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3-PARAMETRIC ROBOT MANIPULATOR WITH INTERSECTING AXES

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Summary. A p-parametric robot manipulator is a mapping g of $\mathbf{R}^p$ into the homogeneous space $P = (C_6 \times C_6)/\operatorname{Diag}(C_6 \times C_6)$ represented by the formula $g(u_1, u_2, \ldots, u_p) = \exp(u_1 X^1) \cdot \ldots \cdot \exp(u_p X^p)$, where C_6 is the Lie group of all congruences of E_3 and $X^1, X^2, \ldots, X^p$ are fixed vectors from the Lie algebra of C_6.

In this paper the 3-parametric robot manipulator will be expressed as a function of rotations around its axes and an invariant of the motion of this robot manipulator will be given.

Most of the results presented here have been obtained during the author's stay at Charles University in Prague.

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INTRODUCTION

A robot manipulator can be defined as a system of links or arms which can rotate or translate with respect to each other and where the movement of each link is independent of the movements of the remaining ones. The number p of moving links is called the degree of freedom of the robot manipulator. The last link of the robot manipulator is called the effector; of special interest here is the position and movement of the effector with respect to the base of the robot manipulator. Points connected with the effector constitute a Euclidean space $\overline{E}_3$ which is called the moving space; the Euclidean space connected with the base of the robot manipulator is called the fixed space and is denoted by E_3.

Of importance here is the motion of $\overline{E}_3$ with respect to E_3 which is generated by the motion of links of the robot manipulator. The actual motion depends on time, so it is a one-parametric motion in the Euclidean space. The links can move independently of each other; the possible positions of the effector depend on p parameters.

Because the focus of this paper is the geometry of the motion of the robot manipulator, a geometric definition will be used. The p-parametric robot manipulator is a product of p rotations or translations of the space.

The following definitions are introduced. Orthonormal frames

$$\bar{R} = (\bar{O}, \bar{f}_1, \bar{f}_2, \bar{f}_3) \text{ in } \bar{E}_3 \quad \text{and} \quad R = (0, f_1, f_2, f_3) \text{ in } E_3$$

are chosen. A p parametric motion in the Euclidean space is given if $\bar{R}$ is given with respect to R as a function of p variables. Thus it can be written as $\bar{R}(u_i) = R \cdot g(u_i)$, where u_i are parameters, $i = 1, 2, \ldots, p$ and $g(u_i) = \begin{pmatrix} 1 & 0 \\ t(u_i) & \gamma(u_i) \end{pmatrix}$, where $t(u_i) = (t_1(u_i), t_2(u_i), t_3(u_i))^T$ and $\gamma(u_i) \in SO(3)$. This means that $g(u_i)$ is an element from the Lie group C_6 of all orientation preserving congruences of the Euclidean space.

Other frames in $\bar{E}_3$ and E_3 can be chosen as basic ones.

Let $\bar{R} = \bar{R}_1 \bar{g}_1$, $R = R_1 g_1$; then $\bar{R} = Rg$ implies $\bar{R}_1 = R_1 g_1 g \bar{g}_1^{-1}$. This means that motions $g(u_i)$ and $g_1 g(u_i) \bar{g}_1^{-1}$ have to be considered as equivalent. In the set of all congruences from $\bar{E}_3$ to E_3, there is an action of the group $C_6 \times C_6$ determined by the rule

$$(g_1, \bar{g}_1) \cdot g = g_1 g \bar{g}_1^{-1}.$$

The set of all congruences from $\bar{E}_3$ to E_3 is the homogeneous space $P = (C_6 \times C_6)/\operatorname{Diag}(C_6 \times C_6)$, where an element $(g, g) \in \operatorname{Diag}(C_6 \times C_6)$ for $g \in G$ can be identified with $g \in C_6$. The natural projection is $\pi\colon (C_6 \times C_6) \to P$, the formula for which is $(g_1, \bar{g}_1) \mapsto g_1 \bar{g}_1^{-1}$. The homogeneous space P is identical with C_6 as a manifold, but the multiplication in C_6 is omitted.

The Lie algebra of C_6 is denoted as L. It can be viewed as the Lie algebra of matrices $X = \begin{pmatrix} 0 & 0 \\ z & y \end{pmatrix}$, where $z = (z_1, z_2, z_3)^T$ and y is a 3×3 skew-symmetric matrix

$$y = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}$$

written in a 3-column form $y = (y_1, y_2, y_3)^T$ and symbolically, $X = (y; z)$.

Let (y, z) be the ordinary scalar product in $\mathbb{R}^n$. On L there are two invariant quadratical forms, the Killing form $K(X, X) = (y, y)$ and the Klein form $Kl(X, X) = (y, z)$. Each $X \in L$ generates a one parametric subgroup $\exp(tX)$, $t \in \mathbb{R}$, which in general is a screw motion around the straight line $A = y \times z + \lambda y$, $\lambda \in \mathbb{R}$, where $K(X, X)$ is the angular velocity and $Kl(X, X)/K(X, X)$ is the pitch. It is shown in paper [1] that $\exp(tX)$ is translation iff $K(X,X) = 0$, and it is a rotation iff $K(X,X) \neq 0$ and $Kl(X,X) = 0$.

We continue with the following two definitions:

Definition 1. A p-dimensional motion in E_3 is an immersion $\hat{g}$ of a p-dimensional manifold M into the homogeneous space P.

Definition 2. A robot manipulator with p degrees of freedom is any map equivalent to the map

$$g\colon \mathbb{R}^p \to P\colon (u_1, \dots, u_p) \mapsto \exp(u_1 X^1) \cdot \ldots \cdot \exp(u_p X^p),$$

where $X^i \in L$, $X^i \neq 0$ and any identification of P with C_6 is used. Paper [1] illustrated the p-parametric robot manipulator as a p-dimensional motion.

We repeat some of the arguments from [1] because of the notions which will be used later in this paper.

Let $g\colon M \to P$ be a p dimensional motion, $\overline{R}_0$ and R_0 are chosen fixed frames in $\overline{E}_3$ and E_3, respectively.

By a lift of the motion $\hat{g}$ we understand any pair of frames $\overline{R}(m)$ in $\overline{E}_3$ and $R(m)$ in E_3 such that $\hat{g}(m)\overline{R}(m) = R(m)$ for each $m \in M$.

For any lift of $\hat{g}$ we obtain

$$\mathrm{d}R = R \cdot \varphi,\ \mathrm{d}\overline{R} = \overline{R} \cdot \psi,$$

where φ and ψ are two copies of the Maurer-Cartan canonical 1-form on C_6. New forms $\omega_0 = \frac{1}{2}(\varphi - \psi)$, $\eta_0 = \frac{1}{2}(\varphi - \psi)$ are introduced.

The integrability conditions then have the form

$$\mathrm{d}\omega_0 + \eta_0 \wedge \omega_0 + \omega_0 \wedge \eta_0 = 0;\ \mathrm{d}\eta_0 + \eta_0 \wedge \eta_0 + \omega_0 \wedge \omega_0 = 0,$$

where $\wedge$ denotes the matrix wedge product.

Further, if $\overline{R}$, R is a lift of $\hat{g}$, then any other lift is $\overline{R}\gamma$, $R\gamma$, where $\gamma \in C_6$.

For the corresponding forms $\tilde{\omega}_0$ and $\tilde{\eta}_0$ we obtain

$$\tilde{\omega}_0 = ad_{\gamma-1}\omega_0; \quad \tilde{\eta}_0 = ad_{\gamma-1}\eta_0 + \gamma^{-1}\,\mathrm{d}\gamma.$$

Additionally, let $\hat{g}(m)$ by any motion given by the matrix representation $g(m)$. Then the pair $\overline{R}_0$, $R_0 \cdot g(m)$ is its lift by definition. Because $\mathrm{d}\overline{R} = 0$ and $\mathrm{d}R = R_0 \cdot \mathrm{d}g(m) = R \cdot g^{-1}\,\mathrm{d}g(m)$, we have $\varphi_0 = g^{-1}\,\mathrm{d}g(m)$, $\psi_0 = 0$.

Therefore this special lift satisfies $\omega_0 = \frac{1}{2}g^{-1}\,\mathrm{d}g$.

The following lemma was proved in paper [1]. Let $X^1, \dots, X^p$ be linearly independent in L. Then there exists a neighbourhood $U(0)$ of 0 in $\mathbb{R}^p$ such that the p-parametric robot $g(u_i)$ is a p-dimensional motion on $U(0)$.

1. Axes of the robot manipulator

In what follows we will consider 3-parametric motions only. In such a case one can write the forms ω_0 and η_0 in the matrix form

$$\omega_0 = \begin{pmatrix} 0 & 0 \\ \vartheta & \omega \end{pmatrix}, \quad \eta_0 = \begin{pmatrix} 0 & 0 \\ \pi & \eta \end{pmatrix}, \tag{1}$$

where

$$\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{pmatrix},$$

$$\vartheta = (\vartheta_1 \vartheta_2 \vartheta_3)^T, \quad \pi = (\pi_1 \pi_2 \pi_3)^T;$$

1-forms ω_i, ϑ_i, η_i, π_i are of the form

$$\begin{aligned} \omega_i &= \sum_{j=1}^{3} a_{ij}\, du_j, \quad & \vartheta_i &= \sum_{j=1}^{3} c_{ij}\, du_j, \\ \eta_i &= \sum_{j=1}^{3} b_{ij}\, du_j, \quad & \pi_i &= \sum_{j=1}^{3} f_{ij}\, du_j. \end{aligned} \tag{2}$$

In [4] it is prove that in the general case we can choose a lift of the motion $\hat{g}$ such that θ is diagonal, $\vartheta_i = \nu_i \omega_i$. Such a lift is called the canonical lift $\hat{g}$.

In the next part we shall compute the forms ω_0, η_0 for the canonical lift of a 3-parametric robot manipulator given by axes O_1, O_2, O_3, where O_1 and O_2 intersect and the unit vectors of axes O_i, $i = (1, 2, 3)$ are linearly independent.

We shall write forms ω, η, ϑ, π in the form

$$\begin{aligned} \omega &= \sum_{i=1}^{3} A^i\, du_i, \quad & \eta &= \sum_{i=1}^{3} B^i\, du_i, \\ \vartheta &= \sum_{i=1}^{3} C^i\, du_i, \quad & \pi &= \sum_{i=1}^{3} F^i\, du_i, \end{aligned} \tag{3}$$

where

$$\begin{aligned} A^i &= (a_{1i}, a_{2i}, a_{3i})^T, \quad & B^i &= (b_{1i}, b_{2i}, b_{3i})^T, \\ C^i &= (c_{1i}, c_{2i}, c_{3i})^T, \quad & F^i &= (f_{1i}, f_{2i}, f_{3i})^T. \end{aligned} \tag{4}$$

Let $\alpha = (\alpha_i)$; $\beta = (\beta_i)$ be two vector forms. One can define the form $\gamma = \alpha \hat{\times} \beta$, where $\gamma_i = \varepsilon_{ijk}\alpha_j \wedge \beta_k$, $i, j, k = 1, 2, 3$ and ε_{ijk} is the sign of the permutation i, j, k. Then the integrability condition can be written in the form

$$
\begin{aligned}
&\mathrm{d}\omega = -\omega \hat{\times} \eta;\ \mathrm{d}\eta = -\tfrac{1}{2}[\eta \hat{\times} \eta + \omega \hat{\times} \omega], \\
&\mathrm{d}\vartheta = -\eta \hat{\times} \vartheta - \omega \hat{\times} \pi;\ \mathrm{d}\pi = -\eta \hat{\times} \pi - \omega \hat{\times} \vartheta
\end{aligned} \tag{5}
$$

where $\omega_0 = (\omega; \vartheta)$, $\eta_0 = (\eta; \pi)$. In view of (3), (4) and (5), the following system of equations is obtained:

$$
\begin{aligned}
&\text{a)} \quad \frac{\partial A^i}{\partial u_j} - \frac{\partial A^i}{\partial u_i} = A^i \times B^j - A^j \times B^i \\
&\text{b)} \quad \frac{\partial B^i}{\partial u_j} - \frac{\partial B^i}{\partial u_i} = B^i \times B^j - A^i \times A^j \\
&\text{c)} \quad \frac{\partial C^i}{\partial u_j} - \frac{\partial C^j}{\partial u_i} = B^i \times C^j - B^j \times C^i + A^i \times F^j - A^j \times F^i \\
&\text{d)} \quad \frac{\partial F^i}{\partial u_j} - \frac{\partial F^i}{\partial u_i} = B^i \times F^j - B^j \times F^i + A^i \times C^j - A^j \times C^i,
\end{aligned} \tag{6}
$$

where $i < j$, $i = 1, 2, 3$; $i = 2, 3$.

Theorem 1. *The position of axes O_1, O_2, O_3 of a 3-parametric robot manipulator depends on the variable u_2 only.*

Proof. It is proved in [1] that the vector (A^i, C^i) yields the Plücker coordinates of the axis O_i of the robot manipulator in the canonical lift $\hat{g}$. A_i is the unit vector of the axis O_i, $i = 1, 2, 3$.

Let u_i be the angle revolution around the axis O_i.

Let X^i be the Plücker coordinates of O_i. Then $X^i \in L$ and

$$
X^i = (A^i; C^i) \quad i = 1, 2, 3. \tag{7}
$$

The Killing form yields

$$
K(X^i, X^j) = (A^i, A^j) = \cos\varphi_k, \tag{8}
$$

where φ_k is the angle of O_i and O_j, $i, j, k = 1, 2, 3$ and $k \neq i$, $k \neq j$, $i \neq j$. Since the Klein form of the vectors X^i is given in the form

$$
Kl(X^i, X^j) = (A^i, C^j) + (A^j, C^i) = \pm d_k \sin\varphi_k, \tag{9}
$$

where $d_k = \varrho(O_i, O_j)$ is the distance between the axes O_i and O_j, $j = 1,2,3$, $i \neq k$, $j \neq k$, $i \neq j$, therefore

$$KI(X^1, X^2) = 0 \quad \text{and} \quad KI(X^2, X^3) = \text{const.} \tag{10}$$

In the canonical lift we have

$$\omega_0 = \begin{pmatrix} \omega_1; & v_1\omega_1 \\ \omega_2; & v_2\omega_2 \\ \omega_3; & v_3\omega_3 \end{pmatrix} = (\omega, \vartheta). \tag{11}$$

It follows from (3) and (11) that $vA^i = C^i$ and

$$v = \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix}. \tag{11'}$$

From $vA^i = C^i$ we obtain

$$c_{ji} = v_j a_{ji}. \tag{12}$$

Since

$$v_j\omega_j = \vartheta_j = \sum_{i=1}^{3} c_{ji}\,\mathrm{d}u_i,$$

it follows that

$$vA^i = (v_1\omega_1, v_2\omega_2, v_3\omega_3)^T.$$

The vectors A^i, $i = 1,2,3$ are linearly independent; therefore, they form a basis and the vectors $C^i = vA^i$ can be expressed as

$$C^i = vA^i = \sum_{j=1}^{3} s^i_j A^j. \tag{13}$$

$\square$

N o t a t i o n.

$$\sin\varphi_k = S_k, \quad \sin 2u_2 = s_2,$$
$$\cos\varphi_k = C_k, \quad \cos 2u_2 = c_2.$$

Computation yields

$$(14) \qquad \text{a)} \qquad \begin{aligned}(vA^i, A^i) &= \tfrac{1}{2}[(vA^i, A^i) + (A^i, vA^i)] \\ &= \tfrac{1}{2}[(C^i, A^i) + (A^i, C^i)] \\ &= \tfrac{1}{2}Kl(X^i, X^i) = 0\end{aligned}$$

and

$$\text{b)} \qquad \begin{aligned}(vA^i, A^j) &= \tfrac{1}{2}[(vA^i, A^j) + (vA^j, A^i)] \\ &= \tfrac{1}{2}[(C^i, A^j) + (C^j, A^i)] \\ &= \tfrac{1}{2}Kl(X^i, X^j) = \tfrac{1}{2}d_k S_k.\end{aligned}$$

Taking (i, j, k) as the permutations of the set $(1, 2, 3)$ we obtain

$$(15) \qquad (vA^i, A^j) = \tfrac{1}{2}d_k S_k.$$

The equalities $d_3 = 0$, $d_1 S_1 = \text{const.}$, $d_2 S_2 = k(u_2)$ together with the formulas (8), (13) and (15) imply that the coordinates s^i_j are functions of one variable u_2 and, further,

$$(16) \qquad (S - v_i E)M^i = 0,$$

where $S = (s^i_j)$ is a 3×3 matrix, the upper index denotes the row of the matrix S, the matrix E is the 3×3 unit matrix and $M^i := (a_{i1}, a_{i2}, a_{i3})^T$.

It is known that the system of linear equations (16) has a non-zero solution iff the characteristic polynomial $\det(S - v_i E) = 0$; therefore, $v_i = v_i\big(s^i_j(u_2)\big) = v_i(u_2)$ and as a consequence,

$$a_{ij} = a_{ij}(u_2) \quad \text{and} \quad c_{ij} = c_{ij}(u_2).$$

For further use we denote

$$\omega^i := \begin{pmatrix} 0 & 0 \\ C^i & A^i \end{pmatrix} \quad \text{and} \quad \eta^i := \begin{pmatrix} 0 & 0 \\ F^i & B^i \end{pmatrix}.$$

In paper [1] the following result proved:

Theorem. *Let a p-dimensional motion $g\colon M \to C_6$ be a robot manipulator. Then there exists a system of coordinates $u_1, \dots, u_p$ such that for*

$$(17) \qquad \omega_0 = \sum_{\alpha=1}^{p} \omega^\alpha \, \mathrm{d}u_\alpha, \quad \eta_0 = \sum_{\alpha=1}^{p} \eta^\alpha \, \mathrm{d}u_\alpha$$

we have

$$\frac{\partial \omega_\alpha}{\partial u_\alpha} = [\omega^\alpha, \eta^\alpha] \quad \textit{for } \alpha = 1, 2, \ldots, p,$$

where the bracket means the Lie bracket in L. Formula (17) *yields the relations*

$$\text{(18)} \qquad \frac{\partial A^i}{\partial u_i} = A^i \times B^j, \quad \frac{\partial C^i}{\partial u_i} = A^i \times F^i - B^i \times C^i.$$

Because the coordinates of vectors A^i and C^i, $i = 1, 2, 3$, depend on the variable u_2 only, we have

$$\frac{\partial A^1}{\partial u_1} = \frac{\partial A^3}{\partial u_3} = 0,$$

and in view of (18) *we obtain*

$$\text{(19)} \qquad A^1 \times B^1 = 0 \quad \textit{and} \quad A^3 \times B^3 = 0.$$

Hence

$$\text{(20)} \qquad B^1 = \mu A^1, \quad B^3 = \kappa A^3,$$

where the functions $\mu = \mu(u_1, u_2, u_3)$ and $\kappa = \kappa(u_1, u_2, u_3)$ are functions of the variables u_i, $i = 1, 2, 3$.

On the other hand, substituting $i = 1$, $j = 3$ in formula (6) a) results in $A^1 \times B^3 - A^3 \times B^1 = 0$ and further, in view of relation (14), we have $(\kappa + \mu)(A^1 \times A^3) = 0$, which means that $\kappa = -\mu$ because A^1, A^3 are linearly independent vectors. Consequently, $B^1 = \mu A^1$ and $B^3 = -\mu A^3$ and differentiation with respect to u_1, u_2 yields

$$\frac{\partial B^1}{\partial u_1} = A^1 \frac{\partial \mu}{\partial u_1}, \quad \frac{\partial B^1}{\partial u_3} = A^1 \frac{\partial \mu}{\partial u_3},$$
$$\frac{\partial B^3}{\partial u_1} = -A^3 \frac{\partial \mu}{\partial u_1}, \quad \frac{\partial B^3}{\partial u_3} = A^3 \frac{\partial \mu}{\partial u_3}.$$

From these equalities together with formula (6) b) one can obtain the quality

$$\text{(21)} \qquad \frac{\partial \mu}{\partial u_3} A^1 + \frac{\partial \mu}{\partial u_1} A^3 = (1 - \mu^2)(A^1, A^3).$$

Multiplying scalarly equality (21) first by A^1 and next by A^3 and taking into consideration the fact that the scalar products $(A^1, A^1) = (A^3, A^3) = 1$ and that the vectors A^1, A^3 are linearly independent we obtain $\frac{\partial \mu}{\partial u_1} = \frac{\partial \mu}{\partial u_3} = 0$, which means that $\mu = \mu(u_2)$ and, consequently the coordinates of vectors B^1 and B^3 depend on the variable u_2 only. Similarly, one can prove that the coordinates of vectors B^2 and F^i, $i = 1, 2, 3$ depend on the variable u_2 only.

Now we will concentrate on the special 3-parametric robot manipulator given by axes O_1, O_2, O_3 where O_1 and O_2 intersect. We will choose a special system of coordinates, in which the expression for the axes O_1, O_2, O_3 will assume a very simple form (see Fig. 1).

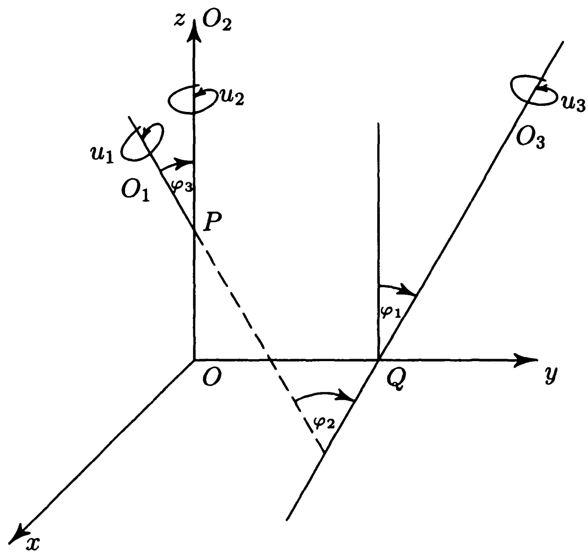

Fig. 1

Here $d_1 = |\overline{OQ}|$ and $\delta = |\overline{OP}|$, where the point O is the origin of the coordinate system, the point P is the intersection point of the axes O_1 and O_2, Q is the intersection point of the axes O_3 and O_y, and d_2 is the distance between the axes O_1 and O_3. The parametric equations of O_1, O_2, O_3 are given as follows:

$$O_1\colon (x, y, z - \delta)^T = (tS_3, O, tC_3)^T;$$
$$O_2\colon (x, y, z)^T = (O, O, t)^T; \quad O_3\colon (x, y - d_1, z) = (-tS_1, O, tC_1)^T$$

and $t \in \mathbb{R}$, where O_3 rotates around O_2 with the angle of rotation u_2.

In the next theorem we express the unit vectors of the axes O_1, O_2, O_3 in the canonical lift $\hat{g}$.

Theorem 2. *The unit vectors A^i, $i = 1, 2, 3$, of the robot manipulator axes in the canonical lift $\hat{g}$ are of the form*

$$A^1 = \begin{pmatrix} \frac{d_2 S_2}{[2l(l-k)]^{\frac{1}{2}}} \\ -\frac{d_2 S_2}{[2l(l+k)]^{\frac{1}{2}}} \\ \frac{a}{(l^2-k^2)^{\frac{1}{2}}} \end{pmatrix}, \quad A^2 = \begin{pmatrix} \frac{d_1 S_1}{[2l(l-k)]^{\frac{1}{2}}} \\ -\frac{d_1 S_1}{[2l(l+k)]^{\frac{1}{2}}} \\ \frac{b}{(l^2-k^2)^{\frac{1}{2}}} \end{pmatrix}, \quad A^3 = \begin{pmatrix} (\frac{l}{2})^{\frac{1}{2}} \frac{S_1 s_2}{(l-k)^{\frac{1}{2}}} \\ (\frac{l}{2})^{\frac{1}{2}} \frac{S_1 s_2}{(l+k)^{\frac{1}{2}}} \\ \frac{c}{(l^2-k^2)^{\frac{1}{2}}} \end{pmatrix} \tag{22}$$

where

$$\begin{aligned} a &= d_1 C_1 C_3 c_2 - d_1 S_1 S_3 - \delta S_1 C_3 s_2, \\ b &= d_1 C_1 c_2 - \delta S_1 s_2, \\ c &= d_1 c_2 - \delta S_1 C_1 s_2, \\ k &= d_1 C_1 s_2 + \delta S_1 c_2, \\ l &= [d_1^2 + (\delta S_1)^2]^{\frac{1}{2}}, \\ C_2 &= C_1 C_3 - S_1 S_3 c_2, \\ d_2 S_2 &= d_1 C_1 S_3 c_2 + d_1 S_1 C_3 - \delta S_1 S_3 s_2; \end{aligned}$$

u_2 is the angle of rotation around the axis O_2.

P r o o f. First we prove

$$\begin{aligned} C_2 &= C_1 C_3 - S_1 S_3 c_2, \\ d_2 S_2 &= d_1 C_1 S_3 c_2 + d_1 S_1 C_3 - \delta S_1 S_3 s_2. \end{aligned} \tag{23}$$

There exists such a frame in E_3 that the vectors of the axes O_1 and O_3 are of the form

$$\tilde{A}^1 = (S_3, O, C_3)^T, \quad \tilde{A}^3 = (-S_1, O, C_1)^T.$$

It has been proved in Theorem 1 that the configuration of axes of the robot manipulator depends only on the rotation around the axis O_2. After making a rotation of angle λu_2 around the axis O_2, it can be seen that the vector A^1 does not change and $\tilde{A}^1 = A^1$,

$$A^3 = U\tilde{A}^3 = (-S_1 \cos \lambda u_2, -S_1 \sin \lambda u_2, C_1)^T,$$

where the matrix U is of the form

$$U = \begin{pmatrix} \cos \lambda u_2 & -\sin \lambda u_2 & 0 \\ \sin \lambda u_2 & \cos \lambda u_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define vectors $r = (0, 0, \delta)^T$, $s = (0, d_1, 0)^T$ and

$$\tilde{q}^1 = r \times \tilde{A}^1, \quad \tilde{q}^3 = s \times \tilde{A}^3;$$

as by the rotation around the axis O_2 we have $Ur = r$ and $\tilde{A}^1 = A^1$ so after this rotation one gets the vector $q^1 = \tilde{q}^1 = r \times \tilde{A}^1 = r \times A^1 = (0, \delta S_3, 0)^T$. Similarly the vector $q^3 = U\tilde{q}^3 = (d_1 C_1 \cos\lambda u_2, d_1 C_1 \sin\lambda u_2, d_1 S_1)^T$ is obtained.

Denote the Plücker coordinates of the axis O_i by p_i, $i = 1, 2, 3$, in the following way:

$$p_1 = (A^1; q^1), \quad p_2 = \frac{1}{d_1}(s; 0), \quad p_3 = (A^3; q^3).$$

Then in view of the relations

$$C_2 = (A^1, A^3) \quad \text{and} \quad d_2 S_2 = (A^1, q^3) + (A^3, q^1)$$

we obtain

$$\begin{aligned} C_2 &= C_1 C_3 - S_1 S_3 \cos\lambda u_2, \\ d_2 S_2 &= d_1 C_1 S_3 \cos\lambda u_2 + d_1 S_1 C_3 - \delta S_1 S_3 \sin\lambda u_2. \end{aligned}$$

In view of (6), (8), (9), (10) and (16), after laborious calculations we obtain $\lambda = 2$ and the formulas for A^i, $i = 1, 2, 3$ of the form (22). □

Theorem 3. *Let us write the axes O_1, O_2, O_3 of the robot manipulator in the form $O_i = A^i \times C^i + tA^i$. Then C^i are given by (25) below.*

Proof. To determine vectors C^i it is necessary and sufficient to calculate the elements of the diagonal matrix determined in (11). In view of formulas (13)–(15) the matrix S assumes the form

$$S = \frac{1}{\det A} \begin{pmatrix} nd_2S_2, & md_2S_2, & d_2S_2S_3^2 \\ nd_1S_1, & md_1S_1, & d_1S_1S_3^2 \\ d_2S_2S_1^2 + pd_1S_1, & d_1S_1S_2^2 + pd_2S_2, & md_1S_1 + nd_2S_2 \end{pmatrix}$$

where A is a symmetric matrix of the form

$$A := \begin{pmatrix} 1 & C_2 & C_1 \\ C_2 & 1 & C_3 \\ C_1 & C_3 & 1 \end{pmatrix},$$

which in view of (23) yields

$$\begin{aligned} \det A = S_1^2 S_3^2 s_2^2; \quad & m = -S_3(C_1S_3 + S_1C_3c_2), \\ n = S_1S_3c_2; \quad & p = -S_1(S_1C_3 + C_1S_3c_2), \\ S_i^2 = \sin^2\varphi_i; \quad & s_2^2 = \sin^2 2u_2 \quad i = 1, 2, 3. \end{aligned}$$

Note that $\det S = 0$, and the characteristic equation $\det(S - v_i E) = 0$, $i = 1,2,3$ implies that

$$v_1 = \frac{l-k}{2S_1 s_2}, \quad v_2 = -\frac{l+k}{2S_1 s_2}, \quad v_3 = 0 \tag{24}$$

where l and k have been determined in (22).

Hence, from $C^i = vA^i$ we conclude

$$C^1 = \frac{d_2 S_2}{2\sqrt{2l} S_1 s_2} \begin{pmatrix} (l-k)^{\frac{1}{2}} \\ (l+k)^{\frac{1}{2}} \\ 0 \end{pmatrix}; \quad C^2 = \frac{d_1}{2\sqrt{2l} s_2} \begin{pmatrix} (l-k)^{\frac{1}{2}} \\ (l+k)^{\frac{1}{2}} \\ 0 \end{pmatrix}, \tag{25}$$
$$C^3 = \frac{\sqrt{l}}{2\sqrt{2}} \begin{pmatrix} (l-k)^{\frac{1}{2}} \\ -(l+k)^{\frac{1}{2}} \\ 0 \end{pmatrix}.$$

In the monograph [2] (p. 68) the following theorem is proved. □

2. Invariants of the 3-parametric motion

Let ω, η, θ, π be 1-form of a 3-parametric motion in the canonical lift $\hat{g}$. Then we have $\theta_i = \nu_i \omega_i$ and we can write

$$\eta_i = \sum_{j=1}^{3} p_{ij} \omega_j; \quad \pi_i = \sum_{j=1}^{3} q_{ij} \omega_j, \qquad \text{where } \nu_i,\ p_{ij},\ q_{ij},$$

$i, j = 1,2,3$ are invariants of the motion. In what follows we shall compute these invariants for the motion determined by our robot manipulator.

To determine the invariants of the 3-parametric motion we prove

Lemma 1. *The vectors B^i, F^i, $i = 1,2,3$, are of the form*

$$\begin{aligned} &B^1 = -A^1,\ B^3 = A^3,\ B^2 = \frac{d_1 S_1}{(2l)^{\frac{1}{2}}} \Big(\frac{(l-k)^{\frac{1}{2}}}{(l+k)}, \frac{(l+k)^{\frac{1}{2}}}{(l-k)}, 0 \Big)^T; \\ &F^1 = -C^1,\ F^3 = C^3; \\ &F^2 = \frac{d_1}{2(2l)^{\frac{1}{2}} s_2} \Big(\frac{l-k}{(l+k)^{\frac{1}{2}}}, \frac{l+k}{(l-k)^{\frac{1}{2}}}, \frac{2(2l)^{\frac{1}{2}} c_2}{(l^2-k^2)^{\frac{1}{2}} S_1 s_2} \Big). \end{aligned} \tag{26}$$

P r o o f. From the formula (6) and after some calculations we obtain

$$B^1 = -A^1,\ B^3 = A^3,\ F^1 = -C^1,\ F^3 = C^3.$$

In the base A^1, A^2, A^3 the vector B^2 is of the form

$$B^2 = \sum_{i=1}^{3} \lambda_i A^i,$$

thus from the formulas (6) a), (18) and from the fact that $B^1 = -A^1$, $B^3 = A^3$ we obtain the following system of equations:

$$\begin{aligned} \frac{\partial A^1}{\partial u_2} &= \lambda_2 A^1 \times A^2 + \lambda_3 A^1 \times A^3 - A^2 \times A^1, \\ \frac{\partial A^2}{\partial u_2} &= \lambda_1 A^2 \times A^1 + \lambda_3 A^2 \times A^3. \end{aligned} \tag{27}$$

Scalar multiplication of the first equation of (27) by A^3 and of the second equation of (27) by A^3 and next by A^1 yields the system of three linear equations with the unknowns λ_i, $i = 1, 2, 3$, of the form

$$\begin{aligned} \left(A^3, \frac{\partial A^1}{\partial u_2}\right) &= (1 + \lambda_2)|A^1, A^2, A^3|, \\ \left(A^3, \frac{\partial A^2}{\partial u_2}\right) &= -\lambda_1 |A^1, A^2, A^3|, \\ \left(A^1, \frac{\partial A^2}{\partial u_2}\right) &= \lambda_3 |A^1, A^2, A^3|, \end{aligned}$$

where $|A^1, A^2, A^3|$ is the mixed product of the vectors A^1, A^2, A^3. Thus

$$\begin{aligned} \lambda_1 &= \frac{-d_1 S_1 (2kc_2 - bs_2)}{(l^2 - k^2) S_3 s_2}, \\ \lambda_2 &= \frac{-d_1 [a S_1 s_2 - 2k(S_1 C_3 c_2 + S_3 C_1)]}{(l^2 - k^2) S_3 s_2}, \\ \lambda_3 &= \frac{-2kd_1 S_3}{(l^2 - k^2) S_3 s_2}, \end{aligned} \tag{28}$$

where a, b, k, l are determined in (22).

In view of (28) and of the fact that $B^2 = \sum_{i=1}^{3} \lambda_i A^i$ one obtains the vector B^2 in the form (26). Similarly the vector F^2 is determined. □

Theorem 4. *The invariants of the* 3*-parametric motion* $g(u_1, u_2, u_3)$ *are given by the following formulas:*

$$P = (p_{ij}), \quad Q = (q_{ij}) \quad \text{and} \quad v = \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix}, \tag{29}$$

$i, j = 1, 2, 3$, where v_1, v_2, v_3 are determined in (24) *and*

$$
\begin{aligned}
p_{11} &= \frac{-d_1 w}{2(l^2-k^2)S_1S_3s_2^2}, \quad p_{12} = \frac{-1}{2S_1S_3s_2}\Big[\frac{d_1\bar{w}}{(l-k)(l^2-k^2)^{\frac12}s_2} - S_3(l^2-k^2)^{\frac12}\Big],\\
p_{13} &= \Big[\frac{1}{2(l+k)}\Big]^{\frac12}\frac{d_1d_2S_2}{(l-k)S_3s_2},\\
p_{21} &= -\frac{1}{2S_1S_3s_2}\Big[\frac{d_1 w}{(l+k)(l^2-k^2)^{\frac12}s_2} + S_3(l^2-k^2)^{\frac12}\Big],\\
p_{22} &= \frac{-d_1\bar{w}}{2(l^2-k^2)S_1S_3s_2^2}, \quad p_{23} = \Big[\frac{1}{2(l-k)}\Big]^{\frac12}\frac{d_1d_2S_2}{(l+k)S_3s_2},\\
p_{31} &= \frac{-d_1cw}{[2l(l+k)]^{\frac12}(l^2-k^2)S_1^2S_3s_2^3}, \quad p_{32} = \frac{-d_1c\bar{w}}{[2l(l-k)]^{\frac12}(l^2-k^2)S_1^2S_3s_2^3},\\
p_{33} &= \frac{d_1cd_2S_2}{(l^2-k^2)S_1S_3s_2^2}, \quad q_{11} = \frac{-d_1w}{(l-k)(l+k)^2S_3s_2},\\
q_{12} &= \Big(\frac{l+k}{l-k}\Big)^{\frac12} - \frac{d_1\bar{w}}{(l^2-k^2)S_3s_2}, \quad q_{13} = \Big(\frac{2l}{l+k}\Big)^{\frac12}\frac{d_1d_2S_1S_2}{(l^2-k^2)S_3},\\
q_{21} &= \Big(\frac{l-k}{l+k}\Big)^{\frac12} + \frac{d_1w}{(l^2-k^2)S_3s_2}, \quad q_{22} = \frac{d_1\bar{w}}{(l-k)^2(l+k)S_3s_2} - 1,\\
q_{23} &= -\Big(\frac{2l}{l-k}\Big)^{\frac12}\frac{d_1d_2S_1S_2}{(l^2-k^2)S_3}, \quad q_{31} = \frac{b\bar{w} - 2acd_1S_1}{[2l(l+k)]^{\frac12}(l^2-k^2)S_1S_3s_2},\\
q_{32} &= \frac{bw - 2acd_1S_1}{[2l(l-k)]^{\frac12}(l^2-k^2)S_1S_3s_2}, \quad q_{33} = \frac{ad_1S_1}{(l^2-k^2)S_3},
\end{aligned}
$$

where $w := cd_2S_2 + laS_1s_2$, $\bar{w} := cd_2S_2 - laS_1s_2$, and c, d_2S_2, l, a, k, b are determined in (22).

Proof. The matrices A, B, C, F will be defined as follows:

$$
\begin{aligned}
A &= (A^1, A^2, A^3), \quad B = (B^1, B^2, B^3),\\
C &= (C^1, C^2, C^3), \quad F = (F^1, F^2, F^3).
\end{aligned}
\tag{30}
$$

Note that the matrices A, B, C, F are square matrices and $\det A \neq 0$. The forms ω, η, θ, π can also be written as

$$
\omega = A\,\mathrm{d}u, \quad \eta = B\,\mathrm{d}u, \quad \theta = C\,\mathrm{d}u, \quad \pi = F\,\mathrm{d}u,
\tag{31}
$$

where $\mathrm{d}u$ is the one-column matrix $\mathrm{d}u = (\mathrm{d}u_1, \mathrm{d}u_2, \mathrm{d}u_3)^T$.

Since $\det A \neq 0$ thus $\mathrm{d}u = A^{-1}\omega$ and we can express the forms η, θ, π with the help of the form ω as follows:

$$
\eta = BA^{-1}\omega = Q\omega, \quad \theta = CA^{-1}\omega = v\omega, \quad \pi = FA^{-1}\omega = P\omega,
\tag{32}
$$

where $Q = (q_{ij}) = BA^{-1}$, $P = (p_{ij}) = FA^{-1}$, $i, j = 1, 2, 3$. Because $C = vA$, we have $\theta = CA^{-1}\omega = v\omega$. From Lemma 1, formulas (22), (26), (32) and [6] the statement of our theorem results. $\square$

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