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ASYMPTOTICALLY NORMAL CONFIDENCE INTERVALS FOR A DETERMINANT IN A GENERALIZED MULTIVARIATE GAUSS-MARKOFF MODEL

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Summary. By using three theorems (Oktaba and Kieloch [3]) and Theorem 2.2 (Srivastava and Khatri [4]) three results are given in formulas (2.1), (2.8) and (2.11). They present asymptotically normal confidence intervals for the determinant $|\sigma^2\Sigma|$ in the MGM model $(U, XB, \sigma^2\Sigma \otimes V)$, $\Sigma > 0$, scalar $\sigma^2 > 0$, with a matrix $V \geqslant 0$. A known $n \times p$ random matrix U has the expected value $E(U) = XB$, where the $n \times d$ matrix X is a known matrix of an experimental design, B is an unknown $d \times p$ matrix of parameters and $\sigma^2\Sigma \otimes V$ is the covariance matrix of U, $\otimes$ being the symbol of the Kronecker product of matrices. A particular case of Srivastava and Khatri's [4] theorem 2.2 was published by Anderson [1], p. 173, Th. 7.5.4, when $V = I$, $\sigma^2 = 1$, $X = \mathbf{1}$ and $B = \mu' = [\mu_1, \ldots, \mu_p]$ is a row vector.

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1. SOME THEOREMS

Theorem 1.1 Multivariate central limit theorem (Anderson [1] pp. 76–77).

Let $\mathbf{Z}(n)$ be an m-component random vector and $\mathbf{b}$ a fixed vector. Assume $p\lim_{n\to\infty} \mathbf{Z}(n) = b$ i.e. $\mathbf{Z}(n)$ converges stochastically to $\mathbf{b}$. Let $\mathbf{a} = \sqrt{n}[f(\mathbf{Z}(n) - \mathbf{b}] \to N(\mathbf{O}, \mathbf{T}^*)$, i.e. $\mathbf{a}$ is asymptotically distributed according to $N(\mathbf{O}, \mathbf{T}^*)$. Let $w = f(\mathbf{z})$ be a function of a vector $\mathbf{z}$ with the first and second derivatives existing in a neighborhood of $\mathbf{z} = \mathbf{b}$. Let $\left.\frac{\partial f(\mathbf{z})}{\partial z_i}\right|_{\mathbf{z}=\mathbf{b}}$ be the i-th component of $\mathbf{\Phi}_b$. Then the limiting distribution of $n^{\frac{1}{2}}[f(\mathbf{Z}(n) - f(\mathbf{b})]$ is

$$N(0, \mathbf{\Phi}_b'\mathbf{T}^*\mathbf{\Phi}_b). \tag{1.1}$$

Srivastava nad Khatri [4] present the following Theorem 1.2 without proof. We give the proof using the idea of Anderson [1] p. 173, who proved it a special case (cf. Summary).

Theorem 1.2. *If* $\nu K \tilde{} W_p(\nu, \Sigma)$, *then*

$$\nu^{\frac{1}{2}}\left[\frac{|K|}{|\Sigma|}-1\right] \to N(0,2p) \tag{1.2}$$

as $\nu \to \infty$, *where* $|K|$ *denotes the determinat of the* $p \times p$ *matrix* K, $2p$ *being the variance.*

Proof. By virtue of $|\nu K| = \nu^p|K|$ and Oktaba [2], (2.1) we obtain

$$\begin{aligned} |W| &= \frac{|K|}{|\Sigma|} = \frac{\nu^p|K|}{|\Sigma|\nu^p} = \frac{|\nu K|}{|\Sigma|\nu^p} = \\ &= \frac{\chi^2_\nu \cdot \chi^2_{\nu-1}\cdots\chi^2_{\nu-p+1}}{\nu^p} = \frac{\chi^2_\nu}{\nu}\cdot\frac{\chi^2_{\nu-1}}{\nu}\cdots\frac{\chi^2_{\nu-p+1}}{\nu} \\ &= V_1(\nu)\cdot V_2(\nu)\dots V_i(\nu)\dots V_p(\nu), \end{aligned} \tag{1.3}$$

where

$$\nu \cdot V_i(\nu) = \chi^2_{\nu-p+i},\ i = 1,\dots,p. \tag{1.4}$$

are independent.
Note that the standardized variate

$$u_i = \frac{\chi^2_{\nu-p+i} - E\chi^2_{\nu-p+i}}{\sqrt{\operatorname{Var}(\chi^2_{\nu-p+i})}} = \frac{\nu V_i(\nu) - (v-p+i)}{\sqrt{2(\nu-p+i)}} = \sqrt{\nu}\frac{V_i(\nu)-1+\frac{p-i}{\nu}}{\sqrt{2}\sqrt{1-\frac{p-i}{\nu}}} \tag{1.5}$$

is asymptotically normal $N(0,1)$. Thus

$$\sqrt{\nu}[V_i(\nu)-1] \to N(0,2).$$

If we replace z_i by v_i and $\mathbf{Z}(n)$ by $\mathbf{V}(\nu) = [V_1(\nu),\dots,V_p(\nu)]$ in the multivariate central limit theorem 1.1 then by virtue of $b' = [1,\dots,1]$ we have

$$|W| = \frac{|K|}{|\Sigma|} = f(\mathbf{V}(\nu)) = V_1(\nu)\dots V_p(\nu), T^* = 2I_p,$$

$$\left.\frac{\partial f}{\partial V_i}\right|_{\mathbf{V}=\mathbf{b}} = 1 \quad \text{and} \quad \Phi_b' T^* \Phi_b = 2p.$$

Hence we get (1.2). □

2. ASYMPTOTICALLY NORMAL CONFIDENCE INTERVALS FOR $|\sigma^2\Sigma|$ IN THE MGM MODEL WITH A SINGULAR COVARIANCE MATRIX

We apply Theorem 1.2 to the MGM model (cf. Summary) with a singular covariance matrix and to three theorems (Oktaba and Kieloch [3]). In this way we get the following three theorems.

Theorem 2.1. *In the MGM model $(U, XB, \sigma^2\Sigma \otimes v)$ (cf. Summary), the $(1-\alpha)$ percent asymptotically normal confidence interval for the determinant $|\sigma^2\Sigma|$ is of the form*

$$(2.1)\qquad \frac{|S_e|}{\nu_e^p + u_\alpha\sqrt{2p}\nu_e^{p-\frac{1}{2}}} < |\sigma^2\Sigma| < \frac{|S_e|}{\nu_e^p - u_\alpha\sqrt{2p}\nu_e^{p-\frac{1}{2}}}$$

where

$$(2.2)\qquad S_e = U'C_1U,$$

$$(2.3)\qquad \nu_e = r(V\vdots X) - r(X),$$

$$(2.4)\qquad C_1 = T^- - T^-X(X'T^-X)^-X'T,$$

$$(2.5)\qquad T = V + XMX',\ M = M' \quad \text{is such that} \quad R(X) \subset R(T),$$

u_α is obtained from the standard normal distribution $N(0,1)$ and given in

$$(2.6)\qquad p(-u_\alpha < u < u_\alpha) = 1 - \alpha$$

where $1-\alpha$ is the confidence.

Proof. Using (1.2), $\nu K = S_e$ and replacing Σ by $\sigma^2\Sigma$ we obtain directly

$$(2.7)\qquad P\left[-u_\alpha < \frac{\nu_e^{\frac{1}{2}}}{\sqrt{2p}}\left(\frac{|S_e|}{\nu_e^p|\sigma^2\Sigma|} - 1\right) < u_\alpha\right] = 1 - \alpha,$$

where $S_e \,\check{}\, W_p(\nu_e, \sigma^2\Sigma)$ (Oktaba and Kieloch [3]). By solving the inequality in (2.7) we get (2.1). □

Theorem 2.2. *In the MGM model $(U, XB, \sigma^2\Sigma \otimes V)$, the $(1-\alpha)$ percent asymptotically normal confidence interval for the determinant $|\sigma^2\Sigma|$, provided the hypothesis H_0; $L^*B = \psi$ is true, is of the form*

$$(2.8)\qquad \frac{|S_H|}{\nu_H^p + u_\alpha\sqrt{2p}\nu_H^{p-\frac{1}{2}}} < |\sigma^2\Sigma| < \frac{|S_H|}{\nu_H^p - u_\alpha\sqrt{2p}\nu_H^{p-\frac{1}{2}}},$$

where (Oktaba nad Kieloch [3]*)*

$$(2.9)\qquad S_H = (L^*\hat{B} - \psi)' L^- (L^*\hat{B} - \psi), \quad \hat{B} = (X'T^-X)^- X'T^-U,$$

$$(2.10)\qquad \nu_H = r(L), L = L^* C_4 L^*, C_4 = (X'T^-X)^- - M,$$

M and T being defined in (2.5).

Proof. We consider $\nu K = S_H$, replace Σ and ν by $\sigma^2\Sigma$ and ν_H, respectively. We know (Oktaba nad Kieloch [3]) that

$$S_H \sim W_p[r(L), \sigma^2\Sigma].$$

Applying Theorem 1.2 we get (2.8). □

Theorem 2.3. *In the MGM model* $(1-\alpha)$ *percent asymptotically normal confidence interval for the determinant* $|\sigma^2\Sigma|$ *can be presented as*

$$(2.11)\qquad \frac{|S_y|}{\nu_y^p + u_\alpha\sqrt{2p}\,\nu_y^{p-\frac{1}{2}}} < |\sigma^2\Sigma| < \frac{|S_y|}{\nu_y^p - u_\alpha\sqrt{2p}\,\nu_y^{p-\frac{1}{2}}},$$

where

$$(2.12)\qquad S_y = S_e + S_H, \quad \nu_y = \nu_e + \nu_H,$$

with S_e and S_H in (2.2) *and* (2.9)*, respectively and ν_e, ν_H in* (2.3) *and* (2.10). *u_α is defined in* (2.6).

Proof. By virtue of $S_e \sim W_p(\nu_e, \sigma^2\Sigma), S_H \sim W_p(\nu_H, \sigma^2\Sigma)$ (Oktaba and Kieloch [3]) and additivity of the Wishart distribution we state that

$$S_e + S_H = S_y \sim W_p(\nu_e + \nu_H, \sigma^2\Sigma).$$

Applying Theorem 1.2 we get (2.11) analogously as in the proofs of Theorems 2.1 and 2.2. □

Particular case (Anderson [1]). In the standard multivariate model $(U, \mathbf{1}\cdot\mu', \Sigma\otimes I_n)$ (cf. Oktaba [2]) the asymptotic normal confidence interval for the determinant $|\Sigma|$ can be obtained if we put: $X = \mathbf{1}$, $S_e = U'C_1U$ in formula (2.1), where $C_1 = I_n - \frac{1}{n}\mathbf{1}\mathbf{1}'$, $\sigma^2 = 1$, $\nu_e = n - \mathbf{1}$. The symbol $\mathbf{1}$ is a column vector with ones, I_n denotes the $n \times n$ identity, μ' is a row vector.

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