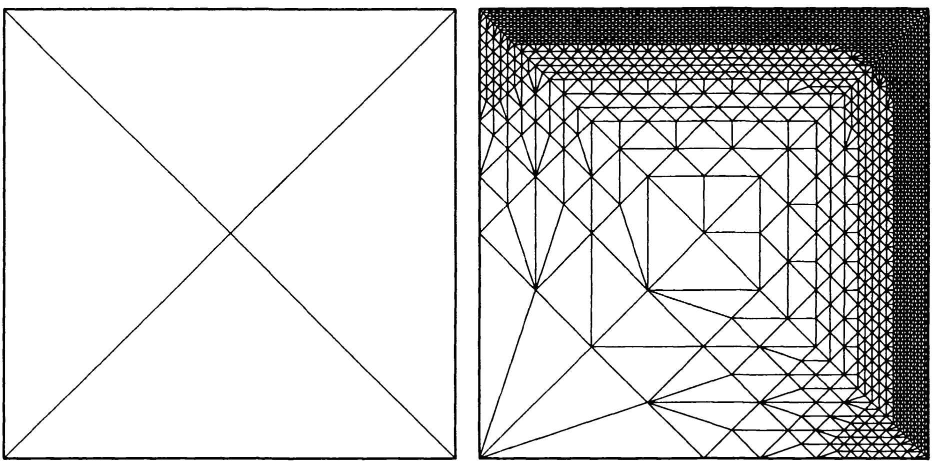
Example 1. Equation (2.1) with $a = 1$, $b = 0$ on $\Omega = (0,1)^2$, homogeneous Dirichlet boundary conditions and right-hand side f according to the solution $u(x,y) = x(x-1)y(y-1)\exp\left(-100(x-0.5)^2 + (y-0.117)^2\right)$. The solution has a peak in $(0.5, 0.0117)$.

Example 2. Equation (2.1) with $a = 1$, $b = 100$ on $\Omega = (0,1)^2$ and Dirichlet boundary data and right-hand side f according to the solution $u(x,y) = (2\cosh 10)^{-1}\left(\cosh(10x) + \cosh(10y)\right)$. The solution has the boundary layer along the lines $x = 1$ and $y = 1$.



Level 0 Level 8

Fig. 5.1. Initial triangulation $\mathcal{T}_0$ and final triangulation $\mathcal{T}_8$ (Example 1)



Level 0 Level 6

Fig. 5.2. Initial triangulation $\mathcal{T}_0$ and final triangulation $\mathcal{T}_6$ (Example 2)

Both examples have been solved by the algorithms described in the previous sections. For presentation we restrict ourselves to the results obtained by the application of Algorithm I to Example 1 and of Algorithm II to Example 2.

The initial coarse triangulation $\mathcal{T}_0$ has been selected as shown in Figures 5.1 and 5.2. The refinement process has been stopped when the square of the ratio of the estimated error and the norm of the iterative approximation was less than the required accuracy TOL times a safety factor α. In particular, we have chosen TOL $= 2.E-3$ and $\alpha = 0.95$.

Figures 5.1 and 5.2 represent the initial and final triangulation for Example 1 and 2. In both cases we observe a significant refinement in the neighborhood of the peak of the solution (Example 1) and the boundary layer (Example 2).

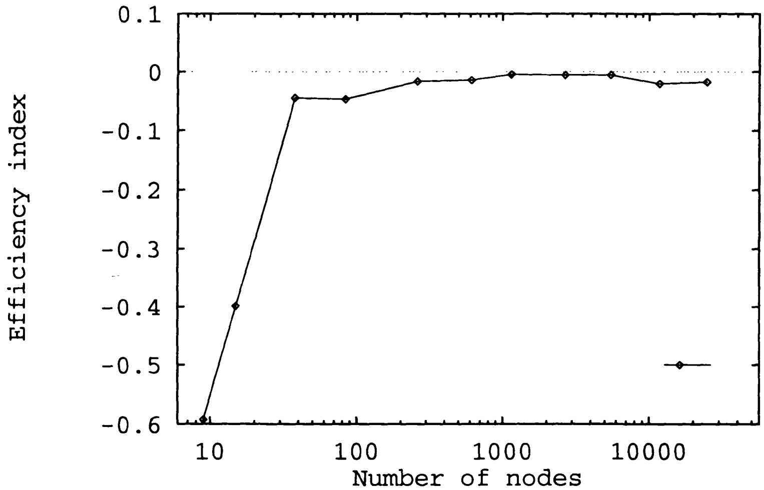


Fig. 5.3. Efficiency index (Example 1)

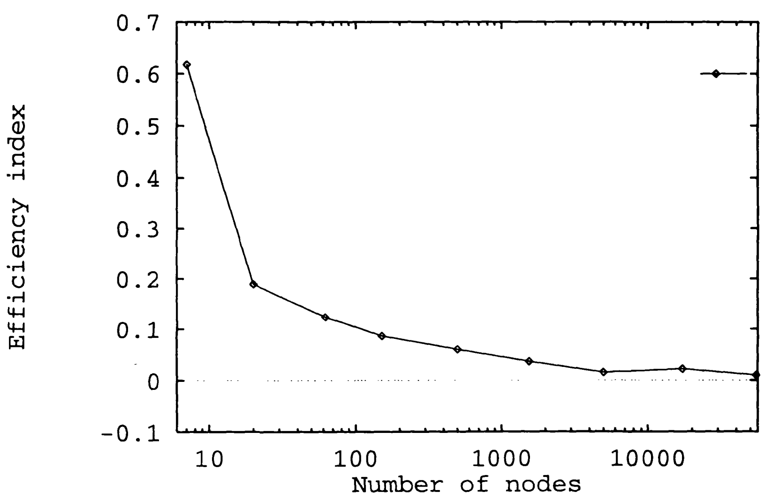


Fig. 5.4. Efficiency index (Example 2)

The performance of both the flux-based a posteriori error estimator (Example 1) and of the a posteriori L^2-error estimator (Example 2) is shown in Figure 5.3 and 5.4 representing the efficiency index $E = \varepsilon_{\text{est}}/\varepsilon_{\text{true}} - 1$ as a function of the total number of nodes where ε_{est} and $\varepsilon_{\text{true}}$ stand for the estimated and true error, respectively. As can be seen, at the beginning of the refinement process we have a slight underestimation for Example 1 and an overestimation for Example 2, but in both cases the estimated error rapidly approaches the true error with increasing refinement.

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